ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

GEORGE MALTESE
GERD NIESTEGGE

A linear Radon-Nikodym type theorem for C^*-algebras with applications to measure theory

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A Linear Radon-Nikodym Type Theorem for C^*-Algebras with Applications to Measure Theory

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0. - Introduction

In a previous paper [10] (see also [11]) the second author defined the notion of absolute continuity for (non-normal) bounded linear forms on C^*-algebras and proved a non-commutative Radon-Nykodym type theorem which generalized the quadratic version of S. Sakai [13]. Here in section 1 we give an extension of Sakai's *linear version* [13] in the context of C^*-algebras. As in [10] the normality of the functionals in question need *not* be assumed and Sakai's condition of strong domination is here replaced by absolute continuity. In contrast to our linear version, the quadratic version of [10] is valid only for *positive* functionals. In commutative C^*-algebras both linear and quadratic versions (essentially) coincide.

Section 2 is devoted to applications of our abstract results to measure theory. We show that the classical Lebesgue-Radon-Nikodym theorem as well as its generalization to finitely additive measures due to S. Bochner [1] and C. Fefferman [7] can be obtained as direct consequences of our results applied to a certain commutative C^*-algebra $B(\Omega, \Sigma)$.

1. - The linear Radon-Nikodym type theorem for C^*-algebras

Let A be a C^*-algebra with positive part A_+ and unit ball S. Let f be a positive bounded linear functional and g an arbitrary bounded linear functional on A. g is said to be *absolutely continuous* with respect to f, if one of the following equivalent conditions is fulfilled (see [10]):

(i) For every $\varepsilon > 0$ there exits $\delta > 0$ such that $|g(x)| < \varepsilon$ whenever $x \in A_+ \cap S$ and $f(x) < \delta$.

(ii) For every sequence $\{x_n\}$ in $A_+ \cap S$ with $\lim_{n\to\infty} f(x_n) = 0$, it follows that $\lim_{n\to\infty} g(x_n) = 0$.

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For $y \in A$, the linear functional $x \rightarrow f(yx + xy)/2$ $(x \in A)$ is denoted by f_y. Since f is continuous, f_y is continuous and $\|f_y\| \leq \|f\| \; \|y\|$. If y is self-adjoint, f_y is self-adjoint (i.e. $f_y(x^*) = \overline{f_y(x)}; x \in A$); but f_y need not be positive, if y is positive.

LEMMA (1.1) *For $y \in A$ and $x \in A_+$ we have the following inequality:*

$$|f_y(x)| \leq \|f\|^{1/2}\|y\|\|x\|^{1/2}f(x)^{1/2}.$$

PROOF. From the Cauchy-Schwarz inequality for positive functionals it follows that

$$\begin{aligned} |f_y(x)| &= \frac{1}{2}|f(yx+xy)| \leq \frac{1}{2}[|f(yx)| + |f(xy)|] \\ &= \frac{1}{2}[|f((yx^{1/2})x^{1/2})| + |f(x^{1/2}(x^{1/2}y))|] \\ &\leq \frac{1}{2}[f(yxy^*)^{1/2}f(x)^{1/2} + f(x)^{1/2}f(y^*xy)^{1/2}] \\ &\leq \|f\|^{1/2}\|y\|\|x\|^{1/2}f(x)^{1/2}. \end{aligned}$$

From Lemma (1.1) we immediately obtain the following:

LEMMA (1.2) *For $y \in A, f_y$ is absolutely continuous with respect to f.*

Since the set of all bounded linear functionals on A which are absolutely continuous with respect to f is a closed linear subspace of the topological dual space A^*, each element of the closure of the set $\{f_y : y \in A\}$ is absolutely continuous with respect to f. Now we will show that the converse is also valid.

