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**Existence and multiplicity results for a semilinear elliptic eigenvalue problem**

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 14, nº 1 (1987), p. 97-121
<http://www.numdam.org/item?id=ASNSP_1987_4_14_1_97_0>

# Existence and Multiplicity Results for a Semilinear Elliptic Eigenvalue Problem

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## 1. - Introduction

The following eigenvalue problem will be considered:

$$
\text{(P)} \qquad \begin{cases} -\Delta u = \lambda f(u) \text{ in } \Omega \in \mathbb{R}^N. \\ u = 0 \text{ on } \partial\Omega = \Gamma \end{cases}
$$

for $\lambda > 0$. The domain $\Omega$ is assumed to be bounded and to have a smooth boundary of class $C^3$.

The function $f$ will satisfy appropriate smoothness conditions. A positive solution of (P) will be a pair $(\lambda, u)$ in $\mathbb{R}^+ \times C^2(\overline{\Omega})$ satisfying (P) with $u > 0$ in $\Omega$. We shall call $u$ a solution of $(P_\lambda)$.

It is a consequence of the strong maximum principle, see [2], that if such a solution exists, then $f(\max u)$ is positive. The main goal of this paper is to study positive solutions having their maximum close to a zero of $f$. Therefore we assume:

(F1) there are two numbers $\rho_1$ and $\rho_2$ such that $\rho_1 < \rho_2$, $0 < \rho_2$,

$$f(\rho_1) = f(\rho_2) = 0 \text{ and } f > 0 \text{ in } (\rho_1, \rho_2)$$

In [13] Hess proves the existence of solutions $(\lambda, u)$ of (P), satisfying $\max u \in (\rho_1, \rho_2)$, when $f(0) > 0$ under the following condition:

$$
\text{(F2)} \qquad J(\rho) = \int_\rho^{\rho_2} f(s) ds > 0 \text{ for every } \rho \in [0, \rho_2).
$$

In Theorem 1 we prove that (F2) is a *necessary* and sufficient condition for the existence of such a solution even without the condition $f(0) \geq 0$.

Pervenuto alla redazione il 4 Aprile 1986 ed in forma definitiva il 29 Settembre 1986.

THEOREM 1. *Let* $f \in C^1$ *satisfy* (F1). *Then problem* (P) *possesses a positive solution* $(\lambda, u)$, *with* $\max u \in (\rho_1, \rho_2)$, *if and only if* (F2) *holds.*

Theorem 1 improves a result of De Figueiredo in [10], since it does not use the inheritance condition or even the starshapedness of $\Omega$.

It also answers a question of Dancer in [9].

Next to this existence result we will prove a uniqueness result for positive solutions having their maximum close to $\rho_2$. We need the following condition:

(F3) there exists an $\varepsilon > 0$ such that $f' \leq 0$ in $(\rho_2 - \varepsilon, \rho_2)$.

THEOREM 2. *Let* $f \in C^{1,\gamma}$, *for some* $\gamma \in (0,1)$, *satisfy* (F1), (F2) *and* (F3). *Let* $\Gamma \in C^3$. *Then there are* $\lambda_0 > 0$ *and a nonnegative function* $z_0 \in C_0^\infty(\Omega)$ *with* $\max z_0 \in (\rho_1, \rho_2)$, *such that for all* $\lambda > \lambda_0$, $(\mathrm{P}_\lambda)$ *possesses exactly one solution* $u_\lambda$ *with* $z_0 < u_\lambda < \rho_2$.

*Moreover,* $\lim\limits_{\lambda \to \infty} \max u_\lambda = \rho_2$.

REMARKS.

1. We will state and prove a sharper version of this theorem in Section 4 (Theorem 2′).

2. If $\rho_1 < 0$, or $\rho_1 = 0$ and $f'(0) > 0$, Theorem 2 was proved in a recent paper, [3], by Angenent. For $\rho_1 \leq 0$ there are also related results in [8].

3. If $\rho_1 = 0$ and $f'(0) = 0$, Rabinowitz showed in [19] the existence of pairs of solutions for $\lambda$ large enough by a degree argument.

When $\rho_1 = 0$ and $f'(0) = 0$ the question arises, whether or not there are exactly two positive solutions of $(\mathrm{P}_\lambda)$, with maximum less than $\rho_2$, for $\lambda$ large enough. We shall consider this problem only for $\Omega = B$, the unit ball in $\mathbb{R}^N$.

It is known, [12], that positive solutions for $\Omega = B$ are radially symmetric, and can be parametrized by $u(0)$. If $f$ satisfies (F1) to (F3), it follows from Theorems 1 and 2′ that $\lambda$ is a monotone increasing function of $u(0)$, for $u(0) \in (\rho_2 - \varepsilon, \rho_2)$, where $\varepsilon$ is some small positive number. Let $\mathcal{C}$ denote the component of solutions of (P) containing these solutions $(\lambda, u)$ with $u(0) \in (\rho_2 - \varepsilon, \rho_2)$.

Set $\rho^* := \inf\{u(0) \colon (\lambda, u) \in \mathcal{C}\}$. If $\rho^* > 0$, it can be shown that more than one component of solutions $(\lambda, u)$, with $u(0) \in (0, \rho_2)$ may exist, implying the existence of at least four solutions for $\lambda$ large enough.

In Theorem 3 we find a sufficient condition on $f$, which guarantees the existence of a component $\mathcal{D}$ of solutions $(\lambda, u)$ of (P) satisfying $\inf\{u(0); (\lambda, u) \in \mathcal{D}\} = 0$.

THEOREM 3. *If in problem* (P), $\Omega$ *is the unit ball in* $\mathbb{R}^N$, *with* $N > 2$, *and* $f$ *satisfies the condition*

(G1) $f(u) = |u|^\alpha \cdot g(u)$ for some $\alpha \in \left(1, \dfrac{N+2}{N-2}\right)$ and $g \in C^{1,\gamma}$ with $g(0) > 0$

*then the following holds.*

*There is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists a positive solution $(\lambda, u)$ of (P) with $u(0) = \varepsilon$.*

*Moreover $\lambda$ is a decreasing function of $\varepsilon$, and $\lim_{\varepsilon \downarrow 0} \lambda(\varepsilon) = \infty$.*

If $f$ satisfies (G1), (F1) and (F3), there is one branch of solutions $\lambda \to (\lambda, \overline{u}_\lambda)$ with $\lim_{\lambda\to\infty} \overline{u}_\lambda(0) = \rho_2$, and one branch of solutions $\lambda \to (\lambda, \underline{u}_\lambda)$ with $\lim_{\lambda\to\infty} \underline{u}_\lambda(0) = 0$. Then, since $u(0) \in (\rho^*, \rho_2)$ parametrizes the solutions of (P) on the ball, which are radially symmetric, [12], one finds the following. For $\lambda$ large enough, ($P_\lambda$) possesses exactly two positive solutions, with maximum less than $\rho_2$, if and only if $\rho^* = 0$. If $\rho^* > 0$, there exists a positive radially symmetric solution of

$$
(P^*) \qquad \begin{cases} -\Delta u = f(u) \text{ in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases}
$$

satisfying $u(0) = \rho^*$.

For the sake of completeness this will be shown in Section 5. Ni and Serrin, in [15], found conditions on $f$ which exclude the existence of a positive solution of ($P^*$).

Combining these results we obtain:

COROLLARY 1. *If in problem* (P) *on the unit ball in* $\mathbb{R}^N$, *with* $N > 2$, $f$ *satisfies conditions* (G1), (F1), (F3) *and*

(G2) *for $\alpha$ and $g$ defined in* (G1) *either* $\alpha \le \frac{N}{N-2}$ *or*

$$
\left(\frac{N+2}{N-2} - \alpha\right) \cdot u^{\alpha+1} \cdot g(u) \ge \frac{2N}{N-2} \cdot \int_0^u s^{\alpha+1} \cdot g'(s) \mathrm{d}s \text{ for all } u \in [0, \rho_2]
$$

*then for $\lambda$ large enough problem* ($P_\lambda$) *possesses exactly two positive solutions with maximum less than* $\rho_2$.

REMARKS.

1. If $N \le 2$, Theorem 3 and Corollary 1 still hold if one replaces in (G1) $\left(1, \dfrac{N+2}{N-2}\right)$ by $(1, \infty)$. Condition (G2) is no longer needed.

2. In [11], Gardner and Peletier prove a similar result when $\rho_1 > 0$, by using different techniques.

3. For every $\alpha \in \left(\dfrac{N}{N-2}, \dfrac{N+2}{N-2}\right)$ a function $f$ exists, for which $\rho^* > 0$. Such an $f$ can be found by using the example on page 2 of [15]. This construction is done in [7].

Concerning the proofs, the main tools will be the sweeping principle of Serrin, see [22], [21], and the construction of appropriate super- and subsolutions. For the sake of completeness we define in the appendix a notion of super- and subsolutions and we prove a suitable version of the sweeping principle. Some basic ideas for the proof of Theorem 2 are contained in [3].

The results of this paper where announced in [6].

We learned that Dancer and Schmitt, [24], have independently found a different proof of the necessity of (F2) in Theorem 1.

## 2. - Preliminary results

In this section we collect some preliminary results, which will be useful in the coming proofs. The first result for $f(0) > 0$ is contained in [13].

LEMMA 2.1. *Let* $f \in C^1$ *satisfy* (F1), (F2) *and* $f(0) \geq 0$. *Then problem* (P) *possesses a positive solution* $(\lambda, u)$, *with* $\max u \in (\rho_1, \rho_2)$.

