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Existence and multiplicity results for a semilinear elliptic eigenvalue problem


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1. - Introduction

The following eigenvalue problem will be considered:

\[ \begin{aligned}
-\Delta u &= \lambda f(u) \quad \text{in } \Omega \subseteq \mathbb{R}^N, \\
u &= 0 \quad \text{on } \partial\Omega = \Gamma
\end{aligned} \]  

for \( \lambda > 0 \). The domain \( \Omega \) is assumed to be bounded and to have a smooth boundary of class \( C^3 \).

The function \( f \) will satisfy appropriate smoothness conditions. A positive solution of (P) will be a pair \((\lambda, u)\) in \( \mathbb{R}^+ \times C^2(\Omega) \) satisfying (P) with \( u > 0 \) in \( \Omega \). We shall call \( u \) a solution of \((P_\lambda)\).

It is a consequence of the strong maximum principle, see [2], that if such a solution exists, then \( f(\max u) \) is positive. The main goal of this paper is to study positive solutions having their maximum close to a zero of \( f \). Therefore we assume:

(F1) there are two numbers \( \rho_1 \) and \( \rho_2 \) such that \( \rho_1 < \rho_2 \), \( 0 < \rho_2 \),

\[ f(\rho_1) = f(\rho_2) = 0 \quad \text{and} \quad f > 0 \quad \text{in} \quad (\rho_1, \rho_2) \]

In [13] Hess proves the existence of solutions \((\lambda, u)\) of (P), satisfying \( \max u \in (\rho_1, \rho_2) \), when \( f(0) > 0 \) under the following condition:

(F2) \[ J(\rho) = \int_0^{\rho_2} f(s)ds > 0 \quad \text{for every} \quad \rho \in [0, \rho_2). \]

In Theorem 1 we prove that (F2) is a necessary and sufficient condition for the existence of such a solution even without the condition \( f(0) \geq 0 \).
THEOREM 1. Let $f \in C^1$ satisfy (F1). Then problem (P) possesses a positive solution $(\lambda, u)$, with $\max u \in (\rho_1, \rho_2)$, if and only if (F2) holds.

Theorem 1 improves a result of De Figueiredo in [10], since it does not use the inheritance condition or even the starshapedness of $\Omega$.

It also answers a question of Dancer in [9].

Next to this existence result we will prove a uniqueness result for positive solutions having their maximum close to $\rho_2$. We need the following condition:

(F3) there exists an $\varepsilon > 0$ such that $f' \leq 0$ in $(\rho_2 - \varepsilon, \rho_2)$.

THEOREM 2. Let $f \in C^{1,\gamma}$, for some $\gamma \in (0, 1)$, satisfy (F1), (F2) and (F3). Let $\gamma \in C^3$. Then there are $\lambda_0 > 0$ and a nonnegative function $z_0 \in C_0^\infty(\Omega)$ with $\max z_0 \in (\rho_1, \rho_2)$, such that for all $\lambda > \lambda_0$, $(P_\lambda)$ possesses exactly one solution $u_\lambda$ with $z_0 < u_\lambda < \rho_2$.

Moreover, $\lim_{\lambda \to \infty} \max u_\lambda = \rho_2$.

REMARKS.

1. We will state and prove a sharper version of this theorem in Section 4 (Theorem 2').

2. If $\rho_1 < 0$, or $\rho_1 = 0$ and $f'(0) > 0$, Theorem 2 was proved in a recent paper, [3], by Angenent. For $\rho_1 \leq 0$ there are also related results in [8].

3. If $\rho_1 = 0$ and $f'(0) = 0$, Rabinowitz showed in [19] the existence of pairs of solutions for $\lambda$ large enough by a degree argument.

When $\rho_1 = 0$ and $f'(0) = 0$ the question arises, whether or not there are exactly two positive solutions of $(P_\lambda)$, with maximum less than $\rho_2$, for $\lambda$ large enough. We shall consider this problem only for $\Omega = B$, the unit ball in $\mathbb{R}^N$.

It is known, [12], that positive solutions for $\Omega = B$ are radially symmetric, and can be parametrized by $u(0)$. If $f$ satisfies (F1) to (F3), it follows from Theorems 1 and 2' that $\lambda$ is a monotone increasing function of $u(0)$, for $u(0) \in (\rho_2 - \varepsilon, \rho_2)$, where $\varepsilon$ is some small positive number. Let $\mathcal{C}$ denote the component of solutions of (P) containing these solutions $(\lambda, u)$ with $u(0) \in (\rho_2 - \varepsilon, \rho_2)$.

Set $\rho^* := \inf\{u(0); (\lambda, u) \in \mathcal{C}\}$. If $\rho^* > 0$, it can be shown that more than one component of solutions $(\lambda, u)$, with $u(0) \in (0, \rho_2)$ may exist, implying the existence of at least four solutions for $\lambda$ large enough.

In Theorem 3 we find a sufficient condition on $f$, which guarantees the existence of a component $\mathcal{D}$ of solutions $(\lambda, u)$ of (P) satisfying $\inf\{u(0); (\lambda, u) \in \mathcal{D}\} = 0$.

THEOREM 3. If in problem (P), $\Omega$ is the unit ball in $\mathbb{R}^N$, with $N > 2$, and $f$ satisfies the condition

$$(G1) \quad f(u) = |u|^\alpha \cdot g(u) \text{ for some } \alpha \in \left(1, \frac{N + 2}{N - 2}\right) \text{ and } g \in C^{1,\gamma} \text{ with } g(0) > 0$$

then the following holds.
There is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists a positive solution $(\lambda, u)$ of (P) with $u(0) = \varepsilon$.

Moreover $\lambda$ is a decreasing function of $\varepsilon$, and $\lim_{\varepsilon \to 0} \lambda(\varepsilon) = \infty$.

If $f$ satisfies (G1), (F1) and (F3), there is one branch of solutions $\lambda \to (\lambda, \overline{u}_\lambda)$ with $\lim_{\lambda \to \infty} \overline{u}_\lambda(0) = \rho_2$, and one branch of solutions $\lambda \to (\lambda, u_\lambda)$ with $\lim_{\lambda \to \infty} u_\lambda(0) = 0$. Then, since $u(0) \in (\rho^*, \rho_2)$ parametrizes the solutions of (P) on the ball, which are radially symmetric, [12], one finds the following. For $\lambda$ large enough, $(\lambda, u_\lambda)$ possesses exactly two positive solutions, with maximum less than $\rho_2$, if and only if $\rho^* = 0$. If $\rho^* > 0$, there exists a positive radially symmetric solution of

$$
\begin{aligned}
-\Delta u &= f(u) \text{ in } \mathbb{R}^N, \\
\lim_{|x| \to \infty} u(x) &= 0
\end{aligned}
$$

satisfying $u(0) = \rho^*$.

For the sake of completeness this will be shown in Section 5. Ni and Serrin, in [15], found conditions on $f$ which exclude the existence of a positive solution of $(P^*)$.

Combining these results we obtain:

**Corollary 1.** If in problem (P) on the unit ball in $\mathbb{R}^N$, with $N > 2$, $f$ satisfies conditions (G1), (F1), (F3) and

$$(G2) \text{ for } \alpha \text{ and } g \text{ defined in (G1) either } \alpha \leq \frac{N}{N-2} \text{ or}$$

$$
\left( \frac{N + 2}{N - 2} - \alpha \right) \cdot u^{\alpha+1} \cdot g(u) \geq \frac{2N}{N - 2} \cdot \int_0^u s^{\alpha+1} \cdot g'(s) ds \text{ for all } u \in [0, \rho_2]
$$

then for $\lambda$ large enough problem $(P_\lambda)$ possesses exactly two positive solutions with maximum less than $\rho_2$.

**Remarks.**

1. If $N \leq 2$, Theorem 3 and Corollary 1 still hold if one replaces in (G1) $\left( 1, \frac{N + 2}{N - 2} \right)$ by $(1, \infty)$. Condition (G2) is no longer needed.

2. In [11], Gardner and Peletier prove a similar result when $\rho_1 > 0$, by using different techniques.

3. For every $\alpha \in \left( \frac{N}{N - 2}, \frac{N + 2}{N - 2} \right)$ a function $f$ exists, for which $\rho^* > 0$. Such an $f$ can be found by using the example on page 2 of [15]. This construction is done in [7].

Concerning the proofs, the main tools will be the sweeping principle of Serrin, see [22], [21], and the construction of appropriate super- and
subsolutions. For the sake of completeness we define in the appendix a notion of super- and subsolutions and we prove a suitable version of the sweeping principle. Some basic ideas for the proof of Theorem 2 are contained in [3].

The results of this paper were announced in [6].

We learned that Dancer and Schmitt, [24], have independently found a different proof of the necessity of (F2) in Theorem 1.

2. - Preliminary results

In this section we collect some preliminary results, which will be useful in the coming proofs. The first result for \( f(0) > 0 \) is contained in [13].

LEMMA 2.1. Let \( f \in C^1 \) satisfy \((F1), (F2)\) and \( f(0) \geq 0 \). Then problem \((P)\) possesses a positive solution \((\lambda, u)\), with \( \max u \in (\rho_1, \rho_2) \).

