

ANNALI DELLA SCUOLA NORMALE SUPERIORE DI PISA *Classe di Scienze*

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elliptic eigenvalue problem**

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 14,
n° 1 (1987), p. 97-121

http://www.numdam.org/item?id=ASNSP_1987_4_14_1_97_0

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Existence and Multiplicity Results for a Semilinear Elliptic Eigenvalue Problem

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1. - Introduction

The following eigenvalue problem will be considered:

$$(P) \quad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \in \mathbb{R}^N. \\ u = 0 & \text{on } \partial\Omega = \Gamma \end{cases}$$

for $\lambda > 0$. The domain Ω is assumed to be bounded and to have a smooth boundary of class C^3 .

The function f will satisfy appropriate smoothness conditions. A positive solution of (P) will be a pair (λ, u) in $\mathbb{R}^+ \times C^2(\overline{\Omega})$ satisfying (P) with $u > 0$ in Ω . We shall call u a solution of (P_λ) .

It is a consequence of the strong maximum principle, see [2], that if such a solution exists, then $f(\max u)$ is positive. The main goal of this paper is to study positive solutions having their maximum close to a zero of f . Therefore we assume:

(F1) there are two numbers ρ_1 and ρ_2 such that $\rho_1 < \rho_2$, $0 < \rho_2$,

$$f(\rho_1) = f(\rho_2) = 0 \text{ and } f > 0 \text{ in } (\rho_1, \rho_2)$$

In [13] Hess proves the existence of solutions (λ, u) of (P), satisfying $\max u \in (\rho_1, \rho_2)$, when $f(0) > 0$ under the following condition:

$$(F2) \quad J(\rho) = \int_{\rho}^{\rho_2} f(s) ds > 0 \text{ for every } \rho \in [0, \rho_2).$$

In Theorem 1 we prove that (F2) is a *necessary* and sufficient condition for the existence of such a solution even without the condition $f(0) \geq 0$.

Pervenuto alla redazione il 4 Aprile 1986 ed in forma definitiva il 29 Settembre 1986.

THEOREM 1. *Let $f \in C^1$ satisfy (F1). Then problem (P) possesses a positive solution (λ, u) , with $\max u \in (\rho_1, \rho_2)$, if and only if (F2) holds.*

Theorem 1 improves a result of De Figueiredo in [10], since it does not use the inheritance condition or even the starshapedness of Ω .

It also answers a question of Dancer in [9].

Next to this existence result we will prove a uniqueness result for positive solutions having their maximum close to ρ_2 . We need the following condition:

(F3) there exists an $\varepsilon > 0$ such that $f' \leq 0$ in $(\rho_2 - \varepsilon, \rho_2)$.

THEOREM 2. *Let $f \in C^{1,\gamma}$, for some $\gamma \in (0, 1)$, satisfy (F1), (F2) and (F3). Let $\Gamma \in C^3$. Then there are $\lambda_0 > 0$ and a nonnegative function $z_0 \in C_0^\infty(\Omega)$ with $\max z_0 \in (\rho_1, \rho_2)$, such that for all $\lambda > \lambda_0$, (P_λ) possesses exactly one solution u_λ with $z_0 < u_\lambda < \rho_2$.*

Moreover, $\lim_{\lambda \rightarrow \infty} \max u_\lambda = \rho_2$.

REMARKS.

1. We will state and prove a sharper version of this theorem in Section 4 (Theorem 2').

2. If $\rho_1 < 0$, or $\rho_1 = 0$ and $f'(0) > 0$, Theorem 2 was proved in a recent paper, [3], by Angenent. For $\rho_1 \leq 0$ there are also related results in [8].

3. If $\rho_1 = 0$ and $f'(0) = 0$, Rabinowitz showed in [19] the existence of pairs of solutions for λ large enough by a degree argument.

When $\rho_1 = 0$ and $f'(0) = 0$ the question arises, whether or not there are exactly two positive solutions of (P_λ) , with maximum less than ρ_2 , for λ large enough. We shall consider this problem only for $\Omega = B$, the unit ball in $\mathbb{R}^N$.

It is known, [12], that positive solutions for $\Omega = B$ are radially symmetric, and can be parametrized by $u(0)$. If f satisfies (F1) to (F3), it follows from Theorems 1 and 2' that λ is a monotone increasing function of $u(0)$, for $u(0) \in (\rho_2 - \varepsilon, \rho_2)$, where ε is some small positive number. Let $\mathcal{C}$ denote the component of solutions of (P) containing these solutions (λ, u) with $u(0) \in (\rho_2 - \varepsilon, \rho_2)$.

Set $\rho^* := \inf\{u(0); (\lambda, u) \in \mathcal{C}\}$. If $\rho^* > 0$, it can be shown that more than one component of solutions (λ, u) , with $u(0) \in (0, \rho_2)$ may exist, implying the existence of at least four solutions for λ large enough.

In Theorem 3 we find a sufficient condition on f , which guarantees the existence of a component $\mathcal{D}$ of solutions (λ, u) of (P) satisfying $\inf\{u(0); (\lambda, u) \in \mathcal{D}\} = 0$.

THEOREM 3. *If in problem (P), Ω is the unit ball in $\mathbb{R}^N$, with $N > 2$, and f satisfies the condition*

$$(G1) \quad f(u) = |u|^\alpha \cdot g(u) \text{ for some } \alpha \in \left(1, \frac{N+2}{N-2}\right) \text{ and } g \in C^{1,\gamma} \text{ with } g(0) > 0$$

then the following holds.

There is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists a positive solution (λ, u) of (P) with $u(0) = \varepsilon$.

Moreover λ is a decreasing function of ε , and $\lim_{\varepsilon \downarrow 0} \lambda(\varepsilon) = \infty$.

If f satisfies (G1), (F1) and (F3), there is one branch of solutions $\lambda \rightarrow (\lambda, \bar{u}_\lambda)$ with $\lim_{\lambda \rightarrow \infty} \bar{u}_\lambda(0) = \rho_2$, and one branch of solutions $\lambda \rightarrow (\lambda, \underline{u}_\lambda)$ with $\lim_{\lambda \rightarrow \infty} \underline{u}_\lambda(0) = 0$. Then, since $u(0) \in (\rho^*, \rho_2)$ parametrizes the solutions of (P) on the ball, which are radially symmetric, [12], one finds the following. For λ large enough, (P_λ) possesses exactly two positive solutions, with maximum less than ρ_2 , if and only if $\rho^* = 0$. If $\rho^* > 0$, there exists a positive radially symmetric solution of

$$(P^*) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N, \\ \lim_{|x| \rightarrow \infty} u(x) = 0 \end{cases}$$

satisfying $u(0) = \rho^*$.

For the sake of completeness this will be shown in Section 5. Ni and Serrin, in [15], found conditions on f which exclude the existence of a positive solution of (P^*) .

Combining these results we obtain:

COROLLARY 1. *If in problem (P) on the unit ball in $\mathbb{R}^N$, with $N > 2$, f satisfies conditions (G1), (F1), (F3) and*

(G2) for α and g defined in (G1) either $\alpha \leq \frac{N}{N-2}$ or

$$\left(\frac{N+2}{N-2} - \alpha \right) \cdot u^{\alpha+1} \cdot g(u) \geq \frac{2N}{N-2} \cdot \int_0^u s^{\alpha+1} \cdot g'(s) ds \text{ for all } u \in [0, \rho_2]$$

then for λ large enough problem (P_λ) possesses exactly two positive solutions with maximum less than ρ_2 .

REMARKS.

1. If $N \leq 2$, Theorem 3 and Corollary 1 still hold if one replaces in (G1) $\left(1, \frac{N+2}{N-2}\right)$ by $(1, \infty)$. Condition (G2) is no longer needed.

2. In [11], Gardner and Peletier prove a similar result when $\rho_1 > 0$, by using different techniques.

3. For every $\alpha \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right)$ a function f exists, for which $\rho^* > 0$. Such an f can be found by using the example on page 2 of [15]. This construction is done in [7].

Concerning the proofs, the main tools will be the sweeping principle of Serrin, see [22], [21], and the construction of appropriate super- and

subsolutions. For the sake of completeness we define in the appendix a notion of super- and subsolutions and we prove a suitable version of the sweeping principle. Some basic ideas for the proof of Theorem 2 are contained in [3].

The results of this paper were announced in [6].

We learned that Dancer and Schmitt, [24], have independently found a different proof of the necessity of (F2) in Theorem 1.

2. - Preliminary results

In this section we collect some preliminary results, which will be useful in the coming proofs. The first result for $f(0) > 0$ is contained in [13].

LEMMA 2.1. *Let $f \in C^1$ satisfy (F1), (F2) and $f(0) \geq 0$. Then problem (P) possesses a positive solution (λ, u) , with $\max u \in (\rho_1, \rho_2)$.*

PROOF. First modify the function f outside of $[0, \rho_2]$ by setting $f(\rho) = 0$ for $\rho > \rho_2$ and $f(\rho) = 2f(0) - f(-\rho)$ for $\rho < 0$. Note that f is bounded on $\mathbb{R}$. As in [13] we want to minimize

$$I(u, \lambda) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \lambda \int_{\Omega} F(u) dx \text{ in } W_0^{1,2}(\Omega),$$

where $F(u) = \int_0^u f(s) ds$.