PROOF. First modify the function $f$ outside of $[0, \rho_2]$ by setting $f(\rho) = 0$ for $\rho > \rho_2$ and $f(\rho) = 2f(0) - f(-\rho)$ for $\rho < 0$. Note that $f$ is bounded on $\mathbb{R}$. As in [13] we want to minimize

$$I(u, \lambda) = \frac{1}{2} \int_\Omega |Du|^2 \mathrm{d}x - \lambda \int_\Omega F(u) \mathrm{d}x \text{ in } W_0^{1,2}(\Omega),$$

where $F(u) = \int_0^u f(s) \mathrm{d}s$.

For $\lambda > 0$, $I(u, \lambda)$ is bounded below.

Let $u_n$ be a minimizing sequence for a fixed $\lambda$, then

$$\begin{aligned} I(|u_n|, \lambda) &= \frac{1}{2} \int_\Omega |D|u_n||^2 \mathrm{d}x - \lambda \int_\Omega F(|u_n|) \mathrm{d}x \leq \\ &\leq \frac{1}{2} \int_\Omega |Du_n|^2 \mathrm{d}x - \lambda \int_\Omega \left\{ \int_0^{|u_n|} (f(s) - f(0)) \mathrm{d}s + \int_0^{|u_n|} f(0) \mathrm{d}s \right\} \mathrm{d}x \leq \\ &\leq \frac{1}{2} \int_\Omega |Du_n|^2 \mathrm{d}x - \lambda \int_\Omega \left\{ \int_0^{u_n} (f(s) - f(0)) \mathrm{d}s + \int_0^{u_n} f(0) \mathrm{d}s \right\} \mathrm{d}x = \\ &= I(u_n, \lambda) \end{aligned}$$

Since $I(\cdot, \lambda)$ is sequentially weakly lower semicontinuous and coercive in $W_0^{1,2}(\Omega)$, $I(\cdot, \lambda)$ possesses a nonnegative minimizer, which we denote by $u_\lambda$.

It is standard that $(\lambda, u_\lambda)$ is a solution of (P), with the modified $f$. By applying the strong maximum principle, we deduce as in [2], that either $f(\|u_\lambda\|_\infty) > 0$ or $u_\lambda = 0$.

Thus $\|u_\lambda\|_\infty < \rho_2$, hence $(\lambda, u)$ is a solution of (P).

Set

$$\alpha = \min\left\{ \int_{\rho}^{\rho_2} f(s)\mathrm{d}s;\ 0 \le \rho \le \max(0, \rho_1) \right\}$$

$$\beta = \max\left\{ \int_{\rho}^{\rho_2} f(s)\mathrm{d}s;\ 0 \le \rho \le \rho_2 \right\}.$$

Suppose that for all positive $\lambda$, $\|u_\lambda\|_\infty \le \rho_1$, then we will obtain a contradiction.

We choose $\delta > 0$ such that $2|\Omega^\delta|\beta < |\Omega|\alpha$, with $\Omega^\delta = \{x \in \Omega;\ \mathrm{d}(x, \Gamma) < \delta\}$ and $|\Omega|$ denoting the Lebesgue-measure of $\Omega$. This is possible since $\alpha > 0$ and $\lim_{\delta \downarrow 0} |\Omega^\delta| = 0$.

Next we choose $w \in C_0^\infty(\Omega)$, satisfying $0 \le w \le \rho_2$ in $\Omega^\delta$ and $w = \rho_2$ in $\Omega - \Omega^\delta$; then

$$\begin{aligned}
&I(w, \lambda) - I(u_\lambda, \lambda) = \\
&= \frac{1}{2}\int_\Omega \left(|Dw|^2 - |Du_\lambda|^2\right) \mathrm{d}x - \lambda \int_\Omega (F(w) - F(u_\lambda))\, \mathrm{d}x \le \\
&\le \frac{1}{2}\int_\Omega |Dw|^2 \mathrm{d}x - \lambda \left( \int_\Omega F(\rho_2)\mathrm{d}x + \int_{\Omega^\delta} (F(w) - F(\rho_2))\, \mathrm{d}x - \int_\Omega F(u_\lambda)\mathrm{d}x \right) \le \\
&\le \frac{1}{2}\int_\Omega |Dw|^2 \mathrm{d}x + 2\lambda|\Omega^\delta|\beta - \lambda \int_\Omega (F(\rho_2) - F(u_\lambda))\, \mathrm{d}x = \\
&= \frac{1}{2}\int_\Omega |Dw|^2 \mathrm{d}x + 2\lambda|\Omega^\delta|\beta - \lambda \int_\Omega \int_{u_\lambda}^{\rho_2} f(s)\mathrm{d}s\mathrm{d}x \le \\
&\le \frac{1}{2}\int_\Omega |Dw|^2 \mathrm{d}x + \lambda(2|\Omega^\delta|\beta - |\Omega|\alpha) < 0
\end{aligned}$$

for $\lambda$ large enough, since $2|\Omega^\delta|\beta - |\Omega|\alpha < 0$.

Then $I(w, \lambda) < I(u_\lambda, \lambda)$, contradicting the fact that $u_\lambda$ is a minimizer. This completes the proof of the lemma.

In what follows it will be convenient to modify $f$ outside of $[0, \rho_2]$ in an appropriate way.

Let $f \in C^1$, respectively $C^{1,\gamma}$ for some $\gamma \in (0, 1)$, satisfy (F1) and (F2). Then there is a function $f^* \in C^1$, respectively $C^{1,\gamma}$, satisfying $f^* = f$ on $[0, \rho_2]$

and

$$
\text{(F}^*\text{)} \qquad \begin{cases} f^* \text{ is bounded,} \\ f^* < 0 \text{ in } (\rho_2, \infty), \\ f^* = 0 \text{ in } (-\infty, -1], \\ \displaystyle\int_u^{\rho_2} f^*(s)\mathrm{d}s > 0 \text{ for } u \in [-1, 0]. \end{cases}
$$

Since we are interested in solutions $(\lambda, u)$ of (P) with $0 \le u \le \rho_2$, we may assume without loss of generality that $f$ satisfies (F*). Then we have

$$
\text{(2.1)} \qquad \inf \left\{ \int_u^{\rho_2} f(s)\mathrm{d}s; \; |\rho_2 - u| > \delta \right\} > 0, \text{ for all } \delta > 0.
$$

LEMMA 2.2. *Let* $f \in C^1$ *satisfy* (F1), (F2) *and* (F*).
*Then there exist* $\mu > 0$ *and* $v \in C^2(\mathbb{R}^N)$, *radially symmetric, which satisfy:*

$$
\begin{cases} -\Delta v = \mu \cdot f(v) \text{ in } \mathbb{R}^N, \\ v(0) \in (\rho_1, \rho_2), \\ v(1) = -1, \\ v'(r) < 0 \text{ for } r > 0. \end{cases}
$$

PROOF. Since $f(u-1)$ satisfies (F1) and (F2) it follows from lemma 2.1 that there exists a positive solution $(\mu, w)$ of

$$
\begin{cases} -\Delta u = \lambda \cdot f(u-1) \text{ in } B, \\ u = 0 \text{ on } \partial B, \end{cases}
$$

where $B$ is the unit ball in $\mathbb{R}^N$, satisfying $\max w \in (\rho_1 + 1, \rho_2 + 2)$. By [12] $w$ is radially symmetric and $w'(r) < 0$ for $r \in (0, 1)$.

Set $v(r) = w(r) - 1$ for $r \in [0, 1]$ and

$$
v(r) = \begin{cases} -1 + (r^{2-N} - 1) \cdot (2-N)^{-1} \cdot w'(1) \text{ for } r \in (1, \infty) \text{ if } N \neq 2, \\ -1 + \log r \cdot w'(1) \text{ for } r \in (1, \infty) \text{ if } N = 2. \end{cases}
$$

Since $f = 0$ on $(-\infty, -1]$ one verifies that $v$ is the required function. This completes the proof of the lemma.

COROLLARY 2.3. *Let* $(\mu, v)$ *be like in Lemma 2.2, and let* $\alpha \in (0, 1)$ *be the unique zero of* $v$.

*Then for $y \in \Omega$ and $\lambda > \mu \cdot \alpha^2 \cdot \mathrm{d}(y, \Gamma)^{-2}$*

$$w(\lambda, y; x) := v\left((\lambda/\mu)^{\frac{1}{2}} \cdot (x - y)\right), \quad x \in \Omega, \tag{2.2}$$

*is a subsolution of* $(\mathrm{P}_\lambda)$.

PROOF. The function $w(\lambda, y) \in C^2(\mathbb{R}^N)$ satisfies $-\Delta w = \lambda \cdot f(w)$ in $\mathbb{R}^N$, hence $\int_\Omega (w(-\Delta\varphi) - \lambda \cdot f(w)\varphi)\,\mathrm{d}x = 0$ for all $\varphi \in \mathcal{D}^+(\Omega)$, where $\mathcal{D}^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$. Since $w(\lambda, y) < 0$ on $\Gamma$ for $\lambda > \mu\alpha^2 \cdot \mathrm{d}(y, \Gamma)^{-2}$, $w(\lambda, y)$ satisfies the definition of subsolution given in the appendix. This proves the corollary.

Next we establish some results for the one-dimensional problem

$$\begin{cases} -u'' = f(u), & x > 0 \\ u(0) = 0, \\ u'(0) = \delta, \end{cases} \tag{2.3}$$

where $f \in C^1$ satisfies (F1), (F2) and (F*).