THEOREM (1.3) *Let f be a positive bounded linear functional and g an arbitrary bounded linear functional on the C^*-algebra A.*

(i) *g is absolutely continuous with respect to f, if and only if there exits a sequence $\{y_n\}$ in A such that*

(1) $$\lim_{n\to\infty} \|g - f_{y_n}\| = 0.$$

(ii) *If g is self-adjoint and absolutely continuous with respect to f, the y_n in (1) can be chosen self-adjoint.*

(iii) *If g is positive and absolutely continuous with respect to f, the y_n in (1) can be chosen positive.*

(iv) *If $0 \leq g \leq f$, the y_n in (1) can be chosen such that $y_n \in A_+ \cap S$.*

Before proceeding to the proof we need to recall some pertinent facts. The second dual A^{**} of the C^*-algebra A is an (abstract) W^*-algebra in a natural manner (with the Arens multiplication). Moreover, A is a $\sigma(A^{**}, A^*)$-dense C^*-subalgebra of A^{**}, when it is canonically embedded into A^{**}, and the continuous linear functionals (positive linear functionals) on A coincide precisely with the restrictions of the normal linear functionals (positive normal functionals) on A^{**} to A. The image of $g, f \in A^*$ under the canonical embedding $A^* \to A^{***}$ will again be denoted by g resp. f. (See [5], [8] and in particular [13]).

In [10], (Lemma (2.2)) it is shown that g is absolutely continuous with respect to f if and only if the image of g under the canonical embedding $A^* \to A^{***}$ is absolutely continuous with respect to the canonical image of f. For this reason we need not distinguish between $g, f \in A^*$ and their canonical images in A^{***}. These facts are very important for the following proof of the theorem.

PROOF of (iv). Let $0 \leq g \leq f$. We consider the set

$$K := \{f_y : y \in A_+ \cap S\}.$$

K is a non-empty convex subset of the dual space A^*. Let $\overline{K}$ be its closure in the norm topology on A^*, and suppose that $g \notin \overline{K}$.

From the Hahn-Banach theorem it follows that there exist $a \in A^{**}$ and $\gamma \in \mathbb{R}, \gamma < 1$, such that

$$\operatorname{Re} g(a) = 1, \operatorname{Re} f_y(a) \leq \gamma \text{ for all } y \in A_+ \cap S.$$

Choose $b := (a + a^*)/2 \in A^{**}$. Then, since g and $f_y (y \in A_+)$ are self-adjoint:

$$\begin{aligned} g(b) &= \operatorname{Re} g(a) = 1, \\ f_b(y) &= f_y(b) = \operatorname{Re} f_y(a) \leq \gamma \text{ for all } y \in A_+ \cap S. \end{aligned}$$

Since f and the mappings $x \to bx$, and $x \to xb$ are $\sigma(A^{**}, A^*)$-continuous on A^{**} (see [13]), f_b is $\sigma(A^{**}, A^*)$-continuous on A^{**}. From Kaplansky's density theorem it follows that $A_+ \cap S$ is $\sigma(A^{**}, A^*)$-dense in the positive part of the unit ball of A^{**} (see [10] Lemma (2.1)). Therefore

$$f_b(y) \leq \gamma \text{ for all } y \in A^{**} \text{ with } y \geq 0 \text{ and } \|y\| \leq 1.$$

The self-adjoint element b has an orthogonal decomposition $b = b^+ - b^-$, where $b^+, b^- \in A^{**}; b^+, b^- \geq 0$ and $b^+ b^- = 0 = b^- b^+$.

Let $q \in A^{**}$ be the support of b^+. Then

$$1 > \gamma \geq f_b(q) = f(bq + qb)/2 = f(b^+) \geq g(b^+) \geq g(b) = 1.$$

This is the desired contradiction.

PROOF of (iii). Let $g \geq 0$ be absolutely continuous with respect to f and consider the set

$$M := \{f_y : y \in A_+\}.$$

M is a non-empty convex cone in the dual space A^*. Let $\overline{M}$ be its closure in the norm topology and suppose that $g \notin \overline{M}$.

As in the above proof of (iv) there exists a self-adjoint element $b \in A^{**}$ such that

$$g(b) = 1,$$
$$f_b(y) = f_y(b) \leq 0 \text{ for all } y \in A_+.$$

Since A_+ is $\sigma(A^{**}, A^*)$-dense in the positive part of the W^*-algebra A^{**} and since f_b is $\sigma(A^{**}, A*)$-continuous, we obtain

$$f_b(y) \leq 0 \text{ for all } y \geq 0, y \in A^{**}.$$

Let $b^+, -b^- \in A^{**}$ be the positive and the negative part of b and let $q \in A^{**}$ be the support of b^+. Then

$$0 \geq f_b(q) = f(bq + qb)/2 = f(b^+) \geq 0, \text{ thus } f(b^+) = 0.$$

From the absolute continuity we conclude that $g(b^+) = 0$.

