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On the Asymptotic Behavior of Solutions of Linear Parabolic Equations in L^1 Space

DONG GUN PARK - HIROKI TANABE (*)

The object of this paper is to investigate the asymptotic behavior of the solution of the initial-boundary value problem for the linear parabolic equation

$$(0.1) \quad \partial u / \partial t + A(x, t, D)u = f(x, t), \text{ in } \Omega \times [0, \infty),$$

$$(0.2) \quad B_j(x, t, D)u = 0, \quad j = 1, \dots, m/2, \text{ on } \partial\Omega \times [0, \infty),$$

$$(0.3) \quad u(x, 0) = u_0(x), \text{ on } \Omega,$$

in $L^1(\Omega)$ as $t \rightarrow \infty$.

This type of problem for an abstract parabolic evolution equation

$$(0.4) \quad du(t)/dt + A(t)u(t) = f(t)$$

was first treated in [9], and the convergence of the solution $u(t)$ to a stationary state was shown under the assumption that the domain $D(A(t))$ of $A(t)$ is independent of t . Pazy [8] established the asymptotic expansion of the solution of (0.4) assuming a certain asymptotic behavior of $A(t)$ and $f(t)$, and as its application he obtained the asymptotic expansion of the solution of the parabolic problem (0.1)-(0.3) in $L^p(\Omega)$, $1 < p < \infty$, in case when the boundary conditions (0.2) are independent of t .

Recently, Guidetti [4] extended the above results to the case when $D(A(t))$ and the boundary conditions (0.2) depend on time. We show that analogous results for the solution of (0.1)-(0.3) hold in $L^1(\Omega)$ using the method of [7], [11] of estimating the Green function of the problem considered.

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1. - Notations

Let Ω be a not necessarily bounded domain in $\mathbb{R}^n$ locally regular of class C^{2m} and uniformly regular of class C^m in the sense of Browder [2]. The boundary of Ω is denoted by $\partial\Omega$. We put

$$D = (\partial/\partial x_1, \dots, \partial/\partial x_n).$$

Let

$$A(x, t, D) = \sum_{|\alpha| \leq m} a_\alpha(x, t) D^\alpha$$

be a linear differential operator of even order m with coefficients defined in $\overline{\Omega}$ for each fixed $t \in [0, \infty)$, and let

$$B_j(x, t, D) = \sum_{|\beta| \leq m_j} b_{j,\beta}(x, t) D^\beta, \quad j = 1, \dots, \frac{m}{2},$$

be a set of linear differential operators of respective orders $m_j < m$ with coefficients defined on $\partial\Omega$ for each fixed $t \in [0, \infty)$.

The principal parts of $A(x, t, D)$ and $B_j(x, t, D)$ are denoted by $A^\#(x, t, D)$ and $B_j^\#(x, t, D)$ respectively.

Let k be a nonnegative integer. For $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ stands for the Banach space consisting of all measurable functions defined in Ω whose distribution derivatives of order up to k belong to $L^p(\Omega)$.

The norm of $W^{k,p}(\Omega)$ is defined by

$$\|u\|_{k,p} = \begin{cases} \left(\sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha u|^p dx \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max_{|\alpha| \leq k} \operatorname{ess\,sup}_{\Omega} |D^\alpha u| & \text{if } p = \infty. \end{cases}$$

We simply write $\|\cdot\|_p$ instead of $\|\cdot\|_{0,p}$ to denote L^p norm for $1 < p \leq \infty$.

We use the notation $\|\cdot\|$ to denote both the norm of $L^1(\Omega)$ and that of bounded linear operators from $L^1(\Omega)$ to itself.

We denote by $B^k(\overline{\Omega})$ the set of all functions which are bounded and uniformly continuous in $\overline{\Omega}$ together with their derivatives of order up to k . $B^k(\overline{\Omega})$ is a Banach space with norm

$$|u|_k = \max_{|\alpha| \leq k} \sup_{x \in \overline{\Omega}} |D^\alpha u(x)|.$$

For $0 < h < 1$, $B^{k+h}(\overline{\Omega})$ is the set of all functions in $B^k(\overline{\Omega})$ whose k th order derivatives are uniformly Hölder continuous of order h . The norm of $B^{k+h}(\overline{\Omega})$ is defined by

$$|u|_{k+h} = |u|_k + \max_{|\alpha| = k} \sup_{\substack{x, y \in \overline{\Omega} \\ x \neq y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^h}.$$

$B^k(\partial\Omega)$ denotes the Banach space consisting of all functions having bounded and uniformly continuous derivatives of order up to k on $\partial\Omega$.

$B^k(\partial\Omega)$ is a Banach space with norm

$$|u|_{k,\partial\Omega} = \max_{|\alpha| \leq k} \sup_{x \in \partial\Omega} |D^\alpha u(x)|.$$

We denote the set of all bounded linear operators from $L^p(\Omega)$ to $L^p(\Omega)$, $W^{m,p}(\Omega)$ by $B(L^p, L^p)$, $B(L^p, W^{m,p})$ respectively.

For a Banach space X we denote by $B^k(I : X)$ the set of all functions with values in X which are bounded and continuous in the interval I together with their derivatives of order up to k .

2. - Convergence as $t \rightarrow \infty$

We assume the following:

(I.1) $A(x, t, D)$ is uniformly strongly elliptic, i.e. there exists an angle $\theta_0 \in (0, \frac{\pi}{2})$ such that for all real vectors $\xi \neq 0$ and all $(x, t) \in \bar{\Omega} \times [0, \infty)$

$$|\arg(-1)^{m/2} A^\#(x, t, \xi)| < \theta_0.$$

(I.2) $\{B_j(x, t, D)\}_{j=1}^{m/2}$ is a normal set of boundary operators, i.e. $\partial\Omega$ is noncharacteristic for each $B_j(x, t, D)$ and $m_j \neq m_k$ for $j \neq k$.

(I.3) For any $(x, t) \in \partial\Omega \times [0, \infty)$ let ν be the normal to $\partial\Omega$ at x and $\xi \neq 0$ be parallel to $\partial\Omega$ at x . The polynomials in τ

$$B_j^\#(x, t, \xi + \tau\nu), \quad j = 1, \dots, m/2$$

are linearly independent modulo the polynomial in τ , $\prod_{j=1}^{m/2} (\tau - \tau_k^+(\xi, \lambda; x, t))$ for any complex number λ with $\theta_0 \leq \arg \lambda \leq 2\pi - \theta_0$ where $\tau_k^+(\xi, \lambda; x, t)$ are the roots with positive imaginary part of the polynomial in τ , $(-1)^{m/2} A^\#(x, t, \xi + \tau\nu) - \lambda$.

(I.4) For each $t \in [0, \infty)$ the formal adjoint of $A(x, t, D)$

$$A'(x, t, D) = \sum_{|\alpha| \leq m} a'_\alpha(x, t) D^\alpha$$

and the adjoint system of boundary operators

$$B'_j(x, t, D) = \sum_{|\beta| \leq m'_j} b'_{j,\beta}(x, t) D^\beta, \quad j = 1, \dots, \frac{m}{2},$$

can be constructed.

(I.5) For $|\alpha| = m$, $a_\alpha \in B^0(\bar{\Omega} \times [0, \infty))$. For $|\alpha| \leq m$, $a_\alpha, a'_\alpha \in B^1([0, \infty); L^\infty(\Omega))$, and

$$\lim_{t \rightarrow \infty} \|\dot{a}_\alpha(\cdot, t)\|_\infty = 0, \quad \lim_{t \rightarrow \infty} \|\dot{a}'_\alpha(\cdot, t)\|_\infty = 0,$$

where $\dot{a}_\alpha = \partial a_\alpha / \partial t$, $\dot{a}'_\alpha = \partial a'_\alpha / \partial t$.

(I.6) For $|\beta| \leq m_j$, $j = 1, \dots, \frac{m}{2}$, $b_{j,\beta} \in B^1([0, \infty); B^{m-m_j}(\partial\Omega))$, and

$$\lim_{t \rightarrow \infty} |\dot{b}_{j,\beta}(\cdot, t)|_{m-m_j, \partial\Omega} = 0.$$

Similarly, for $|\beta| \leq m'_j$, $j = 1, \dots, \frac{m}{2}$, $b'_{j,\beta} \in B^1([0, \infty); B^{m-m'_j}(\partial\Omega))$, and

$$\lim_{t \rightarrow \infty} |\dot{b}'_{j,\beta}(\cdot, t)|_{m-m'_j, \partial\Omega} = 0.$$

(II) For each $p \in (1, \infty)$ there exists a constant C_p such that for $t \in [0, \infty)$, $\theta_0 \leq \arg \lambda \leq 2\pi - \theta_0$, $u, v \in W^{m,p}(\Omega)$

$$(2.1) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|u\|_{j,p} \leq C_p \{ \| (A(\cdot, t, D) - \lambda)u \|_p + \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m} \|g_j\|_p + \sum_{j=1}^{m/2} \|g_j\|_{m-m_j,p} \},$$

$$(2.2) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|v\|_{j,p} \leq C_p \{ \| (A'(\cdot, t, D) - \lambda)v \|_p + \sum_{j=1}^{m/2} |\lambda|^{(m-m'_j)/m} \|h_j\|_p + \sum_{j=1}^{m/2} \|h_j\|_{m-m'_j,p} \},$$

where g_j and h_j are arbitrary functions in $W^{m-m_j,p}(\Omega)$ and $W^{m-m'_j,p}(\Omega)$ such that $B_j(x, t, D)u = g_j$ and $B'_j(x, t, D)v = h_j$ on $\partial\Omega$ respectively.