LEMMA 2.4. *Problem* (2.3) *possesses a unique solution $u_\delta$ in $\mathbb{R}_+$ for all $\delta \in \mathbb{R}$. The function $\delta \to u_\delta \in C[0, r]$ is continuous for every $r > 0$.*

*Moreover, set*

$$\delta_1 = \left(2\int_0^{\rho_2} f(s)\mathrm{d}s\right)^{\frac{1}{2}} \quad \text{and} \quad \delta_2 = \left(\max\left\{-2\int_\rho^0 f(s)\mathrm{d}s;\ \rho \in [-1, 0]\right\}\right)^{\frac{1}{2}},$$

1) *if $\delta > \delta_1$, then $u_\delta(x) > (\delta - \delta_1)x$ for $x \in \mathbb{R}_+$,*
2) *if $\delta = \delta_1$, then $u'_\delta > 0$ on $\mathbb{R}_+$ and $\lim_{x\to\infty} u_\delta(x) = \rho_2$,*
3) *if $-\delta_2 \le \delta < \delta_1$, then $\sup\{u_\nu(x); x \in \mathbb{R}_+, \nu \in [-\delta_2, \delta]\} < \rho_2$,*
4) *if $\delta < -\delta_2$, then $u_\delta < 0$ on $\mathbb{R}_+$.*

PROOF. Since $f$ is $C^1$ and bounded, the first assertion of the lemma is standard.

Note that a solution of (2.3) satisfies

$$\left(u'(x)\right)^2 = \delta^2 - 2\int_0^{u(x)} f(s)\mathrm{d}s. \tag{2.4}$$

1) If $\delta > \delta_1$, then using (2.1) and (2.4) we have

$$\left(u'_\delta(x)\right)^2 > (\delta - \delta_1)^2 + 2\int_{u_\delta(x)}^{\rho_2} f(s)\mathrm{d}s \ge (\delta - \delta_1)^2.$$

Since $u'_\delta(0) > 0$, we obtain $u_\delta(x) > (\delta - \delta_1)x$ for $x \in \mathbb{R}_+$.

2) If $\delta = \delta_1 = \left(2\int_0^{\rho_2} f(s)\mathrm{d}s\right)^{\frac{1}{2}}$, we have

$$(2.5) \qquad \left(u'_\delta(x)\right)^2 = 2\int_{u_\delta(x)}^{\rho_2} f(s)\mathrm{d}s.$$

It follows from (2.5), $f(\rho_2) = 0$ and the uniqueness for the initial value problem that $u_\delta(x) \neq \rho_2$ for all $x \in \mathbb{R}_+$, and thus $u_\delta < \rho_2$ on $\mathbb{R}_+$. Since $u_\delta$ is monotonically increasing and bounded there exists a sequence $\{x_n\}$, with $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} u'_\delta(x_n) = 0$. From (2.1) and (2.5) it follows that $\lim_{x\to\infty} u_\delta(x) = \rho_2$.

3) Note that $\delta_1^2 - \delta_2^2 = 2\int_0^{\rho_2} f(s)\mathrm{d}s - \max\left\{-2\int_\rho^0 f(s)\mathrm{d}s;\ \rho \in [-1,0]\right\} =$
$= 2\min\left\{\int_\rho^{\rho_2} f(s)\mathrm{d}s;\ \rho \in [-1,0]\right\}$.

Hence by (2.1) $\delta_1 > \delta_2$.

If $-\delta_2 \le \nu \le \delta < \delta_1$, one has

$$\begin{aligned} 0 \le \left(u'_\nu(x)\right)^2 &= \nu^2 - 2\int_0^{u_\nu(x)} f(s)\mathrm{d}s \le \\ &\le \max(\delta_2^2, \delta^2) - 2\int_0^{u_\nu(x)} f(s)\mathrm{d}s = \\ &= \max(\delta_2^2 - \delta_1^2, \delta^2 - \delta_1^2) + 2\int_{u_\nu(x)}^{\rho_2} f(s)\mathrm{d}s. \end{aligned}$$

Since $\max(\delta_2^2 - \delta_1^2, \delta^2 - \delta_1^2) < 0$, one finds, by using (2.1) again, that $|u_\nu(x) - \rho_2| \ge m > 0$ for all $x \in \mathbb{R}_+$. From $u_\nu(0) = 0$ it follows $u_\nu < \rho_2 - m$ on $\mathbb{R}_+$.

4) If $\delta < -\delta_2$, then

$$\left(u'_\delta(x)\right)^2 > \max\left\{-2\int_\rho^0 f(s)\mathrm{d}s;\ \rho \in [-1,0]\right\} - 2\int_0^{u_\delta(x)} f(s)\mathrm{d}s \ge 0$$

for all $u_\delta(x) \le 0$.

Since $u'_\delta(0) < 0$, one finds $u'_\delta < 0$ on $\mathbb{R}_+$. Hence $u_\delta < 0$ on $\mathbb{R}_+$.

This completes the proof of Lemma 2.4.

Lemma 2.4. will be used to establish some results for the problem on the halfspace $D = \{(x_1, \dots, x_N) \in \mathbb{R}^N;\ x_1 > 0\}$.

PROPOSITION 2.5. *Let $f \in C^{1,\gamma}$, for some $\gamma \in (0,1)$, satisfy* (F1), (F2) *and* (F3). *Let $u \in C^2(D) \cap C(\overline{D})$ be a solution of*

$$\begin{cases} -\Delta u = f(u) \text{ in } D, \\ u = 0 \text{ on } \partial D, \end{cases}$$

*with $0 \leq u < \rho_2$ in $D$ and $\lim\limits_{x_1 \to \infty} u(x_1, x') = \rho_2$ uniformly for $x' \in \mathbb{R}^{N-1}$.*

*Then $u(x_1, x') = u_{\delta_1}(x_1)$ for $x_1 \geq 0$ and $x' \in \mathbb{R}^{N-1}$, where $u_{\delta_1}$ is defined in Lemma* 2.4.

In order to prove Proposition 2.5 we also need

LEMMA 2.6. *Let $(x_1, u) \to g(x_1, u)$ be a function such that $g$, $\dfrac{\partial}{\partial u} g \in C^{0,\gamma}(\overline{\mathbb{R}}_+ \times \mathbb{R})$, for some $\gamma \in (0,1)$, and $|g(x_1, u)| < h(u)$ for some $h \in C^0(\mathbb{R})$. Let $U \in C^2(D) \cap C^0(\overline{D})$ be a bounded solution of*

$$\begin{cases} -\Delta u = g(x_1, u) \text{ in } D, \\ u = 0 \text{ on } \partial D. \end{cases}$$

*Then $S$, defined by $S(x_1) = \sup\{U(x_1, x');\ x' \in \mathbb{R}^{N-1}\}$, is continuous in* $[0, \infty)$, *with $S(0) = 0$, and satisfies*

$$\int_{\mathbb{R}_+} \big(S \cdot (-\varphi'') - g(x_1, S)\varphi\big)\, dx_1 \leq 0 \text{ for all } \varphi \in \mathcal{D}^+(\mathbb{R}_+). \tag{2.6}$$

*$\mathcal{D}^+(\mathbb{R}_+)$ consists of all nonnegative functions in $C_0^\infty(\mathbb{R}_+)$.*

PROOF OF LEMMA 2.6. Since $U$ and $\Delta U$ are bounded and $U = 0$ on $\partial D$, it follows from standard regularity properties that $U$ and all first-order derivatives are uniformly bounded and uniformly Hölder continuous with exponent $\gamma$. Let $\{\Omega_n\}$ be an increasing sequence of bounded subdomains of $D$, with smooth boundary and such that $\bigcup\limits_{n \in \mathbb{N}} \Omega_n = D$. We first prove that for each $n \in \mathbb{N}$, if $u_1, u_2 \in C(\Omega_n) \cap H^1(\Omega_n)$ satisfy

$$\int_D (u \cdot (-\Delta\varphi) - g(x_1, u) \cdot \varphi)\, dx \leq 0 \text{ for all } \varphi \in \mathcal{D}^+(\Omega_n), \tag{2.7}$$

then $u_3 = \sup(u_1, u_2)$ also satisfies (2.7).

Let $\omega \in \mathbb{R}_+$ be such that $u \to g(x_1, u) + \omega \cdot u$ is increasing on $[\min u_1 \wedge \min u_2, \max u_1 \vee \max u_2]$ for every $x \in \overline{\Omega}_n$.

We obtain

$$\int_D (u_i \cdot (-\Delta\varphi) + \omega \cdot u_i \cdot \varphi)\,\mathrm{d}x \leq \int_D (g(x_1, u_3) + \omega \cdot u_3) \cdot \varphi \mathrm{d}x$$

for all $\varphi \in \mathcal{D}^+(\Omega_n)$, $i = 1, 2$.