PROOF. First modify the function \( f \) outside of \([0, \rho_2]\) by setting \( f(\rho) = 0 \) for \( \rho > \rho_2 \) and \( f(\rho) = 2f(0) - f(-\rho) \) for \( \rho < 0 \). Note that \( f \) is bounded on \( \mathbb{R} \). As in [13] we want to minimize

\[
I(u, \lambda) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \lambda \int_{\Omega} F(u) dx \text{ in } W^{1,2}_0(\Omega),
\]

where \( F(u) = \int_0^u f(s) ds \).

For \( \lambda > 0 \), \( I(u, \lambda) \) is bounded below.
Let \( u_n \) be a minimizing sequence for a fixed \( \lambda \), then

\[
I(|u_n|, \lambda) = \frac{1}{2} \int_{\Omega} |D|u_n|^2 dx - \lambda \int_{\Omega} F(|u_n|) dx \leq
\]

\[
\leq \frac{1}{2} \int_{\Omega} |Du_n|^2 dx - \lambda \int_{\Omega} \left\{ \int_0^{u_n} (f(s) - f(0)) ds + \int_0^{u_n} f(0) ds \right\} dx \leq
\]

\[
\leq \frac{1}{2} \int_{\Omega} |Du_n|^2 dx - \lambda \int_{\Omega} \left\{ \int_0^{u_n} (f(s) - f(0)) ds + \int_0^{u_n} f(0) ds \right\} dx =
\]

\[
= I(u_n, \lambda)
\]

Since \( I(\cdot, \lambda) \) is sequentially weakly lower semicontinuous and coercive in \( W^{1,2}_0(\Omega) \), \( I(\cdot, \lambda) \) possesses a nonnegative minimizer, which we denote by \( u_\lambda \).

It is standard that \((\lambda, u_\lambda)\) is a solution of \((P)\), with the modified \( f \).

By applying the strong maximum principle, we deduce as in [2], that either \( f(\|u_\lambda\|_\infty) > 0 \) or \( u_\lambda = 0 \).

Thus \( \|u_\lambda\|_\infty < \rho_2 \), hence \((\lambda, u)\) is a solution of \((P)\).
Set
\[ \alpha = \min \left\{ \int_{\rho_1}^{\rho_2} f(s) \, ds; \quad 0 \leq \rho \leq \max(0, \rho_1) \right\} \]
\[ \beta = \max \left\{ \int_{\rho_1}^{\rho_2} f(s) \, ds; \quad 0 \leq \rho \leq \rho_2 \right\}. \]

Suppose that for all positive \( \lambda, \|u_\lambda\|_\infty \leq \rho_1 \), then we will obtain a contradiction.

We choose \( \delta > 0 \) such that \( 2|\Omega^\delta|\beta < |\Omega|\alpha \), with \( \Omega^\delta = \{ x \in \Omega; \; d(x, \Gamma) < \delta \} \) and \( |\Omega| \) denoting the Lebesgue-measure of \( \Omega \). This is possible since \( \alpha > 0 \) and \( \lim_{\delta \to 0} |\Omega^\delta| = 0 \).

Next we choose \( w \in C_0^\infty(\Omega) \), satisfying \( 0 \leq w \leq \rho_2 \) in \( \Omega^\delta \) and \( w = \rho_2 \) in \( \Omega - \Omega^\delta \); then

\[
I(w, \lambda) - I(u_\lambda, \lambda) = \]
\[
= \frac{1}{2} \int_{\Omega} \left( |Dw|^2 - |Du_\lambda|^2 \right) \, dx - \lambda \int_{\Omega} (F(w) - F(u_\lambda)) \, dx \leq \]
\[
\leq \frac{1}{2} \int_{\Omega} \left( |Dw|^2 + 2\lambda|\Omega^\delta|\beta - \lambda \int_{\Omega^\delta} (F(\rho_2) - F(u_\lambda)) \, dx \right) \leq \]
\[
\leq \frac{1}{2} \int_{\Omega} \left( |Dw|^2 + 2\lambda|\Omega^\delta|\beta - \lambda \int_{\Omega^\delta} f(s) \, ds \right) \leq \]
\[
\leq \frac{1}{2} \int_{\Omega} \left( |Dw|^2 + \lambda(2|\Omega^\delta|\beta - |\Omega|\alpha) < 0 \right)
\]

for \( \lambda \) large enough, since \( 2|\Omega^\delta|\beta - |\Omega|\alpha < 0 \).

Then \( I(w, \lambda) < I(u_\lambda, \lambda) \), contradicting the fact that \( u_\lambda \) is a minimizer. This completes the proof of the lemma.

In what follows it will be convenient to modify \( f \) outside of \([0, \rho_2]\) in an appropriate way.

Let \( f \in C^1 \), respectively \( C^{1,\gamma} \) for some \( \gamma \in (0, 1) \), satisfy (F1) and (F2). Then there is a function \( f^* \in C^1 \), respectively \( C^{1,\gamma} \), satisfying \( f^* = f \) on \([0, \rho_2]\)
and

\[
\begin{cases}
  f^* \text{ is bounded}, \\
  f^* < 0 \text{ in } (\rho_2, \infty), \\
  f^* = 0 \text{ in } (-\infty, -1], \\
  \int_{\rho_2}^{\rho_2} f^*(s)\,ds > 0 \text{ for } u \in [-1, 0].
\end{cases}
\]

(F*)

Since we are interested in solutions \((\lambda, u)\) of \((P)\) with \(0 \leq u \leq \rho_2\), we may assume without loss of generality that \(f\) satisfies \((F^*)\). Then we have

\[
\inf \left\{ \int_{\rho_2}^{\rho_2} f(s)\,ds; \ |\rho_2 - u| > \delta \right\} > 0, \text{ for all } \delta > 0.
\]

(2.1)

**Lemma 2.2.** Let \(f \in C^1\) satisfy \((F1)\), \((F2)\) and \((F^*)\).

Then there exists \(\mu > 0\) and \(v \in C^2(\mathbb{R}^N)\), radially symmetric, which satisfy:

\[
\begin{cases}
  -\Delta v = \mu \cdot f(v) \text{ in } \mathbb{R}^N, \\
  v(0) \in (\rho_1, \rho_2), \\
  v(1) = -1, \\
  v'(r) < 0 \text{ for } r > 0.
\end{cases}
\]

**Proof.** Since \(f(u - 1)\) satisfies \((F1)\) and \((F2)\) it follows from lemma 2.1 that there exists a positive solution \((\mu, w)\) of

\[
\begin{cases}
  -\Delta u = \lambda \cdot f(u - 1) \text{ in } B, \\
  u = 0 \text{ on } \partial B,
\end{cases}
\]

where \(B\) is the unit ball in \(\mathbb{R}^N\), satisfying \(\max w \in (\rho_1 + 1, \rho_2 + 2)\). By [12] \(w\) is radially symmetric and \(w'(r) < 0\) for \(r \in (0, 1)\).

Set \(v(r) = w(r) - 1\) for \(r \in [0, 1]\) and

\[
v(r) = \begin{cases}
  -1 + (r^2 - N - 1) \cdot (2 - N)^{-1} \cdot w'(1) \text{ for } r \in (1, \infty) \text{ if } N \neq 2, \\
  -1 + \log r \cdot w'(1) \text{ for } r \in (1, \infty) \text{ if } N = 2.
\end{cases}
\]

Since \(f = 0\) on \((-\infty, -1]\) one verifies that \(v\) is the required function. This completes the proof of the lemma.

**Corollary 2.3.** Let \((\mu, v)\) be like in Lemma 2.2, and let \(\alpha \in (0, 1)\) be the unique zero of \(v\).
Then for \( y \in \Omega \) and \( \lambda > \mu \cdot \alpha^2 \cdot d(y, \Gamma)^{-2} \)

\[
(2.2) \quad w(\lambda, y; x) := v \left( \frac{\lambda}{\mu} \frac{1}{2} \cdot (x - y) \right), \quad x \in \Omega,
\]
is a subsolution of \((P_\lambda)\).

**PROOF.** The function \( w(\lambda, y) \in C^2(\mathbb{R}^N) \) satisfies \(-\Delta w = \lambda \cdot f(w)\) in \( \mathbb{R}^N \), hence \( \int_{\Omega} (w(-\Delta \varphi) - \lambda \cdot f(w) \varphi) \, dz = 0 \) for all \( \varphi \in D^+(\Omega) \), where \( D^+(\Omega) \) consists of all nonnegative functions in \( C_0^\infty(\Omega) \). Since \( w(\lambda, y) < 0 \) on \( \Gamma \) for \( \lambda > \mu \alpha^2 \cdot d(y, \Gamma)^{-2} \), \( w(\lambda, y) \) satisfies the definition of subsolution given in the appendix. This proves the corollary.

Next we establish some results for the one-dimensional problem

\[
(2.3) \quad \begin{cases}
- u'' = f(u), \quad x > 0 \\
u(0) = 0, \\
u'(0) = \delta,
\end{cases}
\]

where \( f \in C^1 \) satisfies \((F1), (F2)\) and \((F^*)\).

**LEMMA 2.4.** Problem \((2.3)\) possesses a unique solution \( u_\delta \) in \( \mathbb{R} \), for all \( \delta \in \mathbb{R} \). The function \( \delta \rightarrow u_\delta \in C[0, \rho] \) is continuous for every \( \rho > 0 \).

