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R. DIECKERHOFF
E. ZEHNDER

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Boundedness of Solutions Via the Twist-Theorem

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1. Introduction and results

a) - *Results*

It is well known that the longtime behaviour of a timedependent nonlinear differential equation

(1.1) $$\ddot{x} + f(t, x) = 0,$$

f being periodic in t, can be very intricate. For example, there are equations having unbounded solutions but with infinitely many zeroes and with nearby unbounded solutions having randomly prescribed numbers of zeroes and also periodic solutions, see V.M. Alekseev [8]. K. Sitnikov [7] and J. Moser [4]. In contrast to such unboundedness phenomena one may look for conditions on the nonlinearity, in addition to the condition, that

(1.2) $$\frac{1}{x} f(t, x) \to \infty \quad \text{as } |x| \to \infty,$$

which allow to conclude that all the solutions of the equation are bounded. For example every solution of the equation

(1.3) $$\ddot{x} + x^3 = p(t),$$

$p(t + 1) = p(t)$ being continuous, is bounded. This result, prompted by questions of J.E. Littlewood in [2], is due to G.R. Morris [1]. Our aim is to extend the result to a more general but still very restricted class of equations for which in particular the timedependence is involved in the nonlinearity in x.

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THEOREM 1. *Every solution $x(t)$ of the equation*

$$\ddot{x} + x^{2n+1} + \sum_{j=0}^{2n} x^j p_j(t) = 0, \quad n \geq 1, \tag{1.4}$$

with $p_j(t+1) = p_j(t)$ and $p_j \in C^\infty$, is bounded, i.e. it exists for all $t \in \mathbb{R}$ and

$$\sup_{\mathbb{R}}(|x(t)| + |\dot{x}(t)|) < \infty.$$

We shall not make the bound explicit in terms of the initial conditions $x(0)$ and $\dot{x}(0)$ of the solution. The proof of Theorem 1 is strictly 2-dimensional and based on J. Moser's twist theorem similarly to the proof of Morris' result. The underlying idea is as follows. By means of transformation theory the equation is outside of a large disc D in the $(x,\dot{x})$-plane transformed into a Hamiltonian equation having the following property. Following the solutions from the section $t = 0$ to the section $t = 1$ defines a map, the time 1 map ϕ of the flow, which is close to a so called twist map in $\mathbb{R}^2 \setminus D$. By means of the twist-theorem one finds large invariant curves diffeomorphic to circles and surrounding the origin in the $(x, \dot{x})$-plane. Every such curve is the base of a time periodic and under the flow invariant cylinder in the phase space $(x, \dot{x}, t) \in \mathbb{R}^2 \times \mathbb{R}$, which confines the solutions in its interior and which therefore leads to a bound of these solutions. It turns out that the solutions starting at $t = 0$ on the invariant curves are quasiperiodic.

THEOREM 2. *There is a (large) $\omega^* > 0$ such that for every irrational number $\omega > \omega^*$ satisfying*

$$\left|\omega - \frac{p}{q}\right| \geq c|q|^{-2-\beta} \tag{1.5}$$

for all integers p and $q \neq 0$ with two constants $\beta > 0$ and $c > 0$, there is a quasiperiodic solution of (1.4) *having frequencies* $(\omega, 1)$*; i.e. there is a smooth function $F(\theta_1, \theta_2)$ periodic of period* 1*, such that*

$$x(t) := F(\theta_1 + \omega t, \theta_2 + t)$$

are solutions of the equation.

It is well known that the invariant curves guaranteed by the twist-theorem lead to an abundance of periodic solutions. For example by applying the Poincaré-Birkoff fixed point theorem to the annuli bounded by two suitable invariant curves one finds fixed points and periodic points of the time 1 map of the flow. They give rise to forced oscillations and subharmonic solutions of the equation. One knows, in addition, that these invariant curves are in the closure of the set of periodic points. Since in our case the set of invariant curves has infinite Lebesgue measure in $\mathbb{R}^2 \setminus D$ we shall conclude:

THEOREM 3. *For every integer $m \geq 1$ there are infinitely many periodic solution of* (1.4) *having minimal period m. Moreover the closure of the set of subharmonic solutions of* (1.4) *is of infinite Lebesgue measure in the phase space $\mathbb{R}^2 \times S^1$.*

For related results concerning infinitely many forced oscillations we point out [5] and [6]. We should mention, that up to now there is no genuine generalization of the first part of this statement to higher dimensions, i.e. equations $\dot{x} = J\nabla h(t,x)$, $x \in \mathbb{R}^{2n}$, $n \geq 2$. In the special case of equations of the form $\dot{x} = J\nabla h(x) + p(t)$ Berestycki and Bahri [13] found infinitely many forced oscillations using minimax techniques, under suitable assumptions on the Hamilton function h.

b) - *Remark about the smoothness requirements.*

It is likely that in the general case (1.1) the boundedness of the solution is related to an excessive smoothness requirement in the x-variable. The role of the smoothness in the t-variable is not clear. In the special case of (1.4) where the time is not involved in the nonlinear term, the dependence on t is only required to be continuous. In fact we shall prove:

THEOREM 4. *Every solution of*

$$\ddot{x} + x^{2n+1} + p(t) = 0, \quad n \geq 1, \tag{1.6}$$

with a continuous $p(t+1) = p(t)$ is bounded. Moreover the statements of Theorem 2 and Theorem 3 hold true.