For $\lambda > 0$, $I(u, \lambda)$ is bounded below.

Let u_n be a minimizing sequence for a fixed λ , then

$$\begin{aligned} I(|u_n|, \lambda) &= \frac{1}{2} \int_{\Omega} |D|u_n||^2 dx - \lambda \int_{\Omega} F(|u_n|) dx \leq \\ &\leq \frac{1}{2} \int_{\Omega} |Du_n|^2 dx - \lambda \int_{\Omega} \left\{ \int_0^{|u_n|} (f(s) - f(0)) ds + \int_0^{|u_n|} f(0) ds \right\} dx \leq \\ &\leq \frac{1}{2} \int_{\Omega} |Du_n|^2 dx - \lambda \int_{\Omega} \left\{ \int_0^{u_n} (f(s) - f(0)) ds + \int_0^{u_n} f(0) ds \right\} dx = \\ &= I(u_n, \lambda) \end{aligned}$$

Since $I(\cdot, \lambda)$ is sequentially weakly lower semicontinuous and coercive in $W_0^{1,2}(\Omega)$, $I(\cdot, \lambda)$ possesses a nonnegative minimizer, which we denote by u_λ .

It is standard that (λ, u_λ) is a solution of (P), with the modified f . By applying the strong maximum principle, we deduce as in [2], that either $f(\|u_\lambda\|_\infty) > 0$ or $u_\lambda = 0$.

Thus $\|u_\lambda\|_\infty < \rho_2$, hence (λ, u) is a solution of (P).

Set

$$\alpha = \min \left\{ \int_{\rho}^{\rho_2} f(s) ds; \quad 0 \leq \rho \leq \max(0, \rho_1) \right\}$$

$$\beta = \max \left\{ \int_{\rho}^{\rho_2} f(s) ds; \quad 0 \leq \rho \leq \rho_2 \right\}.$$

Suppose that for all positive λ , $\|u_{\lambda}\|_{\infty} \leq \rho_1$, then we will obtain a contradiction.

We choose $\delta > 0$ such that $2|\Omega^{\delta}|\beta < |\Omega|\alpha$, with $\Omega^{\delta} = \{x \in \Omega; \text{d}(x, \Gamma) < \delta\}$ and $|\Omega|$ denoting the Lebesgue-measure of Ω . This is possible since $\alpha > 0$ and $\lim_{\delta \downarrow 0} |\Omega^{\delta}| = 0$.

Next we choose $w \in C_0^{\infty}(\Omega)$, satisfying $0 \leq w \leq \rho_2$ in Ω^{δ} and $w = \rho_2$ in $\Omega - \Omega^{\delta}$; then

$$\begin{aligned} I(w, \lambda) - I(u_{\lambda}, \lambda) &= \\ &= \frac{1}{2} \int_{\Omega} (|Dw|^2 - |Du_{\lambda}|^2) dx - \lambda \int_{\Omega} (F(w) - F(u_{\lambda})) dx \leq \\ &\leq \frac{1}{2} \int_{\Omega} |Dw|^2 dx - \lambda \left(\int_{\Omega} F(\rho_2) dx + \int_{\Omega^{\delta}} (F(w) - F(\rho_2)) dx - \int_{\Omega} F(u_{\lambda}) dx \right) \leq \\ &\leq \frac{1}{2} \int_{\Omega} |Dw|^2 dx + 2\lambda |\Omega^{\delta}| \beta - \lambda \int_{\Omega} (F(\rho_2) - F(u_{\lambda})) dx = \\ &= \frac{1}{2} \int_{\Omega} |Dw|^2 dx + 2\lambda |\Omega^{\delta}| \beta - \lambda \int_{\Omega} \int_{u_{\lambda}}^{\rho_2} f(s) ds dx \leq \\ &\leq \frac{1}{2} \int_{\Omega} |Dw|^2 dx + \lambda (2|\Omega^{\delta}| \beta - |\Omega| \alpha) < 0 \end{aligned}$$

for λ large enough, since $2|\Omega^{\delta}| \beta - |\Omega| \alpha < 0$.

Then $I(w, \lambda) < I(u_{\lambda}, \lambda)$, contradicting the fact that u_{λ} is a minimizer. This completes the proof of the lemma.

In what follows it will be convenient to modify f outside of $[0, \rho_2]$ in an appropriate way.

Let $f \in C^1$, respectively $C^{1,\gamma}$ for some $\gamma \in (0, 1)$, satisfy (F1) and (F2). Then there is a function $f^* \in C^1$, respectively $C^{1,\gamma}$, satisfying $f^* = f$ on $[0, \rho_2]$

and

$$(F^*) \quad \begin{cases} f^* \text{ is bounded,} \\ f^* < 0 \text{ in } (\rho_2, \infty), \\ f^* = 0 \text{ in } (-\infty, -1], \\ \int_u^{\rho_2} f^*(s) ds > 0 \text{ for } u \in [-1, 0]. \end{cases}$$

Since we are interested in solutions (λ, u) of (P) with $0 \leq u \leq \rho_2$, we may assume without loss of generality that f satisfies (F^*) . Then we have

$$(2.1) \quad \inf \left\{ \int_u^{\rho_2} f(s) ds; \quad |\rho_2 - u| > \delta \right\} > 0, \text{ for all } \delta > 0.$$

LEMMA 2.2. *Let $f \in C^1$ satisfy (F1), (F2) and (F^*) .*

Then there exist $\mu > 0$ and $v \in C^2(\mathbb{R}^N)$, radially symmetric, which satisfy:

$$\begin{cases} -\Delta v = \mu \cdot f(v) \text{ in } \mathbb{R}^N, \\ v(0) \in (\rho_1, \rho_2), \\ v(1) = -1, \\ v'(r) < 0 \text{ for } r > 0. \end{cases}$$

PROOF. Since $f(u - 1)$ satisfies (F1) and (F2) it follows from lemma 2.1 that there exists a positive solution (μ, w) of

$$\begin{cases} -\Delta u = \lambda \cdot f(u - 1) \text{ in } B, \\ u = 0 \text{ on } \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$, satisfying $\max w \in (\rho_1 + 1, \rho_2 + 2)$. By [12] w is radially symmetric and $w'(r) < 0$ for $r \in (0, 1)$.

Set $v(r) = w(r) - 1$ for $r \in [0, 1]$ and

$$v(r) = \begin{cases} -1 + (r^{2-N} - 1) \cdot (2 - N)^{-1} \cdot w'(1) \text{ for } r \in (1, \infty) \text{ if } N \neq 2, \\ -1 + \log r \cdot w'(1) \text{ for } r \in (1, \infty) \text{ if } N = 2. \end{cases}$$

Since $f = 0$ on $(-\infty, -1]$ one verifies that v is the required function. This completes the proof of the lemma.

COROLLARY 2.3. *Let (μ, v) be like in Lemma 2.2, and let $\alpha \in (0, 1)$ be the unique zero of v .*

Then for $y \in \Omega$ and $\lambda > \mu \cdot \alpha^2 \cdot d(y, \Gamma)^{-2}$

$$(2.2) \quad w(\lambda, y; x) := v \left((\lambda/\mu)^{\frac{1}{2}} \cdot (x - y) \right), \quad x \in \Omega,$$

is a subsolution of (P_λ) .

PROOF. The function $w(\lambda, y) \in C^2(\mathbb{R}^N)$ satisfies $-\Delta w = \lambda \cdot f(w)$ in $\mathbb{R}^N$, hence $\int_{\Omega} (w(-\Delta\varphi) - \lambda \cdot f(w)\varphi) dx = 0$ for all $\varphi \in \mathcal{D}^+(\Omega)$, where $\mathcal{D}^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$. Since $w(\lambda, y) < 0$ on Γ for $\lambda > \mu \alpha^2 \cdot d(y, \Gamma)^{-2}$, $w(\lambda, y)$ satisfies the definition of subsolution given in the appendix. This proves the corollary.

Next we establish some results for the one-dimensional problem

$$(2.3) \quad \begin{cases} -u'' = f(u), & x > 0 \\ u(0) = 0, \\ u'(0) = \delta, \end{cases}$$

where $f \in C^1$ satisfies (F1), (F2) and (F*).