REMARK. It is known that under the hypothesis (I.1)-(I.6) the inequalities (2.1), (2.2) hold if we add some positive constant to $A(x, t, D)$ if necessary.

For $1 < p < \infty$ let $A_p(t)$ be the operator defined by

$$D(A_p(t)) = \{u \in W^{m,p}(\Omega) : B_j(x, t, D)u|_{\partial\Omega} = 0, j = 1, \dots, \frac{m}{2}\},$$

for $u \in D(A_p(t))$, $(A_p(t)u)(x) = A(x, t, D)u(x)$ in the distribution sense.

Similarly, the operator $A'_p(t)$ is defined by replacing $A(x, t, D)$ and $\{B_j(x, t, D)\}_{j=1}^{m/2}$ by $A'(x, t, D)$ and $\{B'_j(x, t, D)\}_{j=1}^{m/2}$.

From the assumptions above it follows that $-A_p(t)$, $-A'_p(t)$ generate analytic semigroups in $L^p(\Omega)$, and the resolvent sets $\rho(A_p(t))$, $\rho(A'_p(t))$ contain the closed sector

$$\Sigma = \{\lambda : \theta_0 \leq \arg \lambda \leq 2\pi - \theta_0\} \cup \{0\}.$$

The operator $A(t)$ is defined as follows:

The domain $D(A(t))$ is the totality of functions u satisfying the following three conditions:

- (i) $u \in W^{m-1,q}(\Omega)$ for any q with $1 \leq q < n/(n-1)$,
- (ii) $A(x, t, D)u \in L^1(\Omega)$ in the sense of distributions,
- (iii) for any p with $0 < (n/m)(1 - 1/p) < 1$ and any $v \in D(A'_p(t))$, $p' = p/(p-1)$,

$$(A(x, t, D)u, v) = (u, A'(x, t, D)v).$$

For $u \in D(A(t))$ $(A(t)u)(x) = A(x, t, D)u(x)$ in the distribution sense.

It is known that $-A(t)$ generates an analytic semigroup $\exp(-\tau A(t))$ in $L^1(\Omega)$ ([10], [11]). It can be shown without difficulty that for some positive constant c_0 the inequalities (2.1) and (2.2) hold if we replace $A(x, t, D)$ by $A(x, t, D) - c_0$ and C_p by some other constant. Hence, there exists a constant C_0 such that for $\tau > 0$, $0 \leq t < \infty$

$$(2.3) \quad \|\exp(-\tau A(t))\| \leq C_0 \exp(-c_0 \tau),$$

$$(2.4) \quad A(t)\exp(-\tau A(t))\| \leq C_0 \tau^{-1} \exp(-c_0 \tau).$$

Let $U(t, s)$ be the evolution operator of the evolution equation in $L^1(\Omega)$:

$$(2.5) \quad du(t)/dt + A(t)u(t) = f(t).$$

The existence of such an operator was shown in [6] and it is constructed as follows:

$$(2.6) \quad U(t, s) = \exp(-(t-s)A(t)) + W(t, s),$$

$$(2.7) \quad W(t, s) = \int_s^t \exp(-(t-\tau)A(t))R(\tau, s)d\tau,$$

$$(2.8) \quad R(t, s) = \int_s^t R_1(t, \tau)R(\tau, s)d\tau = R_1(t, s),$$

$$(2.9) \quad R_1(t, s) = -(\partial/\partial t + \partial/\partial s)\exp(-(t-s)A(t)).$$

Our first main result is the following:

THEOREM 2.1. *Suppose that the hypotheses (I.1)-(I.6), (II) are satisfied. Let $f(t)$ be a uniformly Hölder continuous functions with values in $L^1(\Omega)$ defined in $[0, \infty)$:*

$$\|f(t) - f(s)\| \leq C_1(t-s)^h, \quad 0 \leq s < t < \infty,$$

where C_1 and h are constants with $C_1 > 0$, $0 < h \leq 1$. Moreover, assume that the strong limit $f_0 = \lim_{t \rightarrow \infty} f(t)$ exists. Then, for any solution $u(t)$ of the evolution equation (2.5), we have

$$\lim_{t \rightarrow \infty} A(t)u(t) = f_0$$

in the strong topology of $L^1(\Omega)$.

Following the argument of [4] we can prove Theorem 2.1 with the aid of (2.3), (2.4) and the following lemma.

LEMMA 2.1. *For each fixed $s \geq 0$*

$$(2.10) \quad \lim_{t \rightarrow \infty} \|A(t)W(t, s)\| = 0$$

For any $\varepsilon > 0$ there exists a constant $s_0 \geq 0$ such that

$$(2.11) \quad \int_s^t \|A(t)W(t, \sigma)\| d\sigma < \varepsilon \quad \text{for } s_0 \leq s < t < \infty.$$

We plan to prove Lemma 2.1 as follows. First we note that

$$\begin{aligned} A(t)W(t, s) &= A(t) \int_s^t \exp(-(t-\tau)A(t)) R_1(\tau, s) d\tau \\ &\quad + \int_s^t A(t) \int_\sigma^t \exp(-(t-\tau)A(t)) R_1(\tau, \sigma) d\tau R(\sigma, s) d\sigma. \end{aligned}$$

If we have a desired estimate of $A(t)^\rho R_1(\tau, s)$ for some $0 < \rho < 1$, then we can write the first term of the right side of (2.12) as

$$\int_s^t A(t)^{1-\rho} \exp(-(t-\tau)A(t)) A(t)^\rho R_1(\tau, s) d\tau.$$

Let $W^{\theta,1}(\Omega) = (L^1(\Omega), W^{1,1}(\Omega))_{\theta,1}$ be the real interpolation space of $L^1(\Omega)$ and $W^{1,1}(\Omega)$ with norm denoted by $\|\cdot\|_{\theta,1}$.

Then, in view of Grisvard [3] we have for $0 < \theta < 1$

$$(2.13) \quad W^{\theta,1}(\Omega) = (L^1(\Omega), W_0^{m,1}(\Omega))_{\theta/m,1}.$$

It is easy to show that for $0 < \rho < \theta/m$

$$(2.14) \quad (L^1(\Omega), D(A(t)))_{\theta/m,1} \subset D(A(t)^\rho).$$

Clearly,

$$(2.15) \quad W_0^{m,1}(\Omega) \subset D(A(t)).$$

Combining (2.13), (2.14), (2.15) we get

$$(2.16) \quad W^{\theta,1}(\Omega) \subset D(A(t)^\rho).$$

Consequently, in order to establish an estimate of $\|A(t)^\rho R_1(\tau, s)\|$ it suffices to obtain that of $\|R_1(\tau, s)\|_{B(L^1, W^{\theta,1})}$ where $B(L^1, W^{\theta,1})$ is the set of all bounded linear operators from $L^1(\Omega)$ to $W^{\theta,1}(\Omega)$. Since

$$(2.17) \quad \|R_1(\tau, s)f\|_{\theta,1} \leq C\|R_1(\tau, s)f\|_{1,1}^\theta \|R_1(\tau, s)f\|^{1-\theta},$$

the problem is reduced to estimating $\|R_1(t, s)\|_{B(L^1, W^{\theta,1})}$ and $\|R(t, s)\|$ for $0 \leq s < t < \infty$. In view of (2.1) the desired result follows from the estimates of $\partial^2 G(x, y, \tau; t)/\partial x_i \partial t$, $\partial G(x, y, \tau; t)/\partial t$ where $G(x, y, \tau; t)$ is the kernel of $\exp(-\tau A(t))$.

3. - Proof of Lemma 2.1.

In what follows we let the notation C_p stand for constants depending only on the hypothesis (I.1)-(I.6), (II) and $p \in (1, \infty)$.

Arguing as in [7], [11] we see that for each $p \in (1, \infty)$ there exists a positive constant δ_p such that for each $t \in [0, \infty)$, $\lambda \in \Sigma$, a complex vector η with $|\eta| \leq \delta_p |\lambda|^{1/m}$ and $u, v \in W^{m,p}(\Omega)$

$$(3.1) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|u\|_{j,p} \leq C_p \{ \|(A(\cdot, t, D + \eta) - \lambda)u\|_p + \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m} \|g_j\|_p + \sum_{j=1}^{m/2} \|g_j\|_{m-m_j,p} \},$$

$$(3.2) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|v\|_{j,p} \leq C_p \{ \|(A'(\cdot, t, D + \eta) - \lambda)v\|_p + \sum_{j=1}^{m/2} |\lambda|^{(m-m'_j)/m} \|h_j\|_p + \sum_{j=1}^{m/2} \|h_j\|_{m-m'_j,p} \},$$

where g_j and h_j are arbitrary functions satisfying $B_j(x, t, D + \eta)u|_{\partial\Omega} = g_j$ and $B'_j(x, t, D + \eta)v|_{\partial\Omega} = h_j$ for $j = 1, \dots, \frac{m}{2}$.

We define the operator $A_p^\eta(t)$ by

$$D(A_p^\eta(t)) = \{u \in W^{m,p}(\Omega) : B_j(x, t, D + \eta)u|_{\partial\Omega} = 0, \quad j = 1, \dots, \frac{m}{2}\},$$

for $u \in D(A_p^\eta(t))$ $(A_p^\eta(t)u)(x) = A(x, t, D + \eta)u(x)$ in the sense of distributions.

Similarly replacing $A(x, t, D + \eta)$, $\{B_j(x, t, D + \eta)\}_{j=1}^{m/2}$ by $A'(x, t, D + \eta)$, $\{B'_j(x, t, D + \eta)\}_{j=1}^{m/2}$ the operator $A'_p{}^\eta(t)$ is defined.