Moreover, set

\[
\delta_1 = \left( 2 \int_0^{\rho_2} f(s) \, ds \right)^{\frac{1}{2}} \quad \text{and} \quad \delta_2 = \left( \max \left\{ -2 \int_0^\rho f(s) \, ds; \rho \in [-1, 0] \right\} \right)^{\frac{1}{2}},
\]

1) if \( \delta > \delta_1 \), then \( u_\delta(x) > (\delta - \delta_1)x \) for \( x \in \mathbb{R}_+ \),

2) if \( \delta = \delta_1 \), then \( u'_\delta > 0 \) on \( \mathbb{R}_+ \) and \( \lim_{x \rightarrow \infty} u_\delta(x) = \rho_2 \),

3) if \( -\delta_2 \leq \delta < \delta_1 \), then \( \sup \{ u_\nu(x); x \in \mathbb{R}_+ , \nu \in [-\delta_2, \delta] \} < \rho_2 \),

4) if \( \delta < -\delta_2 \), then \( u_\delta < 0 \) on \( \mathbb{R}_+ \).

**PROOF.** Since \( f \) is \( C^1 \) and bounded, the first assertion of the lemma is standard.

Note that a solution of \((2.3)\) satisfies

\[
(2.4) \quad (u'(x))^2 = \delta^2 - 2 \int_0^{u(x)} f(s) \, ds.
\]

1) If \( \delta > \delta_1 \), then using \((2.1)\) and \((2.4)\) we have

\[
(u'(x))^2 > (\delta - \delta_1)^2 + 2 \int_{u_\delta(x)}^{\rho_2} f(s) \, ds \geq (\delta - \delta_1)^2.
\]
Since $u_\delta'(0) > 0$, we obtain $u_\delta(x) > (\delta - \delta_1)x$ for $x \in \mathbb{R}_+$. 

2) If $\delta = \delta_1 = \left(\frac{2}{\rho_2} \int_0^{\rho_2} f(s) ds\right)^{\frac{1}{2}}$, we have

$$
(\frac{u_\delta'(x)}{u_\delta(x)})^2 = 2 \int_{u_\delta(x)}^{\rho_2} f(s) ds.
$$

It follows from (2.5), $f(\rho_2) = 0$ and the uniqueness for the initial value problem that $u_\delta(x) \neq \rho_2$ for all $x \in \mathbb{R}_+$, and thus $u_\delta < \rho_2$ on $\mathbb{R}_+$. Since $u_\delta$ is monotonically increasing and bounded there exists a sequence $\{x_n\}$, with $\lim_{n \to \infty} x_n = \infty$ and $\lim_{n \to \infty} u_\delta(x_n) = 0$. From (2.1) and (2.5) it follows that $\lim_{x \to \infty} u_\delta(x) = \rho_2$.

3) Note that $\delta_1^2 - \delta_2^2 = 2 \int_0^{\rho_2} f(s) ds - \max \left\{-2 \int_0^{\rho_2} f(s) ds; \rho \in [-1, 0]\right\} = 2 \min \left\{\int_0^{\rho_2} f(s) ds; \rho \in [-1, 0]\right\}$.

Hence by (2.1) $\delta_1 > \delta_2$.

If $-\delta_2 \leq \nu \leq \delta < \delta_1$, one has

$$
0 \leq (\frac{u_\nu'(x)}{u_\nu(x)})^2 = \nu^2 - 2 \int_0^{u_\nu(x)} f(s) ds \leq \max(\delta_1^2, \delta_2^2) - 2 \int_0^{u_\nu(x)} f(s) ds = \max(\delta_2^2 - \delta_1^2, \delta_1^2 - \delta_2^2) + 2 \int_0^{\rho_2} f(s) ds.
$$

Since $\max(\delta_2^2 - \delta_1^2, \delta_1^2 - \delta_2^2) < 0$, one finds, by using (2.1) again, that $|u_\nu(x) - \rho_2| \geq m > 0$ for all $x \in \mathbb{R}_+$. From $u_\nu(0) = 0$ it follows $u_\nu < \rho_2 - m$ on $\mathbb{R}_+$.

4) If $\delta < -\delta_2$, then

$$
(\frac{u_\delta'(x)}{u_\delta(x)})^2 > \max \left\{-2 \int_0^{\rho_2} f(s) ds; \rho \in [-1, 0]\right\} - 2 \int_0^{u_\delta(x)} f(s) ds \geq 0
$$

for all $u_\delta(x) \leq 0$.

Since $u_\delta'(0) < 0$, one finds $u_\delta < 0$ on $\mathbb{R}_+$. Hence $u_\delta < 0$ on $\mathbb{R}_+$.

This completes the proof of Lemma 2.4.
Lemma 2.4. will be used to establish some results for the problem on the halfspace \( D = \{(x_1, \ldots, x_N) \in \mathbb{R}^N; \ x_1 > 0\} \).

**PROPOSITION 2.5.** Let \( f \in C^{1,\gamma} \), for some \( \gamma \in (0, 1) \), satisfy (F1), (F2) and (F3). Let \( u \in C^2(D) \cap C(\overline{D}) \) be a solution of

\[
\begin{cases}
- \Delta u = f(u) \text{ in } D, \\
u = 0 \text{ on } \partial D,
\end{cases}
\]

with \( 0 \leq u < \rho_2 \) in \( D \) and \( \lim_{x_1 \to \infty} u(x_1, x') = \rho_2 \) uniformly for \( x' \in \mathbb{R}^{N-1} \).

Then \( u(x_1, x') = u_{\delta_1}(x_1) \) for \( x_1 \geq 0 \) and \( x' \in \mathbb{R}^{N-1} \), where \( u_{\delta_1} \) is defined in Lemma 2.4.

In order to prove Proposition 2.5 we also need

**LEMMA 2.6.** Let \( (a; \beta) \in \mathbb{R}^+ \) be a function such that \( \gamma \in (0, 1) \), and \( h(u) \) for some \( h \in C^0(\mathbb{R}) \).

Let \( U \in C^2(D) \cap C(\overline{D}) \) be a bounded solution of

\[
\begin{cases}
- \Delta u = g(x_1, u) \text{ in } D, \\
u = 0 \text{ on } \partial D.
\end{cases}
\]

Then \( S, \) defined by \( S(x_1) = \sup\{U(x_1, x'); \ x' \in \mathbb{R}^{N-1}\} \), is continuous in \([0, \infty)\), with \( S(0) = 0 \), and satisfies

\[
(2.6) \quad \int_{\mathbb{R}_+} (S \cdot (-\varphi'') - g(x_1, S)\varphi) \, dx_1 \leq 0 \text{ for all } \varphi \in D^*(\mathbb{R}_+).
\]

\( D^*(\mathbb{R}_+) \) consists of all nonnegative functions in \( C^0_0(\mathbb{R}_+) \).

**PROOF OF LEMMA 2.6.** Since \( U \) and \( \Delta U \) are bounded and \( U = 0 \) on \( \partial D \), it follows from standard regularity properties that \( U \) and all first-order derivatives are uniformly bounded and uniformly Hölder continuous with exponent \( \gamma \). Let \( \{\Omega_n\} \) be an increasing sequence of bounded subdomains of \( D \), with smooth boundary and such that \( \bigcup_{n \in \mathbb{N}} \Omega_n = D \). We first prove that for each \( n \in \mathbb{N} \), if \( u_1, u_2 \in C(\Omega_n) \cap H^1(\Omega_n) \) satisfy

\[
(2.7) \quad \int_D (u \cdot (-\Delta \varphi) - g(x_1, u) \cdot \varphi) \, dx \leq 0 \text{ for all } \varphi \in D^*(\Omega_n),
\]

then \( u_3 = \sup(u_1, u_2) \) also satisfies (2.7).

Let \( \omega \in \mathbb{R}_+ \) be such that \( u \to g(x_1, u) + \omega \cdot u \) is increasing on \([\min u_1 \wedge \min u_2, \max u_1 \vee \max u_2] \) for every \( x \in \overline{\Omega}_n \).
We obtain
\[ \int_D (u_i \cdot (-\Delta \varphi) + \omega \cdot u_i \cdot \varphi) \, dx \leq \int_D (g(x_1, u_3) + \omega \cdot u_3) \cdot \varphi \, dx \]
for all \( \varphi \in \mathcal{D}^+(\Omega_n) \), \( i = 1, 2 \).