Finally we obtain the following contradiction:

$$1 = g(b) = g(b^+) - g(b^-) = -g(b^-) \leq 0.$$

PROOF of (ii). Let g be self-adjoint and absolutely continuous with respect to f and consider the set

$$L := \{f_y : y \in A_h\},$$

where A_h denotes the self-adjoint (= hermitian) part of A.

L is a real-linear subspace of A^*. Let $\overline{L}$ be its norm closure and suppose that $g \notin \overline{L}$. Again, as above, there exits a self-adjoint element $b \in A^{**}$ such that

$$g(b) = 1; f_b(y) = f_y(b) = 0 \text{ for all } y \in A_h.$$

Since A_h is $\sigma(A^{**}, A^*)$-dense in the self adjoint part of A^{**} and since f_b is $\sigma(A^{**}, A^*)$- continuous, we have

$$f_b(y) = 0 \text{ for all self-adjoint } y \in A^{**}.$$

Let $b = b^+ - b^-$ be the orthogonal decomposition of b in A^{**}, and let q, p be the supports of b^+, b^- in A^{**}. Then

$$0 = f_b(q) = f(bq + qb)/2 = f(b^+);$$
$$0 = f_b(p) = f(bp + pb)/2 = -f(b^-).$$

From the absolute continuity it follows that

$$g(b^+) = g(b^-) = 0.$$

Thus $g(b) = g(b^+) - g(b^-) = 0$. This contradicts the fact that $g(b) = 1$.

PROOF of (i). The fact that condition (1) implies the absolute continuity of g with respect to f follows from Lemma (1.2). The converse is obtained by applying part (ii) to the real and imaginary parts of g.

In the sequel let A be a W^*-algebra with predual A_*. The linear version of S. Sakai's Radon-Nikodym theorem is an immediate consequence of our Theorem (1.3).

COROLLARY (1.4) (S. Sakai) *Let g, f be positive linear functionals on the W^*-algebra A, where f is normal and $g \leq f$. Then there exists $y_0 \in A, 0 \leq y_0 \leq 1$, such that*

$$g(x) = \frac{1}{2} f(y_0 x + x y_0) \qquad (x \in A).$$

PROOF. From Theorem (1.3) (iv) it follows that there is a sequence $\{y_n\}$ in $A_+ \cap S$, such that

$$\lim_{n \to \infty} \|g - f_{y_n}\| = 0.$$

Since $A_+ \cap S$ is $\sigma(A, A_*)$-compact and since the mapping $y \to f_y$ from A with the $\sigma(A, A_*)$-topology to A^* with the $\sigma(A^*, A)$-topology is continuous, the set

$$K := \{f_y : y \in A_+ \cap S\}$$

is a $\sigma(A^*, A)$-compact subset of A^*. Therefore K is closed in the $\sigma(A^*, A)$-topology and hence in the norm topology on A^*.

From formula (1) of Theorem (1.3) we conclude that $g \in K$; i.e., there is $y_0 \in A_+ \cap S$ such that

$$g(x) = f_{y_0}(x) = \frac{1}{2} f(y_0 x + x y_0) \qquad (x \in A).$$

REMARK. In Corollary (1.4) the element y_0 can be chosen such that $0 \leq y_0 \leq s(f)$, where $s(f)$ is the support of the positive normal functional f. (If need be one can replace the y_0 of Corollary (1.4) by $s(f) y_0 s(f)$.) With this additional restraint y_0 is uniquely determined as we shall prove below. In particular if f is faithful (i.e., $s(f) = 1$), then the y_0 of Corollary (1.4) is uniquely determined.