In this statement, conjectured already in [1], the timedependent term is bounded and is required merely to be continuous. In contrast, the proof of the general case in Theorem 1 with the timedependent term unbounded, requires an excessive amount of derivatives in the t-variable. In fact the number of derivatives in t we need depends on the size of the t-dependent nonlinear term. It will follow from the proof that the statement of Theorem 1 holds true for the equation

$$\ddot{x} + x^{2n+1} + \sum_{j=0}^{\ell} x^j p_j(t) = 0, \tag{1.7}$$

$0 \leq \ell \leq 2n$, under the following smoothness assumption which is weaker than in Theorem 1: $p_j \in C^\nu(S^1)$, where $\nu = \nu(\ell, n)$ is the smallest integer satisfying

$$\begin{aligned} &\nu > \frac{1}{n}(\ell+3) \text{ if } 0 \leq \ell \leq n+1 \\ &\nu > 1 + \frac{4}{n} + \left[\log_2 \frac{n}{2n+1-\ell}\right] \text{ if } n+1 \leq \ell \leq 2n. \end{aligned} \tag{1.8}$$

It is not clear whether the boundedness phenomenon is related to the smoothness in the t-variable or whether this requirement is a shortcoming of our proof.

We point out that related problems for the equation $\ddot{x} + q(t)x^3 = 0$ have been investigated by Coffmann and Ullrich [17]. We should remark that the statement of Theorem 1 holds true for a more general class of equations as can be seen from its proof.

2. - Action and angle-variables

Dropping the timedependent term equation (1.4) becomes $\ddot{x} + x^{2n+1} = 0$. Introducing $\dot{x} = y$ we have vectorfield $\dot{x} = y$ and $\dot{y} = -x^{2n+1}$ which is a time-dependent Hamiltonian system on $\mathbb{R}^2$:

$$X_h : \begin{aligned} \dot{x} &= \frac{\partial}{\partial y} h(x,y) \\ \dot{y} &= -\frac{\partial}{\partial x} h(x,y) \end{aligned} \tag{2.1}$$
$$h(x,y) = \frac{1}{2}y^2 + \frac{1}{2(n+1)}x^{2(n+1)}.$$

Clearly $h > 0$ on $\mathbb{R}^2$ except at the only equilibrium point $(x,y) = (0,0)$ where $h = 0$. All the solutions of (2.1) are periodic, the periods tending to zero as $h = E$ tends to infinity. In fact, since the Hamiltonian function is homogeneous in the x-variable, the solutions are easily described in terms of a single reference solution for which we take $(x^*(t), y^*(t))$ having the initial conditions $(x^*(0), y^*(0)) = (1,0)$. It has the energy $h(x^*(t), y^*(t)) = E = \dfrac{1}{2(n+1)}$. Let $T^* > 0$ be its minimal period and introduce the functions C and S by

$$(C(t), S(t)) = (x^*(t), y^*(t)). \tag{2.2}$$

These analytic functions satisfy

(i) $C(t) = C(t+T^*)$, $S(t) = S(t+T^*)$ and $C(0) = 1$, $S(0) = 0$.

(ii) $\dot{C}(t) = S(t)$ and $\dot{S}(t) = -C(t)^{2n+1}$.

(iii) $(n+1)S(t)^2 + C(t)^{2(n+1)} = 1$

(iv) $C(-t) = C(t)$ and $S(-t) = -S(t)$.

Let $r > 0$ and set $\gamma := \dfrac{1}{n}$, then the solution of (2.1) with the initial conditions $(x(0), y(0)) = (r^\gamma, 0)$ is given by $(x(t), y(t)) = (r^\gamma C(rt), r^{\gamma+1}S(rt))$ as one readily verifies. It has period $T = \dfrac{1}{r}T^*$. In terms of its energy $E = \dfrac{1}{2(n+1)} \cdot r^{\gamma 2(n+1)}$ we find for $T = f(E)$ the function $f(E) = C\cdot E^{-\sigma}$ with $\sigma = \frac{n}{2(n+1)}$. Loosely speaking the solutions spin around faster and faster with increasing energy, which has, as it turns out, a stabilizing effect.

The action and angle variables are now defined by the map $\psi\colon \mathbb{R}^+ \times S^1 \to \mathbb{R}^2 \setminus \{0\}$, where $(x, y) = \psi(\lambda, \theta)$ with $\lambda > 0$ and with θ (mod 1) is given by the formulae:

$$\psi: \quad \begin{aligned} x &= c^\alpha \lambda^\alpha C\left(\frac{\theta}{T^*}\right) \\ y &= c^\beta \lambda^\beta S\left(\frac{\theta}{T^*}\right), \end{aligned} \tag{2.3}$$

$$\alpha = \frac{1}{n+2}, \quad \beta = 1 - \alpha \text{ and } c = \frac{1}{\alpha} T^*.$$

We claim that ψ is a symplectic diffeomorphism from $\mathbb{R}^+ \times S^1$ onto $\mathbb{R}^2 \setminus \{0\}$. Indeed, for the Jacobian Δ of ψ one finds by (ii) and (iii) $|\Delta| = 1$, so that ψ is measure preserving. Moreover since (S, C) is a solution of a differential equation having T^* as minimal period one concludes that ψ is one to one and onto, which proves the claim.