Set $h = g(x_1, u_3) + \omega \cdot u_3$ and let $w$ satisfy

$$\begin{cases} -\Delta w + \omega \cdot w = h \text{ in } \Omega_n, \\ w = 0 \text{ on } \partial\Omega_n. \end{cases}$$

Note that $w \in C(\overline{\Omega}_n) \cap H^1(\Omega_n)$. Then $w_i = u_i - w$, $i = 1, 2$, satisfies

$$\int_D (u \cdot (-\Delta\varphi) + \omega \cdot u \cdot \varphi)\,\mathrm{d}x \leq 0 \text{ for all } \varphi \in \mathcal{D}^+(\Omega_n). \tag{2.8}$$

It is known that $\sup(w_1, w_2)$ also satisfies (2.8), see [23, Th. 28.1]. Therefore $u_3$ satisfies (2.7). Note that $u_3 \in C(\overline{\Omega}_n) \cap H^1(\Omega_n)$. By induction it follows that if $u_i \in C(\overline{\Omega}_n) \cap H^1(\Omega_n)$, $i = 1, \ldots, k$, satisfies (2.7), then $\sup\{u_i;\ i = 1, \ldots, k\}$ also satisfies (2.7). Let $u_i$ be translates of $U$ perpendicular to $(1, 0, \ldots, 0)$. Since $U \in C(\overline{D}) \cap H^1_{\mathrm{loc}}(D)$, $\sup\{u_i;\ i = 1, \ldots, k\}$ will satisfy (2.7). Then by using the Lebesgue dominated convergence theorem and the fact that $U$ is bounded, one shows that

$$S(x_1) = \sup\{U(x_1, x');\ x' \in \mathbb{R}^{N-1}\} = \sup\{U(x_1, x');\ x' \in \mathbb{Q}^{N-1}\}$$

also satisfies (2.7) for each $n$. From the choice of the $\Omega_n$ it follows

$$\int_D (S(-\Delta\varphi) - g(x_1, S) \cdot \varphi)\,\mathrm{d}x \leq 0 \text{ for all } \varphi \in \mathcal{D}^+(D).$$

By choosing $\varphi$ of the form $\varphi_1 \cdot \varphi_2$, with $\varphi_1 \in \mathcal{D}^+(\mathbb{R}_+)$ and $\varphi_2 \in \mathcal{D}^+(\mathbb{R}^{N-1})$, $\varphi_2 \not\equiv 0$, one gets (2.6), since $S$ only depends on $x_1$.

Note that $S$, as the supremum of continuous functions, is lower semicontinuous on $[0, \infty)$. From (2.6) and the fact that $g(x_1, S)$ is bounded, we deduce that $S$ is the sum of a convex function on $(0, \infty)$ and a $C^1$-function on $[0, \infty)$. Hence $S \in C(0, \infty)$. Since $U(0, x') = 0$ and since $\dfrac{\partial}{\partial x} U(0, x')$ is uniformly bounded, $S(0) = 0$ and $S$ is continuous in 0. This completes the proof of Lemma 2.6.

PROOF OF PROPOSITION 2.5. Without loss of generality we assume that $f$ satisfies (F*). Define

$$I(x_1) = \inf\left\{U(x_1, x');\ x' \in \mathbb{R}^{N-1}\right\} \text{ and}$$
$$S(x_1) = \sup\left\{U(x_1, x');\ x' \in \mathbb{R}^{N-1}\right\}.$$

It is sufficient to prove that

$$I \geq u_\delta \text{ on } \mathbb{R}_+, \text{ and} \tag{2.9}$$

$$S \leq u_\delta \text{ on } \mathbb{R}_+, \tag{2.10}$$

for $\delta = \delta_1$.

We first prove (2.9) for $\delta = \delta_1$. By Lemma 2.4, 4), (2.9) holds with $\delta < -\delta_2$, since $I \geq 0$ on $\overline{\mathbb{R}}_+$. We will use a sweeping argument to prove (2.9) for every $\delta \in (-\delta_2 - 1, \delta_1)$. Let $\delta \in (-\delta_2 - 1, \delta_1)$. By Lemma 2.4, 3) and 4), there exists $\rho < \rho_2$ such that

$$\sup\{u_\theta(x_1);\ x_1 \in \mathbb{R}_+ : \theta \leq \delta\} \leq \rho. \tag{2.11}$$

For some $R > 0$ one has $I > \rho$ on $[R, \infty)$. It follows from Lemma 2.6, with $g(x_1, u) = -f(-u)$, that $I \in C[0, \infty)$, $I(0) = 0$ and

$$\int_{\mathbb{R}_+} \big(I \cdot (-\varphi'') - f(I) \cdot \varphi\big)\, \mathrm{d}x \geq 0 \text{ for all } \varphi \in \mathcal{D}^+(\mathbb{R}_+).$$

Hence $I$ is a supersolution of

$$\begin{cases} -u'' = f(u) \text{ in } (0, R), \\ u(0) = 0, \\ u(R) = \rho. \end{cases} \tag{2.12}$$

For $\theta \in [-\delta_2 - 1, \delta]$, (2.11) shows that $u_\theta$ is a subsolution of (2.12). We are now in the position to use Lemma A.2 and we obtain $I \geq u_\delta$ on $(0, R)$, hence on $\mathbb{R}_+$. For $x_1 \geq 0$ one has

$$I(x_1) \geq \lim_{\delta \uparrow \delta_1} u_\delta(x_1) = u_{\delta_1}(x_1).$$

This completes the proof of (2.9), with $\delta = \delta_1$.

Next we give a sketch of the proof of (2.10). Since $\frac{\partial}{\partial x_1} U$ is uniformly bounded, there exists $c > 0$ such that

$$S(x_1) < c \cdot x_1 \text{ for } x_1 \in \mathbb{R}_+.$$

By Lemma 2.4, 1), one has (2.10) with $\delta = \delta_1 + c$. Let $\delta \in (\delta_1, \delta_1 + c)$. Also from Lemma 2.4, 1), if follows

$$u_\theta(x_1) > \rho_2 + 1 \text{ for } x_1 > R := (\delta - \delta_1)^{-1}(\rho_2 + 1) \text{ and } \theta \in [\delta, \delta_1 + c].$$

Note that $S \leq \rho_2$. Then one concludes as above after using a sweeping argument for the problem

$$\begin{cases} -u'' = f(u), \text{ in } (0, R), \\ u(0) = 0, \\ u(R) = \rho_2. \end{cases}$$

This completes the proof of Proposition 2.5.

## 3. - Proof of the first theorem

NECESSITY: With $J(\rho) = \int_{\rho}^{\rho_2} f(s) \mathrm{d}s$, and assuming $\rho_1 > 0$, define

$$J^* := \min \{ J(\rho); \ \rho \in [0, \rho_1] \}.$$

Suppose condition (F2) is not satisfied, that is $J^* \leq 0$. Let $(\lambda, u)$ be a positive solution of (P) satisfying $\max u \in (\rho_1, \rho_2)$. We will obtain a contradiction.

First, if $J^* = 0$, modify $f$ to $f^*$ in $C^1$ such that $f > f^* > 0$ in $(\max u, \rho_2)$ and $f = f^*$ elsewhere. Still $u$ is a solution of $(\mathrm{P}_\lambda)$, but now $J^* < 0$. Hence we may assume without loss of generality that $J^* < 0$.

Consider the initial value problem

$$- v'' = f(v), \tag{3.1}$$

$$\begin{cases} v(0) &= \rho_2, \\ v'(0) &= -(-J^*)^{\frac{1}{2}}. \end{cases} \tag{3.2}$$

For a solution of (3.1), (3.2) one has:

$$\left(v'(r)\right)^2 = -J^* + 2 \int_{v(r)}^{\rho_2} f(s) \mathrm{d}s. \tag{3.3}$$

Set $\rho^* := \max \{ \rho \in [0, \rho_1]; \ J(\rho) = J^* \}$.

Because of (F1), $\left(v'(r)\right)^2 = -J^* + 2 \int_{v(r)}^{\rho_2} f(s) \mathrm{d}s \geq -J^* > 0$ holds for $v(r)$ in $[\rho_1, \rho_2]$, and hence $\inf v < \rho_1$.

Next we show that $v$ remains positive. If not, there exists an $r^*$ such that $v(r^*) = \rho^*$, and since (3.3) holds, one finds

$$\left(v'(r^*)\right)^2 = -J^* + 2 \int_{\rho^*}^{\rho_2} f(s) \mathrm{d}s = +J^* < 0,$$

a contradiction.

So either $v(r) \downarrow \tilde{\rho} \in (\rho^*, \rho_1)$ if $r \to \infty$, or $v$ has a first positive minimum, say in $\tilde{r}$, and $v$ is symmetric with respect to $\tilde{r}$. In the first case define

$$V(r) := \begin{cases} v(r) & \text{for } r > 0 \\ \rho_2 & \text{for } r \leq 0 \end{cases},$$

and in the second case

$$V(r) := \begin{cases} v(r) & \text{for } r \text{ in } (0, 2\tilde{r}) \\ \rho_2 & \text{elsewhere in } \mathbb{R} \end{cases}.$$

Set $w(\lambda, t; x) = V\left(\lambda^{\frac{1}{2}} \cdot (x_1 - t)\right)$, where $x = (x_1, \dots, x_N)$.

Then $\{w(\lambda, t; \cdot);\ t \in \mathbb{R}\}$ is a family of supersolutions, and for $t$ large enough $w(\lambda, t; \cdot) = \rho_2$ in $\Omega$.

By the sweeping principle $u < w(\lambda, t, \cdot)$ for all $t$.

Hence $u(x) \leq \inf\{w(\lambda, t; x);\ t \in \mathbb{R}\} = \inf v < \rho_1$, a contradiction.

REMARK 1. Let $f \in C^1$ satisfy (F1). The proof also shows that, if (F2) is not satisfied, there is no solution $u$ of $(\mathrm{P}_\lambda)$ with $\max u \in (\rho_1, \rho_2)$, even if $u$ changes sign.

REMARK 2. Let $f \in C^1$ satisfy (F1), and let $\Omega \subset \mathbb{R}^N$ be an unbounded domain.

Note that the same technique shows that problem

$$\begin{cases} -\Delta u = f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ \lim\limits_{\substack{|x| \to \infty \\ x \in \Omega}} u(x) = 0 \end{cases}$$

may have a solution $u$, with $\max u \in (\rho_1, \rho_2)$, only if condition (F2) is satisfied.