To show the uniqueness of y_0, let $y_0, y_1 \in A$ be such that $f_{y_0} = f_{y_1} = g$ and $0 \leq y_0 \leq s(f), 0 \leq y_1 \leq s(f)$. Then

$$0 = f_{y_0} - f_{y_1} = f_{y_0 - y_1};$$

$$0 = f_{y_0 - y_1}(y_0 - y_1) = f((y_0 - y_1)^2).$$

Let q be the support of $(y_0 - y_1)^2$; then $f(q) = 0$ (see [14] 5.15), and therefore

$$q \leq 1 - s(f).$$

On the other hand, since $0 \leq y_0, y_1 \leq s(f)$, it follows for $i = 0, 1$ that:

$$0 \leq (1 - s(f))y_i(1 - s(f)) \leq (1 - s(f))s(f)(1 - s(f)) = 0.$$

$$\begin{aligned}\Rightarrow 0 &= (1 - s(f))y_i(1 - s(f)) \\ &= [y_i^{1/2}(1 - s(f))]^*[y_i^{1/2}(1 - s(f))].\end{aligned}$$

$$\Rightarrow 0 = y_i^{1/2}(1 - s(f)).$$

$$\Rightarrow 0 = y_i(1 - s(f)).$$

Then

$$0 = (y_0 - y_1)^2(1 - s(f))$$

$$\Rightarrow q \leq s(f).$$

Thus

$$q = 0, \text{ and hence } (y_0 - y_1)^2 = 0; \text{ i.e., } y_0 = y_1.$$

In Corollary (1.4) we have not required that g be normal. This follows automatically from $0 \leq g \leq f$, when f is normal. It is, in fact, the case that a positive linear functional g is normal if it is absolutely continuous with respect to a positive normal functional f.

The above Theorem (1.3) should be compared with Theorem (2.6) from [10]; in the commutative case they coincide for the most part. But the linear version (1.3) has two advantages: it provides an equivalent characterization of absolute continuity, and the "smaller" functional g need not be positive. It is for this reason that we prefer the linear version for the measure theoretical applications in the next section. However, the quadratic version (2.6) from [10] seems to be more suitable for applications to operator algebras (see section 3 of [10] for a variety of such applications including new proofs of two classical results in the theory of von Neumann algebras due to J. von Neumann and R. Pallu de la Barriére).

2. - Applications to addittive set fuctions

Let Ω be an arbitrary set and let $B(\Omega)$ be the algebra (pointwise operations) of all bounded complex-valued functions on Ω. $B(\Omega)$ is a commutative C^*-algebra for the sup norm $\| \|_\infty$.

Now let Σ be a field of subsets of Ω. The linear combinations of characteristic functions of sets in Ω are called *primitive functions*. The set of all primitive functions is a subalgebra of $B(\Omega)$; it is denoted by $P(\Omega, \Sigma)$. The closure of $P(\Omega, \Sigma)$ in $B(\Omega)$ is a C^*-subalgebra of $B(\Omega)$ and will be denoted by $B(\Omega, \Sigma)$. If Σ is a σ-field, $B(\Omega, \Sigma)$ consists of all bounded measurable complex-valued functions on (Ω, Σ). $B(\Omega) = B(\Omega, \Sigma_0)$, where Σ_0 is the family of *all* subsets of Ω.

The dual space of $B(\Omega, \Sigma)$ is isometrically isomorphic to the Banach space $\text{ba}(\Omega, \Sigma)$ which consists of all bounded (finitely) additive complex set functions on Σ; the norm $\| \ \|_v$ on $\text{ba}(\Omega, \Sigma)$ is given by the total variation. The isomorphism is defined as follows: every $f \in B(\Omega, \Sigma)^*$ is mapped onto $\mu_f \in \text{ba}(\Omega, \Sigma)$ such that the following equation is fulfilled:

$$f(x) = \int x \ \mathrm{d}\mu_f \qquad (x \in B(\Omega, \Sigma)).$$

This isomorphism preserves order; and f is self-adjoint if and only if μ_f is real-valued.

On the linear space $\text{ba}(\Omega, \Sigma)$ a second norm $\| \ \|_\infty$ can be introduced:

$$\|\mu\|_\infty := \sup_{E \in \Sigma} |\mu(E)|.$$

These norms are equivalent: $\| \ \|_\infty \leq \| \ \|_v \leq 4\| \ \|_\infty$.