Set \( h = g(x_1, u_3) + \omega \cdot u_3 \) and let \( w \) satisfy
\[
\begin{align*}
\begin{cases}
-\Delta w + \omega \cdot w &= h \text{ in } \Omega_n, \\
w &= 0 \text{ on } \partial \Omega_n.
\end{cases}
\end{align*}
\]

Note that \( w \in C(\overline{\Omega}_n) \cap H^1(\Omega_n) \). Then \( u_i = u_i - w \), \( i = 1, 2 \), satisfies
\[
\int_D (u \cdot (-\Delta \varphi) + \omega \cdot u \cdot \varphi) \, dx \leq 0 \quad \text{for all } \varphi \in \mathcal{D}^+(\Omega_n).
\]

It is known that \( \sup(w_1, w_2) \) also satisfies \((2.8)\), see [23, Th. 28.1]. Therefore \( u_3 \) satisfies \((2.7)\). Note that \( u_3 \in C(\overline{\Omega}_n) \cap H^1(\Omega_n) \). By induction it follows that if \( u_i \in C(\overline{\Omega}_n) \cap H^1(\Omega_n) \), \( i = 1, \ldots, k \), satisfies \((2.7)\), then \( \sup\{u_i \}; \; i = 1, \ldots, k \} \) also satisfies \((2.7)\). Let \( u_i \) be translates of \( U \) perpendicular to \((1, 0, \ldots, 0) \). Since \( U \in C(\overline{D}) \cap H^1_{\text{loc}}(D) \), \( \sup\{u_i \}; \; i = 1, \ldots, k \} \) will satisfy \((2.7)\). Then by using the Lebesgue dominated convergence theorem and the fact that \( U \) is bounded, one shows that
\[
S(x_1) = \sup\{U(x_1, x'); \; x' \in \mathbb{R}^{N-1}\} = \sup\{U(x_1, x'); \; x' \in \mathbb{Q}^{N-1}\}
\]
also satisfies \((2.7)\) for each \( n \). From the choice of the \( \Omega_n \) it follows
\[
\int_D (S(-\Delta \varphi) - g(x_1, S) \cdot \varphi) \, dx \leq 0 \quad \text{for all } \varphi \in \mathcal{D}^+(D).
\]

By choosing \( \varphi \) of the form \( \varphi_1 \cdot \varphi_2 \), with \( \varphi_1 \in \mathcal{D}^+(\mathbb{R}_+) \) and \( \varphi_2 \in \mathcal{D}^+(\mathbb{R}^{N-1}) \), \( \varphi_2 \neq 0 \), one gets \((2.6)\), since \( S \) only depends on \( x_1 \).

Note that \( S \), as the supremum of continuous functions, is lower semicontinuous on \([0, \infty)\). From \((2.6)\) and the fact that \( g(x_1, S) \) is bounded, we deduce that \( S \) is the sum of a convex function on \((0, \infty)\) and a \( C^1 \)-function on \([0, \infty)\). Hence \( S \in C(0, \infty) \). Since \( U(0, x') = 0 \) and since \( \frac{\partial}{\partial x} U(0, x') \) is uniformly bounded, \( S(0) = 0 \) and \( S \) is continuous in \( 0 \). This completes the proof of Lemma 2.6.

**Proof of Proposition 2.5.** Without loss of generality we assume that \( f \) satisfies \((F^*)\). Define
\[
I(x_1) = \inf \{U(x_1, x'); \; x' \in \mathbb{R}^{N-1}\} \quad \text{and} \quad S(x_1) = \sup \{U(x_1, x'); \; x' \in \mathbb{R}^{N-1}\}.
\]
It is sufficient to prove that

\[(2.9) \quad I \geq u_\delta \text{ on } \mathbb{R}_+, \text{ and} \]

\[(2.10) \quad S \leq u_\delta \text{ on } \mathbb{R}_+, \]

for \( \delta = \delta_1 \).

We first prove (2.9) for \( \delta = \delta_1 \). By Lemma 2.4, 4), (2.9) holds with \( \delta < -\delta_2 \), since \( I \geq 0 \) on \( \mathbb{R}_+ \). We will use a sweeping argument to prove (2.9) for every \( \delta \in (-\delta_2 - 1, \delta_1) \). Let \( \delta \in (-\delta_2 - 1, \delta_1) \). By Lemma 2.4, 3) and 4), there exists \( \rho < \rho_2 \) such that

\[(2.11) \quad \sup \{ u_\theta(x_1); \ x_1 \in \mathbb{R}_+; \theta \leq \delta \} \leq \rho. \]

For some \( R > 0 \) one has \( I > \rho \) on \([R, \infty)\). It follows from Lemma 2.6, with \( g(x_1, u) = -f(u) \), that \( I \in C[0, \infty), I(0) = 0 \) and

\[\int_{\mathbb{R}_+} (I \cdot (g''(u) - f(u)) \, dx \geq 0 \text{ for all } g \in D^+(\mathbb{R}_+).\]

Hence \( I \) is a supersolution of

\[(2.12) \quad \begin{cases} -u'' = f(u) \text{ in } (0, R), \\ u(0) = 0, \\ u(R) = \rho. \end{cases} \]

For \( \theta \in [-\delta_2 - 1, \delta] \), (2.11) shows that \( u_\theta \) is a subsolution of (2.12). We are now in the position to use Lemma A.2 and we obtain \( I \geq u_\delta \) on \((0, R)\), hence on \( \mathbb{R}_+ \). For \( x_1 \geq 0 \) one has

\[I(x_1) \geq \lim_{\delta_1 \delta_1} u_{\delta}(x_1) = u_{\delta_1}(x_1). \]

This completes the proof of (2.9), with \( \delta = \delta_1 \).

Next we give a sketch of the proof of (2.10). Since \( \frac{\partial}{\partial x_1} U \) is uniformly bounded, there exists \( c > 0 \) such that

\[S(x_1) < c \cdot x_1 \text{ for } x_1 \in \mathbb{R}_+. \]

By Lemma 2.4, 1), one has (2.10) with \( \delta = \delta_1 + c \). Let \( \delta \in (\delta_1, \delta_1 + c) \). Also from Lemma 2.4, 1), if follows

\[u_{\theta}(x_1) > \rho_2 + 1 \text{ for } x_1 > R := (\delta - \delta_1)^{-1}(\rho_2 + 1) \text{ and } \theta \in [\delta, \delta_1 + c]. \]
Note that $S \leq \rho_2$. Then one concludes as above after using a sweeping argument for the problem

$$\begin{align*}
-u'' &= f(u), \quad \text{in } (0, R), \\
u(0) &= 0, \\
u(R) &= \rho_2.
\end{align*}$$

This completes the proof of Proposition 2.5.

3. - Proof of the first theorem

**NECESSITY:** With $J(\rho) = \int_{\rho_1}^{\rho_2} f(s)ds$, and assuming $\rho_1 > 0$, define

$$J^* := \min \{J(\rho); \rho \in [0, \rho_1]\}.$$ 

Suppose condition (F2) is not satisfied, that is $J^* \leq 0$. Let $(\lambda, u)$ be a positive solution of $(P)$ satisfying $\max u \in (\rho_1, \rho_2)$. We will obtain a contradiction.

First, if $J^* = 0$, modify $f$ to $f^*$ in $C^1$ such that $f > f^* > 0$ in $(\max u, \rho_2)$ and $f = f^*$ elsewhere. Still $u$ is a solution of $(P\lambda)$, but now $J^* < 0$. Hence we may assume without loss of generality that $J^* < 0$.

Consider the initial value problem

\begin{align}
-(v')' &= f(v), \\
v(0) &= \rho_2, \\
v'(0) &= -(J^*)^{\frac{1}{2}}.
\end{align}

For a solution of (3.1), (3.2) one has:

$$\left(v'(r)\right)^2 = -J^* + 2 \int_{\rho_1}^{\rho_2} f(s)ds.$$ 

Set $\rho^* := \max \{\rho \in [0, \rho_1]; J(\rho) = J^*\}$.

Because of (F1), $\left(v'(r)\right)^2 = -J^* + 2 \int_{\rho_1}^{\rho_2} f(s)ds \geq -J^* > 0$ holds for $v(r)$ in $[\rho_1, \rho_2]$, and hence $\inf v < \rho_1$.

Next we show that $v$ remains positive. If not, there exists an $r^*$ such that $v(r^*) = \rho^*$, and since (3.3) holds, one finds

$$\left(v'(r^*)\right)^2 = -J^* + 2 \int_{\rho_1}^{\rho_2} f(s)ds = +J^* < 0,$$

a contradiction.
So either \( u(r) \downarrow \hat{p} \in (\rho^*, \rho_1) \) if \( r \to \infty \), or \( u \) has a first positive minimum, say in \( \bar{r} \), and \( u \) is symmetric with respect to \( \bar{r} \). In the first case define

\[
V(r) := \begin{cases} 
  v(r) & \text{for } r > 0 \\
  \rho_2 & \text{for } r \leq 0 
\end{cases}
\]

and in the second case

\[
V(r) := \begin{cases} 
  v(r) & \text{for } r \in (0; 2\bar{r}) \\
  \rho_2 & \text{elsewhere in } \mathbb{R} 
\end{cases}
\]

Set \( w(\lambda, t; x) = V \left( \lambda^{\frac{1}{2}} \cdot (x_1 - t) \right) \), where \( x = (x_1, \ldots, x_N) \).

Then \( \{w(\lambda, t; \cdot); t \in \mathbb{R}\} \) is a family of supersolutions, and for \( t \) large enough \( w(\lambda, t; \cdot) = \rho_2 \) in \( \Omega \).

By the sweeping principle \( u \leq w(\lambda, t, \cdot) \) for all \( t \).

Hence \( u(x) \leq \inf \{w(\lambda, t; x); t \in \mathbb{R}\} = \inf v < \rho_1 \), a contradiction.

**REMARK 1.** Let \( f \in C^1 \) satisfy (F1). The proof also shows that, if (F2) is not satisfied, there is no solution \( u \) of \( (P_\lambda) \) with \( \max u \in (\rho_1, \rho_2) \), even if \( u \) changes sign.

**REMARK 2.** Let \( f \in C^1 \) satisfy (F1), and let \( \Omega \subset \mathbb{R}^N \) be an unbounded domain.

Note that the same technique shows that problem

\[
\begin{cases} 
- \Delta u = f(u) & \text{in } \Omega, \\
  u = 0 & \text{on } \partial \Omega, \\
  \lim_{|x| \to \infty} u(x) = 0 
\end{cases}
\]

may have a solution \( u \), with \( \max u \in (\rho_1, \rho_2) \), only if condition (F2) is satisfied.