LEMMA 2.4. *Problem (2.3) possesses a unique solution u_δ in $\mathbb{R}_+$ for all $\delta \in \mathbb{R}$. The function $\delta \rightarrow u_\delta \in C[0, r]$ is continuous for every $r > 0$.*

Moreover, set

$$\delta_1 = \left(2 \int_0^{\rho_2} f(s) ds \right)^{\frac{1}{2}} \quad \text{and} \quad \delta_2 = \left(\max \left\{ -2 \int_{\rho}^0 f(s) ds; \quad \rho \in [-1, 0] \right\} \right)^{\frac{1}{2}},$$

- 1) if $\delta > \delta_1$, then $u_\delta(x) > (\delta - \delta_1)x$ for $x \in \mathbb{R}_+$,
- 2) if $\delta = \delta_1$, then $u'_\delta > 0$ on $\mathbb{R}_+$ and $\lim_{x \rightarrow \infty} u_\delta(x) = \rho_2$,
- 3) if $-\delta_2 \leq \delta < \delta_1$, then $\sup \{u_\nu(x); x \in \mathbb{R}_+, \nu \in [-\delta_2, \delta]\} < \rho_2$,
- 4) if $\delta < -\delta_2$, then $u_\delta < 0$ on $\mathbb{R}_+$.

PROOF. Since f is C^1 and bounded, the first assertion of the lemma is standard.

Note that a solution of (2.3) satisfies

$$(2.4) \quad (u'(x))^2 = \delta^2 - 2 \int_0^{u(x)} f(s) ds.$$

- 1) If $\delta > \delta_1$, then using (2.1) and (2.4) we have

$$(u'_\delta(x))^2 > (\delta - \delta_1)^2 + 2 \int_{u_\delta(x)}^{\rho_2} f(s) ds \geq (\delta - \delta_1)^2.$$

Since $u'_\delta(0) > 0$, we obtain $u_\delta(x) > (\delta - \delta_1)x$ for $x \in \mathbb{R}_+$.

2) If $\delta = \delta_1 = \left(2 \int_0^{\rho_2} f(s)ds\right)^{\frac{1}{2}}$, we have

$$(2.5) \quad (u'_\delta(x))^2 = 2 \int_{u_\delta(x)}^{\rho_2} f(s)ds.$$

It follows from (2.5), $f(\rho_2) = 0$ and the uniqueness for the initial value problem that $u_\delta(x) \neq \rho_2$ for all $x \in \mathbb{R}_+$, and thus $u_\delta < \rho_2$ on $\mathbb{R}_+$. Since u_δ is monotonically increasing and bounded there exists a sequence $\{x_n\}$, with $\lim_{n \rightarrow \infty} x_n = \infty$ and $\lim_{n \rightarrow \infty} u'_\delta(x_n) = 0$. From (2.1) and (2.5) it follows that $\lim_{x \rightarrow \infty} u_\delta(x) = \rho_2$.

$$\begin{aligned} 3) \text{ Note that } \delta_1^2 - \delta_2^2 &= 2 \int_0^{\rho_2} f(s)ds - \max \left\{ -2 \int_\rho^0 f(s)ds; \rho \in [-1, 0] \right\} = \\ &= 2 \min \left\{ \int_\rho^{\rho_2} f(s)ds; \rho \in [-1, 0] \right\}. \end{aligned}$$

Hence by (2.1) $\delta_1 > \delta_2$.

If $-\delta_2 \leq \nu \leq \delta < \delta_1$, one has

$$\begin{aligned} 0 \leq (u'_\nu(x))^2 &= \nu^2 - 2 \int_0^{u_\nu(x)} f(s)ds \leq \\ &\leq \max(\delta_2^2, \delta^2) - 2 \int_0^{u_\nu(x)} f(s)ds = \\ &= \max(\delta_2^2 - \delta_1^2, \delta^2 - \delta_1^2) + 2 \int_{u_\nu(x)}^{\rho_2} f(s)ds. \end{aligned}$$

Since $\max(\delta_2^2 - \delta_1^2, \delta^2 - \delta_1^2) < 0$, one finds, by using (2.1) again, that $|u_\nu(x) - \rho_2| \geq m > 0$ for all $x \in \mathbb{R}_+$. From $u_\nu(0) = 0$ it follows $u_\nu < \rho_2 - m$ on $\mathbb{R}_+$.

4) If $\delta < -\delta_2$, then

$$(u'_\delta(x))^2 > \max \left\{ -2 \int_\rho^0 f(s)ds; \rho \in [-1, 0] \right\} - 2 \int_0^{u_\delta(x)} f(s)ds \geq 0$$

for all $u_\delta(x) \leq 0$.

Since $u'_\delta(0) < 0$, one finds $u'_\delta < 0$ on $\mathbb{R}_+$. Hence $u_\delta < 0$ on $\mathbb{R}_+$.

This completes the proof of Lemma 2.4.

Lemma 2.4. will be used to establish some results for the problem on the halfspace $D = \{(x_1, \dots, x_N) \in \mathbb{R}^N; x_1 > 0\}$.

PROPOSITION 2.5. *Let $f \in C^{1,\gamma}$, for some $\gamma \in (0, 1)$, satisfy (F1), (F2) and (F3). Let $u \in C^2(D) \cap C(\overline{D})$ be a solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

with $0 \leq u < \rho_2$ in D and $\lim_{x_1 \rightarrow \infty} u(x_1, x') = \rho_2$ uniformly for $x' \in \mathbb{R}^{N-1}$.

Then $u(x_1, x') = u_{\delta_1}(x_1)$ for $x_1 \geq 0$ and $x' \in \mathbb{R}^{N-1}$, where u_{δ_1} is defined in Lemma 2.4.

In order to prove Proposition 2.5 we also need

LEMMA 2.6. *Let $(x_1, u) \rightarrow g(x_1, u)$ be a function such that $g, \frac{\partial}{\partial u}g \in C^{0,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R})$, for some $\gamma \in (0, 1)$, and $|g(x_1, u)| < h(u)$ for some $h \in C^0(\mathbb{R})$. Let $U \in C^2(D) \cap C^0(\overline{D})$ be a bounded solution of*

$$\begin{cases} -\Delta u = g(x_1, u) & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

Then S , defined by $S(x_1) = \sup\{U(x_1, x'); x' \in \mathbb{R}^{N-1}\}$, is continuous in $[0, \infty)$, with $S(0) = 0$, and satisfies

$$(2.6) \quad \int_{\mathbb{R}_+} (S \cdot (-\varphi'') - g(x_1, S)\varphi) dx_1 \leq 0 \text{ for all } \varphi \in \mathcal{D}^+(\mathbb{R}_+).$$

$\mathcal{D}^+(\mathbb{R}_+)$ consists of all nonnegative functions in $C_0^\infty(\mathbb{R}_+)$.

PROOF OF LEMMA 2.6. Since U and ΔU are bounded and $U = 0$ on ∂D , it follows from standard regularity properties that U and all first-order derivatives are uniformly bounded and uniformly Hölder continuous with exponent γ . Let $\{\Omega_n\}$ be an increasing sequence of bounded subdomains of D , with smooth boundary and such that $\bigcup_{n \in \mathbb{N}} \Omega_n = D$. We first prove that for each $n \in \mathbb{N}$, if $u_1, u_2 \in C(\Omega_n) \cap H^1(\Omega_n)$ satisfy

$$(2.7) \quad \int_D (u \cdot (-\Delta \varphi) - g(x_1, u) \cdot \varphi) dx \leq 0 \text{ for all } \varphi \in \mathcal{D}^+(\Omega_n),$$

then $u_3 = \sup(u_1, u_2)$ also satisfies (2.7).

Let $\omega \in \mathbb{R}_+$ be such that $u \rightarrow g(x_1, u) + \omega \cdot u$ is increasing on $[\min u_1 \wedge \min u_2, \max u_1 \vee \max u_2]$ for every $x \in \overline{\Omega_n}$.

We obtain

$$\int_D (u_i \cdot (-\Delta\varphi) + \omega \cdot u_i \cdot \varphi) dx \leq \int_D (g(x_1, u_3) + \omega \cdot u_3) \cdot \varphi dx$$

for all $\varphi \in \mathcal{D}^+(\Omega_n)$, $i = 1, 2$.

Set $h = g(x_1, u_3) + \omega \cdot u_3$ and let w satisfy

$$\begin{cases} -\Delta w + \omega \cdot w = h & \text{in } \Omega_n, \\ w = 0 & \text{on } \partial\Omega_n. \end{cases}$$

Note that $w \in C(\overline{\Omega}_n) \cap H^1(\Omega_n)$. Then $w_i = u_i - w$, $i = 1, 2$, satisfies

$$(2.8) \quad \int_D (u \cdot (-\Delta\varphi) + \omega \cdot u \cdot \varphi) dx \leq 0 \text{ for all } \varphi \in \mathcal{D}^+(\Omega_n).$$

It is known that $\sup(w_1, w_2)$ also satisfies (2.8), see [23, Th. 28.1]. Therefore u_3 satisfies (2.7). Note that $u_3 \in C(\overline{\Omega}_n) \cap H^1(\Omega_n)$. By induction it follows that if $u_i \in C(\overline{\Omega}_n) \cap H^1(\Omega_n)$, $i = 1, \dots, k$, satisfies (2.7), then $\sup\{u_i; i = 1, \dots, k\}$ also satisfies (2.7). Let u_i be translates of U perpendicular to $(1, 0, \dots, 0)$. Since $U \in C(\overline{D}) \cap H_{\text{loc}}^1(D)$, $\sup\{u_i; i = 1, \dots, k\}$ will satisfy (2.7). Then by using the Lebesgue dominated convergence theorem and the fact that U is bounded, one shows that

$$S(x_1) = \sup\{U(x_1, x'); x' \in \mathbb{R}^{N-1}\} = \sup\{U(x_1, x'); x' \in \mathbb{Q}^{N-1}\}$$

also satisfies (2.7) for each n . From the choice of the Ω_n it follows

$$\int_D (S(-\Delta\varphi) - g(x_1, S) \cdot \varphi) dx \leq 0 \text{ for all } \varphi \in \mathcal{D}^+(D).$$

By choosing φ of the form $\varphi_1 \cdot \varphi_2$, with $\varphi_1 \in \mathcal{D}^+(\mathbb{R}_+)$ and $\varphi_2 \in \mathcal{D}^+(\mathbb{R}^{N-1})$, $\varphi_2 \neq 0$, one gets (2.6), since S only depends on x_1 .