The notion of absolute continuity for measures (= countably additive set functions) is extended to (finitely) additive set functions in the following way (see [1], [2], [6], [7]):

DEFINITION (2.1) Let $\nu, \mu \in \text{ba}(\Omega, \Sigma)$, $\mu \geq 0$. Then ν is said to be absolutely continuous with respect to μ, if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\mu(E) < \delta$ for $E \in \Sigma$ implies that $|\nu(E)| < \varepsilon$.

REMARKS 2.2 *Let* $\nu, \mu \in \text{ba}(\Omega, \Sigma)$, *and* $\mu \geq 0$.

(i) ν *is absolutely continuous with respect to* ν, *iff for every sequence* $\{E_n\}$ *in* $\Sigma, \lim \mu(E_n) = 0$ *implies* $\lim \nu(E_n) = 0$.

(ii) ν *is absolutely continuous with respect to* μ, *iff the variation,* $|\nu|$, *is absolutely continuous with respect to* μ.

(iii) *Let* Σ *be* a σ*-field and let* ν, μ *be countably addittive*; *then* ν *is absolutely continuous with respect to* μ, *iff* $\mu(E) = 0$ *for* $E \in \Sigma$ *implies that* $\nu(E) = 0$. (For the proofs see [6] chap.III.)

The following proposition illustrates the relationship between absolutely continuous functionals on a C^*-algebra and absolutely continuous set functions.

PROPOSITION (2.3) *Let* g, f *be bounded linear functionals on the* C^**-algebra* $B(\Omega, \Sigma)$ *and suppose that* $f \geq 0$. *Then* g *is absolutely continuous with respect to* f, *iff* μ_g *is absolutely continuous with respect to* μ_f.

PROOF. The necessity of the condition is obvious, since the characteristic functions are positive elements of $B(\Omega, \Sigma)$ of norm 1. To prove the sufficiency let μ_g be absolutely continuous with respect to μ_f; then the variation $|\mu_g|$ is absolutely continuous with respect to μ_f as well.

Since $P(\Omega, \Sigma)$ is dense in $B(\Omega, \Sigma)$, it is sufficient to consider only primitive functions. Let $\{x_n\}$ be a sequence in $P(\Omega, \Sigma)$ with $0 \leq x_n \leq 1$ and $\lim f(x_n) = 0$. We will show: $\lim g(x_n) = 0$. Let $\varepsilon > 0$. Since x_n is primitive, the sets $E_n := \{t \in \Omega : x_n(t) \geq \varepsilon\}$ are elements of Σ. From $f \geq 0$ it follows that for every $n \in \mathbb{N}$:

$$f(x_n) \geq f(\varepsilon \chi_{E_n}) = \varepsilon \mu_f(E_n) \geq 0,$$

where χ_{E_n} denotes the characteristic function of E_n.

Therefore

$$\lim_{n \to \infty} \mu_f(E_n) = 0.$$

Since $|\mu_g|$ is absolutely continuous with respect to μ_f, it follows that

$$\lim_{n \to \infty} |\mu_g|(E_n) = 0.$$

Thus there is an $n_0 \in \mathbb{N}$ such that $|\mu_g|(E_n) < \varepsilon$ for all $n \geq n_0$, and since $0 \leq x_n \leq 1$, we get for all $n \geq n_0$:

$$\begin{aligned} |g(x_n)| &= |\int_\Omega x_n \ \mathrm{d}\mu_g| \leq \int_\Omega x_n \ \mathrm{d}|\mu_g| = \int_{E_n} x_n \ \mathrm{d}|\mu_g| + \int_{E_n^c} x_n \ \mathrm{d}|\mu_g| \\ &\leq |\mu_g|(E_n) + \varepsilon|\mu_g|(E_n^c) < \varepsilon + \varepsilon \|\mu_g\|_v = \varepsilon(1 + \|\mu_g\|_v), \end{aligned}$$

where E_n^c denotes the complement. Hence $\lim g(x_n) = 0$.

Next we apply our Theorem (1.3) to the C^*-algebra $B(\Omega, \Sigma)$ and obtain a generalization of the classical Lebesgue-Radon-Nikodym theorem for (finitely) additive set functions due to S. Bochner [1].