In the new coordinates the Hamiltonian function (2.1) becomes

$$h \circ \psi(\lambda, \theta) = d \cdot \lambda^{2\beta} = h_0(\lambda), \tag{2.4}$$

$d = \frac{c^{2\beta}}{2(n+1)}$, which is independent of the angle variable θ so that the system (2.1) becomes very simple:

$$\begin{aligned} \dot{\theta} &= \frac{\partial}{\partial \lambda} h_0 = 2\beta d \lambda^{2\beta - 1} \\ \dot{\lambda} &= -\frac{\partial}{\partial \theta} h_0 = 0. \end{aligned} \tag{2.5}$$

The full equation (1.4) has the Hamiltonian function:

$$h(x, y, t) = \frac{1}{2} y^2 + \frac{1}{2(n+1)} x^{2(n+1)} + \sum_{\ell=1}^{2n+1} \frac{x^\ell}{\ell} p_\ell(t). \tag{2.6}$$

Under the symplectic transformation ψ it is transformed into:

$$h_1(\lambda, \theta, t) = h(\psi(\lambda, \theta), t) = d\lambda^{2\beta} + G(\lambda, \theta, t), \tag{2.7}$$

where G is of the form:

$$G(\lambda, \theta, t) = \sum_{j=1}^{2n+1} G_j(\theta, t) \lambda^{\alpha j},$$

with $G_j \in C^\infty(T^2)$ and $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The Hamiltonian system X_{h_1}, is now more complicated:

$$X_{h_1}: \quad \begin{aligned} \dot{\theta} &= \frac{\partial}{\partial \lambda} h_1 = 2\beta d \lambda^{2\beta-1} + \frac{\partial G}{\partial \lambda} \\ \dot{\lambda} &= -\frac{\partial}{\partial \theta} h_1 = -\frac{\partial G}{\partial \theta}. \end{aligned} \tag{2.8}$$

In the following we shall transform this system, for large λ, into a simpler system for which the θ-depending terms are small, so that in the new variables λ is close to an integral. To this effect an iterative sequence of finitely many canonical transformations of $\mathbb{R}^+ \times S^1$ which depend periodically on time t will be carried out in the next section.

3. - More canonical transformations

First we introduce a space of functions $\mathcal{F}(r)$ which behave for large $\lambda > 0$, with all their derivatives, like $\lambda^r f_r(\theta, t)$. Given $r \in \mathbb{R}$ we denote by $\mathcal{F}(r)$ the set of C^∞ functions in $(\lambda, \theta, t) \in \mathbb{R}^+ \times T^2$ which are defined in $\lambda \geq \lambda_0$ for some $\lambda_0 > 0$ and for which there is a sequence $\lambda_{j\ell k} > 0$ such that

$$\sup_{\substack{\lambda \geq \lambda_{j\ell k} \\ (\theta,t) \in T^2}} (\lambda^{j-r} |(D_\lambda)^j (D_\theta)^\ell (D_t)^k f(\lambda, \theta, t)|) < \infty.$$

We summarize some properties readily verified from the definition:

LEMMA 1.
(i) *if* $r_1 < r_2$ *then* $\mathcal{F}(r_1) \subset \mathcal{F}(r_2)$
(ii) *if* $f \in \mathcal{F}(r)$ *then* $(D_\lambda)^j f \in \mathcal{F}(r - j)$
(iii) *if* $f_1 \in \mathcal{F}(r_1)$ *and* $f_2 \in \mathcal{F}(r_2)$ *then* $f_1 \cdot f_2 \in \mathcal{F}(r_1 + r_2)$
(iv) *if* $f \in \mathcal{F}(r)$ *satisfies* $|f(\lambda, \bullet)| \geq c\lambda^r$ *for* $\lambda > \lambda_0$ *then* $\frac{1}{f} \in \mathcal{F}(-r)$.

For $f \in \mathcal{F}(r)$ we denote the meanvalue over the θ-variable by $[f]$:

$$[f](\lambda, t) := \int_0^1 f(\lambda, \theta, t) d\theta.$$

If $\lambda_0 > 0$, then $A_{\lambda_0} \subset \mathbb{R}^+ \times T^2$ denotes the annulus

$$A_{\lambda_0} := \{(\lambda, \theta, t) | \lambda \geq \lambda_0 \text{ and } (\theta, t) \in T^2\}.$$

PROPOSITION 1. *Let*

$$H = \lambda^a + h_1(\lambda, t) + h_2(\lambda, \theta, t)$$

with $h_1 \in \mathcal{F}(c)$ *and* $h_2 \in \mathcal{F}(b)$. *Assume* $a > 1$, $b < a$ *and* $c < a$. *Then there is a canonical diffeomorphism* ψ *depending periodically on* t *of the form*

$$\psi : \begin{array}{l} \lambda = \mu + u(\mu, \phi, t) \\ \theta = \phi + v(\mu, \phi, t), \end{array}$$

with $u \in \mathcal{F}(1 - (a - b))$ *and* $v \in \mathcal{F}(-(a - b))$ *such that* $A_{\mu_+} \subset \psi(A_{\mu_0}) \subset A_{\mu_-}$ *for some large* $\mu_- < \mu_0 < \mu_+$. *Moreover the transformed Hamiltonian vectorfield* $\psi^*(X_H) = X_{\hat{H}}$ *is of the form:*

$$\hat{H} = \mu^a + \hat{h}_1(\mu, t) + \hat{h}_2(\mu, \phi, t),$$

where $\hat{h}_1 \in \mathcal{F}(c_1)$, *with* $c_1 = \max\{c, b\}$, *is given by*

$$\hat{h}_1(\mu, t) = h_1 + [h_2],$$

and where $\hat{h}_2 \in \mathcal{F}(b_1)$. *The constant* b_1 *is smaller that* b *and is given by*

$$b_1 = \begin{cases} b - (a - b) \text{ if } b \geq 1 \\ b - (a - 1) \text{ if } b \leq 1. \end{cases}$$