Note that S , as the supremum of continuous functions, is lower semicontinuous on $[0, \infty)$. From (2.6) and the fact that $g(x_1, S)$ is bounded, we deduce that S is the sum of a convex function on $(0, \infty)$ and a C^1 -function on $[0, \infty)$. Hence $S \in C(0, \infty)$. Since $U(0, x') = 0$ and since $\frac{\partial}{\partial x} U(0, x')$ is uniformly bounded, $S(0) = 0$ and S is continuous in 0. This completes the proof of Lemma 2.6.

PROOF OF PROPOSITION 2.5. Without loss of generality we assume that f satisfies (F*). Define

$$\begin{aligned} I(x_1) &= \inf \{U(x_1, x'); x' \in \mathbb{R}^{N-1}\} \text{ and} \\ S(x_1) &= \sup \{U(x_1, x'); x' \in \mathbb{R}^{N-1}\}. \end{aligned}$$

It is sufficient to prove that

$$(2.9) \quad I \geq u_\delta \text{ on } \mathbb{R}_+, \text{ and}$$

$$(2.10) \quad S \leq u_\delta \text{ on } \mathbb{R}_+,$$

for $\delta = \delta_1$.

We first prove (2.9) for $\delta = \delta_1$. By Lemma 2.4, 4), (2.9) holds with $\delta < -\delta_2$, since $I \geq 0$ on $\mathbb{R}_+$. We will use a sweeping argument to prove (2.9) for every $\delta \in (-\delta_2 - 1, \delta_1)$. Let $\delta \in (-\delta_2 - 1, \delta_1)$. By Lemma 2.4, 3) and 4), there exists $\rho < \rho_2$ such that

$$(2.11) \quad \sup \{u_\theta(x_1); x_1 \in \mathbb{R}_+; \theta \leq \delta\} \leq \rho.$$

For some $R > 0$ one has $I > \rho$ on $[R, \infty)$. It follows from Lemma 2.6, with $g(x_1, u) = -f(-u)$, that $I \in C[0, \infty)$, $I(0) = 0$ and

$$\int_{\mathbb{R}_+} (I \cdot (-\varphi'') - f(I) \cdot \varphi) dx \geq 0 \text{ for all } \varphi \in \mathcal{D}^+(\mathbb{R}_+).$$

Hence I is a supersolution of

$$(2.12) \quad \begin{cases} -u'' = f(u) & \text{in } (0, R), \\ u(0) = 0, \\ u(R) = \rho. \end{cases}$$

For $\theta \in [-\delta_2 - 1, \delta]$, (2.11) shows that u_θ is a subsolution of (2.12). We are now in the position to use Lemma A.2 and we obtain $I \geq u_\delta$ on $(0, R)$, hence on $\mathbb{R}_+$. For $x_1 \geq 0$ one has

$$I(x_1) \geq \lim_{\delta \uparrow \delta_1} u_\delta(x_1) = u_{\delta_1}(x_1).$$

This completes the proof of (2.9), with $\delta = \delta_1$.

Next we give a sketch of the proof of (2.10). Since $\frac{\partial}{\partial x_1} U$ is uniformly bounded, there exists $c > 0$ such that

$$S(x_1) < c \cdot x_1 \text{ for } x_1 \in \mathbb{R}_+.$$

By Lemma 2.4, 1), one has (2.10) with $\delta = \delta_1 + c$. Let $\delta \in (\delta_1, \delta_1 + c)$. Also from Lemma 2.4, 1), it follows

$$u_\theta(x_1) > \rho_2 + 1 \text{ for } x_1 > R := (\delta - \delta_1)^{-1}(\rho_2 + 1) \text{ and } \theta \in [\delta, \delta_1 + c].$$

Note that $S \leq \rho_2$. Then one concludes as above after using a sweeping argument for the problem

$$\begin{cases} -u'' = f(u), & \text{in } (0, R), \\ u(0) = 0, \\ u(R) = \rho_2. \end{cases}$$

This completes the proof of Proposition 2.5.

3. - Proof of the first theorem

NECESSITY: With $J(\rho) = \int_{\rho}^{\rho_2} f(s)ds$, and assuming $\rho_1 > 0$, define

$$J^* := \min \{J(\rho); \rho \in [0, \rho_1]\}.$$

Suppose condition (F2) is not satisfied, that is $J^* \leq 0$. Let (λ, u) be a positive solution of (P) satisfying $\max u \in (\rho_1, \rho_2)$. We will obtain a contradiction.

First, if $J^* = 0$, modify f to f^* in C^1 such that $f > f^* > 0$ in $(\max u, \rho_2)$ and $f = f^*$ elsewhere. Still u is a solution of (P_λ) , but now $J^* < 0$. Hence we may assume without loss of generality that $J^* < 0$.

Consider the initial value problem

$$(3.1) \quad -v'' = f(v),$$

$$(3.2) \quad \begin{cases} v(0) = \rho_2, \\ v'(0) = -(-J^*)^{\frac{1}{2}}. \end{cases}$$

For a solution of (3.1), (3.2) one has:

$$(3.3) \quad (v'(r))^2 = -J^* + 2 \int_{v(r)}^{\rho_2} f(s)ds.$$

Set $\rho^* := \max \{\rho \in [0, \rho_1]; J(\rho) = J^*\}$.

Because of (F1), $(v'(r))^2 = -J^* + 2 \int_{v(r)}^{\rho_2} f(s)ds \geq -J^* > 0$ holds for $v(r)$ in $[\rho_1, \rho_2]$, and hence $\inf v < \rho_1$.

Next we show that v remains positive. If not, there exists an r^* such that $v(r^*) = \rho^*$, and since (3.3) holds, one finds

$$(v'(r^*))^2 = -J^* + 2 \int_{\rho^*}^{\rho_2} f(s)ds = +J^* < 0,$$

a contradiction.

So either $v(r) \downarrow \tilde{\rho} \in (\rho^*, \rho_1)$ if $r \rightarrow \infty$, or v has a first positive minimum, say in $\tilde{r}$, and v is symmetric with respect to $\tilde{r}$. In the first case define

$$V(r) := \begin{cases} v(r) & \text{for } r > 0 \\ \rho_2 & \text{for } r \leq 0 \end{cases},$$

and in the second case

$$V(r) := \begin{cases} v(r) & \text{for } r \text{ in } (0, 2\tilde{r}) \\ \rho_2 & \text{elsewhere in } \mathbb{R} \end{cases}.$$

Set $w(\lambda, t; x) = V\left(\lambda^{\frac{1}{2}} \cdot (x_1 - t)\right)$, where $x = (x_1, \dots, x_N)$.

Then $\{w(\lambda, t; \cdot); t \in \mathbb{R}\}$ is a family of supersolutions, and for t large enough $w(\lambda, t; \cdot) = \rho_2$ in Ω .

By the sweeping principle $u < w(\lambda, t, \cdot)$ for all t .

Hence $u(x) \leq \inf \{w(\lambda, t; x); t \in \mathbb{R}\} = \inf v < \rho_1$, a contradiction.

REMARK 1. Let $f \in C^1$ satisfy (F1). The proof also shows that, if (F2) is not satisfied, there is no solution u of (P_λ) with $\max u \in (\rho_1, \rho_2)$, even if u changes sign.

REMARK 2. Let $f \in C^1$ satisfy (F1), and let $\Omega \subset \mathbb{R}^N$ be an unbounded domain.

Note that the same technique shows that problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \lim_{\substack{|x| \rightarrow \infty \\ x \in \Omega}} u(x) = 0 \end{cases}$$

may have a solution u , with $\max u \in (\rho_1, \rho_2)$, only if condition (F2) is satisfied.

SUFFICIENCY: We will prove a stronger result, which will be used later on.

Let $x^* \in \Omega$. Then define $\lambda^* = \mu\alpha^2 d(x^*, \Gamma)^{-2}$ and $z_\lambda = w(\lambda, x^*)$, where μ, α and w are defined in Corollary 2.3.