**SUFFICIENCY:** We will prove a stronger result, which will be used later on.

Let \( x^* \in \Omega \). Then define \( \lambda^* = \mu \alpha^2 d(x^*, \Gamma)^{-2} \) and \( z_\lambda = w(\lambda, x^*) \), where \( \mu, \alpha \) and \( w \) are defined in Corollary 2.3.

**LEMMA 3.1.** Let \( f \) satisfy (F1), (F2) and (F*). Then

1) \( \lambda > \lambda^* \) problem \( (P_\lambda) \) possesses a solution \( u_\lambda \in [z_\lambda, \rho_2] \),

2) there exists \( \lambda^{**} > \lambda^* \), \( \epsilon > 0 \) and \( \tau \in (\rho_1, \rho_2) \), such that for \( \lambda > \lambda^{**} \) every solution \( u \in [z_\lambda, \rho_2] \) of \( (P_\lambda) \) satisfies

\[
(3.4) \quad u(x) > \min \left( \epsilon \lambda^{\frac{1}{2}} d(x, \Gamma), \tau \right) \text{ for all } x \in \Omega.
\]

**REMARK 3.** It follows from (3.4) that \( u_\lambda > 0 \) for \( \lambda > \lambda^{**} \), and that \( \max u_\lambda \in (\rho_1, \rho_2) \), for \( \lambda \) large enough.
Remark 4. Lemma 3.1, 2), shows $\frac{\partial}{\partial n} u_\lambda < 0$ on $\Gamma$ for $\lambda > \lambda^{**}$, even when $f(0) < 0$. ($\frac{\partial}{\partial n}$ denotes the outward normal derivative)

Proof of Lemma 3.1. By Corollary 2.3 one knows that for $\lambda > \lambda^*$, $z_\lambda$ is a subsolution of $(P_\lambda)$, with $z_\lambda < \rho_2$. Since $\rho_2$ is a supersolution of $(P_\lambda)$, Lemma A.1 yields a solution $u_\lambda \in [z_\lambda, \rho_2]$ of $(P_\lambda)$, for $\lambda > \lambda^*$. This completes the proof of the first assertion.

Since $\Omega$ satisfies a uniform interior sphere condition, there exists $\varepsilon_0 > 0$ such that $\Omega = \bigcup \{B(x, \varepsilon); x \in \Omega_\varepsilon\}$ for $\varepsilon \in (0, \varepsilon_0]$, where $\Omega_\varepsilon = \{x \in \Omega; d(x, \Gamma) > \varepsilon\}$. Set

$$
\lambda^{**} = \max(\lambda^*, \mu \alpha^2 \varepsilon_0^{-2}),
$$

$$
c = \mu^{-\frac{1}{2}} \inf \{(\alpha - r)^{-1} \cdot v(r); r \in [0, \alpha]\} \quad \text{and}
$$

$$
\tau = v(0)
$$

with $\mu$, $\nu$ and $\alpha$ defined in Corollary 2.3.

Note that $c > 0$, since $v > 0$ on $[0, \alpha)$ and $v'(\alpha) < 0$.

Let $(\lambda, u)$ be a solution of $(P)$ with $\lambda > \lambda^{**}$ and $u \in [z_\lambda, \rho_2]$. Since for $\lambda > \lambda^{**}$, $\Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}$ is arcwise connected and since $w(\lambda, y)$ is a subsolution for $y \in \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}$, with $w(\lambda, y) < 0$ on $\Gamma$, one finds by Lemma A.2 that

$$
u > w(\lambda, y) \text{ in } \Omega \text{ for all } y \in \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}.
$$

Hence

$$
u(x) > c \lambda^{\frac{1}{2}} d(x, \Gamma) \text{ for all } x \in \Omega \setminus \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}, \text{ and}
$$

$$
u(x) > \tau \text{ for all } x \in \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}},
$$

which completes the proof.

4. - Proof of the second theorem

As mentioned in the introduction Theorem 2 will be a consequence of a sharper version, Theorem 2'.

Theorem 2'. Let $\Gamma \subset C^3$ and let $f \in C^{1,\gamma}$, for some $\gamma \in (0, 1)$, satisfy (F1), (F2) and (F3). Then for some $\lambda_i > 0$,

1) there exists $\varphi \in C^1 ([\lambda_1, \infty); C^2(\Omega))$, such that $(\lambda, \varphi(\lambda))$ is a solution of $(P)$ for $\lambda \geq \lambda_1$, with $\varphi(\lambda) > 0$ in $\Omega$, $\max \varphi(\lambda) \in (\rho_1, \rho_2)$ and $\lim_{\lambda \to \infty} \max \varphi(\lambda) = \rho_2$;

2) if $\mu_0(\lambda, u)$ denotes the principal eigenvalue of

\[
\begin{align*}
\left\{ \begin{array}{l}
- \lambda^{-1} \cdot \Delta h - f'(u) \cdot h = \mu h \text{ in } \Omega, \\
h = 0 \text{ on } \Gamma,
\end{array} \right.
\]

(LP)
then \( \mu_0(\lambda, \varphi(\lambda)) > 0 \) for \( \lambda > \lambda_1 \);

3) for all nonnegative \( z \in C_0^\infty(\Omega) \) with \( \max z \in (\rho_1, \rho_2) \), there exists \( \lambda(z) > \lambda_1 \), such that, if \((\lambda, u)\) is a solution of \((P)\) with \( \lambda > \lambda(z) \) and \( u \in [z, \rho_2] \), then \( u = \varphi(\lambda) \).

**Remark 1.** Theorem 2 follows from theorem 2’ by choosing a nonnegative function \( z_0 \in C_0^\infty(\Omega) \) and setting \( \lambda_0 = \lambda(z_0) \) in the third assertion of Theorem 2’.

**Remark 2.** If \( \rho_1 > 0 \), let \( C \) denote the component of solutions of \((P)\) in \( \mathbb{R}_+ \times C^2(\overline{\Omega}) \) containing \( \{(\lambda, \varphi(\lambda)); \lambda \geq \lambda_1\} \). Since \( C \) is connected, one has for \((\lambda, u) \in C \) that \( \max u \in (\rho_1, \rho_2) \) (see [2]) and \( \lambda > 0 \). By using degree arguments as in [19], [20], one can show that for \( \lambda \) large enough, \( C \cap \{\lambda \times C^2(\overline{\Omega})\} \) contains at least two solutions of \((P)\). The proof of this assertion will appear elsewhere.

For the proof of Theorem 2’ we need the following lemmas.

**Lemma 4.1.** Let \( f \in C^1 \) satisfy (F1), (F2) and (F*). For every \( \delta > 0 \) there is a \( c(\delta) > 0 \), such that for all solutions \((\lambda, u)\) of \((P)\), with \( \lambda > \lambda^{**} \) and \( u \in [z_\lambda, \rho_2] \), the following holds

\[
(4.1) \quad u(x) > \min \left( c(\delta) \lambda^{1/2}d(x, \Gamma), \rho_2 - \delta \right) \quad \text{for all} \quad x \in \Omega,
\]

with \( \lambda^{**} \) and \( z_\lambda \) as in Lemma 3.1.

**Proof of Lemma 4.1.** If \( \rho_2 - \delta < \tau \), we are done with \( c(\delta) = c \) as in Lemma 3.1. Otherwise, by (F1) there exists \( \sigma > 0 \) such that \( \sigma(u - \tau) < f(u) \) for all \( u \in [\tau, \rho_2 - \delta] \).

Let \( \nu \) denote the principal eigenvalue of

\[
\begin{cases}
-\Delta \psi = \nu \psi \text{ in } B, \\
\psi = 0 \text{ on } \partial B,
\end{cases}
\]

where \( B \) denotes the unit ball in \( \mathbb{R}^N \).

Then by using Lemma A.3 with \( \Omega' = \Omega_{k \lambda^{-1/2}}, k = c^{-1} \tau \), one finds

\[
(4.2) \quad u(x) > \rho_2 - \delta \quad \text{for all} \quad x \in \Omega \setminus (\sigma/\nu + k) \lambda^{-1/2},
\]

since \( (\Omega')_{(\sigma/\nu + k) \lambda^{-1/2}} = \Omega \setminus (\sigma/\nu + k) \lambda^{-1/2} \).

By (3.4) one finds

\[
(4.3) \quad u(x) > c(\delta) \lambda^{-1/2}d(x, \Gamma) \quad \text{for all} \quad x \in \Omega \setminus \Omega \setminus (\sigma/\nu + k) \lambda^{-1/2}
\]
with \( c(\delta) = \tau \left( (\nu/\sigma)^{1/2} + k \right)^{-1} \)

This completes the proof of the lemma.

**Lemma 4.2.** Let \( f \in C^{1,\gamma} \), for some \( \gamma \in (0,1) \), satisfy (F1), (F2), (F3) and (F*). Then there exists \( \lambda_1 > \lambda_{**} \), such that for every solution \( u \) of \((P_\lambda)\), with \( \lambda > \lambda_1 \) and \( u \in [z_\lambda, \rho_2] \), one finds \( \mu_0(\lambda, u) > 0 \).