PROOF. The proof will follow from several Lemmata. We shall look for the requires reansformation ψ given by means of a generating function $S(\mu, \theta, t)$, so that $\psi$is implicitely defined by

$$\psi : \begin{array}{l} \lambda = \mu + \dfrac{\partial}{\partial \theta} S \\ \phi = \theta + \dfrac{\partial}{\partial \mu} S. \end{array} \tag{3.1}$$

In the following we shall often supress the variable t in the formulae. Abbreviating:

$$\begin{array}{l} h_0(\lambda) := \lambda^a + h_1(\lambda, t) \\ \nu := \dfrac{\partial}{\partial \theta} S \end{array} \tag{3.2}$$

we have for the transformed Hamiltonian vectorfield $\psi^*(X_H) = X_{\hat{H}}$, $\hat{H}$ expressed in the variables (μ, θ) instead of (μ, ϕ):

$$\hat{H}(\mu, \theta) = h_0(\mu + \nu) + h_2(\mu + \nu, \theta) + \frac{\partial}{\partial t} S. \tag{3.3}$$

By Taylor's formula we can write

$$\hat{H}(\mu, \theta) = h_0(\mu) + h_0'(\mu)\nu + h_2(\mu, \theta) + R, \tag{3.4}$$

with

$$R = \frac{\partial}{\partial t} S + \int_0^1 (1 - \tau) h_0''(\mu + \tau\nu)\nu^2 \mathrm{d}\tau + \int_0^1 h_2'(\mu + \tau\nu, \theta) \cdot \nu \mathrm{d}\tau, \tag{3.5}$$

where $'$ stands for the derivative in λ. We now determine ν from the equation

$$h_0'(\mu)\nu + h_2(\mu, \theta) - [h_2] = 0$$

so that

$$\nu = \frac{1}{h_0'(\mu)}(h_2 - [h_2]) \tag{3.6}$$

and therefore

$$S = \int_0^\theta \nu \mathrm{d}\theta. \tag{3.7}$$

Consequently:

$$\hat{H}(\mu, \theta) = h_0(\mu) + [h_2] + R. \tag{3.8}$$

LEMMA 2. *Let S be defined by* (3.7) *and* (3.6). *Then the formula* (3.1) *defines a symplectic diffeomorphism ψ depending periodically on time t of the form*

$$\psi : \quad \begin{aligned} \lambda &= \mu + u(\mu, \phi, t) \\ \theta &= \phi + v(\mu, \phi, t), \end{aligned}$$

with $u \in \mathcal{F}(1 - (a - b))$ and $v \in \mathcal{F}(-(a - b))$.

PROOF. We first claim that

$$S \in \mathcal{F}(1 - (a - b)). \tag{3.9}$$

Indeed, since $a > c$ we find, for μ sufficiently large,

$$\frac{1}{2} a\mu^{a-1} \leq h_0'(\mu) \leq 2a\mu^{a-1}.$$

Set $f(\mu) = h_0'(\mu)$. Then $f \in \mathcal{F}(a - 1)$ and therefore $\frac{1}{f} \in \mathcal{F}(1 - a)$ by Lemma 1 (iv). Now set $g_1 = \frac{1}{f}$ and $g_2 = h_2 - [h_2] \in \mathcal{F}(b)$. Then in view of Lemma 1

(iii) $g_1 \cdot g_2 = \nu \in \mathcal{F}(1-(a-b))$ and the claim (3.9) follows. Next we solve the second equation of (3.1) for θ and write

$$\phi = \theta + \frac{\partial}{\partial \mu} S = \theta + g(\mu, \theta, t). \tag{3.10}$$

By (3.9), $g \in \mathcal{F}(-(a-b))$. For the inverse we set

$$\theta = \phi + v(\mu, \phi, t).$$

In view of (3.10) we find for v the equation

$$v = -g(\mu, \phi + v). \tag{3.11}$$

If μ is large, then $|D_\theta g| \leq \frac{1}{2}$ so that v is uniquely determined by the contraction principle. Moreover, by the implicit function theorem $v \in C^\infty(A_{\mu_0})$ for some large μ_0. We claim

$$v \in \mathcal{F}(-(a-b)). \tag{3.12}$$

Indeed, apply $(D_\mu)^n$ to the equation (3.11). The right hand side is a sum of terms

$$(D_\theta^s D_\mu^k g) \cdot D_\mu^{j_1} v \cdot D_\mu^{j_2} v \cdots D_\mu^{j_s} v \tag{3.13}$$

with $1 \leq s + k \leq n$ and $\sum_{k=1}^{s} j_k = n - k$. The highest order term is the one with $s = 0$ and $k = 0$, namely $(D_\theta g) \cdot (D_\mu)^n v$, which we put to the left hand side of the equation. Inductively assuming that for $j \leq n-1$ the estimates $|D_\nu^j h| \leq C\mu^{-(a-b)-j}$ hold true we conclude the same estimate for $j = n$, since $g \in \mathcal{F}(-(a-b))$ and therefore $|D_\theta^s D_\mu^k g| \leq C\mu^{-(a-b)-k}$. The claim (3.12) follows. We next insert $\theta = \phi + v$ into the first equation of (3.1) and define u to be

$$\lambda = \mu + \nu(\mu, \phi + v, t) = \mu + u(\mu, \phi, t). \tag{3.14}$$

Since $\nu \in \mathcal{F}(1-(a-b))$ in view of (3.9) and since $v \in \mathcal{F}(-(a-b))$ in view of (3.12) one concludes using (3.13) that $u \in \mathcal{F}(1-(a-b))$.