SUFFICIENCY: We will prove a stronger result, which will be used later on.

Let $x^* \in \Omega$. Then define $\lambda^* = \mu\alpha^2 \mathrm{d}(x^*, \Gamma)^{-2}$ and $z_\lambda = w(\lambda, x^*)$, where $\mu$, $\alpha$ and $w$ are defined in Corollary 2.3.

LEMMA 3.1. *Let $f$ satisfy* (F1), (F2) *and* (F*). *Then*

1) *for $\lambda > \lambda^*$ problem* $(\mathrm{P}_\lambda)$ *possesses a solution $u_\lambda \in [z_\lambda, \rho_2]$,*

2) *there exists $\lambda^{**} > \lambda^*$, $c > 0$ and $\tau \in (\rho_1, \rho_2)$, such that for $\lambda > \lambda^{**}$ every solution $u \in [z_\lambda, \rho_2]$ of* $(\mathrm{P}_\lambda)$ *satisfies*

$$u(x) > \min\left(c\lambda^{\frac{1}{2}} \mathrm{d}(x, \Gamma), \tau\right) \text{ for all } x \in \Omega. \tag{3.4}$$

REMARK 3. It follows from (3.4) that $u_\lambda > 0$ for $\lambda > \lambda^{**}$, and that $\max u_\lambda \in (\rho_1, \rho_2)$, for $\lambda$ large enough.

REMARK 4. Lemma 3.1, 2), shows $\frac{\partial}{\partial n} u_\lambda < 0$ on $\Gamma$ for $\lambda > \lambda^{**}$, even when $f(0) < 0$. ($\frac{\partial}{\partial n}$ denotes the outward normal derivative)

PROOF OF LEMMA 3.1. By Corollary 2.3 one knows that for $\lambda > \lambda^*$, $z_\lambda$ is a subsolution of $(\mathrm{P}_\lambda)$, with $z_\lambda < \rho_2$. Since $\rho_2$ is a supersolution of $(\mathrm{P}_\lambda)$, Lemma A.1 yields a solution $u_\lambda \in [z_\lambda, \rho_2]$ of $(\mathrm{P}_\lambda)$, for $\lambda > \lambda^*$. This completes the proof of the first assertion.

Since $\Omega$ satisfies a uniform interior sphere condition, there exists $\varepsilon_0 > 0$ such that $\Omega = \bigcup \{B(x,\varepsilon);\ x \in \Omega_\varepsilon\}$ for $\varepsilon \in (0, \varepsilon_0]$, where $\Omega_\varepsilon = \{x \in \Omega;\ \mathrm{d}(x,\Gamma) > \varepsilon\}$. Set

$$\begin{aligned} \lambda^{**} &= \max(\lambda^*, \mu\alpha^2\varepsilon_0^{-2}),\\ c &= \mu^{-\frac{1}{2}} \inf\left\{(\alpha - r)^{-1} \cdot v(r);\ r \in [0,\alpha)\right\} \text{ and}\\ \tau &= v(0) \end{aligned}$$

with $\mu$, $v$ and $\alpha$ defined in Corollary 2.3.

Note that $c > 0$, since $v > 0$ on $[0,\alpha)$ and $v'(\alpha) < 0$.

Let $(\lambda, u)$ be a solution of (P) with $\lambda > \lambda^{**}$ and $u \in [z_\lambda, \rho_2]$. Since for $\lambda > \lambda^{**}$, $\Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}$ is arcwise connected and since $w(\lambda, y)$ is a subsolution for $y \in \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}$, with $w(\lambda, y) < 0$ on $\Gamma$, one finds by Lemma A.2 that

$$u > w(\lambda, y) \text{ in } \Omega \text{ for all } y \in \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}.$$

Hence

$$\begin{aligned} &u(x) > c\lambda^{\frac{1}{2}}\mathrm{d}(x,\Gamma) \text{ for all } x \in \Omega \setminus \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}, \text{ and}\\ &u(x) > \tau \text{ for all } x \in \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}, \end{aligned}$$

which completes the proof.

## 4. - Proof of the second theorem

As mentioned in the introduction Theorem 2 will be a consequence of a sharper version, Theorem 2'.

THEOREM 2'. *Let* $\Gamma \in C^3$ *and let* $f \in C^{1,\gamma}$, *for some* $\gamma \in (0,1)$, *satisfy* (F1), (F2) *and* (F3). *Then for some* $\lambda_1 > 0$,

1) *there exists* $\varphi \in C^1\left([\lambda_1, \infty); C^2(\overline{\Omega})\right)$, *such that* $(\lambda, \varphi(\lambda))$ *is a solution of* (P) *for* $\lambda \geq \lambda_1$, *with* $\varphi(\lambda) > 0$ *in* $\Omega$, $\max \varphi(\lambda) \in (\rho_1, \rho_2)$ *and* $\lim_{\lambda \to \infty} \max \varphi(\lambda) = \rho_2$;
2) *if* $\mu_0(\lambda, u)$ *denotes the principal eigenvalue of*

$$\text{(LP)} \qquad \begin{cases} -\lambda^{-1} \cdot \Delta h - f'(u) \cdot h = \mu h \text{ in } \Omega,\\ h = 0 \text{ on } \Gamma, \end{cases}$$

*then* $\mu_0(\lambda, \varphi(\lambda)) > 0$ *for* $\lambda > \lambda_1$*;*

3) *for all nonnegative* $z \in C_0^\infty(\Omega)$ *with* $\max z \in (\rho_1, \rho_2)$*, there exists* $\lambda(z) > \lambda_1$*, such that, if* $(\lambda, u)$ *is a solution of* (P) *with* $\lambda > \lambda(z)$ *and* $u \in [z, \rho_2]$*, then* $u = \varphi(\lambda)$.

REMARK 1. Theorem 2 follows from theorem 2′ by choosing a nonnegative function $z_0 \in C_0^\infty(\Omega)$ and setting $\lambda_0 = \lambda(z_0)$ in the third assertion of Theorem 2′.

REMARK 2. If $\rho_1 > 0$, let $\mathcal{C}$ denote the component of solutions of (P) in $\mathbb{R}_+ \times C^2(\overline{\Omega})$ containing $\{(\lambda, \varphi(\lambda)); \lambda \geq \lambda_1\}$. Since $\mathcal{C}$ is connected, one has for $(\lambda, u) \in \mathcal{C}$ that $\max u \in (\rho_1, \rho_2)$ (see [2]) and $\lambda > 0$. By using degree arguments as in [19], [20], one can show that for $\lambda$ large enough, $\mathcal{C} \cap \big(\{\lambda\} \times C^2(\overline{\Omega})\big)$ contains at least two solutions of (P). The proof of this assertion will appear elsewhere.

For the proof of Theorem 2′ we need the following lemmas.

LEMMA 4.1. *Let* $f \in C^1$ *satisfy* (F1), (F2) *and* (F*). *For every* $\delta > 0$ *there is a* $c(\delta) > 0$*, such that for all solutions* $(\lambda, u)$ *of* (P)*, with* $\lambda > \lambda^{**}$ *and* $u \in [z_\lambda, \rho_2]$*, the following holds*

$$u(x) > \min\left(c(\delta)\lambda^{\frac{1}{2}}\mathrm{d}(x, \Gamma),\ \ \rho_2 - \delta\right) \text{ for all } x \in \Omega, \tag{4.1}$$

*with* $\lambda^{**}$ *and* $z_\lambda$ *as in Lemma* 3.1.

PROOF OF LEMMA 4.1. If $\rho_2 - \delta < \tau$, we are done with $c(\delta) = c$ as in Lemma 3.1. Otherwise, by (F1) there exists $\sigma > 0$ such that $\sigma(u - \tau) < f(u)$ for all $u \in [\tau, \rho_2 - \delta]$.

Let $\nu$ denote the principal eigenvalue of

$$\begin{cases} -\Delta\psi = \nu\psi \text{ in } B, \\ \quad \psi = 0 \text{ on } \partial B, \end{cases}$$

where $B$ denotes the unit ball in $\mathbb{R}^N$.

Then by using Lemma A.3 with $\Omega' = \Omega_{k\lambda^{-\frac{1}{2}}}$, $k = c^{-1}\tau$, one finds

$$u(x) > \rho_2 - \delta \text{ for all } x \in \Omega_{\left((\nu/\sigma)^{\frac{1}{2}} + k\right)\lambda^{-\frac{1}{2}}}, \tag{4.2}$$

since $(\Omega')_{(\nu/\sigma\lambda)^{\frac{1}{2}}} = \Omega_{\left((\nu/\sigma)^{\frac{1}{2}} + k\right)\lambda^{-\frac{1}{2}}}$.

By (3.4) one finds

$$u(x) > c(\delta)\lambda^{-\frac{1}{2}}\mathrm{d}(x, \Gamma) \text{ for all } x \in \Omega \setminus \Omega_{\left((\nu/\sigma)^{\frac{1}{2}} + k\right)\lambda^{-\frac{1}{2}}} \tag{4.3}$$

with $c(\delta) = \tau\left((\nu/\sigma)^{\frac{1}{2}} + k\right)^{-1}$

This completes the proof of the lemma.