It follows from (3.1), (3.2) that if $|\eta| \leq \delta_p |\lambda|^{1/m}$, $\lambda \in \Sigma$, then

$$(3.3) \quad \left\{ \begin{aligned} \|(A_p^\eta(t) - \lambda)^{-1}\|_{B(L^p, L^p)} \\ \|(A'_p{}^\eta(t) - \lambda)^{-1}\|_{B(L^p, L^p)} \end{aligned} \right\} \leq \frac{C_p}{|\lambda|},$$

$$(3.4) \quad \left\{ \begin{aligned} \|(A_p^\eta(t) - \lambda)^{-1}\|_{B(L^p, W^{m,p})} \\ \|(A'_p{}^\eta(t) - \lambda)^{-1}\|_{B(L^p, W^{m,p})} \end{aligned} \right\} \leq C_p.$$

Let $\omega(t)$ be a function defined in $[0, \infty)$ such that

$$\lim_{t \rightarrow \infty} \omega(t) = 0$$

$$\|\dot{a}_\alpha(\cdot, t)\|_\infty \leq \omega(t), \quad \|\dot{a}'_\alpha(\cdot, t)\|_\infty \leq \omega(t) \quad \text{for } |\alpha| \leq m,$$

$$|\dot{b}_{j,\beta}(\cdot, t)|_{m-m_j, \partial\Omega} \leq \omega(t) \quad \text{for } |\beta| \leq m_j, \quad j = 1, \dots, \frac{m}{2},$$

$$|\dot{b}'_{j,\beta}(\cdot, t)|_{m-m'_j, \partial\Omega} \leq \omega(t) \quad \text{for } |\beta| \leq m'_j, \quad j = 1, \dots, \frac{m}{2}.$$

Since the derivative $\dot{w} = \partial w / \partial t$ of the function $w(t) = (A_p^\eta(t) - \lambda)^{-1}f$ satisfies

$$(A(x, t, D + \eta) - \lambda)\dot{w}(x, t) = -\dot{A}(x, t, D + \eta)w(x, t) \quad x \in \Omega,$$

$$B_j(x, t, D + \eta)\dot{w}(x, t) = -\dot{B}_j(x, t, D + \eta)w(x, t), \quad j = 1, \dots, \frac{m}{2}, \quad x \in \partial\Omega,$$

it follows from (3.1), (3.3) that

$$(3.6) \quad \|(\partial/\partial t)(A_p^\eta(t) - \lambda)^{-1}\|_{B(L^p, L^p)} \leq \frac{C_p \omega(t)}{|\lambda|},$$

$$(3.7) \quad \|(\partial/\partial t)(A_p^\eta(t) - \lambda)^{-1}\|_{B(L^p, W^{m,p})} \leq C_p \omega(t)$$

for $|\eta| \leq \delta_p |\lambda|^{1/m}$, $\lambda \in \Sigma$. Similarly for those values of η , λ

$$(3.8) \quad \|(\partial/\partial t)(A'_p{}^\eta(t) - \lambda)^{-1}\|_{B(L^p, L^p)} \leq \frac{C_p \omega(t)}{|\lambda|},$$

$$(3.9) \quad \|(\partial/\partial t)(A'_p{}^\eta(t) - \lambda)^{-1}\|_{B(L^p, W^{m,p})} \leq C_p \omega(t).$$

We choose natural numbers ℓ , s and exponents $2 = q_1 < q_2 < \dots < q_s < q_{s+1} = \infty$, $2 = r_1 < r_2 < \dots < r_{\ell-s} < r_{\ell-s+1} = \infty$ as follows (Beals [1]):

- (i) in case $n/2m < 1 - 1/m$: $\ell = 2$ and $s = 1$, hence $2 = q_1 < q_2 = \infty$, $2 = r_1 < r_2 = \infty$, $q_1^{-1} < (m-1)/n$.
- (ii) in case $1 - 1/m \leq n/2m < 1$: $\ell = 3$ and $s = 2$, $2 = q_1 < q_2 < q_3 = \infty$, $q_1^{-1} - q_2^{-1} < m/n$, $q_2^{-1} < (m-1)/n$, $2 = r_1 < r_2 = \infty$.
- (iii) in case $n/2m > 1$: $s > n/2m + 1/m$, $\ell - s > n/2m$, $q_j^{-1} - q_{j+1}^{-1} < m/n$ for $j = 1, \dots, s-1$, $q_{s-1}^{-1} > m/n$, $q_s^{-1} < (m-1)/n$, $m - n/q_s$ is not a nonnegative integer, $r_j^{-1} - r_{j+1}^{-1} < m/n$ for $j = 1, \dots, \ell - s - 1$, $r_{\ell-s-1}^{-1} > m/n > r_{\ell-s}^{-1}$, $m - n/r_{\ell-s}$ is not a nonnegative integer.
- (iv) in case $n/2m = 1$: $\ell = 4$ and $s = 2$, $2 = q_1 < q_2 < q_3 = \infty$, $2 = r_1 < r_2 < r_3 = \infty$, $q_2^{-1} < 1/2 - 1/n = (m-1)/n$.

In what follows we consider only the case (iii).

According to Sobolev's imbedding theorem there exists a positive constant γ such that for $j = 1, \dots, s-1$

$$(3.10) \quad W^{m, q_j}(\Omega) \subset L^{q_{j+1}}(\Omega), \quad \|u\|_{q_{j+1}} \leq \gamma \|u\|_{m, q_j}^{a_j} \|u\|_{q_j}^{1-a_j},$$

where $0 < a_j = (n/m)(q_j^{-1} - q_{j+1}^{-1}) < 1$,

$$(3.11) \quad \begin{aligned} W^{m, q_s}(\Omega) &\subset B^{m-n/q_s}(\bar{\Omega}) \subset W^{1, \infty}(\Omega), \\ \|u\|_{1, \infty} &\leq \gamma \|u\|_{m, q_s}^{a_s+1/m} \|u\|_{q_s}^{1-a_s-1/m}, \end{aligned}$$

where $0 < a_s = (n/m)q_s^{-1} < 1 - 1/m$, and for $j = 1, \dots, \ell - s$

$$(3.12) \quad W^{m, r_j}(\Omega) \subset L^{r_{j+1}}(\Omega), \quad \|u\|_{r_{j+1}} \leq \gamma \|u\|_{m, r_j}^{a_{s+j}} \|u\|_{r_j}^{1-a_{s+j}},$$

where $0 < a_{s+j} = (n/m)(r_j^{-1} - r_{j+1}^{-1}) < 1$.

Let $\delta = \min\{\delta_p : p = q_1, \dots, q_s, r_1, \dots, r_{\ell-s}\}$. By virtue of (3.3), (3.4), (3.6), (3.7), (3.10), (3.11) for $\lambda \in \Sigma$ and $|\eta| \leq \delta|\lambda|^{1/m}$

$$(3.13) \quad \|(A_{q_j}^\eta(t) - \lambda)^{-1}\|_{B(L^{q_j}, L^{q_{j+1}})} \leq C|\lambda|^{a_j-1}, \quad j = 1, \dots, s-1,$$

$$(3.14) \quad \|(A_{q_s}^\eta(t) - \lambda)^{-1}\|_{B(L^{q_s}, W^{1, \infty})} \leq C|\lambda|^{a_s+1/m-1},$$

$$(3.15) \quad \|(\partial/\partial t)(A_{r_j}^\eta(t) - \lambda)^{-1}\|_{B(L^{r_j}, L^{r_{j+1}})} \leq C\omega(t)|\lambda|^{a_{s+j}-1}, \quad j = 1, \dots, s-1,$$

$$(3.16) \quad \|(\partial/\partial t)(A_{r_s}^\eta(t) - \lambda)^{-1}\|_{B(L^{r_s}, W^{1, \infty})} \leq C\omega(t)|\lambda|^{a_s+1/m-1}.$$

Similarly, by virtue of (3.3), (3.4), (3.8), (3.9), (3.12) we obtain

$$(3.17) \quad \|(A_{r_j}^\eta(t) - \lambda)^{-1}\|_{B(L^{r_j}, L^{r_{j+1}})} \leq C|\lambda|^{a_{s+j}-1},$$

$$(3.18) \quad \|(\partial/\partial t)(A_{r_j}^\eta(t) - \lambda)^{-1}\|_{B(L^{r_j}, L^{r_{j+1}})} \leq C\omega(t)|\lambda|^{a_{s+j}-1}$$

for $j = 1, \dots, \ell - s$.

As is easily seen

$$(3.19) \quad \exp(-\ell\tau A_2(t)) = (1/2\pi i)^\ell \int_{\Gamma} \dots \int_{\Gamma} e^{-\lambda_1\tau - \dots - \lambda_\ell\tau} \\ (A_2(t) - \lambda_1)^{-1} \dots (A_2(t) - \lambda_\ell)^{-1} d\lambda_1 \dots d\lambda_\ell,$$

where Γ is a smooth contour running in $\Sigma \setminus \{0\}$ from $\infty e^{-i\theta_0}$ to $\infty e^{i\theta_0}$.