To finish the proof of the Lemma one verifies easily that the map ψ has a right inverse of the same type as ψ defined on A_{μ_-} for some large μ_- and that it is injective on A_{μ_0} for μ_0 large. •

For the transformed Hamiltonian function $\hat{H}$, now expressed in the variables μ, ϕ we have in view of (3.8), (3.5) and (3.14):

$$\hat{H}(\mu, \phi, t) = h_0(\mu) + [h_2] + R, \tag{3.15}$$

where $R = R_1 + R_2 + R_3$, with

$$R_1 = \left(\frac{\partial}{\partial t} S\right)(\mu, \phi + v, t)$$

$$R_2 = \int_0^1 (1-\tau) h_0''(\mu + \tau u) u^2 \mathrm{d}\tau$$

$$R_3 = \int_0^1 h_2'(\mu + \tau u, \phi + v) u \mathrm{d}\tau.$$

LEMMA 3.
(i) $R_1 \in \mathcal{F}(1-(a-b))$,
(ii) $R_2 \in \mathcal{F}(b-(a-b))$,
(iii) $R_3 \in \mathcal{F}(b-(a-b))$.

PROOF. ad(i) $S \in \mathcal{F}(1-(a-b))$ by (3.9) and $v \in \mathcal{F}(-(a-b))$ by Lemma 2, hence $R_1 \in \mathcal{F}(1-(a-b))$.

ad(ii) Set $f = h_0''(\mu)$, then $f \in \mathcal{F}(a-2)$. Setting $w := \mu + \tau u$, $D = D_\mu$, then $D^n(f(w))$ is a sum of terms:

$$(D^s f)(w) \cdot D^{j_1} w \cdot D^{j_2} w \cdots D^{j_s} w$$

with $\sum_{k=1}^{s} j_k = n$ and $1 \leq j_k \leq n$ amd $1 \leq s \leq n$. Since $f \in \mathcal{F}(a-2)$ and $u \in \mathcal{F}(1-(a-b))$ we have $|(D^s f)(w)| \leq C\mu^{a-2-s}$. Moreover, if $j = 1$ then $|D^j w| \leq 2$ for μ sufficiently large, since $Du \in \mathcal{F}(-(a-b))$. If $j > 1$, then $D^j w = \tau D^j u \in \mathcal{F}(1-(a-b)-j)$. Therefore the above term is estimated by $\leq \mu^{a-2-n}$. Hence $g_1 := f(w) \in \mathcal{F}(a-2)$. Since $g_2 := u^2 \in \mathcal{F}(2-2(a-b))$ we conclude in view of Lemma 1 (iii) that $g_1 \cdot g_2 \in \mathcal{F}(b-(a-b))$ as claimed.

ad(iii) Recall that $D^n(f(A,B))$ is a sum of terms

$$(D_2^s D_1^k f) \cdot (D^{i_1} A) \cdots (D^{i_k} A) \cdot (D^{j_1} B) \cdots (D^{j_s} B),$$

with $1 \leq s+k \leq n$ and $\sum_{r=1}^{k} i_r = n-s$ and $\sum_{r=1}^{s} j_r = n-k$. Since $h_2' \in \mathcal{F}(b-1)$, $u \in \mathcal{F}(1-(a-b))$ and $v \in \mathcal{F}(-(a-b))$ one concludes that $g := h_2'(\mu + \tau u, \phi + v) \in \mathcal{F}(b-1)$. Therefore $g \cdot u \in \mathcal{F}(b-(a-b))$ as claimed. •

In view of Lemma 2 and Lemma 3 and in view of (3.15) the proof of Proposition 1 is finished setting $\hat{h}_2 = R$.

4. - Proof of the Theorems 1-3

If H satisfies the assumptions of Proposition 1 then for any given number $d < 0$ there is an integer $j = j(a, b, d)$ so that after j successive applications of the proposition we find that the corresponding perturbation term $\hat{h}_2$ belongs to $\mathcal{F}(b_j)$ with $b_j < d$. We shall make the number j of steps needed precise for our special case (1.7), which in angle and action variables is a Hamiltonian system of the form:

$$(4.1) \qquad \begin{aligned} &H(\lambda, \theta, t) = \lambda^a + h_2(\lambda, \theta, t) \\ &\text{with } h_2 \in \mathcal{F}(b), \quad a = \alpha(2n+2) \text{ and } b = \alpha(\ell + 1), \end{aligned}$$

where $0 \leq \ell \leq 2n$ and $\alpha = \dfrac{1}{n+2}$. Therefore $1 < a < 2$ and $b < a$ so that the assumptions of Proposition 1 are met.

PROPOSITION 2. *Let H be as in* (4.1). *Then there is a canonical transformation, periodic in time $t : \psi = \psi_j \circ \psi_{j-1} \circ \cdots \circ \psi_1$ with $A_{\mu_+} \subset \psi(A_{\mu_0}) \subset A_{\mu_-}$ for some $\mu_- < \mu_0 < \mu_+$, which transforms the Hamiltonian system into $\psi^*(X_H) = H_{\hat{H}}$, where*

$$(4.2) \qquad \hat{H}(\lambda, \theta, t) = \lambda^a + \hat{h}_1(\lambda, t) + \hat{h}_2(\lambda, \theta, t)$$

with $\hat{h}_1 \in \mathcal{F}(b)$ and $\hat{h}_2 \in \mathcal{F}(-\varepsilon)$ for $\varepsilon > 2 - a$. The number j of transformations is smaller or equal to $j^* = j^*(n, \ell)$ where j^* is the smallest integer $> \frac{1}{n}(\ell + 3)$ in the case that $\ell \leq n + 1$, respectively $> 1 + \dfrac{4}{n} + \left[\log_2 \dfrac{n}{2n + 1 - \ell}\right]$ in the case that $\ell \geq n + 1$.