LEMMA 4.2. *Let $f \in C^{1,\gamma}$, for some $\gamma \in (0,1)$, satisfy* (F1), (F2), (F3) *and* (F*). *Then there exists $\lambda_1 > \lambda^{**}$, such that for every solution $u$ of* ($P_\lambda$), *with $\lambda > \lambda_1$ and $u \in [z_\lambda, \rho_2]$, one finds $\mu_0(\lambda, u) > 0$.*

PROOF. Suppose this is not the case. Then there exists a sequence $\{(\lambda_n, u_n);\ n \in \mathbb{N}\}$ of solutions of (P), with $u_n \in [z_{\lambda_n}, \rho_2]$, $\mu_n := \mu_0(\lambda_n, u_n) \leq 0$ for all $n$, and $\lim_{n\to\infty} \lambda_n = \infty$.

Let $\varepsilon$ be defined by (F3). Since $\mu_n \leq 0$, for all $n$, the associated eigenfunctions $v_n$, normalized by $\max v_n = 1$, satisfy

$$
-\lambda_n^{-1}\Delta v_n(x) = \left(f'(u_n(x)) + \mu_n\right) v_n(x) \leq 0 \text{ for } x \in \Omega_{K\lambda_n^{-\frac{1}{2}}}, \tag{4.4}
$$

where $K = (c(\varepsilon))^{-1}(\rho_2 - \varepsilon)$.

The constant $c(\varepsilon)$ is defined in the previous lemma.

Hence the function $v_n$ is subharmonic in $\Omega_{K\lambda_n^{-\frac{1}{2}}}$, and $v_n$ attains its maximum outside of $\Omega_{K\lambda_n^{-\frac{1}{2}}}$. Like in [3] let $y^n \in \Omega \setminus \Omega_{K\lambda_n^{-\frac{1}{2}}}$ be a point where $v_n$ attains its maximum and let $x^n \in \Gamma$ be a point which minimizes $\{\mathrm{d}(x, y^n);\ x \in \Gamma\}$. Since $\{x^n\}$ and $\{\mu_n\}$ are bounded, there exists a subsequence, still denoted $\{(\lambda_n, u_n)\}$, such that $\lim_{n\to\infty} x^n = \overline{x} \in \Gamma$ and $\lim_{n\to\infty} \mu_n = \overline{\mu} \leq 0$. Let $\mathcal{O}$ be an open neighbourhood of $\overline{x}$ in $\mathbb{R}^N$, chosen so small that it permits $C^3$ local coordinates $(\xi_1, \ldots, \xi_N)$: $\mathcal{O} \to \mathbb{R}^N$, such that $x \in \Omega \cap \mathcal{O}$ if and only if $\xi_1(x) > 0$, and $\xi(\overline{x}) = 0$. In these coordinates the Laplacian is given by

$$
\Delta u = \sum_{i,j} a_{ij}(\xi) \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_j} \tilde{u} + \sum_j b_j(\xi) \frac{\partial}{\partial \xi_j} \tilde{u},
$$

where $a_{ij} \in C^2$, $b_j \in C^1$ and $u(x) = \tilde{u}(\xi(x))$.

Moreover we choose the local coordinates such that $a_{ij}(0) = \delta_{ij}$. Next define the functions

$$
\begin{aligned}
U_n(\eta) &= \tilde{u}_n\left(\xi(x^n) + \lambda_n^{-\frac{1}{2}}\eta\right), \\
V_n(\eta) &= \tilde{v}_n\left(\xi(x^n) + \lambda_n^{-\frac{1}{2}}\eta\right), \quad \eta \in D.
\end{aligned}
$$

Since $\{U_n\}$ and $\{V_n\}$ are precompact in $C^2_{\mathrm{loc}}$, there exists a convergent subsequence. Hence there are $U, V \in C^2(\overline{D})$, bounded and positive in $D =$

$\{(x_1, x');\ x_1 > 0, x' \in \mathbb{R}^{N-1}\}$, satisfying respectively

$$\begin{cases} -\Delta U = f(U) \text{ in } D, \\ U = 0 \text{ on } \partial D, \end{cases}$$

$$\begin{cases} -\Delta V - f'(U)V = \overline{\mu}V \text{ in } D, \\ V = 0 \text{ on } \partial D. \end{cases}$$

Moreover by Lemma 4.1 the following inequalities,

$$\min(c(\delta)x_1, \rho_2 - \delta) \le U(x_1, x') \le \rho_2 \text{ for all } x_1 > 0, x' \in \mathbb{R}^{N-1}, \tag{4.5}$$

hold for every $\delta > 0$. From Proposition 2.5 we have

$$U(x_1, x') = u_{\delta_1}(x_1) \text{ for } x_1 \ge 0, x' \in \mathbb{R}^{N-1}.$$

Set $S(x_1) = \sup\{V(x_1, x'); x' \in \mathbb{R}^{N-1}\}$. Then $0 < S \le 1$ in $\mathbb{R}_+$ and we obtain by using Lemma 2.6 that $S \in C[0, \infty)$, $S(0) = 0$ and

$$\int_{\mathbb{R}_+} \left(S \cdot (-\varphi'') - \left(f'(u_{\delta_1}) + \overline{\mu}\right) S\varphi\right) \mathrm{d}x \le 0 \text{ for all } \varphi \in \mathcal{D}^+(\mathbb{R}_+). \tag{4.6}$$

Since $u'_{\delta_1} > 0$ on $\overline{\mathbb{R}}_+$, there exists a smallest $C > 0$ such that $W := Cu'_{\delta_1} - S \ge 0$ on $[0, K + 1]$, where $K$ is defined in (4.4). Then one finds by using (4.6) and $-(u'_{\delta_1})'' = f'(u_{\delta_1})u'_{\delta_1}$ in $\mathbb{R}_+$, that

$$\int_{\mathbb{R}_+} \left(W \cdot (-\varphi'') - f'(u_{\delta_1})W\varphi\right) \mathrm{d}x \ge 0 \text{ for all } \varphi \in \mathcal{D}^+(\mathbb{R}_+). \tag{4.7}$$

Since $W$ is nonnegative in $[0, K + 1]$, there is $\omega > 0$ such that

$$\int_{\mathbb{R}_+} \left(W \cdot (-\varphi'') + \omega W\varphi\right) \mathrm{d}x \ge 0 \text{ for all } \varphi \in \mathcal{D}^+((0, K + 1)).$$

By [5, Corollary p. 581] and the fact that $W \not\equiv 0$, one obtains

$$W \ge bx(K + 1 - x) \text{ for all } x \in [0, K + 1] \text{ and some } b > 0. \tag{4.8}$$

By construction $W$ vanishes somewhere in $[0, K + 1]$. Since $W(0) > 0$ one finds $W(K + 1) = 0$. Moreover $f'(u_{\delta_1}) \le 0$ on $(K, \infty)$. Hence (4.6) yields that $S$ is convex on $(K, \infty)$. Since $W$ is the sum of a $C^1$ and a concave function on $(K, \infty)$, (4.8) shows $0 > \frac{\mathrm{d}^-}{\mathrm{d}x}W(K + 1) \ge \frac{\mathrm{d}^+}{\mathrm{d}x}W(K + 1)$, and therefore $W(x) < 0$ on $(K + 1, K + 1 + c)$ for some $c > 0$. Moreover $W$ cannot vanish on $(K + 1, \infty)$.

Otherwise there would be $c > 0$ such that $W < 0$ on $(K+1, K+1+c)$ and $W(K+1) = W(K+1+c) = 0$. But this cannot happen since by (4.7) $W$ is concave as long as $W$ is negative on $(K, \infty)$.

Hence $W$ is concave on $(K+1, \infty)$. Since $\frac{d^+}{dx} W(K+1) < 0$, $W$ is not bounded below, contradicting $W = Cu'_{\delta_1} - S \geq -1$ on $\mathbb{R}_+$. This completes the proof of Lemma 4.2.

It follows from Lemma 4.2 that for $\lambda > \lambda_1$ $(P_\lambda)$ possesses at most one solution in $[z_\lambda, \rho_2]$. Indeed, choose $\omega > 0$ such that $\lambda f'(u) + \omega > 0$ for $u \in [0, \rho_2]$, and define the mapping $K: C(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$K(u) := (-\Delta + \omega)^{-1} (\lambda f(u) + \omega u),$$

where $(-\Delta+\omega)^{-1}$ is the inverse of $-\Delta+\omega$ with homogeneous Dirichlet boundary conditions. By our choice of $\omega$, $K$ maps $[z_\lambda, \rho_2]$ into itself and $K$ has no fixed point on its boundary. Since $K$ is compact, the Leray-Schauder degree on $(z_\lambda, \rho_2)$ is well defined. Because $(z_\lambda, \rho_2)$ is convex one finds

$$\text{degree } (I - K, (z_\lambda, \rho_2), 0) = 1.$$

If $(\lambda, u)$ is a solution of (P), with $u \in [z_\lambda, \rho_2]$ and $\mu_0(\lambda, u) > 0$, it follows that $u$ is an isolated fixed point of $K$. Moreover, the local degree of $I - K$ at $u$ is +1. From the additivity of degree it follows that $K$ possesses at most one fixed point in $(z_\lambda, \rho_2)$. We denote this solution by $\varphi(\lambda)$. Since $\mu_0(\lambda, \varphi(\lambda)) > 0$, for $\lambda > \lambda_1$, one finds by the implicit function theorem and Schauder estimates, that $\lambda \to \varphi(\lambda) \in C^1\left([\lambda_1, \infty); C^{2,\gamma}(\overline{\Omega})\right)$. The estimate (4.1) implies that $\lim\limits_{\lambda \to \infty} \max \varphi(\lambda) = \rho_2$.