For $\lambda_1, \dots, \lambda_\ell \in \Sigma$ and η with

$$(3.20) \quad |\eta| \leq \delta \min\{|\lambda_1|^{1/m}, \dots, |\lambda_\ell|^{1/m}\}$$

set

$$S(t) = (A_2^\eta(t) - \lambda_s)^{-1} \dots (A_2^\eta(t) - \lambda_1)^{-1}, \\ T(t) = (A_2^\eta(t) - \lambda_{s+1})^{-1} \dots (A_2^\eta(t) - \lambda_\ell)^{-1}.$$

Let $K_{\lambda_1, \dots, \lambda_\ell}^\eta(x, y; t)$ be the kernel of

$$S(t)T(t) = (A_2^\eta(t) - \lambda_1)^{-1} \dots (A_2^\eta(t) - \lambda_\ell)^{-1}.$$

Then $(\partial^2/\partial x_j \partial t) K_{\lambda_1, \dots, \lambda_\ell}^\eta(x, y; t)$ is the kernel of

$$D_j(d/dt)S(t) \cdot T(t) + D_jS(t) \cdot (d/dt)T(t).$$

By an elementary calculus

$$(3.21) \quad D_j(d/dt)S(t) = D_j(\partial/\partial t)(A_2^\eta(t) - \lambda_s)^{-1} \\ \cdot (A_2^\eta(t) - \lambda_{s-1})^{-1} \dots (A_2^\eta(t) - \lambda_1)^{-1} \\ + \dots + D_j(A_2^\eta(t) - \lambda_s)^{-1} \\ \dots (A_2^\eta(t) - \lambda_2)^{-1} (\partial/\partial t)(A_2^\eta(t) - \lambda_1)^{-1}.$$

With the aid of (3.13), (3.14), (3.15), (3.16)

$$\|D_j(\partial/\partial t)(A_2^\eta(t) - \lambda_s)^{-1} \cdot (A_2^\eta(t) - \lambda_{s-1})^{-1} \dots (A_2^\eta(t) - \lambda_1)^{-1}\|_{B(L^2, L^\infty)} \\ \leq \|(\partial/\partial t)(A_{q_s}^\eta(t) - \lambda_s)^{-1}\|_{B(L^q, W^{1, \infty})} \\ \cdot \prod_{j=1}^{s-1} \|(A_{q_j}^\eta(t) - \lambda_j)^{-1}\|_{B(L^{q_j}, L^{q_{j+1}})} \\ \leq C\omega(t)|\lambda_s|^{1/m} \prod_{j=1}^s |\lambda_j|^{q_j-1}.$$

Estimating other terms of the right side of (3.21) analogously we obtain

$$(3.22) \quad \|D_j(d/dt)S(t)\|_{B(L^2, L^\infty)} \leq C\omega(t)|\lambda_s|^{1/m} \prod_{j=1}^s |\lambda_j|^{q_j-1}.$$

Similarly we get

$$(3.23) \quad \|T^*(t)\|_{B(L^2, L^\infty)} \leq C \prod_{j=s+1}^{\ell} |\lambda_j|^{a_j-1},$$

$$(3.24) \quad \|D_j S(t)\|_{B(L^2, L^\infty)} \leq C |\lambda_s|^{1/m} \prod_{j=1}^s |\lambda_j|^{a_j-1},$$

$$(3.25) \quad \|(d/dt)T^*(t)\|_{B(L^2, L^\infty)} \leq C\omega(t) \prod_{j=s+1}^{\ell} |\lambda_j|^{a_j-1}.$$

Hence

$$(3.26) \quad |(\partial^2/\partial x_j \partial t) K_{\lambda_1, \dots, \lambda_\ell}^\eta(x, y; t)| \leq C |\lambda_s|^{1/m} \prod_{j=1}^{\ell} |\lambda_j|^{a_j-1}.$$

It is easy to show

$$(3.27) \quad |(\partial/\partial t) K_{\lambda_1, \dots, \lambda_\ell}^\eta(x, y; t)| \leq C\omega(t) \prod_{j=1}^{\ell} |\lambda_j|^{a_j-1}.$$

Let $K_{\lambda_1, \dots, \lambda_\ell}(x, y; t)$ be the kernel of $(A_2(t) - \lambda_1)^{-1} \dots (A_2(t) - \lambda_\ell)^{-1}$. Then as was shown in [7], [11]

$$(3.28) \quad K_{\lambda_1, \dots, \lambda_\ell}(x, y; t) = e^{(x-y)\eta} K_{\lambda_1, \dots, \lambda_\ell}^\eta(x, y; t).$$

With the aid of (3.20), (3.26), (3.27), (3.28) we obtain

$$|(\partial^2/\partial x_j \partial t) K_{\lambda_1, \dots, \lambda_\ell}(x, y; t)| \leq C\omega(t) e^{(x-y)\eta} |\lambda_s|^{1/m} \prod_{j=1}^{\ell} |\lambda_j|^{a_j-1}.$$

Minimizing the right side of the above inequality with respect to η we get (Hörmander [5])

$$(3.29) \quad \begin{aligned} & |(\partial^2/\partial x_j \partial t) K_{\lambda_1, \dots, \lambda_\ell}(x, y; t)| \\ & \leq C\omega(t) |\lambda_s|^{1/m} \prod_{j=1}^{\ell} |\lambda_j|^{a_j-1} \exp\{-\delta \min(|\lambda_1|^{1/m}, \dots, |\lambda_\ell|^{1/m}) |x-y|\} \\ & \leq C\omega(t) |\lambda_s|^{1/m} \prod_{j=1}^{\ell} |\lambda_j|^{a_j-1} \sum_{k=1}^{\ell} \exp(-\delta |\lambda_k|^{1/m} |x-y|) \end{aligned}$$

In view of (3.19) we have

$$(3.30) \quad \begin{aligned} & G(x, y, \ell\tau; t) \\ & = (1/2\pi i)^\ell \int_{\Gamma} \dots \int_{\Gamma} e^{-\lambda_1 \tau - \dots - \lambda_\ell \tau} K_{\lambda_1, \dots, \lambda_\ell}(x, y; t) d\lambda_1 \dots d\lambda_\ell. \end{aligned}$$

For any fixed $x, y \in \Omega, \tau > 0$ let $\Gamma_{x,y,\tau}$ be the contour defined by

$$\Gamma_{x,y,\tau} = \{\lambda : |\arg \lambda| = \theta_0, |\lambda| \geq a\} \cup \{\lambda : \lambda = ae^{i\theta}, \theta_0 \leq \theta \leq 2\pi - \theta_0\}$$

where $a = \varepsilon(|x - y|/\tau)^{m/(m-1)} = \varepsilon\rho/\tau$, $\rho = |x - y|^{m/(m-1)}/\tau^{1/(m-1)}$ and ε is a positive constant which will be fixed later. Differentiating both sides of (3.30) with respect to x_j and t , deforming the integral path Γ to $\Gamma_{x,y,\tau}$, and using (3.29) yield

$$\begin{aligned} & |(\partial^2/\partial x_j \partial t)G(x, y, \ell\tau; t)| \\ & \leq C\omega(t) \sum_{k=1}^{\ell} \int_{\Gamma_{x,y,\tau}} \dots \int_{\Gamma_{x,y,\tau}} e^{-Re\lambda_1\tau - \dots - Re\lambda_{\ell}\tau} |\lambda_s|^{1/m} \\ & \times \prod_{j=1}^{\ell} |\lambda_j|^{a_j-1} \exp(-\delta|\lambda_k|^{1/m}|x-y|) |d\lambda_1 \dots d\lambda_{\ell}|. \end{aligned}$$

Estimating the right side of the above inequality as in section 5 of [6] we conclude

$$(3.31) \quad |(\partial^2/\partial x_j \partial t)G(x, y, \tau; t)| \leq \frac{C\omega(t)}{\tau^{(n+1)/m}} \exp(-c \frac{|x-y|^{m/(m-1)}}{\tau^{1/(m-1)}}).$$

Similarly

$$(3.32) \quad |(\partial/\partial t)G(x, y, \tau; t)| \leq \frac{C\omega(t)}{\tau^{n/m}} \exp(-c \frac{|x-y|^{m/(m-1)}}{\tau^{1/(m-1)}}).$$

If we denote the kernel of $R_1(t, s)$ by $R_1(x, y, t, s)$, then in view of (2.9)

$$R_1(x, y, t, s) = -(\partial/\partial t)G(x, y, \tau; t)|_{\tau=t-s}.$$

By virtue of (3.31) and (3.32)

$$\begin{aligned} (3.33) \quad & |(\frac{\partial}{\partial x_j})R_1(x, y, t, s)| = |(\frac{\partial^2}{\partial x_j \partial t})G(x, y, \tau; t)|_{\tau=t-s}| \\ & \leq \frac{C\omega(t)}{(t-s)^{(n+1)/m}} \exp(-c \frac{|x-y|^{m/(m-1)}}{(t-s)^{1/(m-1)}}), \end{aligned}$$

$$(3.34) \quad |R_1(x, y, t, s)| \leq \frac{C\omega(t)}{(t-s)^{n/m}} \exp(-c \frac{|x-y|^{m/(m-1)}}{(t-s)^{1/(m-1)}}).$$

With the aid of (3.33), (3.34) we conclude

$$(3.35) \quad \left\| \left(\frac{\partial}{\partial x_j} \right) R_1(t, s) f \right\| \leq C \omega(t) (t-s)^{-1/m} \|f\|,$$

$$(3.36) \quad \|R_1(t, s) f\| \leq C \omega(t) \|f\|$$

for any $f \in L^1(\Omega)$.

We choose constants ρ and θ so that $0 < \rho < \theta/m$, $0 < \theta < 1$.