PROOF. Assume $\ell \geq n + 1$, so that $b = \alpha(\ell + 1) \leq 1$. Then j applications of Proposition 1 lead to a perturbation term $\hat{h}_2 \in \mathcal{F}(b_j)$ with $b_j = b - j(a - 1) = \alpha(\ell + 1) - j\alpha n$, which is smaller than -2α if $j > \dfrac{1}{n}(\ell + 3)$. On the other hand, if $\ell > n + 1$, then $b = \alpha(\ell + 1) > 1$. Set $\ell + 1 = 2n + 2 - s$, then after r applications of Proposition 1 we have $b_r = \alpha(2n + 2) - 2^r s\alpha$ as long as $b_r \geq 1$. Now $b_r > 1$ if and only if $2^r s < n$. Let k be so that $2^k s < n \leq 2^{k+1} s$. Then $b_{k+1} = \alpha(2n+2) - 2^{k+1} s\alpha \leq 1$ and therefore $b_{k+1+q} = b_{k+1} - q\alpha n$, hence $b_{k+1+q} < -2\alpha$ if $q > \dfrac{1}{n}(2n + 4 - 2^{k+1}s)$, hence in particular if $q > 1 + \dfrac{4}{n}$. We find in this case $j = k + 1 + q > 1 + \dfrac{4}{n} + \left[\log_2 \dfrac{n}{s}\right]$ hence the result. •

Let now H be as in (4.2) i.e.

$$(4.3) \qquad H = \lambda^a + h_1(\lambda, t) + h_2(\lambda, \theta, t)$$

on A_{λ_0} for some large λ_0, with $h_1 \in \mathcal{F}(b)$ and $h_2 \in \mathcal{F}(-\varepsilon)$ for $\varepsilon > 2 - a$. The Hamiltonian equations are

$$(4.4) \qquad X_H : \quad \begin{aligned} \dot{\theta} &= \frac{\partial}{\partial \lambda} H \\ \dot{\lambda} &= -\frac{\partial}{\partial \theta} H = -\frac{\partial}{\partial \theta} h_2. \end{aligned}$$

Since $h_2 \in \mathcal{F}(-\varepsilon)$ one verifies easily that the solutions do exist for $0 \leq t \leq 1$, if the initial value $\lambda(0) = \lambda$ is sufficiently large. In fact we shall conclude that the time 1 map is close to a twist map.

LEMMA 4. *The time* 1 *map* ϕ^1 *of the flow* ϕ^t *of the vectorfield* X_H *given by* (4.3) *is of the form:*

$$(4.5) \qquad \phi^1 : \quad \begin{aligned} \theta_1 &= \theta + r(\lambda) + f(\lambda, \theta) \\ \lambda_1 &= \lambda + g(\lambda, \theta), \end{aligned}$$

with $r(\lambda) = \alpha \lambda^{a-1} + \int_0^1 \frac{\partial}{\partial \lambda} h_1(\lambda, s) \mathrm{d}s$. *Moreover for every pair* (r, s)*:*

$$|D_\lambda^r D_\theta^s f(\lambda, \bullet)|, \quad |D_\lambda^r D_\theta^s g(\lambda, \bullet)| \leq \lambda^{-\varepsilon - r(2-a)}$$

if $\lambda > \lambda^*(r, s)$.

PROOF. Set

$$r(\lambda, t) = t a \lambda^{a-1} + \int_0^t \frac{\partial}{\partial \lambda} h_1(\lambda, s) \mathrm{d}s,$$

and set for the flow $(\lambda(t), \theta(t)) = \phi^t(\lambda, \theta)$ with $\phi^0 = \mathrm{id}$

$$\begin{aligned} \lambda(t) &= \lambda + B(\lambda, \theta, t) \\ \theta(t) &= \theta + r(\lambda, t) + A(\lambda, \theta, t). \end{aligned}$$

Then the integral-equation

$$\phi^t(\lambda, \theta) = \phi^0(\lambda, \theta) + \int_0^t X_H \circ \phi^s \mathrm{d}s$$

for the flow is equivalent to the following equations for A and B:

$$\begin{aligned} A(\lambda,\theta,t) &= a(a-1)\int_0^t\int_0^1 (\lambda+\tau B)^{a-2}\cdot B\mathrm{d}\tau\mathrm{d}s \\ &+\int_0^t\int_0^1\left(\frac{\partial^2}{\partial\lambda^2}h_1\right)(\lambda+\tau B)\cdot B\mathrm{d}\tau\mathrm{d}s \\ &+\int_0^t\left(\frac{\partial}{\partial\lambda}h_2\right)(\lambda+B,\theta+r+A)\mathrm{d}\tau\mathrm{d}s \\ B(\lambda,\theta,t) &= -\int_0^t\left(\frac{\partial}{\partial\theta}h_2\right)(\lambda+B,\theta+r+A)\mathrm{d}\tau\mathrm{d}s. \end{aligned} \tag{4.6}$$

One verifies easily that for $\lambda\geq\lambda_0$ these equations have a unique solution in the space $|A|$, $|B|\leq 1$ using the contraction principle, moreover A and B are smooth. The required estimates can then inductively be verified from (4.6) in view of $h_1\in\mathcal{F}(b)$, hence $\frac{\partial^2}{\partial\lambda^2}h_1\in\mathcal{F}(b-2)$, and $h_2\in\mathcal{F}(-\varepsilon)$. •

For the completness we include a proof of the following well known fact (see f.e. [14]).

