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On the Coincidence Set in Biharmonic Variational Inequalities with Thin Obstacles (*).

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1. – Introduction and results.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and set $\Omega' := \Omega \cap \mathbb{R}\times\{0\} \neq \emptyset$. For $g \in H^{2,2}(\Omega)$ and $\Psi' \in C^0(\Omega')$ fixed we define by

$$\mathbb{K}_{\Psi'} := \{v \in g + H_0^{2,2}(\Omega) \colon v \geqslant \Psi' \text{ on } \Omega'\} \tag{1.1}$$

a subset of admissible variations of the Sobolev space $H^{2,2}(\Omega)$ assuming that $\mathbb{K}_{\Psi'} \neq \emptyset$. Then for a $f \in L^p(\Omega)$ with $p > 1$ we introduce the following biharmonic variational inequality

Find $u \in \mathbb{K}_{\Psi'}$ *such that*

$$\langle \Delta^2 u - f, u - v\rangle \leqslant 0 \qquad \text{for all } v \in \mathbb{K}_{\Psi'}, \tag{1.2}$$

where the brackets $\langle\cdot,\cdot\rangle$ denote the natural pairing of the Sobolev space $H_0^{2,2}(\Omega)$ with its dual $H^{-2,2}(\Omega)$ and as usual $\Delta = \partial_{11} + \partial_{22}$ the Laplacian. The problem (1.2) is sometimes called the « one-sided thin obstacle problem for the biharmonic operator » and equivalent to the following variational problem

Find $u \in K_{\Psi'}$ *such that*

$$\int_\Omega |\Delta u|^2 - 2\cdot f\cdot u\,dx \leqslant \int_\Omega |\Delta v|^2 - 2\cdot f\cdot v\,dx\,, \qquad v \in \mathbb{K}_{\Psi'}, \tag{1.3}$$

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which means that both problems possess the same solutions. The problem (1.3) arises in Kirchhoff's linearized plate bending theory when considering the vertical deflection of an isotropic thin plate which is clamped at the boundary $\partial\Omega$ and satisfies an unilateral restriction on Ω'. Then the minimization in (1.3) is with respect to the potential energy. It is well-known from the direct methods in the calculus of variations or from other theories that (1.2) resp. (1.3) possesses a unique solution u provided that $K_{\Psi'} \neq \emptyset$. Choosing suitable variations in $K_{\Psi'}$ and setting them into (1.2), one shows that

$$(1.4) \qquad \mu := \Delta^2 u - f \geqslant 0 \text{ \textit{is a measure with} } \operatorname{supp}(\mu) \subset [u/\Omega' = \Psi'],$$

$$\textit{which means } \Delta^2 u - f = 0 \textit{ on } \Omega - [u|\Omega' = \Psi'].$$

So one is interested in the topological properties of the coincidence set $[u|\Omega' = \Psi']$, but up to now there is nothing known apart from the trivial fact that it is a closed set in Ω. Let us now besides assume that our data is analytic,

$$(1.5) \qquad \Psi' \in C^{\omega}(\Omega') \quad \textit{and } f \in C^{\omega}(\Omega)\,.$$

Then we show in § 2 of the present work the following theorem which is an analogue to Lewy's result [5] for the corresponding second order problem.

THEOREM 1.1. *Assume* (1.5) *and let u solve* (1.2). *Then for an arbitrary set $\Omega_0 \subset\subset \Omega$ there exists $j^0 \in \mathbf{N}$ such that*

$$[u|\Omega' = \Psi'] \cap \Omega_0 = \bigcup_{j=1}^{j^0} [a^j, b^j] \times \{0\} \cap \Omega_0$$

where $a^j \leqslant b^j$ for $j = 1, \ldots, j^0$.

REMARK. i) It suffices to assume that $\Psi' \in C^1(\Omega')$ is piecewise analytic, see theorem 1.1″ in § 2.

ii) The assertion is in general false in case of $\Psi' \in C^{\infty}(\Omega')$. In fact, given an arbitrary problem of type (1.2) with solution u and obstacle Ψ' satisfying $[u|\Omega' = \Psi'] \neq \Omega'$, one can enlarge Ψ' outside of $[u|\Omega' = \Psi']$ to a $\tilde{\Psi}' \in C^{\infty}(\Omega')$ with $\Psi' \leqslant \tilde{\Psi}' \leqslant u|\Omega'$ such that $[u|\Omega' = \tilde{\Psi}'] - [u|\Omega' = \Psi']$ is an arbitrary closed subset of $\Omega' - [u|\Omega' = \Psi']$ and again u solves (1.2) associated to $\tilde{\Psi}'$.