**Proof.** Suppose this is not the case. Then there exists a sequence \( \{ (\lambda_n, u_n); \ n \in \mathbb{N} \} \) of solutions of \((P)\), with \( u_n \in [z_{\lambda_n}, \rho_2] \), \( \mu_n := \mu_0(\lambda_n, u_n) \leq 0 \) for all \( n \), and \( \lim_{n \to \infty} \lambda_n = \infty \).

Let \( \varepsilon \) be defined by (F3). Since \( \mu_n \leq 0 \), for all \( n \), the associated eigenfunctions \( v_n \), normalized by \( \max v_n = 1 \), satisfy

\[
-\lambda_n^{-1} \Delta v_n(x) = \left( f' (u_n(x)) + \mu_n \right) v_n(x) \leq 0 \quad \text{for} \quad x \in \Omega_{K\lambda_n^{-1/2}},
\]

where \( K = (c(\varepsilon))^{-1} (\rho_2 - \varepsilon) \).

The constant \( c(\varepsilon) \) is defined in the previous lemma. Hence the function \( v_n \) is subharmonic in \( \Omega_{K\lambda_n^{-1/2}} \), and \( v_n \) attains its maximum outside of \( \Omega_{K\lambda_n^{-1/2}} \). Like in [3] let \( y^n \in \Omega \setminus \Omega_{K\lambda_n^{-1/2}} \) be a point where \( v_n \) attains its maximum and let \( x^n \in \Gamma \) be a point which minimizes \( \{ d(x, y^n); \ x \in \Gamma \} \). Since \( \{ x^n \} \) and \( \{ \mu_n \} \) are bounded, there exists a subsequence, still denoted \( \{ (\lambda_n, u_n) \} \), such that \( \lim_{n \to \infty} x^n = \bar{x} \in \Gamma \) and \( \lim_{n \to \infty} \mu_n = \bar{\mu} \leq 0 \). Let \( \mathcal{O} \) be an open neighbourhood of \( \bar{x} \) in \( \mathbb{R}^N \), chosen so small that it permits \( C^3 \) local coordinates \( (\xi_1, \ldots, \xi_N): \mathcal{O} \to \mathbb{R}^N \), such that \( x \in \Omega \cap \mathcal{O} \) if and only if \( \xi_1(x) > 0 \), and \( \xi(\bar{x}) = 0 \). In these coordinates the Laplacian is given by

\[
\Delta u = \sum_{i,j} a_{ij}(\xi) \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_j} \bar{u} + \sum_j b_j(\xi) \frac{\partial}{\partial \xi_j} \bar{u},
\]

where \( a_{ij} \in C^2 \), \( b_j \in C^1 \) and \( u(x) = \bar{u}(\xi(x)) \).

Moreover we choose the local coordinates such that \( a_{ij}(0) = \delta_{ij} \). Next define the functions

\[
U_n(\eta) = \bar{u}_n \left( \xi(x^n) + \lambda_n^{-1/2} \eta \right),
\]

\[
V_n(\eta) = \bar{v}_n \left( \xi(x^n) + \lambda_n^{-1/2} \eta \right), \quad \eta \in D.
\]

Since \( \{ U_n \} \) and \( \{ V_n \} \) are precompact in \( C^2_{\text{loc}} \), there exists a convergent subsequence. Hence there are \( U, V \in C^2(\bar{D}) \), bounded and positive in \( D = \)
\{(x_1, x') : x_1 > 0, x' \in \mathbb{R}^{N-1}\}, satisfying respectively

\[-\Delta U = f(U) \text{ in } D,
\begin{cases}
U = 0 & \text{on } \partial D, \\
-\Delta V - f'(U)V = \nu V & \text{in } D, \\
V = 0 & \text{on } \partial D.
\end{cases}

Moreover by Lemma 4.1 the following inequalities,

\[\min(c(\delta)x_1, \rho_2 - \delta) \leq U(x_1, x') \leq \rho_2 \text{ for all } x_1 > 0, x' \in \mathbb{R}^{N-1},\]

hold for every \(\delta > 0\). From Proposition 2.5 we have

\[U(x_1, x') = u_{\delta_1}(x_1) \text{ for } x_1 \geq 0, x' \in \mathbb{R}^{N-1}.\]

Set \(S(x_1) = \sup\{V(x_1, x') : x' \in \mathbb{R}^{N-1}\}\). Then \(0 < S \leq 1\) in \(\mathbb{R}_+\) and we obtain by using Lemma 2.6 that \(S \in C[0, \infty), S(0) = 0\) and

\[\int_{\mathbb{R}_+} \left( S \cdot (-\varphi'' - (f'(u_{\delta_1}) + \mu) S \varphi) \right) dx \leq 0 \text{ for all } \varphi \in D^+(\mathbb{R}_+).\]  

Since \(u_{\delta_1}' > 0\) on \(\mathbb{R}_+\), there exists a smallest \(C > 0\) such that \(W := Cu_{\delta_1}' - S \geq 0\) on \([0, K + 1]\), where \(K\) is defined in (4.4). Then one finds by using (4.6) and \(-u_{\delta_1}'') = f'(u_{\delta_1})u_{\delta_1}'\) in \(\mathbb{R}_+\), that

\[\int_{\mathbb{R}_+} \left( W \cdot (-\varphi'' - f'(u_{\delta_1})W \varphi) \right) dx \geq 0 \text{ for all } \varphi \in D^+(\mathbb{R}_+).\]  

Since \(W\) is nonnegative in \([0, K + 1]\), there is \(\omega > 0\) such that

\[\int_{\mathbb{R}_+} \left( W \cdot (-\varphi'' + \omega W \varphi) \right) dx \geq 0 \text{ for all } \varphi \in D^+((0, K + 1)).\]  

By [5, Corollary p. 581] and the fact that \(W \neq 0\), one obtains

\[W \geq bx(K + 1 - x) \text{ for all } x \in [0, K + 1] \text{ and some } b > 0.\]  

By construction \(W\) vanishes somewhere in \([0, K + 1]\). Since \(W(0) > 0\) one finds \(W(K + 1) = 0\). Moreover \(f'(u_{\delta_1}) \leq 0\) on \((K, \infty)\). Hence (4.6) yields that \(S\) is convex on \((K, \infty)\). Since \(W\) is the sum of a \(C^1\) and a concave function on \((K, \infty)\), (4.8) shows \(0 > \frac{d}{dx} W(K + 1) \geq \frac{d^r}{dx} W(K + 1)\), and therefore \(W(x) < 0\) on \((K + 1, K + 1 + c)\) for some \(c > 0\). Moreover \(W\) cannot vanish on \((K + 1, \infty)\).
Otherwise there would be \( c > 0 \) such that \( W < 0 \) on \((K + 1, K + 1 + c)\) and \( W(K + 1) = W(K + 1 + c) = 0\). But this cannot happen since by (4.7) \( W \) is concave as long as \( W \) is negative on \((K, +\infty)\).

Hence \( W \) is concave on \((K + 1, +\infty)\). Since \( \frac{d^+}{dx} W(K + 1) < 0 \), \( W \) is not bounded below, contradicting \( W = C u_0' - S \geq -1 \) on \( \mathbb{R}_+ \). This completes the proof of Lemma 4.2.

It follows from Lemma 4.2 that for \( \lambda > \lambda_1 \) \((P_\lambda)\) possesses at most one solution in \([z_\lambda, \rho_2]\). Indeed, choose \( \omega > 0 \) such that \( \lambda f'(u) + \omega > 0 \) for \( u \in [0, \rho_2] \), and define the mapping \( K: C(\overline{\Omega}) \rightarrow C(\overline{\Omega}) \) by

\[
K(u) := (-\Delta + \omega)^{-1} (\lambda f(u) + \omega u),
\]

where \((-\Delta + \omega)^{-1}\) is the inverse of \(-\Delta + \omega\) with homogeneous Dirichlet boundary conditions. By our choice of \( \omega \), \( K \) maps \([z_\lambda, \rho_2]\) into itself and \( K \) has no fixed point on its boundary. Since \( K \) is compact, the Leray-Schauder degree on \((za, P_2)\) is well defined. Because \((z_\lambda, \rho_2)\) is convex one finds

\[
\text{degree } (I - K, (z_\lambda, \rho_2), 0) = 1.
\]

If \((\lambda, u)\) is a solution of \((P)\), with \( u \in [z_\lambda, \rho_2] \) and \( \mu_0(\lambda, u) > 0 \), it follows that \( u \) is an isolated fixed point of \( K \). Moreover, the local degree of \( I - K \) at \( u \) is +1. From the additivity of degree it follows that \( K \) possesses at most one fixed point in \((z_\lambda, \rho_2)\). We denote this solution by \( \varphi(\lambda) \). Since \( \mu_0(\lambda, \varphi(\lambda)) > 0 \), for \( \lambda > \lambda_1 \), one finds by the implicit function theorem and Schauder estimates, that \( \lambda \rightarrow \varphi(\lambda) \in C^1 ([\lambda_1, \infty); C^{2, \gamma}(\overline{\Omega})) \). The estimate (4.1) implies that \( \lim_{\lambda \rightarrow \infty} \max \varphi(\lambda) = \rho_2 \).