LEMMA 5. *The map $\mathcal{F}=\phi^1$ has the intersection property on A_{λ_0}, i.e. if C is an embedded circle in A_{λ_0} homotopic to a circle λ =const in A_{λ_0}, then $\mathcal{F}(C)\cap C\neq\emptyset$.*

PROOF. Since $\mathcal{F}=\phi^1$ is the time 1 map of an exact Hamiltonian vectorfield, X_H it is exact symplectic, i.e. $\mathcal{F}^*\omega-\omega=dW$ for some W on A_{λ_0}, where $\omega=\lambda d\theta$. In fact, since ϕ^0 =id:

$$\begin{aligned} \mathcal{F}^*\omega-\omega &= \int_0^1\frac{\mathrm{d}}{\mathrm{d}t}(\phi^t)^*\omega \\ &= \int_0^1\phi_t^*\{d(X_H\llcorner\omega)+X_H\llcorner\mathrm{d}\omega\}=\mathrm{d}W \end{aligned}$$

where $W=\int_0^1\{\omega(X_H)+H\}\circ\phi^t\mathrm{d}t$. Therefore, if $i_C: C\to A_{\lambda_0}$ is the injection map, we have $\int_C i_C^*(\mathcal{F}^*\omega-\omega)=0$ and hence

$$\int_{\mathcal{F}(C)} i_{\mathcal{F}(C)}^*\omega=\int_C i_C^*\mathcal{F}^*\omega=\int_C i_C^*\omega. \tag{4.7}$$

Assume now, by contradiction, that $\mathcal{F}(C) \cap C = \emptyset$, then $\mathcal{F}(C) \cup C$ bounds an annulus in A_{λ_0} which has positive measure. Consequently, by Stokes formula, $\int_{\mathcal{F}(C)} i^*_{\mathcal{F}(C)}\omega - \int_C i^*_C\omega \neq 0$, in contradiction to (4.7). This proves the Lemma. •

Finally, in the new coordinates

$$\mu = r(\lambda) \tag{4.8}$$

the time 1 map $\mathcal{F}$ is expressed as follows:

$$\hat{\mathcal{F}}: \quad \begin{aligned} \theta_1 &= \theta + \mu + \hat{f}(\mu,\theta) \\ \mu_1 &= \mu + \hat{g}(\mu,\theta). \end{aligned} \tag{4.9}$$

From Lemma 4 one concludes the following estimates for the perturbation:

$$\begin{aligned} |D^r_\mu D^s_\theta \hat{f}(\mu,\bullet)| &\leq \mu^{-\frac{\varepsilon}{a-1}} \\ |D^r_\mu D^s_\theta \hat{g}(\mu,\bullet)| &\leq \mu^{-\frac{\varepsilon}{a-1}-\frac{2-a}{a-1}}, \end{aligned} \tag{4.10}$$

if $\mu \geq \mu^*(r,s)$. Therefore, if μ is sufficiently large, the map $\hat{\mathcal{F}}$ is, with its derivatives, close to a standard twist map. Moreover, it has the intersection property in view of Lemma 5, so that the assumptions of the twist-theorem [3] are met.

It follows that for $\omega \geq \omega^*$, ω^* sufficiently large, and

$$\left|\omega - \frac{p}{q}\right| \geq c|q|^{-2-\beta} \tag{4.11}$$

for two constants $\beta > 0$ and $c > 0$ and for all integers p and $q \neq 0$ there is an embedding $\psi : S^1 \to A_{\mu_0}$ of a circle, which is differentiably close to the injection map j of the circle $\{\omega\} \times S^1 \to A_{\mu_0}$, and which is invariant under the map $\hat{\mathcal{F}}$. Moreover, on this invariant curve the map $\hat{\mathcal{F}}$ is conjugated to a rotation with rotation number ω:

$$\hat{\mathcal{F}} \circ \psi(s) = \psi(s+\omega) \text{ with } s (\text{mod } 1). \tag{4.12}$$

The solutions of the Hamitonian equation starting at time $t = 0$ on this invariant curve determine a 1-periodic cylinder in the space $(\mu,\theta,t) \in A_{\mu_0} \times \mathbb{R}$. Since the Hamiltonian vectorfield X_H is timeperiodic, the phase space is $A_{\mu_0} \times S^1$. Let Φ^t with Φ^0 =id be the flow of the time-independent vectorfield $(X_H, 1)$ on $A_{\mu_0} \times S^1$ and define the embedded torus $\Psi : T^2 \to A_{\mu_0} \times S^1$ by setting

$$\Psi(s,\tau) = \Phi^\tau(\psi(s-\tau\omega),0) = (\phi^\tau \circ \psi(s-\tau\omega),\tau). \tag{4.13}$$

In view of (4.12) we have with $\hat{\mathcal{F}} = \phi^1$ indeed $\Psi(s+1,\tau) = \Psi(1,\tau+1) = \Psi(s,\tau)$. Moreover $\Phi^t \circ \Psi(s,\tau) = \Psi(s+\omega t, \tau+t)$, so that the torus $\Psi(T^2)$ is quasiperiodic having the frequencies $(\omega, 1)$. This proves the statement of Theorem 2. In order to prove the statement of Theorem 1 just observed that, in the original coordinates, every point $(x, y) \in \mathbb{R}^2$ is in the interior of some invariant curve of the time 1 map of the flow which goes around the origin. Its solution is therefore confined in the interior of the time periodic cylinder above the invariant curve and hence is bounded. This ends the proof of Theorem 1. •