It remains to prove the third assertion of theorem 2′. Let $z \in \mathcal{D}^+(\Omega)$ with $\max z \in (\rho_1, \rho_2)$. It follows from the first part of the proof, that it is sufficient to show that there exists $\lambda(z) > \lambda_1$, such that any solution $u$ of $(P_\lambda)$, with $\lambda > \lambda(z)$ and $u \in [z, \rho_2]$, is larger than $z_\lambda$. This will be done in two steps.

First note that, from the definition of $z$, there exist $s \in (\rho_1, \rho_2)$ and a ball $B(x_0, r) \subset \Omega$, such that $z > s$ in $B(x_0, r)$. Let $\sigma > 0$ be such that $f(u) > \sigma \cdot (u - s)$ for $u \in [s, \tau]$, where $\tau = \max z_\lambda$. For $\lambda > \lambda_1(z) := \left((\nu/\sigma)^{\frac{1}{2}} + \mu^{\frac{1}{2}}\right)^2 r^{-2}$, with $\mu$ defined in Lemma 2.2, we can apply Lemma A.3 in order to get

$$u(x) > \tau \text{ for } x \in B\left(x_0, (\mu/\lambda)^{\frac{1}{2}}\right) \subset B\left(x_0, r - (\nu/\sigma\lambda)^{\frac{1}{2}}\right).$$

Observe that $w(\lambda, x_0) < u$ in $\Omega$ for $\lambda > \lambda_1(z)$. By Corollary 2.3 $w(\lambda, x_0)$ is a subsolution of $(P_\lambda)$ for $\lambda > \lambda_1(z)$.

Finally, like in proof of Lemma 3.1 part 2), one uses Lemma A.2 to show that if $u > w(\lambda, x_0)$ in $\Omega$ and $\lambda > \lambda(z) := \max(\lambda_1(z), \lambda^{**})$ also the following estimate holds,

$$u > w(\lambda, x^*) = z_\lambda.$$

This completes the proof of Theorem 2′.

## 5. - Proof of the third theorem

Note that, if $(\lambda, u)$ is a positive solution of (P), then $v := (u(0))^{-1}u$ satisfies

$$\begin{cases} -\Delta v = (u(0))^{\alpha-1}\lambda v^\alpha g(u(0)v) \text{ in } B \\ v = 0 \text{ on } \partial B. \end{cases}$$

Moreover by defining $w(r) := v(R^{-1}r)$ with $\varepsilon = u(0)$ and

(5.1) $$R = u(0)^{\frac{1}{2}(\alpha-1)}\lambda^{\frac{1}{2}} \text{ one gets}$$

(5.2) $$-w'' - \frac{N-1}{r}w' = w^\alpha g(\varepsilon w)$$

(5.3) $$\begin{cases} w(0) = 1 \\ w'(0) = 0 \end{cases}$$

$$\begin{cases} w(R) = 0 \\ \quad w > 0 \text{ on } [0, R). \end{cases}$$

Let $w(\varepsilon, \cdot)$ denote the unique solution of the initial value problem (5.2-5.3)

LEMMA 5. *There exists $\varepsilon_1 > 0$ such that for $\varepsilon$ in $[0, \varepsilon_1)$, $w(\varepsilon, \cdot)$ possesses a first zero, which we denote by $R(\varepsilon)$. Moreover $R$ as a function of $\varepsilon$ is $C^1(0, \varepsilon_1) \cap C[0, \varepsilon_1)$ and $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}R$ is bounded on $(0, \frac{1}{2}\varepsilon_1)$.*

We first show that the assertion of Theorem 3 is an easy consequence of this lemma. By (5.1) we have $\lambda(\varepsilon) = R(\varepsilon)^2\varepsilon^{1-\alpha}$, and hence

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda(\varepsilon) = R(\varepsilon)\varepsilon^{-\alpha}\left(2\varepsilon\frac{\mathrm{d}}{\mathrm{d}\varepsilon}R(\varepsilon) + (1-\alpha)R(\varepsilon)\right), 0 < \varepsilon < \varepsilon_1.$$

Since $\alpha - 1 > 0$, $R(0) > 0$ and $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}R$ is bounded on $\left(0, \frac{1}{2}\varepsilon_1\right)$, it follows that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda(\varepsilon) < 0$ on some interval $(0, \varepsilon_0)$.

Then for $\lambda > \lambda(\varepsilon_0)$, $u_\lambda(r) = \varepsilon(\lambda)w(R(\varepsilon(\lambda))r)$ is a solution of $(\mathrm{P}_\lambda)$ on the unit ball, where $\varepsilon(\lambda)$ is the inverse of the function $\lambda(\varepsilon)$. This function $\varepsilon(\lambda)$ is well defined on $(\lambda(\varepsilon_0), \infty)$, decreasing and satisfies $\lim\limits_{\lambda\to\infty} \varepsilon(\lambda) = 0$. This completes the proof of the theorem.

PROOF OF LEMMA 5. It is known, see [17], that (5.2-5.3) with $\varepsilon = 0$ possesses a solution $w$, having a first positive zero which we denoted by $R(0)$. We want to obtain the function $w(\varepsilon, \cdot)$ by a perturbation argument.

Since we are only interested in bounded positive solutions, we modify the right-hand-side of (5.2) by setting $h(\varepsilon, w) = k(w)g(\varepsilon w)$ where $k$ is a $C^1$-function satisfying

$$k(w) = \begin{cases} 0 & \text{for } w \leq 0 \\ w^\alpha & \text{for } 0 < w < 1 \\ 0 & \text{for } w \geq 2. \end{cases}$$

The function $h$ is $C^1((-1,1)\times\mathbb{R})$ and has bounded derivatives. The initial value problem

$$-w'' - \frac{N-1}{r}w' = h(\varepsilon, w), \ \ \varepsilon \text{ in } (-1,1), \tag{5.4}$$

$$\begin{cases} w(0) = 1 \\ w'(0) = 0, \end{cases} \tag{5.5}$$

possesses a unique solution $w(\varepsilon, \cdot)$ on $[0,\infty)$.

For $\varepsilon$ in $[0,1)$, since $w(\varepsilon, \cdot)$ is decreasing until it possibly becomes zero, this function $w(\varepsilon, \cdot)$ is identical with the one in the lemma, as long as it is positive.

We claim, for every $r > 0$, $w(\cdot, r)$ is a $C^1$-function of $\varepsilon$. First this will be proved for $r \in (0,\delta)$, with $\delta$ small enough. Note that (5.4-5.5) can be rewritten as $w = T(\varepsilon, w)$, where $T(\varepsilon, z)(r) = 1 - \int_0^r t^{1-N} \int_0^t s^{N-1} h(\varepsilon, z(s))\mathrm{d}s\mathrm{d}t$, for $z$ in $C[0,\delta]$. For every $\delta > 0$, $T : (-1,1)\times C[0,\delta] \to C[0,\delta]$, where $C[0,\delta]$ is equipped with the supremum-norm, is continuously Fréchet-differentiable. For $\delta$ small enough, $T(\varepsilon, \cdot) : C[0,\delta] \to C[0,\delta]$ is a strict contraction with a unique fixed point $z(\varepsilon)$ such that $\varepsilon \to z(\varepsilon)$ is continuously differentiable.

Since $w(\varepsilon, r) = z(\varepsilon)(r)$, the claim is proved for $r < \delta$.

By repeating the argument it can be shown that $\varepsilon \to w(\varepsilon, r)$ is continuously differentiable for every $r > 0$.

Since $w(0, R(0)) = 0$ and $w_r(0, R(0)) < 0$ it follows from the implicit function theorem, that there exists $\varepsilon_1 > 0$ and a continuously differentiable function $R(\cdot)$, defined on $(-\varepsilon_1, \varepsilon_1)$, such that $w(\varepsilon, R(\varepsilon)) = 0$. From (5.4) it follows that $R(\varepsilon)$ is the unique zero of $w(\varepsilon, \cdot)$ on $\mathbb{R}^+$. This completes the proof.

PROOF OF THE COROLLARY. Since $u(0)$ parametrizes the solutions $(\lambda, u)$ of (P), $\rho^* = \inf\{\sigma > 0;$ (P) has a solution $(\lambda, u)$, with $u(0) = \rho$, for all $\rho \in [\sigma, \rho_2)\}$. Suppose $\rho^* > 0$ and let $v$ be the solution of the initial value problem

$$-v'' - \frac{N-1}{r}v' = f(v), \tag{5.6}$$

$$\begin{cases} v(0) = \rho^*, \\ v'(0) = 0. \end{cases} \tag{5.7}$$

Since $f(\rho) > 0$ on $(0, \rho^*]$, $v$ is strictly decreasing while $v$ is positive. If $v$ has a (first) positive zero $R$, then $(R^2, v(R^{-1}, \cdot))$ is a solution of (P), which contradicts the definition of $\rho^*$. If $v$ stays positive, then

$$\lim_{r\to\infty} v(r) = 0. \tag{5.8}$$

Otherwise, there are $c > 0$ and $R > 0$ such that $f(v(s)) > c$ for $s > R$. By integrating (5.6), one finds

$$v'(r) = (R/r)^{N-1}v'(R) - r^{1-N}\int_R^r s^{N-1}f(v(s))\mathrm{d}s \leq$$
$$\leq (R/r)^{N-1}v'(R) - (c/N)(r - R(R/r)^{N-1}) < -1,$$

for $r$ large enough, contradicting the fact that $v$ stays positive. The existence of a positive function satisfying (5.6-5.8), is contradicted by Theorem 2.2 of [15], if $\alpha \leq N/(N-2)$, and by Theorem 3.1 of [15], if the integral condition of (G2) is satisfied. Therefore $\rho^* = 0$.