It remains to prove the third assertion of theorem 2'. Let \( z \in \partial^+(\Omega) \) with \( \max z \in (\rho_1, \rho_2) \). It follows from the first part of the proof, that it is sufficient to show that there exists \( \lambda(z) > \lambda_1 \), such that any solution \( u \) of \((P_\lambda)\), with \( \lambda > \lambda(z) \) and \( u \in [z, \rho_2] \), is larger than \( z \). This will be done in two steps.

First note that, from the definition of \( z \), there exist \( s \in (\rho_1, \rho_2) \) and a ball \( B(x_0, r) \subset \Omega \), such that \( z > s \) in \( B(x_0, r) \). Let \( \sigma > 0 \) be such that \( f(u) > \sigma \cdot (u - s) \) for \( u \in [s, r] \), where \( r = \max z_\lambda \). For \( \lambda > \lambda_1(z) := \left(\frac{\nu}{\sigma}\right)^{\frac{1}{2}} + \mu^{\frac{1}{2}} \), \( \mu \) defined in Lemma 2.2, we can apply Lemma A.3 in order to get

\[
u(x) > \tau \text{ for } x \in B \left( x_0, (\mu/\lambda)^{\frac{1}{2}} \right) \subset B \left( x_0, r - (\nu/\sigma) \lambda^{\frac{1}{2}} \right).
\]

Observe that \( w(\lambda, x_0) < u \) in \( \Omega \) for \( \lambda > \lambda_1(z) \). By Corollary 2.3 \( w(\lambda, x_0) \) is a subsolution of \((P_\lambda)\) for \( \lambda > \lambda_1(z) \).

Finally, like in proof of Lemma 3.1 part 2), one uses Lemma A.2 to show that if \( u > w(\lambda, x_0) \) in \( \Omega \) and \( \lambda > \lambda(z) := \max (\lambda_1(z), \lambda^{**}) \) also the following estimate holds,

\[
u > w(\lambda, x^*) = z_\lambda.
\]
This completes the proof of Theorem 2'.

5. - Proof of the third theorem

Note that, if \((\lambda, u)\) is a positive solution of \((P)\), then \(v := (u(0))^{-1} u\) satisfies

\[
\begin{aligned}
-\Delta v &= (u(0))^{\alpha - 1} \lambda v^\alpha g(u(0)v) \quad \text{in } B \\
v &= 0 \quad \text{on } \partial B.
\end{aligned}
\]

Moreover by defining \(w(r) := v(R^{-1}r)\) with \(\varepsilon = u(0)\) and

\[
\begin{aligned}
R &= u(0)^{\frac{1}{\alpha}(\alpha - 1)} \lambda \frac{1}{\lambda} \quad \text{one gets} \\
- w'' - \frac{N - 1}{r} w' &= w^\alpha g(\varepsilon w) \\
\end{aligned}
\]

\[
\begin{aligned}
\left\{
\begin{array}{l}
w(0) = 1 \\
w'(0) = 0 \\
w(R) = 0 \\
w > 0 \quad \text{on } [0, R).
\end{array}
\right.
\]

Let \(w(\varepsilon, \cdot)\) denote the unique solution of the initial value problem (5.2-5.3)

**LEMMA 5.** There exists \(\varepsilon_1 > 0\) such that for \(\varepsilon\) in \([0, \varepsilon_1)\), \(w(\varepsilon, \cdot)\) possesses a first zero, which we denote by \(R(\varepsilon)\). Moreover \(R\) as a function of \(\varepsilon\) is \(C^1(0, \varepsilon_1) \cap C[0, \varepsilon_1)\) and \(\frac{d}{d\varepsilon} R\) is bounded on \((0, \frac{1}{2}\varepsilon_1)\).

We first show that the assertion of Theorem 3 is an easy consequence of this lemma. By (5.1) we have \(\lambda(\varepsilon) = R(\varepsilon)^{2\varepsilon^{1-a}},\) and hence

\[
\frac{d}{d\varepsilon} \lambda(\varepsilon) = R(\varepsilon)^{-\alpha} \left(2\varepsilon \frac{d}{d\varepsilon} R(\varepsilon) + (1 - \alpha)R(\varepsilon)\right),\quad 0 < \varepsilon < \varepsilon_1.
\]

Since \(\alpha - 1 > 0,\) \(R(0) > 0\) and \(\frac{d}{d\varepsilon} R\) is bounded on \(\left(0, \frac{1}{2}\varepsilon_1\right)\), it follows that \(\frac{d}{d\varepsilon} \lambda(\varepsilon) < 0\) on some interval \((0, \varepsilon_0)\).

Then for \(\lambda > \lambda(\varepsilon_0),\) \(u_\lambda(r) = \varepsilon(\lambda)w(R(\varepsilon(\lambda))r)\) is a solution of \((P_\lambda)\) on the unit ball, where \(\varepsilon(\lambda)\) is the inverse of the function \(\lambda(\varepsilon)\). This function \(\varepsilon(\lambda)\) is well defined on \((\lambda(\varepsilon_0), \infty),\) decreasing and satisfies \(\lim_{\lambda \to \infty} \varepsilon(\lambda) = 0.\) This completes the proof of the theorem.

**PROOF OF LEMMA 5.** It is known, see [17], that (5.2-5.3) with \(\varepsilon = 0\) possesses a solution \(w,\) having a first positive zero which we denoted by \(R(0).\) We want to obtain the function \(w(\varepsilon, \cdot)\) by a perturbation argument.
Since we are only interested in bounded positive solutions, we modify the right-hand-side of (5.2) by setting \( h(\varepsilon, w) = k(w)g(\varepsilon w) \) where \( k \) is a \( C^1 \)-function satisfying

\[
k(w) = \begin{cases} 
0 & \text{for } w \leq 0 \\
\varepsilon^\alpha & \text{for } 0 < w < 1 \\
0 & \text{for } w \geq 2.
\end{cases}
\]

The function \( h \) is \( C^1(\varepsilon, 1) \times \mathbb{R} \) and has bounded derivatives. The initial value problem

\[
-w'' - \frac{N-1}{r}w' = h(\varepsilon, w), \ \varepsilon \text{ in } (-1, 1),
\]

(5.4)

\[
\begin{cases}
  w(0) = 1 \\
  w'(0) = 0,
\end{cases}
\]

(5.5)

possesses a unique solution \( w(\varepsilon, r) \) on \([0, \infty)\).

For \( \varepsilon \) in \([0, 1)\), since \( w(\varepsilon, r) \) is decreasing until it possibly becomes zero, this function \( w(\varepsilon, r) \) is identical with the one in the lemma, as long as it is positive.

We claim, for every \( r > 0 \), \( w(\varepsilon, r) \) is a \( C^1 \)-function of \( \varepsilon \). First this will be proved for \( r \in (0, \delta) \), with \( \delta \) small enough. Note that (5.4-5.5) can be rewritten as \( w = T(\varepsilon, w) \), where \( T(\varepsilon, z)(r) = 1 - \int_0^r \int_0^t s^{N-1} h(z(s), \varepsilon) \, ds \, dt \), for \( z \) in \( C[0, \delta] \). For every \( \delta > 0 \), \( T(\varepsilon, \cdot) : C[0, \delta] \to C[0, \delta] \), where \( C[0, \delta] \) is equipped with the supremum-norm, is continuously Fréchet-differentiable. For \( \delta \) small enough, \( T(\varepsilon, \cdot) : C[0, \delta] \to C[0, \delta] \) is a strict contraction with a unique fixed point \( z(\varepsilon) \) such that \( \varepsilon \to z(\varepsilon) \) is continuously differentiable.

Since \( w(\varepsilon, r) = z(\varepsilon)(r) \), the claim is proved for \( r < \delta \).

By repeating the argument it can be shown that \( \varepsilon \to w(\varepsilon, r) \) is continuously differentiable for every \( r > 0 \).

Since \( w(0, R(0)) = 0 \) and \( w_r(0, R(0)) \leq 0 \) it follows from the implicit function theorem, that there exists \( \varepsilon_1 > 0 \) and a continuously differentiable function \( R(\varepsilon) \), defined on \((-\varepsilon_1, \varepsilon_1)\), such that \( w(\varepsilon, R(\varepsilon)) = 0 \). From (5.4) it follows that \( R(\varepsilon) \) is the unique zero of \( w(\varepsilon, \cdot) \) on \( \mathbb{R}^+ \). This completes the proof.

**Proof of the Corollary.** Since \( u(0) \) parametrizes the solutions \((\lambda, u)\) of (P), \( \rho^* = \inf\{\sigma > 0; (P) \text{ has a solution } (\lambda, u), \text{ with } u(0) = \rho, \text{ for all } \rho \in [\sigma, \rho_2]\} \). Suppose \( \rho^* > 0 \) and let \( v \) be the solution of the initial value problem

\[
-w'' - \frac{N-1}{r}w' = f(v),
\]

(5.6)

\[
\begin{cases}
  v(0) = \rho^* \\
  v'(0) = 0.
\end{cases}
\]

(5.7)
Since \( f(p) > 0 \) on \((0, p^*)\], \( v \) is strictly decreasing while \( v \) is positive. If \( v \) has a (first) positive zero \( R \), then \((R^2, v(R^{-1}, \cdot))\) is a solution of \((P)\), which contradicts the definition of \( p^* \). If \( v \) stays positive, then

\[
\lim_{r \to \infty} v(r) = 0.
\]

Otherwise, there are \( c > 0 \) and \( R > 0 \) such that \( f(v(s)) > c \) for \( s > R \). By integrating (5.6), one finds

\[
v'(r) = (R/r)^{N-1}v'(R) - r^{1-N} \int_R^r s^{N-1}f(v(s))ds \leq \leq (R/r)^{N-1}v'(R) - (c/N)(r - R(r/r)^{N-1}) < -1,
\]

for \( r \) large enough, contradicting the fact that \( v \) stays positive. The existence of a positive function satisfying (5.6-5.8), is contradicted by Theorem 2.2 of [15], if \( \alpha \leq N/(N - 2) \), and by Theorem 3.1 of [15], if the integral condition of \((G2)\) is satisfied. Therefore \( p^* = 0 \).