Combining (2.16), (2.17), (3.35), (3.36) we obtain

$$(3.37) \quad \|A(t)^\rho R_1(\tau, s)\| \leq C \omega(\tau) \{1 + (\tau-s)^{-1/m}\}^\theta.$$

By virtue of (2.8), (3.36) and Gronwall's inequality we get

$$(3.38) \quad \|R(t, s)\| \leq C \omega(t) \exp \left(C \int_s^t \omega(\tau) d\tau \right).$$

Using (3.37) and the inequality

$$\|A(t)^{1-\rho} \exp(-(t-\tau)A(t))\| \leq C(t-\tau)^{\rho-1}$$

we get

$$(3.39) \quad \begin{aligned} & \left\| A(t) \int_s^t \exp(-(t-\tau)A(t)) R_1(\tau, s) d\tau \right\| \\ &= \left\| \int_s^t A(t)^{1-\rho} \exp(-(t-\tau)A(t)) A(t)^\rho R_1(\tau, s) d\tau \right\| \\ &\leq C \int_s^t (t-\tau)^{\rho-1} \{1 + (\tau-s)^{-\theta/m}\} \omega(\tau) d\tau \\ &\leq C \{(t-s)^\rho + (t-s)^{\rho-\theta/m}\} \sup_{s \leq \tau \leq t} \omega(\tau). \end{aligned}$$

Making use of (3.37) and (3.39) yields

$$(3.40) \quad \begin{aligned} & \left\| \int_s^t A(t) \int_\sigma^t \exp(-(t-\tau)A(t)) R_1(\tau, \sigma) d\tau R(\sigma, s) d\sigma \right\| \\ &\leq C \int_s^t \{(t-\sigma)^\rho + (t-\sigma)^{\rho-\theta/m}\} \sup_{s \leq \tau \leq t} \omega(\tau) \omega(\sigma) \exp \left(C \int_s^\sigma \omega(\tau) d\tau \right) d\sigma \\ &\leq C \{(t-s)^{\rho+1} + (t-s)^{\rho-\theta/m+1}\} \sup_{s \leq \tau \leq t} \omega(\tau)^2 \exp \left(C \int_s^t \omega(\tau) d\tau \right). \end{aligned}$$

Combining (2.12), (3.39), (3.40) we get

$$\begin{aligned} \|A(t)W(t, s)\| &\leq C\{(t-s)^\rho + (t-s)^{\rho-\theta/m}\} \sup_{s \leq \tau \leq t} \omega(\tau) \\ &+ C\{(t-s)^{\rho+1} + (t-s)^{\rho-\theta/m+1}\} \sup_{s \leq \tau \leq t} \omega(\tau)^2 \exp \left(C \int_s^t \omega(\tau) d\tau \right) \end{aligned}$$

With the aid of (2.7) and (3.38) we get

$$(3.42) \quad \|W(t, s)\| \leq \{\exp (C \int_s^t \omega(\tau) d\tau) - 1\} \sup \|\exp (-\tau A(t))\|$$

$$(3.43) \quad \|U(t, s)\| \leq \exp (C \int_s^t \omega(\tau) d\tau) \sup \|\exp (-\tau A(t))\|.$$

As was mentioned in section 1 all the hypothesis (I.1)-(I.6), (II) are satisfied by $A(x, t, D) - c_0$, $\{B_j(x, t, D)\}_{j=1}^{m/2}$ for some $c_0 > 0$ if we replace C_p by some other constant. If we denote by $U^0(t, s)$, $W^0(t, s)$, $R_1^0(t, s)$, $R^0(t, s)$ operators obtained by replacing $A(t)$ by $A^0(t) = A(t) - c_0$ in the definition of $U(t, s)$, $W(t, s)$, $R_1(t, s)$, $R(t, s)$, then

$$\begin{aligned} U(t, s) &= U^0(t, s)e^{-c_0(t-s)}, \quad W(t, s) = W^0(t, s)e^{-c_0(t-s)}, \\ R_1(t, s) &= R_1^0(t, s)e^{-c_0(t-s)}, \quad R(t, s) = R^0(t, s)e^{-c_0(t-s)}, \end{aligned}$$

and (3.41), (3.42) hold with $A^0(t)$, $W^0(t, s)$ in place of $A(t)$, $W(t, s)$.

Hence

$$\begin{aligned} (3.44) \quad \|A(t)W(t, s)\| &= \|((A^0(t) + c_0)W^0(t, s)e^{-c_0(t-s)})\| \\ &\leq \|A^0(t)W^0(t, s)\|e^{-c_0(t-s)} + c_0\|W^0(t, s)\|e^{-c_0(t-s)} \\ &\leq C\{(t-s)^\rho + (t-s)^{\rho-\theta/m}\} \sup_{s \leq \tau \leq t} \omega(\tau)e^{-c_0(t-s)} \\ &+ C\{(t-s)^{\rho+1} + (t-s)^{\rho-\theta/m+1}\} \sup_{s \leq \tau \leq t} \omega(\tau)^2 \\ &\times \exp (C \int_s^t \omega(\tau) d\tau - c_0(t-s)) \\ &+ c_0C_0\{\exp (C \int_s^t \omega(\tau) d\tau) - 1\}e^{-c_0(t-s)}, \end{aligned}$$

which implies (2.10).

With the aid of (3.44) we have

$$\begin{aligned} \int_s^t \|A(t)W(t, \sigma)\| d\sigma &\leq C \int_s^t \{(t-\sigma)^\rho + (t-\sigma)^{\rho-\theta/m}\} e^{-c_0(t-\sigma)} d\sigma \sup_{s \leq \tau \leq t} \omega(\tau) \\ &+ C \int_s^t \{(t-\sigma)^{\rho+1} + (t-\sigma)^{\rho-\theta/m+1}\} \exp(C \int_s^t \omega(\tau) d\tau - c_0(t-\sigma)) d\sigma \sup_{s \leq \tau \leq t} \omega(\tau)^2 \\ &+ c_0 C_0 \int_s^t \{\exp(C \int_s^t \omega(\tau) d\tau) - 1\} e^{-c_0(t-\sigma)} d\sigma. \end{aligned}$$

Let ε be an arbitrary positive number. If s is so large that $\sup_{s \leq \tau \leq \infty} \omega(\tau) < \varepsilon$, then the right side of (3.45) does not exceed

$$\begin{aligned} C \int_0^\infty (\tau^\rho + \tau^{\rho-\theta/m}) e^{-c_0\tau} d\tau \varepsilon + C \int_0^\infty (\tau^{\rho+1} + \tau^{\rho-\theta/m+1}) e^{-(c_0-C\varepsilon)\tau} d\tau \varepsilon^2 \\ + c_0 C_0 \int_0^\infty (e^{-(c_0-C\varepsilon)\tau} - e^{-c_0\tau}) d\tau, \end{aligned}$$

from which the second half of the assertion of Lemma 2.1 follows.

Thus the proof of Lemma 2.1 is complete.

4. - Asymptotic expansion at $t = \infty$

In this section we consider the asymptotic expansion at $t = \infty$.

In addition to (I.1)-(I.6), (II) we make the following assumptions.

(III.1) For $|\alpha| \leq m$

$$(4.1) \quad a_\alpha(x, t) = \sum_{k=0}^{\nu} t^{-k} a_{\alpha,k}(x) + t^{-\nu} r_\alpha(x, t),$$

$$(4.2) \quad a'_\alpha(x, t) = \sum_{k=0}^{\nu} t^{-k} a'_{\alpha,k}(x) + t^{-\nu} r'_\alpha(x, t)$$

with $a_{\alpha,k}, a'_{\alpha,k} \in L^\infty(\Omega)$ for $k = 0, \dots, \nu$ and $r_\alpha, r'_\alpha \in B^1([0, \infty) : L^\infty(\Omega))$. If $|\alpha| = m$, $a_{\alpha,k} \in B^0(\bar{\Omega})$ and $r_\alpha \in B^0(\bar{\Omega} \times [0, \infty))$, and hence so do $a'_{\alpha,k}$ and r'_α .

(III.2) For $|\alpha| \leq m$

$$(4.3) \quad \lim_{t \rightarrow \infty} \|r_\alpha(\cdot, t)\|_\infty = 0, \quad \lim_{t \rightarrow \infty} \|\dot{r}_\alpha(\cdot, t)\|_\infty = 0,$$

$$(4.4) \quad \lim_{t \rightarrow \infty} \|r'_\alpha(\cdot, t)\|_\infty = 0, \quad \lim_{t \rightarrow \infty} \|\dot{r}'_\alpha(\cdot, t)\|_\infty = 0,$$

where $\dot{r}_\alpha = \partial r_\alpha / \partial t$, $\dot{r}'_\alpha = \partial r'_\alpha / \partial t$.

(III.3) For $|\beta| \leq m_j$, $j = 1, \dots, \frac{m}{2}$,

$$(4.5) \quad b_{j,\beta}(x, t) = \sum_{k=0}^{\nu} t^{-k} b_{j,\beta,k}(x) + \rho_{j,\beta}(x, t)$$

with $b_{j,\beta,k} \in B^{m-m_j}(\partial\Omega)$ for $k = 0, \dots, \nu$ and $\rho_{j,\beta} \in B^1([0, \infty); B^{m-m_j}(\partial\Omega))$.

For $|\beta| \leq m'_j$, $j = 1, \dots, \frac{m}{2}$,

$$(4.6) \quad b'_{j,\beta}(x, t) = \sum_{k=0}^{\nu} t^{-k} b'_{j,\beta,k}(x) + \rho'_{j,\beta}(x, t)$$

with $b'_{j,\beta,k} \in B^{m-m'_j}(\partial\Omega)$ for $k = 0, \dots, \nu$ and $\rho'_{j,\beta} \in B^1([0, \infty); B^{m-m'_j}(\partial\Omega))$.