Now the question arises, whether theorem 1.1 has consequences for the regularity of the solution u. In [6], [7] we show that

$$\text{(1.6)} \qquad \textit{for } \Psi' \in C^2(\Omega') \textit{ there is } u \in C^2(\Omega) \cap H^{3,2}_{\mathrm{loc}}(\Omega)\,.$$

But theorem 1.1 allows us to show in § 3

THEOREM 1.2. *Under the assumption* (1.5) *the solution* u *of* (1.2) *has the optimal regularity* $u \in C^{2,\frac{1}{2}}_{\mathrm{loc}}(\Omega)$.

REMARK. i) It suffices to assume that $\Psi' \in C^2(\Omega')$ is piecewise analytic, see theorem 1.2′ in § 3.

ii) It is not possible to give any $C^{2,\frac{1}{2}}$-a priori estimates on compact subsets of Ω for u in terms of the data. In fact, the $C^{2,\frac{1}{2}}$-norm depends crucially on the length of the nondegenerated components of $[u|\Omega' = \Psi']$ as we demonstrate by means of an example in § 1.

iii) The $C^{2,\frac{1}{2}}$-regularity is optimal with respect to the $C^{2,\alpha}$-norms. Moreover, no a priori estimates for the modulus of continuity of $\nabla^2 u$ on compact subsets of Ω are generally possible to establish in terms of the data, see the example in § 1.

Recalling the remark of theorem 1.2 one is faced with the fact that the modulus of continuity of $\nabla^2 u$ depends on the coincidence set. So in order to obtain stability results for u in the C^2-norm, one first has to study the behaviour of the coincidence set of perturbated problems. Now we require the additional assumption that the boundary

$$\text{(1.7)} \qquad \partial\Omega \textit{ is } C^{0,1} \textit{ and } \Psi' \in C^0(\overline{\Omega'})\,, \qquad \Psi'|\partial\Omega' < g|\partial\Omega'\,.$$

Assumption (1.7) assures that

$$\text{(1.8)} \qquad [u|\Omega' = \Psi'] \subset\subset \Omega\,.$$

where dist $([u|\Omega' = \Psi'], \partial\Omega) > 0$ can be estimated from below by a positive bound depending only on the given data. Considering the coincidence set of the perturbated problem, we can not exclude that a nondegenerated connected component I of $[u|\Omega' = \Psi']$ splits into several pieces. But in § 4 we show that for a given I there exists only a finite number of points at which the interval can split after perturbation. The splitting is continuous with respect to analytic changes of the obstacle Ψ' and $H^{2,2}(\Omega)$-changes of the boundary value g. Note, these points are determined only by the measure μ and independent of the perturbations.

THEOREM 1.3. *Let u resp. $\check{u}$ be the solution of* (1.2) *associated to* g, Ψ' *resp.* $\check{g}, \check{\Psi}'$ *under the assumption* (1.5), (1.7), $\delta > 0$ *given and* $[a, b]\times\{0\}$ *a connected component of* $[u|\Omega' = \Psi']$ *with* $a < b$. *Then there exist* $\varepsilon_0 = \varepsilon_0([a, b]\times\{0\}, g, \Psi', \delta) > 0$ *and* $x_1^1, \dots, x_1^{i_0} \in [a, b]$ *with* $i_0 \in \mathbf{N}$ *such that if* $\|g - \check{g}\|_{2,2} + \sum_{i=0}^{\infty} \|d^i(\Psi' - \check{\Psi}')/ds^i\|_{\infty,\Omega'} \cdot \delta^i/i! \leqslant \varepsilon \leqslant \varepsilon_0$, *then*

$$[a, b]\times\{0\} - \bigcup_{i=1}^{i_0} (x_1^i - \eta, x_1^i + \eta)\times\{0\} \subset [\check{u}|\Omega' = \check{\Psi}']$$

where $\eta = \eta(\varepsilon) > 0$, $\eta \to 0$ $(\varepsilon \to 0)$.