This completes the proof.

6. - Appendix

In this section we state, for the sake of completeness, a definition and some lemmas concerning sub- and supersolutions of problem

\[
(H) \quad \left\{ \begin{array}{l}
-\Delta u = h(u) \text{ in } \Omega \subset \mathbb{R}^N, \\
u = g \text{ on } \Gamma,
\end{array} \right.
\]

where \( \Omega \) is a bounded domain with \( C^2\)-boundary, \( h \in C^1 \) and \( g \in C^0 \).

**DEFINITION.** We call a function \( v \) a subsolution (supersolution) of \((H)\) if:

i) \( v \in C(\overline{\Omega}) \),

ii) \( v \geq (\geq) g \) on \( \partial \Omega \), and

iii) \( \int (v \cdot (-\Delta \varphi) - h(v)\varphi)dx \leq (\geq) 0 \) for every \( \varphi \in D^+(\Omega) \), where \( D^+(\Omega) \) consists of all nonnegative functions in \( C_0^\infty(\Omega) \).

**LEMMA A.1.** Let \( v \) and \( w \) be respectively a sub- and supersolution of \((H)\) with \( g = 0 \). If \( v \leq w \) in \( \Omega \), then there exists a solution \( u \in C^2(\overline{\Omega}) \) of \((H)\) with \( g = 0 \), which satisfies \( v \leq u \leq w \).

**PROOF.** We essentially follow the proof in [21] on page 24. Choose a number \( \omega > 0 \) such that \( h'(u) + \omega \geq 0 \) for \( \min v \leq u \leq \max w \), and define the
nonlinear map $T$ by $u = Tu$, where
\[
\begin{cases}
-\Delta u_1 + \omega u_1 = h(u) + \omega u & \text{in } \Omega, \\
u_1 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Clearly $T : C(\Omega) \to C(\Omega)$ is compact. (Where $C(\Omega)$ is equipped with the supremum-norm)

It is standard that $T$ is monotone on $[v, w]$. Next we show that $v_1 := Tv \geq v$ in $\Omega$.

By the definition of a subsolution and by the construction of $v_1$, we have
\[
\int_{\Omega} (v \cdot (-\Delta \varphi) + \omega v \varphi) \, dx \leq \int_{\Omega} (h(v) + \omega v) \varphi \, dx = \int_{\Omega} (v_1 \cdot (-\Delta \varphi) + \omega v_1 \varphi) \, dx \quad \text{for every } \varphi \in \mathcal{D}^+(\Omega).
\]

Thus $z = v_1 - v$ satisfies $z \geq 0$ on $\partial \Omega$, and
\[
\int_{\Omega} (z \cdot (-\Delta \varphi) + \omega z \varphi) \, dx \geq 0 \quad \text{for every } \varphi \in \mathcal{D}^+(\Omega).
\]

We claim that $z$ is nonnegative in $\Omega$.

Otherwise there exists a ball $B(x_0, r) \subset \Omega$, such that $z$ is negative in $B(x_0, r)$ and achieves its minimum in $x_0$.

Hence
\[
\int_{\Omega} z \cdot (-\Delta \varphi) \, dx \geq 0 \quad \text{for every } \varphi \in \mathcal{D}^+(B(x_0, r)).
\]

This shows $z$ is superharmonic on $B(x_0, r)$, and from the minimum principle we get $z(x) = z(x_0)$ on $B(x_0, r)$.

Then
\[
\int_{B(x_0, r)} (z \cdot (-\Delta \varphi) + \omega z \varphi) \, dx = \omega z(x_0) \int_{B(x_0, r)} \varphi \, dx < 0
\]
for every nontrivial $\varphi \in \mathcal{D}^+(B(x_0, r))$, a contradiction. Thus $Tv = v_1 \geq v$ on $\Omega$. Similarly, one proves $T \omega \leq w$ on $\Omega$. Now it is standard, see [1], that $T$ possesses a fixed point in $[v, w]$, which is a solution of (H) with $g = 0$.

Next we prove an appropriate version of the sweeping principle of Serrin, [22], [21].

Let $\Gamma = \partial \Omega$ be the union of two disjoint closed subsets $\Gamma_1$ and $\Gamma_2$, where $\Gamma_1$ or $\Gamma_2$ may be empty. Let $e \in C^1(\Omega)$ satisfy $e > 0$ on $\Omega \cap \Gamma_1$ and $e = 0$, $\frac{\partial e}{\partial n} < 0$ on
LEMMA A.2. Let $u$ be a supersolution of (H) and let $A = \{v_t; \ t \in [0, 1]\}$ be a family of subsolutions of (H) satisfying $v_t < g$ on $\Gamma_1$ and $v_t = g$ on $\Gamma_2$, for all $t \in [0, 1]$. If
1) $t \to (v_t - v_0) \in C_c(\Omega)$ is continuous with respect to the $\| \cdot \|_c$-norm,
2) $u \geq v_0$ in $\Omega$, and
3) $u \neq v_t$, for all $t \in [0, 1]$,
then there exists $\alpha > 0$, such that for all $t \in [0, 1]$ $u - v_t \geq \alpha e$ in $\Omega$.

PROOF. Set $E = \{t \in [0, 1]; u \geq v_t \in \Omega\}$. By 2) $E$ is not empty. Moreover $E$ is closed. For $t \in E$ $v_t := u - v_t$ satisfies
\[ \int_\Omega (w \cdot (-\Delta \varphi) + \omega w \varphi) \, dx \geq 0 \quad \text{for all } \varphi \in \mathcal{D}^+(\Omega) \text{ and some } \omega > 0. \]

Since $w_t \neq 0$ it follows from [5, Corollary p. 581] that there is $\beta > 0$, such that $w_t \geq \beta u_0$, for some $u_0 \in C^1(\Omega)$, which satisfies $u_0 > 0$ in $\Omega$, $u_0 = 0$ and $\frac{\partial}{\partial n} u_0 < 0$ on $\Gamma$. The function $w_t$ is positive on $\Gamma_1$, which is compact, and continuous on $\Omega$. Hence there exists $\gamma > 0$ such that $w_t \geq \gamma e$. Since $t \to (w_t - w_0)$ is continuous with respect to the $\| \cdot \|_c$-norm, $E$ is also open. Hence $E = [0, 1]$ and there is $\alpha > 0$, such that $w_t \geq \alpha e$ in $\Omega$ for all $t \in [0, 1]$.

This completes the proof of Lemma A.2.

Let $\psi$ be the principal eigenfunction, with eigenvalue $\nu$, of
\[ \begin{cases} -\Delta v = \lambda v \text{ in } B, \\ v = 0 \text{ on } \partial B, \end{cases} \]
where $B$ denotes the unit ball in $\mathbb{R}^N$.

Let $\psi$ be normalized such that $\max \psi = 1$.

LEMMA A.3. Let $u$ satisfy $-\Delta u = \lambda f(u)$ in an open $\Omega' \subset \Omega$, such that $u(x) > a$ for $x \in \Omega'$. Let $\sigma > 0$ be such that $f(u) > \sigma (u - a)$ for $u \in [a, b]$.
If $x_0 \in (\Omega')_{(\nu/\sigma \lambda)^{1/2}}$, then $u(x_0) > b$.

PROOF. Set $\theta(x_0, \lambda, t; x) = a + t\psi((\sigma \lambda/\nu)^{1/2}(x - x_0))$ for $x \in B(\ )$ and $t \in [0, b - a]$, where $B(\ ) = B(x_0, (\nu/\sigma \lambda)^{1/2})$. The set $\{\theta(x_0, \lambda, t; t \in [0, b - a]\}$ is a family of subsolutions of the problem

\[ \begin{cases} -\Delta v = \lambda f(v) \text{ in } B(\ ) \\ v = u \text{ on } \partial B(\ ), \end{cases} \]
and $\overline{B(\ )} \subset \Omega'$. 

Existance and multiplicity results for a semilinear, etc.
By using Lemma A.2 one finds \( u(x_0) > b \).

It remains to show that \( \theta(x_0, \lambda, t) \) is a subsolution of \( (\text{Pb}_\lambda) \). By the assumption of the lemma \( u > a = \theta(x_0, \lambda, t) \) on \( \partial B(\cdot) \).

The integral condition is also satisfied:

\[
\int_{B(\cdot)} \left( \theta(-\Delta \varphi) - \lambda f(\theta) \varphi \right) dx = \int_{B(\cdot)} (-\Delta \theta - \lambda f(\theta)) \varphi dx \\
\leq \int_{B(\cdot)} (-\Delta \theta - \lambda \sigma(\theta - a)) \varphi dx = 0 \text{ for all } \varphi \in D^+(B(\cdot))
\]

This completes the proof of the lemma.

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