(III.4) For $|\beta| \leq m_j$, $j = 1, \dots, \frac{m}{2}$,

$$(4.7) \quad \lim_{t \rightarrow \infty} |\rho_{j,\beta}(x, t)|_{m-m_j, \partial\Omega} = 0, \quad \lim_{t \rightarrow \infty} |\dot{\rho}_{j,\beta}(x, t)|_{m-m_j, \partial\Omega} = 0,$$

and for $|\beta| \leq m'_j$, $j = 1, \dots, \frac{m}{2}$,

$$(4.8) \quad \lim_{t \rightarrow \infty} |\rho'_{j,\beta}(\cdot, t)|_{m-m'_j, \partial\Omega} = 0, \quad \lim_{t \rightarrow \infty} |\dot{\rho}'_{j,\beta}(\cdot, t)|_{m-m'_j, \partial\Omega} = 0,$$

where $\dot{\rho}_{j,\beta} = \partial \rho_{j,\beta} / \partial t$, $\dot{\rho}'_{j,\beta} = \partial \rho'_{j,\beta} / \partial t$.

THEOREM 4.1. Suppose that the hypotheses (I.1)-(I.6), (II), (III.1)-(III.4) are satisfied. Let $f(t)$ be such that

$$f(t) = \sum_{k=0}^{\nu} t^{-k} f_k + t^{-\nu} r(t)$$

with $f_k \in L^1(\Omega)$ for $k = 0, \dots, \nu$ and $r \in B^0([0, \infty); L^1(\Omega))$, $\lim_{t \rightarrow \infty} \|r(t)\| = 0$.
Then for any mild solution of (1.5)

$$u(t) = \sum_{k=0}^{\nu} t^{-k} u_k + t^{-\nu} \rho(t)$$

with $u_k \in L^1(\Omega)$ for $k = 0, \dots, \nu$ and $\rho \in B^0([0, \infty); L^1(\Omega))$, $\lim_{t \rightarrow \infty} \|\rho(t)\| = 0$.

According to the argument of theorem 1.4 of [4] it suffices to show that

$$(4.9) \quad A(t)^{-1} = \sum_{k=0}^{\nu} t^{-k} T_k + t^{-\nu} R(t)$$

with $T_k, R(t) \in B(L^1, L^1)$ for $k = 0, \dots, \nu$ and $t \in [0, \infty)$, $\lim_{t \rightarrow \infty} R(t) = 0$, $\lim_{t \rightarrow \infty} (d/dt)R(t) = 0$ in the strong operator topology. Actually we shall prove this convergence in the uniform operator topology.

In what follows we assume that $b_{j,\beta,k}, \rho_{j,\beta}, b'_{j,\beta,k}, \rho'_{j,\beta}$ are extended to the whole of $\bar{\Omega}$ or $\bar{\Omega} \times [0, \infty)$ so that

$$\begin{aligned} |b_{j,\beta,k}|_{m-m_j} &\leq 2|b_{j,\beta,k}|_{m-m_j, \partial\Omega}, \\ |\rho_{j,\beta}(\cdot, t)|_{m-m_j} &\leq 2|\rho_{j,\beta}(\cdot, t)|_{m-m_j, \partial\Omega}, \\ |b'_{j,\beta,k}|_{m-m'_j} &\leq 2|b'_{j,\beta,k}|_{m-m'_j, \partial\Omega}, \\ |\rho'_{j,\beta}(\cdot, t)|_{m-m'_j} &\leq 2|\rho'_{j,\beta}(\cdot, t)|_{m-m'_j, \partial\Omega}. \end{aligned}$$

We put

$$\begin{aligned} A(x, D) &= \sum_{|\alpha| \leq m} a_{\alpha,0}(x) D^\alpha, & B_j(x, D) &= \sum_{|\beta| \leq m_j} b_{j,\beta,0}(x) D^\beta \\ A_k(x, D) &= \sum_{|\alpha| \leq m} a_{\alpha,k}(x) D^\alpha, & B_{j,k}(x, D) &= \sum_{|\beta| \leq m_j} b_{j,\beta,k}(x) D^\beta \end{aligned}$$

for $k = 1, \dots, \nu$, and

$$\tilde{A}(x, t, D) = \sum_{|\alpha| \leq m} r_\alpha(x, t) D^\alpha, \quad \tilde{B}_j(x, t, D) = \sum_{|\beta| \leq m_j} \rho_{j,\beta}(x, t) D^\beta.$$

Analogously, the operators $A'(x, D), A'_k(x, D), \tilde{A}'(x, t, D), B'_j(x, D), B'_{j,k}(x, D), \tilde{B}'_j(x, t, D)$ are defined.

It is obvious that (3.1) holds also for $A(x, D + \eta), \{B_j(x, D + \eta)\}_{j=1}^{m/2}$ in place of $A(x, t, D + \eta), \{B_j(x, t, D + \eta)\}_{j=1}^{m/2}$: for $\lambda \in \Sigma, \eta \in \mathbb{C}^n$ with $|\eta| \leq \delta_p |\lambda|^{1/m}$ and $u \in w^{m,p}(\Omega)$

$$(4.10) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|u\|_{j,p} \leq C_p \{ \| (A(x, D + \eta) - \lambda) u \|_p + \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m} \|g_j\|_p + \sum_{j=1}^{m/2} \|g_j\|_{m-m_j} \},$$

where g_j is an arbitrary function in $W^{m-m_j,p}(\Omega)$ satisfying $B_j(x, D + \eta)u|_{\partial\Omega} = g_j$ for each $j = 1, \dots, m/2$. This is the same for the inequality (3.2).

Let A_p be the operator defined by

$$D(A_p) = \{u \in W^{m,p}(\Omega); B_j(x, D)u|_{\partial\Omega} = 0, j = 1, \dots, \frac{m}{2}\},$$

$(A_p u)(x) = A(x, D)u(x)$ for $u \in D(A_p)$ in the distribution sense, and A_p^η be the operator defined analogously with $A(x, D + \eta)$ and $\{B_j(x, D + \eta)\}_{j=1}^{m/2}$ in place of $A(x, D)$ and $\{B_j(x, D)\}_{j=1}^{m/2}$. Similarly, the operator A'_p , A'^η_p are defined.

For $\lambda \in \sum \setminus \{0\}$, $\eta \in \mathbb{C}^n$ with $|\eta| \leq \delta_p |\lambda|^{1/m}$ and $f \in L^p(\Omega)$, $1 < p < \infty$, we put $v(t) = (A_p^\eta(t) - \lambda)^{-1}f$ and $v_0 = (A_p^\eta - \lambda)^{-1}f$. In view of (3.1) and (4.10)

$$(4.11) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|v(t)\|_{j,p} \leq C_p \|f\|_p,$$

$$(4.12) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|v_0\|_{j,p} \leq C_p \|f\|_p.$$

We define a finite sequence of functions v_i , $i = 1, \dots, \nu$, successively as the solutions of the following boundary value problems:

$$(A(x, D + \eta) - \lambda)v_i(x) = - \sum_{k=0}^{i-1} A_{i-k}(x, D + \eta)v_k(x), \quad x \in \Omega$$

$$B_j(x, D + \eta)v_i(x) = - \sum_{k=0}^{i-1} B_{j,i-k}(x, D + \eta)v_k(x), \quad j = 1, \dots, \frac{m}{2}, \quad x \in \partial\Omega.$$

Since the functions v_i , $i = 1, \dots, \nu$, are uniquely determined by f , we may denote them as

$$v_i = H_{i,\lambda,p}^\eta f, \quad i = 1, \dots, \nu.$$

We put

$$H_{0,\lambda,p}^\eta = (A_p^\eta - \lambda)^{-1}, \quad H_{0,\lambda,p} = (A_p - \lambda)^{-1},$$

$$R_{\lambda,p}^\eta(t) = t^\nu (A_p^\eta(t) - \lambda)^{-1} - \sum_{i=0}^{\nu} t^{\nu-i} H_{i,\lambda,p}^\eta, \quad R_{\lambda,p}(t) = R_{\lambda,p}^0(t).$$

Clearly

$$(4.13) \quad (A_p^\eta(t) - \lambda)^{-1} = \sum_{i=0}^{\nu} t^{-i} H_{i,\lambda,p}^\eta + t^{-\nu} R_{\lambda,p}^\eta(t).$$

Applying (4.10) to v_i yields

$$\begin{aligned} \sum_{j=0}^m |\lambda|^{(m-j)/m} \|v_i\|_{j,p} &\leq C_p \left\{ \left\| \sum_{k=0}^{i-1} A_{i-k}(x, D + \eta) v_k \right\|_p \right. \\ &\quad + \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m} \left\| \sum_{k=0}^{i-1} B_{j,i-k}(x, D + \eta) v_k \right\|_p \\ &\quad \left. + \sum_{j=1}^{m/2} \left\| \sum_{k=0}^{i-1} B_{j,i-k}(x, D + \eta) v_k \right\|_{m-m_j,p} \right\} \\ &\leq C_p \sum_{i=0}^{i-1} \sum_{j=0}^m |\lambda|^{(m-j)/m} \|v_k\|_{j,p}. \end{aligned}$$

It follows from (4.12) and the above inequality that

$$(4.14) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|H_{i,\lambda,p}^\eta f\|_{j,p} \leq C_p \|f\|_p, \quad i = 0, \dots, \nu.$$

We put

$$\omega_1(t) = \max \{ \|r_\alpha(\cdot, t)\|_\infty, \|\dot{r}_\alpha(\cdot, t)\|_\infty, |\rho_{j,\beta}(\cdot, t)|_{m-m_j}, \\ |\dot{\rho}_{j,\beta}(\cdot, t)|_{m-m_j}; |\alpha| \leq m, |\beta| \leq m_j, j = 1, \dots, \frac{m}{2} \}.$$

$$\omega_2(t) = \max \{ \|r'_\alpha(\cdot, t)\|_\infty, \|\dot{r}'_\alpha(\cdot, t)\|_\infty, |\rho'_{j,\beta}(\cdot, t)|_{m-m'_j}, \\ |\dot{\rho}'_{j,\beta}(\cdot, t)|_{m-m'_j}; |\alpha| \leq m, |\beta| \leq m'_j, j = 1, \dots, \frac{m}{2} \}.$$