LEMMA 3.1. *Let f satisfy (F1), (F2) and (F*). Then*

- 1) *for $\lambda > \lambda^*$ problem (P_λ) possesses a solution $u_\lambda \in [z_\lambda, \rho_2]$,*
- 2) *there exists $\lambda^{**} > \lambda^*$, $c > 0$ and $\tau \in (\rho_1, \rho_2)$, such that for $\lambda > \lambda^{**}$ every solution $u \in [z_\lambda, \rho_2]$ of (P_λ) satisfies*

$$(3.4) \quad u(x) > \min \left(c\lambda^{\frac{1}{2}} d(x, \Gamma), \tau \right) \text{ for all } x \in \Omega.$$

REMARK 3. It follows from (3.4) that $u_\lambda > 0$ for $\lambda > \lambda^{**}$, and that $\max u_\lambda \in (\rho_1, \rho_2)$, for λ large enough.

REMARK 4. Lemma 3.1, 2), shows $\frac{\partial}{\partial n} u_\lambda < 0$ on Γ for $\lambda > \lambda^{**}$, even when $f(0) < 0$. ($\frac{\partial}{\partial n}$ denotes the outward normal derivative)

PROOF OF LEMMA 3.1. By Corollary 2.3 one knows that for $\lambda > \lambda^*$, z_λ is a subsolution of (P_λ) , with $z_\lambda < \rho_2$. Since ρ_2 is a supersolution of (P_λ) , Lemma A.1 yields a solution $u_\lambda \in [z_\lambda, \rho_2]$ of (P_λ) , for $\lambda > \lambda^*$. This completes the proof of the first assertion.

Since Ω satisfies a uniform interior sphere condition, there exists $\varepsilon_0 > 0$ such that $\Omega = \bigcup \{B(x, \varepsilon); x \in \Omega_\varepsilon\}$ for $\varepsilon \in (0, \varepsilon_0]$, where $\Omega_\varepsilon = \{x \in \Omega; d(x, \Gamma) > \varepsilon\}$. Set

$$\begin{aligned} \lambda^{**} &= \max(\lambda^*, \mu \alpha^2 \varepsilon_0^{-2}), \\ c &= \mu^{-\frac{1}{2}} \inf \{(\alpha - r)^{-1} \cdot v(r); r \in [0, \alpha]\} \text{ and} \\ \tau &= v(0) \end{aligned}$$

with μ , v and α defined in Corollary 2.3.

Note that $c > 0$, since $v > 0$ on $[0, \alpha]$ and $v'(\alpha) < 0$.

Let (λ, u) be a solution of (P) with $\lambda > \lambda^{**}$ and $u \in [z_\lambda, \rho_2]$. Since for $\lambda > \lambda^{**}$, $\Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}$ is arcwise connected and since $w(\lambda, y)$ is a subsolution for $y \in \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}$, with $w(\lambda, y) < 0$ on Γ , one finds by Lemma A.2 that

$$u > w(\lambda, y) \text{ in } \Omega \text{ for all } y \in \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}.$$

Hence

$$\begin{aligned} u(x) &> c \lambda^{\frac{1}{2}} d(x, \Gamma) \text{ for all } x \in \Omega \setminus \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}, \text{ and} \\ u(x) &> \tau \text{ for all } x \in \Omega_{\alpha(\mu/\lambda)^{\frac{1}{2}}}, \end{aligned}$$

which completes the proof.

4. - Proof of the second theorem

As mentioned in the introduction Theorem 2 will be a consequence of a sharper version, Theorem 2'.

THEOREM 2'. Let $\Gamma \in C^3$ and let $f \in C^{1,\gamma}$, for some $\gamma \in (0, 1)$, satisfy (F1), (F2) and (F3). Then for some $\lambda_1 > 0$,

- 1) there exists $\varphi \in C^1([\lambda_1, \infty); C^2(\bar{\Omega}))$, such that $(\lambda, \varphi(\lambda))$ is a solution of (P) for $\lambda \geq \lambda_1$, with $\varphi(\lambda) > 0$ in Ω , $\max \varphi(\lambda) \in (\rho_1, \rho_2)$ and $\lim_{\lambda \rightarrow \infty} \max \varphi(\lambda) = \rho_2$;
- 2) if $\mu_0(\lambda, u)$ denotes the principal eigenvalue of

$$(LP) \quad \begin{cases} -\lambda^{-1} \cdot \Delta h - f'(u) \cdot h = \mu h & \text{in } \Omega, \\ h = 0 & \text{on } \Gamma, \end{cases}$$

- then $\mu_0(\lambda, \varphi(\lambda)) > 0$ for $\lambda > \lambda_1$;
- 3) for all nonnegative $z \in C_0^\infty(\Omega)$ with $\max z \in (\rho_1, \rho_2)$, there exists $\lambda(z) > \lambda_1$, such that, if (λ, u) is a solution of (P) with $\lambda > \lambda(z)$ and $u \in [z, \rho_2]$, then $u = \varphi(\lambda)$.

REMARK 1. Theorem 2 follows from theorem 2' by choosing a nonnegative function $z_0 \in C_0^\infty(\Omega)$ and setting $\lambda_0 = \lambda(z_0)$ in the third assertion of Theorem 2'.

REMARK 2. If $\rho_1 > 0$, let $\mathcal{C}$ denote the component of solutions of (P) in $\mathbb{R}_+ \times C^2(\bar{\Omega})$ containing $\{(\lambda, \varphi(\lambda)); \lambda \geq \lambda_1\}$. Since $\mathcal{C}$ is connected, one has for $(\lambda, u) \in \mathcal{C}$ that $\max u \in (\rho_1, \rho_2)$ (see [2]) and $\lambda > 0$. By using degree arguments as in [19], [20], one can show that for λ large enough, $\mathcal{C} \cap (\{\lambda\} \times C^2(\bar{\Omega}))$ contains at least two solutions of (P). The proof of this assertion will appear elsewhere.

For the proof of Theorem 2' we need the following lemmas.

LEMMA 4.1. *Let $f \in C^1$ satisfy (F1), (F2) and (F*). For every $\delta > 0$ there is a $c(\delta) > 0$, such that for all solutions (λ, u) of (P), with $\lambda > \lambda^{**}$ and $u \in [z_\lambda, \rho_2]$, the following holds*

$$(4.1) \quad u(x) > \min \left(c(\delta) \lambda^{\frac{1}{2}} d(x, \Gamma), \rho_2 - \delta \right) \text{ for all } x \in \Omega,$$

with λ^{**} and z_λ as in Lemma 3.1.

PROOF OF LEMMA 4.1. If $\rho_2 - \delta < \tau$, we are done with $c(\delta) = c$ as in Lemma 3.1. Otherwise, by (F1) there exists $\sigma > 0$ such that $\sigma(u - \tau) < f(u)$ for all $u \in [\tau, \rho_2 - \delta]$.

Let ν denote the principal eigenvalue of

$$\begin{cases} -\Delta \psi = \nu \psi & \text{in } B, \\ \psi = 0 & \text{on } \partial B, \end{cases}$$

where B denotes the unit ball in $\mathbb{R}^N$.

Then by using Lemma A.3 with $\Omega' = \Omega_{k\lambda^{-\frac{1}{2}}}$, $k = c^{-1}\tau$, one finds

$$(4.2) \quad u(x) > \rho_2 - \delta \text{ for all } x \in \Omega_{\left(\frac{\nu}{\sigma}\right)^{\frac{1}{2}} + k} \lambda^{-\frac{1}{2}},$$

since $(\Omega')_{\left(\frac{\nu}{\sigma}\right)^{\frac{1}{2}}} = \Omega_{\left(\frac{\nu}{\sigma}\right)^{\frac{1}{2}} + k} \lambda^{-\frac{1}{2}}$.

By (3.4) one finds

$$(4.3) \quad u(x) > c(\delta) \lambda^{-\frac{1}{2}} d(x, \Gamma) \text{ for all } x \in \Omega \setminus \Omega_{\left(\frac{\nu}{\sigma}\right)^{\frac{1}{2}} + k} \lambda^{-\frac{1}{2}}$$

with $c(\delta) = \tau \left((\nu/\sigma)^{\frac{1}{2}} + k \right)^{-1}$

This completes the proof of the lemma.

LEMMA 4.2. *Let $f \in C^{1,\gamma}$, for some $\gamma \in (0, 1)$, satisfy (F1), (F2), (F3) and (F*). Then there exists $\lambda_1 > \lambda^{**}$, such that for every solution u of (P_λ) , with $\lambda > \lambda_1$ and $u \in [z_\lambda, \rho_2]$, one finds $\mu_0(\lambda, u) > 0$.*

PROOF. Suppose this is not the case. Then there exists a sequence $\{(\lambda_n, u_n); n \in \mathbb{N}\}$ of solutions of (P), with $u_n \in [z_{\lambda_n}, \rho_2]$, $\mu_n := \mu_0(\lambda_n, u_n) \leq 0$ for all n , and $\lim_{n \rightarrow \infty} \lambda_n = \infty$.