We next prove Theorem 3. Following at first the arguments of G. Morris [1] we shall establish fixed points of the iterated map $\hat{\mathcal{F}}^m$ for every integer m applying the Poincaré-Birkoff fixed point theorem [9] to the map $\hat{\mathcal{F}}^m$ in the annulus R bounded by two invariant curves with rotation numbers $\omega_1 = \omega < \omega_2 = \omega + 1$ satisfying (4.11). First we map this annulus R onto an annulus $R_0 = \{(\xi, \eta) | 0 \leq \eta \leq 1,\ \xi \bmod 1\}$ bounded by concentric circles. If ψ_j are the embeddings of the invariant curves (see (4.12)) having rotation numbers ω_j we define the map $\chi : R_0 \to R$ by

$$\chi : (\mu, \theta) = \eta\psi_1(\xi) + (1 - \eta)\psi_2(\xi). \tag{4.14}$$

Since the embeddings ψ_j are differentiably close to the injections of the circles ω_j =const, χ is a diffeomorphism. The induced map $G : \chi^{-1} \circ \hat{\mathcal{F}} \circ \chi$, expressed by

$$G : \quad \begin{aligned} \xi_1 &= f(\xi, \eta) \\ \eta_1 &= g(\xi, \eta), \end{aligned} \tag{4.15}$$

satisfies in view of (4.12) $f(\xi + 1, \eta) = f(\xi, n)$ and $g(\xi + 1, \eta) = g(\xi, \eta)$. It leaves the boundaries $\eta = 0$ and $\eta = 1$ invariant and satisfies on these boundaries:

$$f(\xi, 0) = \xi + \omega_2 \text{ and } f(\xi, 1) = \xi + \omega_1. \tag{4.16}$$

Define the map $s : R_0 \to R_0$ by $s(\xi, \eta) = (\xi + 1, \eta)$. To find a periodic point of minimal period $q \geq 1$ of G, for any given integer q, we let $[q\omega]$ be the integral part of $q\omega$, i.e. $q\omega - 1 < [q\omega] \leq q\omega$, and choose an integer $0 < p \leq q$ such that $p + [q\omega]$ and q are relative prime. The map $G_q := s^{-p-[q\omega]} \circ G^q$ on R_0, expressed by

$$G_q : \quad \begin{aligned} \xi_1 &= f_q(\xi, \eta) \\ \eta_1 &= g_q(\xi, \eta), \end{aligned} \tag{4.17}$$

satisfies the required twist condition at the boundaries. Indeed with $0 < \alpha = q\omega - [q\omega] < 1$ we conclude from (4.16):

$$f_q(\xi, 1) - \xi = -p + \alpha < 0 < (q - p) + \alpha = f_q(\xi, 0) - \xi.$$

Since G_q leaves a regular measure invariant we conclude that G_q possesses a fixed point in R_0. One verifies readily that it corresponds to a periodic point of G, hence of $\hat{f}$, with minimal period q. This proves the first statement of Theorem 3.

As for the second statement of Theorem 3 we observe that the invariant curves found by J. Moser's twist-theorem are in the closure of the set of periodic points. This is well known and follows from an application of the Birkoff-Lewis fixed point theorem, see e.g. [15]. On the other hand it is also well known, that the set $E \subset A$ covered by the invariant curves contained in an annulus A fill out a set of relatively large measure $m(E) \geq (1-\varepsilon)\cdot m(A)$ with $0 < \varepsilon < 1$, we refer to J. Pöschel [16] and the references therein. Applying these observations to the annuli $A : k \leq \mu \leq k+1$ for all large integers one sees that the set of subharmonic solutions of (1.4) is indeed of infinite Lebesgue measure, since $d\lambda \simeq \mu^d \mathrm{d}\mu$ with $d = \dfrac{2-a}{a-1} = \dfrac{2}{n} > 0$. This proves the statement of Theorem 3. •

5. - Proof of Theorem 4

In case $\ell = 0$ the equation (1.7), expressed in action and angle-variables (λ, θ), (2.3) becomes, after the transformation $\mu = \lambda^a$ with $a = \dfrac{n}{n+2}$:

$$\dot{\theta} = \mu + \mu^{-\varepsilon-1} f_1(\theta, t)$$
$$\dot{\mu} = \mu^{-\varepsilon} f_2(\theta, t),$$

$\varepsilon = \dfrac{1}{n}$, and f_j being continuous in t. One verifies readily that the time 1 flow of this equation satisfies

$$\theta_1 = \theta + \mu + g_1(\theta, \mu)$$
$$\mu_1 = \mu + g_2(\theta, \mu)$$

with g_j being analytic and $|D^r_\mu D^s_\theta g_j(\mu, \bullet)| \leq \mu^{-\varepsilon}$ for $\mu \geq \mu^*(r,s)$. Therefore the assumptions of the twist-theorem are met and Theorem 4 follows readily arguing as at the very end of the proof of Theorem 1.

As for the Remark (1.8) we observe that in Proposition 2 one loses one derivative in t at every successive iteration step, which is due to the derivative in the term $\dfrac{\partial}{\partial t} S$ occurring in the transformation formula. This explains our smoothness requirement. As for the smoothness in the x variable we should remark that the twist-theorem requires merely $C^{3+\varepsilon}$-small perturbations for those rotation numbers which we have considered above [11], and even C^3-small perturbations for certain other irrational rotation numbers [12].

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