An elementary calculus yields

$$\begin{aligned} (A(x, t, D + \eta) - \lambda)(t^\nu v(x, t) - \sum_{i=0}^\nu t^{\nu-i} v_i(x)) \\ = -\tilde{A}(x, t, D + \eta) v_0(x), \quad x \in \Omega \\ B_j(x, t, D + \eta)(t^\nu v(x, t) - \sum_{i=0}^\nu t^{\nu-i} v_i(x)) \\ = -\tilde{B}_j(x, t, D + \eta) v_0(x), \quad j = 1, \dots, \frac{m}{2}, \quad x \in \partial\Omega. \end{aligned}$$

Hence, applying (4.10) to

$$R_{\lambda,p}^\eta(t)f = t^\nu v(t) - \sum_{i=0}^\nu t^{\nu-i} v_i \quad \text{and} \quad (\partial/\partial t) R_{\lambda,p}^\eta(t)f,$$

we easily get

$$(4.15) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|R_{\lambda,p}^\eta(t)f\|_{j,p} \leq C_p \omega_1(t) \|f\|_p,$$

$$(4.16) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|(\partial/\partial t)R_{\lambda,p}^\eta(t)f\|_{j,p} \leq C_p \omega_1(t) \|f\|_p.$$

The inequality (4.15) implies

$$(4.17) \quad \lim_{t \rightarrow \infty} \|R_{\lambda,p}^\eta(t)f\|_{m,p} = 0, \quad \lim_{t \rightarrow \infty} \|(\partial/\partial t)R_{\lambda,p}^\eta(t)f\|_{m,p} = 0.$$

Similarly, replacing $(A, \{B_j\})$ by its adjoint $(A', \{B'_j\})$ we define operators $H_{i,\lambda,p}^{\prime\eta}$, $i = 0, \dots, \nu$, and $R_{\lambda,p}^{\prime\eta}(t)$ so that

$$(A_p^{\prime\eta}(t) - \lambda)^{-1} = \sum_{i=0}^{\nu} t^{-\nu} H_{i,\lambda,p}^{\prime\eta} + t^{-\nu} R_{\lambda,p}^{\prime\eta}(t).$$

We obtain

$$(4.18) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|H_{i,\lambda,p}^{\prime\eta} f\|_{j,p} \leq C_p \|f\|_p, \quad i = 0, \dots, \nu,$$

$$(4.19) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|R_{\lambda,p}^{\prime\eta}(t)f\|_{j,p} \leq C_p \omega_2(t) \|f\|_p,$$

$$(4.20) \quad \sum_{j=0}^m |\lambda|^{(m-j)/m} \|(\partial/\partial t)R_{\lambda,p}^{\prime\eta}(t)f\|_{j,p} \leq C_p \omega_2(t) \|f\|_p.$$

Since $((A_p^\eta(t) - \lambda)^{-1})^* = (A_{p'}^{\prime\bar{\eta}}(t) - \bar{\lambda})^{-1}$ we see that

$$(4.21) \quad (H_{i,\lambda,p}^\eta)^* = H_{i,\bar{\lambda},p'}^{\prime\bar{\eta}}, \quad (R_{\lambda,p}^\eta(t))^* = R_{\bar{\lambda},p'}^{\prime\bar{\eta}}(t).$$

We first establish the asymptotic expansion of the kernels of the semigroup $\exp(-\tau A_2(t))$ at $t = \infty$. We choose natural numbers ℓ , s and exponents $2 = q_1 < q_2 < \dots < q_s < q_{s+1} = \infty$, $2 = r_1 < r_2 < \dots < r_{\ell-s} < r_{\ell-s+1} = \infty$ as in [7], [11] (Beals [1]) i.e.

- (i) in case $m > n/2$. $\ell = 2$ and $s = 1$, hence $2 = q_1 < q_2 = \infty$ and $2 = r_1 < r_2 = \infty$;
- (ii) in case $m < n/2$. $s > n/2m$, $\ell - s > n/2m$, $q_j^{-1} - q_{j+1}^{-1} < m/n$ for $j = 1, \dots, s-1$, $q_{s-1}^{-1} > m/n > q_s^{-1}$, $m - n/q_s$ is not a nonnegative integer, $r_j^{-1} - r_{j+1}^{-1} < m/n$ for $j = 1, \dots, \ell-s-1$, $r_{\ell-s-1}^{-1} > m/n > r_{\ell-s}^{-1}$, $m - n/r_{\ell-s}$ is not a nonnegative integer;

- (iii) in case $m = n/2$. $\ell = 4$, $s = 2$, $2 = q_1 < q_2 < q_3 = \infty$, $2 = r_1 < r_2 < r_3 = \infty$.

In what follows we only consider the case (ii). According to Sobolev's imbedding theorem the inequalities (3.10) and (3.12) hold for $j = 1, \dots, s$ and $j = 1, \dots, \ell - s$ respectively.

Put $\delta = \min\{\delta_p; p = q_1, \dots, q_s, r_1, \dots, r_{\ell-s}\}$. Let $\lambda_1, \dots, \lambda_\ell \in \Sigma \setminus \{0\}$ and η be a complex n -vector satisfying

$$(4.22) \quad |\eta| \leq \delta \min\{|\lambda_1|^{1/m}, \dots, |\lambda_\ell|^{1/m}\}.$$

In view of (4.13)

$$\begin{aligned} (4.23) \quad & (A_2^\eta(t) - \lambda_1)^{-1} \dots (A_2^\eta(t) - \lambda_\ell)^{-1} \\ &= (A_2^\eta(t) - \lambda_s)^{-1} \dots (A_2^\eta(t) - \lambda_1)^{-1} (A_2^\eta(t) - \lambda_{s+1})^{-1} \dots \\ & \quad \dots (A_2^\eta(t) - \lambda_\ell)^{-1} \\ &= \sum_{i_1, \dots, i_\ell=0}^{\nu} t^{-i_1 - \dots - i_\ell} H_{i_s, \lambda_s, 2}^\eta \dots H_{i_1, \lambda_1, 2}^\eta H_{i_{s+1}, \lambda_{s+1}, 2}^\eta \dots \\ & \quad \dots H_{i_\ell, \lambda_\ell, 2}^\eta + t^{-\nu} R_{\lambda_1, \dots, \lambda_\ell, 2}^\eta(t), \end{aligned}$$

where $R_{\lambda_1, \dots, \lambda_\ell, 2}^\eta(t)$ is the sum of terms which contain at least one of $R_{\lambda_k, 2}^\eta(t)$'s as a factor. With the aid of (4.14) and (3.10) we see that

$$(4.24) \quad \|H_{i_k, \lambda_k, q_k}^\eta f\|_{q_{k+1}} \leq C |\lambda_k|^{a_k-1} \|f\|_{q_k}, \quad k = 1, \dots, s.$$

Hence

$$\begin{aligned} (4.25) \quad & \|H_{i_s, \lambda_s, 2}^\eta \dots H_{i_1, \lambda_1, 2}^\eta\|_{B(L^2, L^\infty)} \\ & \leq \prod_{k=1}^s \|H_{i_k, \lambda_k, q_k}^\eta\|_{B(L^{q_k}, L^{q_{k+1}})} \leq C \prod_{k=1}^s |\lambda_k|^{a_k-1}. \end{aligned}$$

Analogously

$$\begin{aligned} (4.26) \quad & \|(H_{i_{s+1}, \lambda_{s+1}, 2}^\eta \dots H_{i_\ell, \lambda_\ell, 2}^\eta)^*\|_{B(L^2, L^\infty)} \\ &= \|H_{i_\ell, \lambda_\ell, 2}^{\prime - \bar{\eta}} \dots H_{i_{s+1}, \lambda_{s+1}, 2}^{\prime - \bar{\eta}}\|_{B(L^2, L^\infty)} \\ & \leq C \prod_{k=s+1}^{\ell} |\lambda_k|^{a_k-1}. \end{aligned}$$

Therefore, if we denote the kernel of

$$H_{i_s, \lambda_s, 2}^\eta \dots H_{i_1, \lambda_1, 2}^\eta H_{i_{s+1}, \lambda_{s+1}, 2}^\eta \dots H_{i_\ell, \lambda_\ell, 2}^\eta$$

by $M^n(x, y)$, then in view of (4.25), (4.26) and Lemma 2 of [1]

$$(4.27) \quad |M^n(x, y)| \leq C \prod_{k=1}^{\ell} |\lambda_k|^{a_k-1}$$

As was shown in [7], [11] if η is pure imaginary, then

$$(4.28) \quad (A_2 - \lambda)^{-1} = e^{\cdot \eta} (A_2^n - \lambda)^{-1} e^{-\cdot \eta}$$

$$(4.29) \quad (A_2(t) - \lambda)^{-1} = e^{\cdot \eta} (A_2^n(t) - \lambda)^{-1} e^{-\cdot \eta}$$

(if Ω is bounded, η need not be pure imaginary). Since

$$H_{1,\lambda,2}^\eta = \lim_{t \rightarrow \infty} \{(A_2^n(t) - \lambda)^{-1} - (A_2^n - \lambda)^{-1}\},$$

it follows from (4.28), (4.29) that $H_{1,\lambda,2} = e^{\cdot \eta} H_{1,\lambda,2}^\eta e^{-\cdot \eta}$.