Let ε be defined by (F3). Since $\mu_n \leq 0$, for all n , the associated eigenfunctions v_n , normalized by $\max v_n = 1$, satisfy

$$(4.4) \quad -\lambda_n^{-1} \Delta v_n(x) = (f'(u_n(x)) + \mu_n) v_n(x) \leq 0 \text{ for } x \in \Omega_{K\lambda_n^{-\frac{1}{2}}},$$

where $K = (c(\varepsilon))^{-1}(\rho_2 - \varepsilon)$.

The constant $c(\varepsilon)$ is defined in the previous lemma.

Hence the function v_n is subharmonic in $\Omega_{K\lambda_n^{-\frac{1}{2}}}$, and v_n attains its maximum outside of $\Omega_{K\lambda_n^{-\frac{1}{2}}}$. Like in [3] let $y^n \in \Omega \setminus \Omega_{K\lambda_n^{-\frac{1}{2}}}$ be a point where v_n attains its maximum and let $x^n \in \Gamma$ be a point which minimizes $\{d(x, y^n); x \in \Gamma\}$. Since $\{x^n\}$ and $\{\mu_n\}$ are bounded, there exists a subsequence, still denoted $\{(\lambda_n, u_n)\}$, such that $\lim_{n \rightarrow \infty} x^n = \bar{x} \in \Gamma$ and $\lim_{n \rightarrow \infty} \mu_n = \bar{\mu} \leq 0$. Let $\mathcal{O}$ be an open neighbourhood of $\bar{x}$ in $\mathbb{R}^N$, chosen so small that it permits C^3 local coordinates $(\xi_1, \dots, \xi_N): \mathcal{O} \rightarrow \mathbb{R}^N$, such that $x \in \Omega \cap \mathcal{O}$ if and only if $\xi_1(x) > 0$, and $\xi(\bar{x}) = 0$. In these coordinates the Laplacian is given by

$$\Delta u = \sum_{i,j} a_{ij}(\xi) \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_j} \tilde{u} + \sum_j b_j(\xi) \frac{\partial}{\partial \xi_j} \tilde{u},$$

where $a_{ij} \in C^2$, $b_j \in C^1$ and $u(x) = \tilde{u}(\xi(x))$.

Moreover we choose the local coordinates such that $a_{ij}(0) = \delta_{ij}$. Next define the functions

$$U_n(\eta) = \tilde{u}_n \left(\xi(x^n) + \lambda_n^{-\frac{1}{2}} \eta \right),$$

$$V_n(\eta) = \tilde{v}_n \left(\xi(x^n) + \lambda_n^{-\frac{1}{2}} \eta \right), \quad \eta \in D.$$

Since $\{U_n\}$ and $\{V_n\}$ are precompact in C_{loc}^2 , there exists a convergent subsequence. Hence there are $U, V \in C^2(\overline{D})$, bounded and positive in $D =$

$\{(x_1, x'); x_1 > 0, x' \in \mathbb{R}^{N-1}\}$, satisfying respectively

$$\begin{cases} -\Delta U = f(U) & \text{in } D, \\ U = 0 & \text{on } \partial D, \end{cases} \quad \begin{cases} -\Delta V - f'(U)V = \bar{\mu}V & \text{in } D, \\ V = 0 & \text{on } \partial D. \end{cases}$$

Moreover by Lemma 4.1 the following inequalities,

$$(4.5) \quad \min(c(\delta)x_1, \rho_2 - \delta) \leq U(x_1, x') \leq \rho_2 \text{ for all } x_1 > 0, x' \in \mathbb{R}^{N-1},$$

hold for every $\delta > 0$. From Proposition 2.5 we have

$$U(x_1, x') = u_{\delta_1}(x_1) \text{ for } x_1 \geq 0, x' \in \mathbb{R}^{N-1}.$$

Set $S(x_1) = \sup \{V(x_1, x'); x' \in \mathbb{R}^{N-1}\}$. Then $0 < S \leq 1$ in $\mathbb{R}_+$ and we obtain by using Lemma 2.6 that $S \in C[0, \infty)$, $S(0) = 0$ and

$$(4.6) \quad \int_{\mathbb{R}_+} (S \cdot (-\varphi'') - (f'(u_{\delta_1}) + \bar{\mu}) S \varphi) dx \leq 0 \text{ for all } \varphi \in \mathcal{D}^+(\mathbb{R}_+).$$

Since $u'_{\delta_1} > 0$ on $\overline{\mathbb{R}_+}$, there exists a smallest $C > 0$ such that $W := Cu'_{\delta_1} - S \geq 0$ on $[0, K+1]$, where K is defined in (4.4). Then one finds by using (4.6) and $-(u'_{\delta_1})'' = f'(u_{\delta_1})u'_{\delta_1}$ in $\mathbb{R}_+$, that

$$(4.7) \quad \int_{\mathbb{R}_+} (W \cdot (-\varphi'') - f'(u_{\delta_1})W\varphi) dx \geq 0 \text{ for all } \varphi \in \mathcal{D}^+(\mathbb{R}_+).$$

Since W is nonnegative in $[0, K+1]$, there is $\omega > 0$ such that

$$\int_{\mathbb{R}_+} (W \cdot (-\varphi'') + \omega W\varphi) dx \geq 0 \text{ for all } \varphi \in \mathcal{D}^+((0, K+1)).$$

By [5, Corollary p. 581] and the fact that $W \not\equiv 0$, one obtains

$$(4.8) \quad W \geq bx(K+1-x) \text{ for all } x \in [0, K+1] \text{ and some } b > 0.$$

By construction W vanishes somewhere in $[0, K+1]$. Since $W(0) > 0$ one finds $W(K+1) = 0$. Moreover $f'(u_{\delta_1}) \leq 0$ on (K, ∞) . Hence (4.6) yields that S is convex on (K, ∞) . Since W is the sum of a C^1 and a concave function on (K, ∞) , (4.8) shows $0 > \frac{d^-}{dx}W(K+1) \geq \frac{d^+}{dx}W(K+1)$, and therefore $W(x) < 0$ on $(K+1, K+1+c)$ for some $c > 0$. Moreover W cannot vanish on $(K+1, \infty)$.

Otherwise there would be $c > 0$ such that $W < 0$ on $(K+1, K+1+c)$ and $W(K+1) = W(K+1+c) = 0$. But this cannot happen since by (4.7) W is concave as long as W is negative on (K, ∞) .

Hence W is concave on $(K+1, \infty)$. Since $\frac{d^+}{dx}W(K+1) < 0$, W is not bounded below, contradicting $W = Cu'_{\delta_1} - S \geq -1$ on $\mathbb{R}_+$. This completes the proof of Lemma 4.2.

It follows from Lemma 4.2 that for $\lambda > \lambda_1$ (P_λ) possesses at most one solution in $[z_\lambda, \rho_2]$. Indeed, choose $\omega > 0$ such that $\lambda f'(u) + \omega > 0$ for $u \in [0, \rho_2]$, and define the mapping $K: C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ by

$$K(u) := (-\Delta + \omega)^{-1}(\lambda f(u) + \omega u),$$

where $(-\Delta + \omega)^{-1}$ is the inverse of $-\Delta + \omega$ with homogeneous Dirichlet boundary conditions. By our choice of ω , K maps $[z_\lambda, \rho_2]$ into itself and K has no fixed point on its boundary. Since K is compact, the Leray-Schauder degree on (z_λ, ρ_2) is well defined. Because (z_λ, ρ_2) is convex one finds

$$\text{degree } (I - K, (z_\lambda, \rho_2), 0) = 1.$$

If (λ, u) is a solution of (P), with $u \in [z_\lambda, \rho_2]$ and $\mu_0(\lambda, u) > 0$, it follows that u is an isolated fixed point of K . Moreover, the local degree of $I - K$ at u is $+1$. From the additivity of degree it follows that K possesses at most one fixed point in (z_λ, ρ_2) . We denote this solution by $\varphi(\lambda)$. Since $\mu_0(\lambda, \varphi(\lambda)) > 0$, for $\lambda > \lambda_1$, one finds by the implicit function theorem and Schauder estimates, that $\lambda \rightarrow \varphi(\lambda) \in C^1([\lambda_1, \infty); C^{2,\gamma}(\overline{\Omega}))$. The estimate (4.1) implies that $\lim_{\lambda \rightarrow \infty} \max \varphi(\lambda) = \rho_2$.

It remains to prove the third assertion of theorem 2'. Let $z \in \mathcal{D}^+(\Omega)$ with $\max z \in (\rho_1, \rho_2)$. It follows from the first part of the proof, that it is sufficient to show that there exists $\lambda(z) > \lambda_1$, such that any solution u of (P_λ) , with $\lambda > \lambda(z)$ and $u \in [z, \rho_2]$, is larger than z_λ . This will be done in two steps.