REMARK. i) There exists by theorem 4.1 a $\mathbf{R}^1$-Lebesgue density $\Theta' \in C^\omega(a, b) \cap L^1(a, b)$ possessing a finite number of zeros such that $\mu|I = \Theta' \cdot dx_1|[a, b]$ where dx_1 denotes the Lebesgue measure in $\mathbf{R}^1$, and we choose $\{x_1^1, \dots, x_1^{i_0}\} := \{a, b\} \cup [\Theta' = 0]$.

ii) The condition for $\Psi' - \check{\Psi}'$ has to be understood in the sense of majorants known from the theory of power series for a given radius of convergence $\delta > 0$. As possible perturbations of Ψ' we can consider for example a suitable small lowering, lifting or scaling of the obstacle Ψ'. Further, we can add to Ψ' an analytic function multiplied by a suitable small number provided that the $\| \; \|_{\infty,\Omega'}$-norm of the i-th derivative grows like M^i for a fixed $M > 0$.

Comparing the assertion of theorem 1.3 with the result of Athanasopoulos [1] for the related second order problem replacing $H_0^{2,2}(\Omega)$, $H^{2,2}(\Omega)$ by $H_0^{1,2}(\Omega)$, $H^{1,2}(\Omega)$ in (1.1) and Δ^2 by the Laplacian Δ in (1.2), our result is weaker, since we do not show a continuous deformation of the interval under perturbation. But our result is strong enough to prove in § 5 the following C^2-resp. $H^{3,2}$-norm stability result.

THEOREM 1.4. *Let u resp. $\check{u}$ be the solution of* (1.2) *associated to* g, Ψ' *resp.* $\check{g}, \check{\Psi}'$ *under the assumptions* (1.5), (1.7) *and for a given* $\delta > 0$ *the estimate that*

$$\|g - \check{g}\|_{2,2} + \sum_{i=0}^{\infty} \|d^i(\Psi' - \check{\Psi}')/ds^i\|_{\infty,\Omega'} \cdot \delta^i/i! \leqslant \varepsilon .$$

i) *Let* $[a, b]\times\{0\}$ *be a connected component of* $[u|\Omega' = \Psi']$ *with* $a < b$ *and* $U \subset\subset \Omega$ *an open* $\mathbf{R}^2$*-neighborhood of* $[a, b]\times\{0\}$ *with* $U \cap ([u|\Omega' = \Psi'] - [a, b]\times\{0\}) = \emptyset$. *Then there exists* $\varepsilon_0 = \varepsilon_0([a, b]\times\{0\}, U, g, \Psi', \delta) > 0$ *such that for* ε *with* $0 < \varepsilon \leqslant \varepsilon_0$

$$\|u - \check{u}\|_{2,\infty,U} + \|u - \check{u}\|_{3,2,U} \leqslant A(\varepsilon) \to 0 \quad (\varepsilon \to 0) .$$

ii) *Let* $(x_1^0, 0)$ *be an isolated point of* $[u|\Omega' = \Psi]$ *such that* $(u|\Omega' - \Psi')(x_1) = O((x_1 - x_1^0)^3)(x_1 \to x_1^0)$ *and* $U \subset\subset \Omega$ *an open* $\mathbb{R}^2$*-neighborhood of* $(x_1^0, 0)$ *with* $U \cap ([u|\Omega' = \Psi'] - \{(x_1^0, 0)\}) = \emptyset$. *Then there exists* $\varepsilon_0 = \varepsilon_0((x_1^0, 0), U, g, \Psi', \delta) > 0$ *such that for* ε *with* $0 < \varepsilon \leqslant \varepsilon_0$

$$\|u - \breve{u}\|_{2,\infty,U} + \|u - \breve{u}\|_{3,2,U} \leqslant A(\varepsilon) \to 0 \qquad (\varepsilon \to 0)\,.$$

REMARK. i) See remark ii) of theorem 1.3.

ii) The assertion ii) of theorem 1.4 is not generally true for isolated points of $[u|\Omega' = \Psi']$, this situation may happen when the solution has only second order contact with the obstacle as in the example in § 1.