Similarly we obtain

$$H_{i,\lambda,2} = e^{\cdot \eta} H_{i,\lambda,2}^\eta e^{-\cdot \eta}, \quad i = 2, \dots, \nu, \quad R_{\lambda,2}(t) = e^{\cdot \lambda} R_{\lambda,2}^\eta e^{-\cdot \eta}.$$

Hence, arguing as in [7], [11] if we denote the kernel of

$$H_{i_s, \lambda_s, 2} \dots H_{i_1, \lambda_1, 2} H_{i_{s+1}, \lambda_{s+1}, 2} \dots H_{i_\ell, \lambda_\ell, 2}$$

by $M(x, y)$, then we have

$$(4.30) \quad M(x, y) = e^{(x-y)\eta} M^n(x, y)$$

for any complex n -vector satisfying (4.22). Combining (4.27), (4.30) and arguing as in section 4 of [7] we get

$$(4.31) \quad |M(x, y)| \leq C \prod_{k=1}^{\ell} |\lambda_k|^{a_k-1} \exp \{-\delta \min(|\lambda_1|^{1/m}, \dots, |\lambda_\ell|^{1/m}) |x - y|\}.$$

Analogously, if we denote the kernel of $R_{\lambda_1, \dots, \lambda_\ell}^\eta(t)$ by $\tilde{M}(x, y; t)$ we get

$$(4.32) \quad |\tilde{M}(x, y; t)|, \quad |(\partial/\partial t)\tilde{M}(x, y; t)| \\ \leq C\tilde{\omega}(t) \prod_{k=1}^{\ell} |\lambda_k|^{a_k-1} \exp \{-\delta \min(|\lambda_1|^{1/m}, \dots, |\lambda_\ell|^{1/m}) |x - y|\},$$

where $\tilde{\omega}(t) = \max \{\omega_1(t), \omega_2(t), t^{-1}\}$.

It follows from (4.23), (4.31), (4.32) that

$$(4.33) \quad (A_2(t) - \lambda_1)^{-1} \dots (A_2(t) - \lambda_\ell)^{-1} = (A_2 - \lambda_1)^{-1} \dots (A_2 - \lambda_\ell)^{-1} \\ + \sum t^{-i} H_{i, \lambda_1, \dots, \lambda_\ell, 2} + t^{-\nu} R_{\lambda_1, \dots, \lambda_\ell, 2}(t),$$

where $H_{i,\lambda_1,\dots,\lambda_\ell,2}$, $R_{\lambda_1,\dots,\lambda_\ell,2}(t)$ are operators with kernels $K_{i,\lambda_1,\dots,\lambda_\ell}(x,y)$, $\tilde{K}_{\lambda_1,\dots,\lambda_\ell}(x,y;t)$ satisfying

$$(4.34) \quad |K_{i,\lambda_1,\dots,\lambda_\ell}(x,y)| \leq C \prod_{k=1}^{\ell} |\lambda_k|^{a_k-1} \exp \{-\delta \min(|\lambda_1|^{1/m}, \dots, |\lambda_\ell|^{1/m})|x-y|\},$$

$$(4.35) \quad |\tilde{K}_{\lambda_1,\dots,\lambda_\ell}(x,y;t)|, |(\partial/\partial t)\tilde{K}_{\lambda_1,\dots,\lambda_\ell}(x,y;t)| \leq C\tilde{\omega}(t) \prod_{k=1}^{\ell} |\lambda_k|^{a_k-1} \exp\{-\delta \min(|\lambda_1|^{1/m}, \dots, |\lambda_\ell|^{1/m})|x-y|\}.$$

If we denote by $K_{\lambda_1,\dots,\lambda_\ell}(x,y;t)$, $K_{\lambda_1,\dots,\lambda_\ell}(x,y)$ the kernels of $(A_2(t) - \lambda_1)^{-1} \dots (A_2(t) - \lambda_\ell)^{-1}$, $(A_2 - \lambda_1)^{-1} \dots (A_2 - \lambda_\ell)^{-1}$, then in view of (4.33)

$$(4.36) \quad K_{\lambda_1,\dots,\lambda_\ell}(x,y;t) = K_{\lambda_1,\dots,\lambda_\ell}(x,y) + \sum_{i=1}^{\nu} t^{-i} K_{i,\lambda_1,\dots,\lambda_\ell}(x,y) + t^{-\nu} \tilde{K}_{\lambda_1,\dots,\lambda_\ell}(x,y;t).$$

We denote the kernels of the operators $\exp(-\tau A_2(t))$, $\exp(-\tau A_2)$,

$$(1/2\pi i)^{\ell} \int_{\Gamma} \dots \int_{\Gamma} \exp(-\lambda_1 \tau - \dots - \lambda_{\ell} \tau) H_{i,\lambda_1,\dots,\lambda_{\ell},2} d\lambda_1 \dots d\lambda_{\ell},$$

$$(1/2\pi i)^{\ell} \int_{\Gamma} \dots \int_{\Gamma} \exp(-\lambda_1 \tau - \dots - \lambda_{\ell} \tau) R_{\lambda_1,\dots,\lambda_{\ell},2}(t) d\lambda_1 \dots d\lambda_{\ell}$$

by $G(x,y,\tau;t)$, $G(x,y,\tau)$, $G_i(x,y,\ell\tau)$, $\tilde{G}(x,y,\ell\tau;t)$. Then by virtue of the equality (3.19) and that with A_2 in place of $A_2(t)$ we have

$$(4.37) \quad G(x,y,\tau;t) = G(x,y,\tau) + \sum_{i=1}^{\nu} t^{-i} G_i(x,y,\tau) + t^{-\nu} \tilde{G}(x,y,\tau;t).$$

Arguing as in [7], [11] we get

$$\left. \begin{aligned} &|G(x,y,\tau;t)| \\ &|G(x,y,\tau)| \\ &|G_i(x,y,\tau)| \end{aligned} \right\} \leq \frac{C}{|\tau|^{n/m}} \exp\left(-c \frac{|x-y|^{m/(m-1)}}{|\tau|^{1/(m-1)}}\right)$$

$$\left. \begin{aligned} &|\tilde{G}(x,y,\tau;t)| \\ &|(\partial/\partial t)\tilde{G}(x,y,\tau;t)| \end{aligned} \right\} \leq \frac{C\tilde{\omega}(t)}{|\tau|^{n/m}} \exp\left(-c \frac{|x-y|^{m/(m-1)}}{|\tau|^{1/(m-1)}}\right)$$

for τ in the region $|\arg \tau| < \frac{\pi}{2} - \theta_0$.

Let $K_\lambda(x, y; t)$, $K_\lambda(x, y)$ be the kernels of $(A_2(t) - \lambda)^{-1}$, $(A_2 - \lambda)^{-1}$, and put

$$K_{i,\lambda}(x, y) = \int_0^\infty e^{\lambda\tau} G_i(x, y, \tau) d\tau, \quad \tilde{K}_\lambda(x, y; t) = \int_0^\infty e^{\lambda\tau} \tilde{G}(x, y, \tau; t) d\tau.$$

With the aid of the argument of [7], [11] we obtain

$$(4.38) \quad \begin{aligned} & |K_\lambda(x, y; t)|, |K_\lambda(x, y)|, |K_{i,\lambda}(x, y)| \\ & \leq C \exp(-c|\lambda|^{1/m}|x-y|) \\ & \times \begin{cases} |x-y|^{m-n} & \text{if } m < n, \\ |\lambda|^{n/m-1} & \text{if } m > n, \\ (1 + \log^+(|\lambda|^{-1/m}|x-y|^{-1})) & \text{if } m = n. \end{cases} \end{aligned}$$

$$(4.39) \quad \begin{aligned} & |\tilde{K}_\lambda(x, y; t)|, |(\partial/\partial t)\tilde{K}_\lambda(x, y; t)| \\ & \leq C\tilde{\omega}(t)\exp(-c|\lambda|^{1/m}|x-y|) \\ & \times \begin{cases} |x-y|^{m-n} & \text{if } m < n, \\ |\lambda|^{n/m-1} & \text{if } m > n, \\ (1 + \log^+(|\lambda|^{-1/m}|x-y|^{-1})) & \text{if } m = n. \end{cases} \end{aligned}$$

From (4.13) with $\eta = 0$ and the equality

$$K_\lambda(x, y; t) = K_\lambda(x, y) + \sum_{i=1}^{\nu} t^{-i} K_{i,\lambda}(x, y) + t^{-\nu} \tilde{K}_\lambda(x, y; t),$$

it follows that $K_{i,\lambda}(x, y)$, and $K_\lambda(x, y; t)$ are the kernels of $H_{i,\lambda,p}$ and $R_{\lambda,p}(t)$ respectively for $1 < p < \infty$. If we define the operators $H_{i,\lambda,1}$, $R_{\lambda,1}(t)$ as integral operators in $L^1(\Omega)$ with kernels $K_{i,\lambda}(x, y)$, $K_\lambda(x, y; t)$, then

$$(4.40) \quad (A_1(t) - \lambda)^{-1} = (A_1 - \lambda)^{-1} + \sum_{i=1}^{\nu} t^{-i} H_{i,\lambda,1} + t^{-\nu} R_{\lambda,1}(t).$$

In view of (4.39) we have

$$\lim_{t \rightarrow \infty} \|R_{\lambda,1}(t)\| = 0, \quad \lim_{t \rightarrow \infty} \|(\partial/\partial t)R_{\lambda,1}(t)\| = 0.$$

Since the above argument remains valid if we replace $A(x, t, D)$ by $A(x, t, D) - c_0$ for some $c_0 > 0$ (section 1), the expansion (4.40) also holds for $\lambda = 0$. Thus the proof of (4.9) is complete.

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