First note that, from the definition of z , there exist $s \in (\rho_1, \rho_2)$ and a ball $B(x_0, r) \subset \Omega$, such that $z > s$ in $B(x_0, r)$. Let $\sigma > 0$ be such that $f(u) > \sigma \cdot (u - s)$ for $u \in [s, \tau]$, where $\tau = \max z_\lambda$. For $\lambda > \lambda_1(z) := \left((\nu/\sigma)^{\frac{1}{2}} + \mu^{\frac{1}{2}} \right)^2 r^{-2}$, with μ defined in Lemma 2.2, we can apply Lemma A.3 in order to get

$$u(x) > \tau \text{ for } x \in B\left(x_0, (\mu/\lambda)^{\frac{1}{2}}\right) \subset B\left(x_0, r - (\nu/\sigma\lambda)^{\frac{1}{2}}\right).$$

Observe that $w(\lambda, x_0) < u$ in Ω for $\lambda > \lambda_1(z)$. By Corollary 2.3 $w(\lambda, x_0)$ is a subsolution of (P_λ) for $\lambda > \lambda_1(z)$.

Finally, like in proof of Lemma 3.1 part 2), one uses Lemma A.2 to show that if $u > w(\lambda, x_0)$ in Ω and $\lambda > \lambda(z) := \max(\lambda_1(z), \lambda^{**})$ also the following estimate holds,

$$u > w(\lambda, x^*) = z_\lambda.$$

This completes the proof of Theorem 2'.

5. - Proof of the third theorem

Note that, if (λ, u) is a positive solution of (P), then $v := (u(0))^{-1}u$ satisfies

$$\begin{cases} -\Delta v = (u(0))^{\alpha-1} \lambda v^\alpha g(u(0)v) & \text{in } B \\ v = 0 & \text{on } \partial B. \end{cases}$$

Moreover by defining $w(r) := v(R^{-1}r)$ with $\varepsilon = u(0)$ and

$$(5.1) \quad R = u(0)^{\frac{1}{2}(\alpha-1)} \lambda^{\frac{1}{2}} \quad \text{one gets}$$

$$(5.2) \quad -w'' - \frac{N-1}{r} w' = w^\alpha g(\varepsilon w)$$

$$(5.3) \quad \begin{cases} w(0) = 1 \\ w'(0) = 0 \\ w(R) = 0 \\ w > 0 \text{ on } [0, R). \end{cases}$$

Let $w(\varepsilon, \cdot)$ denote the unique solution of the initial value problem (5.2-5.3)

LEMMA 5. *There exists $\varepsilon_1 > 0$ such that for ε in $[0, \varepsilon_1)$, $w(\varepsilon, \cdot)$ possesses a first zero, which we denote by $R(\varepsilon)$. Moreover R as a function of ε is $C^1(0, \varepsilon_1) \cap C[0, \varepsilon_1)$ and $\frac{d}{d\varepsilon} R$ is bounded on $(0, \frac{1}{2}\varepsilon_1)$.*

We first show that the assertion of Theorem 3 is an easy consequence of this lemma. By (5.1) we have $\lambda(\varepsilon) = R(\varepsilon)^2 \varepsilon^{1-\alpha}$, and hence

$$\frac{d}{d\varepsilon} \lambda(\varepsilon) = R(\varepsilon) \varepsilon^{-\alpha} \left(2\varepsilon \frac{d}{d\varepsilon} R(\varepsilon) + (1-\alpha) R(\varepsilon) \right), \quad 0 < \varepsilon < \varepsilon_1.$$

Since $\alpha - 1 > 0$, $R(0) > 0$ and $\frac{d}{d\varepsilon} R$ is bounded on $\left(0, \frac{1}{2}\varepsilon_1\right)$, it follows that

$\frac{d}{d\varepsilon} \lambda(\varepsilon) < 0$ on some interval $(0, \varepsilon_0)$.

Then for $\lambda > \lambda(\varepsilon_0)$, $u_\lambda(r) = \varepsilon(\lambda) w(R(\varepsilon(\lambda))r)$ is a solution of (P_λ) on the unit ball, where $\varepsilon(\lambda)$ is the inverse of the function $\lambda(\varepsilon)$. This function $\varepsilon(\lambda)$ is well defined on $(\lambda(\varepsilon_0), \infty)$, decreasing and satisfies $\lim_{\lambda \rightarrow \infty} \varepsilon(\lambda) = 0$. This completes the proof of the theorem.

PROOF OF LEMMA 5. It is known, see [17], that (5.2-5.3) with $\varepsilon = 0$ possesses a solution w , having a first positive zero which we denoted by $R(0)$. We want to obtain the function $w(\varepsilon, \cdot)$ by a perturbation argument.

Since we are only interested in bounded positive solutions, we modify the right-hand-side of (5.2) by setting $h(\varepsilon, w) = k(w)g(\varepsilon w)$ where k is a C^1 -function satisfying

$$k(w) = \begin{cases} 0 & \text{for } w \leq 0 \\ w^\alpha & \text{for } 0 < w < 1 \\ 0 & \text{for } w \geq 2. \end{cases}$$

The function h is $C^1((-1, 1) \times \mathbb{R})$ and has bounded derivatives. The initial value problem

$$(5.4) \quad -w'' - \frac{N-1}{r}w' = h(\varepsilon, w), \quad \varepsilon \text{ in } (-1, 1),$$

$$(5.5) \quad \begin{cases} w(0) = 1 \\ w'(0) = 0, \end{cases}$$

possesses a unique solution $w(\varepsilon, \cdot)$ on $[0, \infty)$.

For ε in $[0, 1)$, since $w(\varepsilon, \cdot)$ is decreasing until it possibly becomes zero, this function $w(\varepsilon, \cdot)$ is identical with the one in the lemma, as long as it is positive.

We claim, for every $r > 0$, $w(\cdot, r)$ is a C^1 -function of ε . First this will be proved for $r \in (0, \delta)$, with δ small enough. Note that (5.4-5.5) can be rewritten as $w = T(\varepsilon, w)$, where $T(\varepsilon, z)(r) = 1 - \int_0^r t^{1-N} \int_0^t s^{N-1} h(\varepsilon, z(s)) ds dt$, for z in $C[0, \delta]$. For every $\delta > 0$, $T : (-1, 1) \times C[0, \delta] \rightarrow C[0, \delta]$, where $C[0, \delta]$ is equipped with the supremum-norm, is continuously Fréchet-differentiable. For δ small enough, $T(\varepsilon, \cdot) : C[0, \delta] \rightarrow C[0, \delta]$ is a strict contraction with a unique fixed point $z(\varepsilon)$ such that $\varepsilon \rightarrow z(\varepsilon)$ is continuously differentiable.

Since $w(\varepsilon, r) = z(\varepsilon)(r)$, the claim is proved for $r < \delta$.

By repeating the argument it can be shown that $\varepsilon \rightarrow w(\varepsilon, r)$ is continuously differentiable for every $r > 0$.

Since $w(0, R(0)) = 0$ and $w_r(0, R(0)) < 0$ it follows from the implicit function theorem, that there exists $\varepsilon_1 > 0$ and a continuously differentiable function $R(\cdot)$, defined on $(-\varepsilon_1, \varepsilon_1)$, such that $w(\varepsilon, R(\varepsilon)) = 0$. From (5.4) it follows that $R(\varepsilon)$ is the unique zero of $w(\varepsilon, \cdot)$ on $\mathbb{R}^+$. This completes the proof.

PROOF OF THE COROLLARY. Since $u(0)$ parametrizes the solutions (λ, u) of (P), $\rho^* = \inf\{\sigma > 0; \text{(P) has a solution } (\lambda, u), \text{ with } u(0) = \rho, \text{ for all } \rho \in [\sigma, \rho_2]\}$. Suppose $\rho^* > 0$ and let v be the solution of the initial value problem

$$(5.6) \quad -v'' - \frac{N-1}{r}v' = f(v),$$

$$(5.7) \quad \begin{cases} v(0) = \rho^*, \\ v'(0) = 0. \end{cases}$$

Since $f(\rho) > 0$ on $(0, \rho^*]$, v is strictly decreasing while v is positive. If v has a (first) positive zero R , then $(R^2, v(R^{-1}, \cdot))$ is a solution of (P), which contradicts the definition of ρ^* . If v stays positive, then

$$(5.8) \quad \lim_{r \rightarrow \infty} v(r) = 0.$$

Otherwise, there are $c > 0$ and $R > 0$ such that $f(v(s)) > c$ for $s > R$. By integrating (5.6), one finds

$$\begin{aligned} v'(r) &= (R/r)^{N-1} v'(R) - r^{1-N} \int_R^r s^{N-1} f(v(s)) ds \leq \\ &\leq (R/r)^{N-1} v'(R) - (c/N)(r - R(R/r)^{N-1}) < -1, \end{aligned}$$

for r large enough, contradicting the fact that v stays positive. The existence of a positive function satisfying (5.6-5.8), is contradicted by Theorem 2.2 of [15], if $\alpha \leq N/(N-2)$, and by Theorem 3.1 of [15], if the integral condition of (G2) is satisfied. Therefore $\rho^* = 0$.