In order to justify the remarks of theorem 1.2 as well as the remark ii) of theorem 1.4, we present the following expample. The associated construction is completely described in [6], [7]. By

$$E(x) := (8\cdot\pi)^{-1}\cdot|x|^2\cdot(\log|x| - 1)\,, \qquad x \in \mathbb{R}^2\,, \tag{1.9}$$

we denote the fundamental solution of the biharmonic operator Δ^2 in $\mathbb{R}^2$, that means in the distributional sense

$$\Delta^2 E = \delta \qquad (\delta \textit{ Dirac measure})\,, \tag{1.10}$$

where

$$-\Delta E(x) = -(2\cdot\pi)^{-1}\cdot\log|x|\,, \qquad x \in \mathbb{R}^2\,, \tag{1.11}$$

is the fundamental solution of the Laplacian Δ times -1 in $\mathbb{R}^2$. Further, let for ε with $0 < \varepsilon \leqslant \frac{1}{4}$

(1.12) ν_ε *be the logarithmic capacitary measure of the straight line segment* $L'_\varepsilon := [-\varepsilon, \varepsilon] \times \{0\}$ *in* $\mathbb{R}^2$,

which is the unique

$$\textit{positive measure } \nu_\varepsilon \geqslant 0 \textit{ with } \operatorname{supp}(\nu_\varepsilon) \subset L'_\varepsilon \tag{1.13}$$

such that the logarithmic capacity potential $-\Delta E * \nu_\varepsilon$ satisfies the conditions

$$-\Delta E * \nu_\varepsilon \equiv 1 \textit{ on } L'_\varepsilon \textit{ and } -\Delta E * \nu_\varepsilon < 1 \textit{ on } \mathbb{R}^2 - L'_\varepsilon\,. \tag{1.14}$$

For the details consult Landkof's book [4], but keep in mind that what we call logarithmic capacitary measure is there called equilibrium measure. Further, one has to take into account that every point of L'_ε is regular relative to $\mathbf{R}^2 - L'_\varepsilon$ for the Dirichlet problem for the Laplace equation which means that $-\Delta E * \nu_\varepsilon$ continuously attains the value 1 at every point of L'_ε.

Now let $B_1(0)$ denote the circle with radius 1 and center $0 \in \mathbf{R}^2$. Let $\tau \in C_0^\infty(B_1(0))$ be a cut-off function with $0 \leqslant \tau \leqslant 1$ and $\tau \equiv 1$ on $B_{\frac{3}{4}}(0)$. Then define for $0 < \varepsilon \leqslant \frac{1}{4}$

$$g_\varepsilon := (1-\tau)\cdot(E * \nu_\varepsilon)|B_1(0) \in C^\infty\big(\overline{B_1(0)}\big) \tag{1.15}$$

and the quadratic polynom Ψ'_ε by

$$\begin{aligned}\Psi'_\varepsilon(x_1) :={}& \qquad a_{2,\varepsilon} \qquad\qquad \cdot x_1^2 + a_{0,\varepsilon}\\ :={}& \big(-1 + (4\cdot\pi)^{-1}\cdot\nu_\varepsilon(\mathbf{R}^2)\big)\cdot x_1^2 + E * \nu_\varepsilon(0)\end{aligned} \tag{1.16}$$

for $x_1 \in \mathbf{R}$. Then we have the following example.

EXAMPLE. *Let $\Omega := B_1(0)$, $\Omega' := \Omega \cap \mathbf{R}\times\{0\}$ and $f \equiv 0$. Then for ε with $0 < \varepsilon \leqslant \frac{1}{4}$ there is $u_\varepsilon := (E * \nu_\varepsilon)|\Omega$ the solution of* (1.2) *associated to g_ε, Ψ'_ε and for the coincidence set of the problem $[u_\varepsilon|\Omega' = \Psi'_\varepsilon] = L'_\varepsilon$.*

Note, that by construction there is

$$\mu_\varepsilon := \Delta^2 u_\varepsilon = \nu_\varepsilon \geqslant 0\,, \tag{1.17}$$

and for the logarithmic capacitary measure ν_ε we show in § 3

$$\begin{aligned}&\nu_\varepsilon = q_\varepsilon\cdot\nu_{\frac{1}{4}}\big((4\cdot\varepsilon)^{-1}\cdot.\big)\,, \qquad \textit{where}\\ &q_\varepsilon := 2\cdot\pi/\big(2\cdot\pi - \nu_{\frac{1}{4}}(\mathbf{R}^2)\cdot\log|4\cdot\varepsilon|\big) > 0\,.\end{aligned} \tag{1.18}$$