This completes the proof.

## 6. - Appendix

In this section we state, for the sake of completeness, a definition and some lemmas concerning sub- and supersolutions of problem

$$\begin{cases} -\Delta u = h(u) \text{ in } \Omega \subset \mathbb{R}^N, \\ u = g \text{ on } \Gamma, \end{cases} \tag{H}$$

where $\Omega$ is a bounded domain with $C^3$-boundary, $h \in C^1$ and $g \in C^0$.

DEFINITION. *We call a function $v$ a subsolution (supersolution) of* (H) *if:*

i) $v \in C(\overline{\Omega})$,

ii) $v \leq (\geq)\ g$ *on* $\partial\Omega$, *and*

iii) $\int_\Omega (v\cdot(-\Delta\varphi) - h(v)\varphi)\mathrm{d}x \leq (\geq)\ 0$ *for every* $\varphi \in \mathcal{D}^+(\Omega)$, *where* $\mathcal{D}^+(\Omega)$ *consists of all nonnegative functions in* $C_0^\infty(\Omega)$.

LEMMA A.1. *Let $v$ and $w$ be respectively a sub- and supersolution of* (H) *with $g = 0$. If $v \leq w$ in $\Omega$, then there exists a solution $u \in C^2(\overline{\Omega})$ of* (H) *with $g = 0$, which satisfies $v \leq u \leq w$.*

PROOF. We essentially follow the proof in [21] on page 24. Choose a number $\omega > 0$ such that $h'(u) + \omega \geq 0$ for $\min v \leq u \leq \max w$, and define the nonlinear map $T$ by $u_1 = Tu$, where

$$\begin{cases} -\Delta u_1 + \omega u_1 = h(u) + \omega u \text{ in } \Omega, \\ u_1 = 0 \text{ on } \partial\Omega. \end{cases}$$

Clearly $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ is compact. (Where $C(\overline{\Omega})$ is equipped with the supremum-norm)

It is standard that $T$ is monotone on $[v, w]$. Next we show that $v_1 := Tv \geq v$ in $\Omega$.

By the definition of a subsolution and by the construction of $v_1$, we have

$$\int_\Omega (v \cdot (-\Delta\varphi) + \omega v\varphi)\mathrm{d}x \leq \int_\Omega (h(v) + \omega v)\varphi \mathrm{d}x =$$

$$= \int_\Omega (v_1 \cdot (-\Delta\varphi) + \omega v_1 \varphi)\mathrm{d}x \text{ for every } \varphi \in \mathcal{D}^+(\Omega).$$

Thus $z = v_1 - v$ satisfies $z \geq 0$ on $\partial\Omega$, and

$$\int_\Omega (z \cdot (-\Delta\varphi) + \omega z\varphi)\mathrm{d}x \geq 0 \text{ for every } \varphi \in \mathcal{D}^+(\Omega).$$

We claim that $z$ is nonnegative in $\Omega$.

Otherwise there exists a ball $B(x_0, r) \subset \Omega$, such that $z$ is negative in $B(x_0, r)$ and achieves its minimum in $x_0$.

Hence

$$\int_\Omega z \cdot (-\Delta\varphi)\mathrm{d}x \geq 0 \text{ for every } \varphi \in \mathcal{D}^+(B(x_0, r)).$$

This shows $z$ is superharmonic on $B(x_0, r)$, and from the minimum principle we get $z(x) = z(x_0)$ on $B(x_0, r)$.

Then

$$\int_{B(x_0,r)} (z \cdot (-\Delta\varphi) + \omega z\varphi)\mathrm{d}x = \omega z(x_0) \int_{B(x_0,r)} \varphi \mathrm{d}x < 0$$

for every nontrivial $\varphi \in \mathcal{D}^+(B(x_0, r))$, a contradiction. Thus $Tv = v_1 \geq v$ on $\overline{\Omega}$. Similarly, one proves $Tw \leq w$ on $\overline{\Omega}$. Now it is standard, see [1], that $T$ possesses a fixed point in $[v, w]$, which is a solution of (H) with $g = 0$.

Next we prove an appropriate version of the sweeping principle of Serrin, [22], [21].

Let $\Gamma = \partial\Omega$ be the union of two disjoint closed subsets $\Gamma_1$ and $\Gamma_2$, where $\Gamma_1$ or $\Gamma_2$ may be empty. Let $e \in C^1(\overline{\Omega})$ satisfy $e > 0$ on $\Omega \cap \Gamma_1$ and $e = 0$, $\dfrac{\partial e}{\partial n} < 0$ on

$\Gamma_2$. ($n$ is the outward normal) Set $C_e(\overline{\Omega}) = \{u \in C(\overline{\Omega});\ |u| \leq \alpha e$ for some $\alpha > 0\}$ and for $u \in C_e(\overline{\Omega})$ define $\|u\|_e = \inf\{\alpha > 0;\ |u| \leq \alpha e\}$.

LEMMA A.2. *Let $u$ be a supersolution of* (H) *and let $A = \{v_t;\ t \in [0,1]\}$ be a family of subsolutions of* (H) *satisfying $v_t < g$ on $\Gamma_1$ and $v_t = g$ on $\Gamma_2$, for all $t \in [0,1]$. If*
1) $t \to (v_t - v_0) \in C_e(\overline{\Omega})$ *is continuous with respect to the* $\|\cdot\|_e$*-norm,*
2) $u \geq v_0$ *in* $\overline{\Omega}$, *and*
3) $u \not\equiv v_t$, *for all* $t \in [0,1]$,

*then there exists $\alpha > 0$, such that for all $t \in [0,1]$ $u - v_t \geq \alpha e$ in $\overline{\Omega}$.*

PROOF. Set $E = \{t \in [0,1];\ u \geq v_t$ in $\overline{\Omega}\}$. By 2) $E$ is not empty. Moreover $E$ is closed. For $t \in E$ $w_t := u - v_t$ satisfies

$$\int_\Omega (w \cdot (-\Delta\varphi) + \omega w \varphi)\mathrm{d}x \geq 0 \text{ for all } \varphi \in \mathcal{D}^+(\Omega) \text{ and some } \omega > 0.$$

Since $w_t \not\equiv 0$ it follows from [5, Corollary p. 581] that there is $\beta > 0$, such that $w_t \geq \beta u_0$, for some $u_0 \in C^1(\overline{\Omega})$, which satisfies $u_0 > 0$ in $\Omega, u_0 = 0$ and $\dfrac{\partial}{\partial n}u_0 < 0$ on $\Gamma$. The function $w_t$ is positive on $\Gamma_1$, which is compact, and continuous on $\overline{\Omega}$. Hence there exists $\gamma > 0$ such that $w_t \geq \gamma e$. Since $t \to (w_t - w_0)$ is continuous with respect to the $\|\cdot\|_e$-norm, $E$ is also open. Hence $E = [0,1]$ and there is $\alpha > 0$, such that $w_t \geq \alpha e$ in $\overline{\Omega}$ for all $t \in [0,1]$.

This completes the proof of Lemma A.2.

Let $\psi$ be the principal eigenfunction, with eigenvalue $\nu$, of

$$\begin{cases} -\Delta v = \lambda v \text{ in } B, \\ v = 0 \text{ on } \partial B, \end{cases}$$

where $B$ denotes the unit ball in $\mathbb{R}^N$.

Let $\psi$ be normalized such that $\max \psi = 1$.

LEMMA A.3. *Let $u$ satisfy $-\Delta u = \lambda f(u)$ in an open $\Omega' \subset \Omega$, such that $u(x) > a$ for $x \in \Omega'$. Let $\sigma > 0$ be such that $f(u) > \sigma(u-a)$ for $u \in [a,b]$.*

*If $x_0 \in (\Omega')_{(\nu/\sigma\lambda)^{\frac{1}{2}}}$, then $u(x_0) > b$.*

PROOF. Set $\theta(x_0, \lambda, t; x) = a + t\psi((\sigma\lambda/\nu)^{\frac{1}{2}}(x - x_0))$ for $x \in B(\ )$ and $t \in [0, b-a]$, where $B(\ ) = B(x_0, (\nu/\sigma\lambda)^{\frac{1}{2}})$. The set $\{\theta(x_0, \lambda, t); t \in [0, b-a]\}$ is a family of subsolutions of the problem

$$\text{(Pb)} \qquad \begin{cases} -\Delta v = \lambda f(v) \text{ in } B(\ ) \\ v = u \text{ on } \partial B(\ ), \end{cases}$$

and $\overline{B(\ )} \subset \Omega'$.

By using Lemma A.2 one finds $u(x_0) > b$.

It remains to show that $\theta(x_0, \lambda, t)$ is a subsolution of $(\mathrm{Pb}_\lambda)$. By the assumption of the lemma $u > a = \theta(x_0, \lambda, t)$ on $\partial B(\ )$.

The integral condition is also satisfied:

$$\int_{B(\ )} (\theta(-\Delta\varphi) - \lambda f(\theta)\varphi)\mathrm{d}x = \int_{B(\ )} (-\Delta\theta - \lambda f(\theta))\varphi \mathrm{d}x \leq$$

$$\leq \int_{B(\ )} (-\Delta\theta - \lambda\sigma(\theta - a))\varphi \mathrm{d}x = 0 \text{ for all } \varphi \in \mathcal{D}^+(B(\ ))$$

This completes the proof of the lemma.

**Acknowledgement**

We would like to thank S.B. Angenent and L.A. Peletier for some helpful discussions, and one referee for pointing out several errors.

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