THEOREM (2.4) *Let Ω be a set, and let Σ be a field of subsets of Ω. Let $\nu, \mu \in \mathrm{ba}(\Omega, \Sigma)$ be such that μ is positive and ν is absolutely continuous with respect to μ.*

(i) *Then there is a sequence $\{y_n\}$ of primitive functions on Ω such that:*

(1) $\nu(E) = \lim_{n \to \infty} \int_E y_n \mathrm{d}\mu$ *uniformly for* $E \in \Sigma$

(2) $\lim_{n,m \to \infty} \int_\Omega |y_n - y_m| \ \mathrm{d}\mu = 0.$

(ii) *If ν is real-valued (positive), the y_n in* (i) *can be chosen as real-valued (non- negative) primitive functions.*

PROOF. We consider the commutative C^*-algebra $B(\Omega, \Sigma)$ and the linear functionals $g := \int \mathrm{d}\nu, f := \int \mathrm{d}\mu$. By Proposition (2.3) g is absolutely continuous with respect to f, and we can therefore apply our Theorem (1.3). Thus there exists a sequence $\{y'_n\}$ in $B(\Omega, \Sigma)$ such that

$$\lim_{n \to \infty} \|g - f_{y'_n}\| = 0.$$

Since $P(\Omega, \Sigma)$ is dense in $B(\Omega, \Sigma)$, we can find $y_n \in P(\Omega, \Sigma)$ such that $\|y_n - y'_n\|_\infty < \frac{1}{n}$ $(n \in \mathbb{N})$. Then

$$\begin{aligned} \|g - f_{y_n}\| &\leq \|g - f_{y'_n}\| + \|f_{y'_n} - f_{y_n}\| \\ &\leq \|g - f_{y'_n}\| + \|f\| \cdot \|y'_n - y_n\|_\infty \\ &\leq \|g - f_{y'_n}\| + \frac{\|f\|}{n} \end{aligned}$$

and hence we conclude that

$$\lim_{n\to\infty} \|g - f_{y_n}\| = 0. \tag{3}$$

For $E \in \Sigma$ we have

$$\nu(E) = g(\chi_E) = \lim_{n\to\infty} f_{y_n}(\chi_E) = \lim_{n\to\infty} \int_E y_n \, d\mu. \tag{1}$$

Moreover from (3) we have

$$\begin{aligned} 0 &= \lim_{n,m\to\infty} \|f_{y_n} - f_{y_m}\| = \lim_{n,m\to\infty} \|(y_n - y_m)\mu\|_v \\ &= \lim_{n,m\to\infty} \int_\Omega |y_n - y_m| \, d\mu. \end{aligned} \tag{2}$$

(2) implies uniform convergence for $E \in \Sigma$ in (1).

(ii) follows in the same way from Theorem (1.3) parts (ii) and (iii).

Finally let Σ be a σ-field and let $\nu, \mu \in \mathrm{ba}(\Omega, \Sigma)$ be countably additive, where μ is positive and ν is absolutely continuous with respect to μ. Then the space $L^1(\mu)$ is complete and from (2) it follows that there is an $h \in L^1(\mu)$ such that:

$$0 = \lim_{n\to\infty} \int_\Omega |h - y_n| \, d\mu.$$

Hence $\nu(E) = \lim_{n\to\infty} \int_E y_n \, d\mu = \int_E h \, d\mu$ for all $E \in \Sigma$, wich is the classical Lebesgue-Radon-Nikodym theorem for finite measures.

REMARKS. (a) Different proofs of Theorem (2.4) may be found in [1], [6], or [7]. C. Fefferman generalizes this theorem in [7] for an arbitrary (not necessarily positive) $\mu \in \mathrm{ba}(\Omega, \Sigma)$.

(b) In this section we have applied our results to the C^*-algebra $B(\Omega, \Sigma)$. Similarly we could consider the commutative C^*-algebra $C_0(T)$ consisting of all continuous complex-valued functions on some locally compact Hausdorff space T which vanish at infinity; but since the continuous functionals on $C_0(T)$ correspond presisely to the regular complex Borel measures on T, we would obtain the Lebesgue-Radon-Nikodym theorem only for regular measures, whereas the example $B(\Omega, \Sigma)$ leads us to much more general results.

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Matematisches Institut
der Universität Münster
Einsteinstraße 62
D-4400 Münster

Siemens AG
Hofmannstr. 51
D-8000 München 70