Now we study the properties of u_ε for varying ε and in particular the second order derivatives $\nabla^2 u_\varepsilon$. First, we treat the $C^{0,\alpha}$-regularity of $\nabla^2 u_\varepsilon$ where by virtue of the Schauder theory for the Poisson equation we only have to consider Δu_ε. Recalling (1.13), (1.14), we have the usual boundary regularity for the Laplace equation where the line segment L'_ε is the boundary, and further analyticity on $\mathbf{R}^2 - L'_\varepsilon$ so that

$$\Delta u_\varepsilon \in C^{0,\frac{1}{2}}(\bar\Omega)\,, \quad \notin C^{0,\beta}(\Omega) \quad \text{for } \beta > \tfrac{1}{2}\,. \tag{1.19}$$

One proves (1.19) usually by means of suitable comparison functions as we do in § 3 and obtains for the $C^{0,\alpha}$-norms for $0 < \alpha \leqslant \frac{1}{2}$ an upper estimate depending on ε which means

$$(1.20)\qquad \|\Delta u_\varepsilon\|_{C^{0,\alpha}(\bar{\Omega})} \leqslant F(\varepsilon) \quad \textit{with } F \in C_0(0, \tfrac{1}{4}] \quad \text{and} \quad F(\varepsilon) \to \infty \quad (\varepsilon \to 0) .$$

But resting on (1.18) we also show in § 3 the lower estimate

$$(1.21)\qquad \|\Delta u_\varepsilon\|_{C^{0,\alpha}(\overline{B_\varepsilon(0)})} \geqslant C/(-\varepsilon^\alpha \cdot \log|\varepsilon|) , \qquad C = C(\alpha) > 0 ,$$

which means that there can not exist a uniform bound for the $C^{0,\alpha}$-norm on a compact subset of Ω which contains the point 0 for $\varepsilon \to 0$. But at least one has the following fact which is also shown in § 3 using (1.14), (1.18). There is

$$(1.22)\qquad \begin{aligned} &\|u_\varepsilon\|_{2,\infty,\Omega} \leqslant C \quad \textit{with } C > 0 , \\ &\|u_\varepsilon\|_{1,\infty,\Omega},\ \|\partial_{12} u_\varepsilon\|_{\infty,\Omega} \to 0 \quad (\varepsilon \to 0) , \\ &\partial_{ii} u_\varepsilon \to 0 \textit{ uniformly on compact subsets of } \Omega - \{0\} \textit{ but } \partial_{ii} u_\varepsilon(0) \to \tfrac{1}{2} \\ &(\varepsilon \to 0) \textit{ for } i = 1, 2 . \end{aligned}$$

Note, that from (1.22) follows that

$$(1.23)\qquad u_\varepsilon \to u_0 := 0 \quad \textit{in } H^{2,2}(\Omega) \quad (\varepsilon \to 0) .$$

Next, we study the behaviour of g_ε, Ψ'_ε for $\varepsilon \to 0$. Again from (1.18) we see that $\nu_3(\mathbf{R}^2) \to 0$ $(\varepsilon \to 0)$ what yields

$$(1.24)\qquad \|E * \nu_\varepsilon\|_{i,\infty,\Omega - B_{\frac{1}{2}}(0)} \to 0 \quad (\varepsilon \to 0) \textit{ for every } i \in \mathbf{N} ,$$

thus setting $g_0 := 0$ we have that

$$(1.25)\qquad \|g_\varepsilon - g_0\|_{i,\infty,\Omega} \to 0 \quad (\varepsilon \to 0) \textit{ for every } i \in \mathbf{N} .$$

Further, from (1.18), (1.22) follows that in (1.16)

$$(1.26)\qquad a_{2,\varepsilon} \to -1 , \quad a_{0,\varepsilon} \to 0 \quad (\varepsilon \to 0) ,$$

thus defining Ψ'_0 by $\Psi'_0(x_1) := (-1) \cdot x_1^2$ for $x_1 \in \mathbf{R}$ we even have for an arbitrary but fixed $\delta > 0$ that

$$\sum_{i=0}^{\infty} \|d^i(\Psi'_\varepsilon - \Psi'_0)/ds^i\|_{\infty,\mathbf{R}} \cdot \delta^i/i! \to 0 \quad (\varepsilon \to 0) . \tag{1.27}$$

Now we mention that trivially there is

(1.28) *u_0 the solution of* (1.2) *associated to g_0, Ψ'_0 with* $[u_0|\Omega' = \Psi'_0] = \{0\}$ *where Ω, Ω', f are the same as in our example.*