This completes the proof.

6. - Appendix

In this section we state, for the sake of completeness, a definition and some lemmas concerning sub- and supersolutions of problem

$$(H) \quad \begin{cases} -\Delta u = h(u) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = g & \text{on } \Gamma, \end{cases}$$

where Ω is a bounded domain with C^3 -boundary, $h \in C^1$ and $g \in C^0$.

DEFINITION. We call a function v a *subsolution* (*supersolution*) of (H) if:

- i) $v \in C(\overline{\Omega})$,
- ii) $v \leq (\geq) g$ on $\partial\Omega$, and
- iii) $\int_{\Omega} (v \cdot (-\Delta\varphi) - h(v)\varphi) dx \leq (\geq) 0$ for every $\varphi \in \mathcal{D}^+(\Omega)$, where $\mathcal{D}^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$.

LEMMA A.1. Let v and w be respectively a sub- and supersolution of (H) with $g = 0$. If $v \leq w$ in Ω , then there exists a solution $u \in C^2(\overline{\Omega})$ of (H) with $g = 0$, which satisfies $v \leq u \leq w$.

PROOF. We essentially follow the proof in [21] on page 24. Choose a number $\omega > 0$ such that $h'(u) + \omega \geq 0$ for $\min v \leq u \leq \max w$, and define the

nonlinear map T by $u_1 = Tu$, where

$$\begin{cases} -\Delta u_1 + \omega u_1 = h(u) + \omega u & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly $T : C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ is compact. (Where $C(\overline{\Omega})$ is equipped with the supremum-norm)

It is standard that T is monotone on $[v, w]$. Next we show that $v_1 := Tv \geq v$ in Ω .

By the definition of a subsolution and by the construction of v_1 , we have

$$\begin{aligned} \int_{\Omega} (v \cdot (-\Delta\varphi) + \omega v\varphi) dx &\leq \int_{\Omega} (h(v) + \omega v)\varphi dx = \\ &= \int_{\Omega} (v_1 \cdot (-\Delta\varphi) + \omega v_1\varphi) dx \text{ for every } \varphi \in \mathcal{D}^+(\Omega). \end{aligned}$$

Thus $z = v_1 - v$ satisfies $z \geq 0$ on $\partial\Omega$, and

$$\int_{\Omega} (z \cdot (-\Delta\varphi) + \omega z\varphi) dx \geq 0 \text{ for every } \varphi \in \mathcal{D}^+(\Omega).$$

We claim that z is nonnegative in Ω .

Otherwise there exists a ball $B(x_0, r) \subset \Omega$, such that z is negative in $B(x_0, r)$ and achieves its minimum in x_0 .

Hence

$$\int_{\Omega} z \cdot (-\Delta\varphi) dx \geq 0 \text{ for every } \varphi \in \mathcal{D}^+(B(x_0, r)).$$

This shows z is superharmonic on $B(x_0, r)$, and from the minimum principle we get $z(x) = z(x_0)$ on $B(x_0, r)$.

Then

$$\int_{B(x_0, r)} (z \cdot (-\Delta\varphi) + \omega z\varphi) dx = \omega z(x_0) \int_{B(x_0, r)} \varphi dx < 0$$

for every nontrivial $\varphi \in \mathcal{D}^+(B(x_0, r))$, a contradiction. Thus $Tv = v_1 \geq v$ on $\overline{\Omega}$. Similarly, one proves $Tw \leq w$ on $\overline{\Omega}$. Now it is standard, see [1], that T possesses a fixed point in $[v, w]$, which is a solution of (H) with $g = 0$.

Next we prove an appropriate version of the sweeping principle of Serrin, [22], [21].

Let $\Gamma = \partial\Omega$ be the union of two disjoint closed subsets Γ_1 and Γ_2 , where Γ_1 or Γ_2 may be empty. Let $e \in C^1(\overline{\Omega})$ satisfy $e > 0$ on $\Omega \cap \Gamma_1$ and $e = 0$, $\frac{\partial e}{\partial n} < 0$ on

Γ_2 . (n is the outward normal) Set $C_e(\overline{\Omega}) = \{u \in C(\overline{\Omega}); |u| \leq \alpha e \text{ for some } \alpha > 0\}$ and for $u \in C_e(\overline{\Omega})$ define $\|u\|_e = \inf\{\alpha > 0; |u| \leq \alpha e\}$.

LEMMA A.2. *Let u be a supersolution of (H) and let $A = \{v_t; t \in [0, 1]\}$ be a family of subsolutions of (H) satisfying $v_t < g$ on Γ_1 and $v_t = g$ on Γ_2 , for all $t \in [0, 1]$. If*

- 1) $t \rightarrow (v_t - v_0) \in C_e(\overline{\Omega})$ is continuous with respect to the $\|\cdot\|_e$ -norm,
- 2) $u \geq v_0$ in $\overline{\Omega}$, and
- 3) $u \neq v_t$, for all $t \in [0, 1]$,

then there exists $\alpha > 0$, such that for all $t \in [0, 1]$ $u - v_t \geq \alpha e$ in $\overline{\Omega}$.

PROOF. Set $E = \{t \in [0, 1]; u \geq v_t \text{ in } \overline{\Omega}\}$. By 2) E is not empty. Moreover E is closed. For $t \in E$ $w_t := u - v_t$ satisfies

$$\int_{\Omega} (w \cdot (-\Delta \varphi) + \omega w \varphi) dx \geq 0 \text{ for all } \varphi \in \mathcal{D}^+(\Omega) \text{ and some } \omega > 0.$$

Since $w_t \not\equiv 0$ it follows from [5, Corollary p. 581] that there is $\beta > 0$, such that $w_t \geq \beta u_0$, for some $u_0 \in C^1(\overline{\Omega})$, which satisfies $u_0 > 0$ in Ω , $u_0 = 0$ and $\frac{\partial}{\partial n} u_0 < 0$ on Γ . The function w_t is positive on Γ_1 , which is compact, and continuous on $\overline{\Omega}$. Hence there exists $\gamma > 0$ such that $w_t \geq \gamma e$. Since $t \rightarrow (w_t - w_0)$ is continuous with respect to the $\|\cdot\|_e$ -norm, E is also open. Hence $E = [0, 1]$ and there is $\alpha > 0$, such that $w_t \geq \alpha e$ in $\overline{\Omega}$ for all $t \in [0, 1]$.

This completes the proof of Lemma A.2.

Let ψ be the principal eigenfunction, with eigenvalue ν , of

$$\begin{cases} -\Delta v = \lambda v & \text{in } B, \\ v = 0 & \text{on } \partial B, \end{cases}$$

where B denotes the unit ball in $\mathbb{R}^N$.

Let ψ be normalized such that $\max \psi = 1$.

LEMMA A.3. *Let u satisfy $-\Delta u = \lambda f(u)$ in an open $\Omega' \subset \Omega$, such that $u(x) > a$ for $x \in \Omega'$. Let $\sigma > 0$ be such that $f(u) > \sigma(u - a)$ for $u \in [a, b]$.*

If $x_0 \in (\Omega')_{(\nu/\sigma\lambda)^{\frac{1}{2}}}$, then $u(x_0) > b$.

PROOF. Set $\theta(x_0, \lambda, t; x) = a + t\psi((\sigma\lambda/\nu)^{\frac{1}{2}}(x - x_0))$ for $x \in B(\cdot)$ and $t \in [0, b - a]$, where $B(\cdot) = B(x_0, (\nu/\sigma\lambda)^{\frac{1}{2}})$. The set $\{\theta(x_0, \lambda, t); t \in [0, b - a]\}$ is a family of subsolutions of the problem

$$(Pb) \quad \begin{cases} -\Delta v = \lambda f(v) & \text{in } B(\cdot) \\ v = u & \text{on } \partial B(\cdot), \end{cases}$$

and $\overline{B(\cdot)} \subset \Omega'$.

By using Lemma A.2 one finds $u(x_0) > b$.

It remains to show that $\theta(x_0, \lambda, t)$ is a subsolution of (Pb_λ) . By the assumption of the lemma $u > a = \theta(x_0, \lambda, t)$ on $\partial B(\cdot)$.

The integral condition is also satisfied:

$$\begin{aligned} \int_{B(\cdot)} (\theta(-\Delta\varphi) - \lambda f(\theta)\varphi) dx &= \int_{B(\cdot)} (-\Delta\theta - \lambda f(\theta))\varphi dx \leq \\ &\leq \int_{B(\cdot)} (-\Delta\theta - \lambda\sigma(\theta - a))\varphi dx = 0 \text{ for all } \varphi \in \mathcal{D}^+(B(\cdot)) \end{aligned}$$

This completes the proof of the lemma.

Acknowledgement

We would like to thank S.B. Angenent and L.A. Peletier for some helpful discussions, and one referee for pointing out several errors.

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