For small $\varepsilon > 0$ one can consider our example as a perturbation of (1.28) in the sense which is stated in the assumption in theorem 1.3, 1.4. So by virtue of (1.19), (1.20), (1.21), (1.22), all the remarks of theorem 1.2, 1.4 are justified. Moreover, we see that the regularity of the solution u_0 does not guarantee the stability of the solutions u_ε in the C^2-norm in every neighborhood of the coinc. set $[u_0|\Omega' = \Psi'_0]$, although the obstacle is a quadratic polynomial. Now we indicate how to treat a perturbation of f in the variational inequality (1.2). Therefore, we present a transformation of the problem as in [6], [7] which leaves all the regularity assumptions of the data valid. Recalling that at least $E * f \in C^2(\mathbf{R}^2)$ due to our assumptions for f, we can set

$$\begin{aligned} &\hat{u} := u - E * f , \quad \hat{g} := g - E * f , \quad \hat{f} :\equiv 0 , \\ &\hat{\Psi}' := \Psi' - (E * f)|\Omega' , \end{aligned} \tag{1.29}$$

and arrive at an equivalent problem of type (1.2) where

(1.30) *$\hat{u}$ solves* (1.2) *associated to $\hat{g}$, $\hat{\Psi}'$ and $\hat{f}$ with the property* $[\hat{u}|\Omega' = \hat{\Psi}'] = [u|\Omega' = \Psi']$.

This transformation enables us to include a perturbation of f in the assumptions of theorem 1.3, 1.4 which then causes a perturbation of $\hat{\Psi}'$ and $\hat{g}$ in the transformed problem. So one has to add for $(f - \check{f})$ a condition like

$$\sum_{i,j=0}^{\infty} \|\partial_1^i \partial_2^j (f - \check{f})\|_\infty \cdot \delta^{i+j}/(i! \cdot j!) \leqslant \varepsilon \tag{1.31}$$

to the assumptions of theorem 1.3, 1.4 in order to obtain the same conclusions. Further, in view of theorem 1.2 one has to recall that $E * f \in C^\omega(\Omega)$ due to (1.5). We see that by virtue of the transformation the assumption

$f \equiv 0$ is no real restriction for our analysis of problem (1.2). Therefore, for our present work we propose the following

CONVENTION. *Without loss of generality, we assume that $f \equiv 0$ for simplicity in the sequel.*

Finally, we give a

COMMENT *on the obstacle Ψ' given on a more general Ω'*:

Assuming that Ω' is a C^ω-Jordan curve, one might expect that all the results and in particular theorem 1.1, 1.2 remain valid. Recalling lemma 2.1, we have to use $d^2u|\Omega'/ds^2$, s denoting the arc length with respect to Ω', in order to cause a suitable alternating of Δu plus a harmonic remainder, compare the remark on lemma 2.2. Since $d^2u|\Omega'/ds^2 \equiv \partial_{tt}u + \iota\cdot\partial_{\mathcal{N}}u$ where ι denotes the curvature of Ω' and ∂_t, $\partial_{\mathcal{N}}$ the tangential resp. normal derivative with respect to Ω', further on account of representation formulas in [7] $\partial_{tt}u \equiv \frac{1}{2}\cdot\Delta u|\Omega' + T$ with $T \in C^\omega(\Omega')$, we must in case of $\iota \not\equiv 0$ require that $\partial_{\mathcal{N}}u \in C^\omega(\Omega')$ in order to proceed as in lemma 2.2. But regarding the detailed representation of $\partial_{\mathcal{N}}u$ in [7], $\partial_{\mathcal{N}}u \in C^\omega(\Omega')$ can only happen in some particular cases, so our proof breaks down for a general C^ω-Jordan curve Ω'. Under the assumption $\partial_{\mathcal{N}}u \in C^\omega(\Omega')$, we can indeed show theorem 1.1, 1.2 for a C^ω-Jordan curve Ω'.

Finally, we should point out that for symmetry reasons in case of $\Omega' = \Omega \cap \mathbf{R}\times\{0\}$ there is $\partial_{\mathcal{N}}(E * \mu^*)|\Omega' \equiv \partial_2(E * \mu^*)|\mathbf{R}\times\{0\} \equiv 0$, and moreover $\partial_{\mathcal{N}}u \in C^\omega(\Omega')$. So for a straight line Ω', we have C^ω-Dirichlet data on $[u|\Omega' = \Psi']$ for u contrary to almost all more general C^ω-Jordan curves Ω'.

NOTATIONS. By « C », we always mean a generic constant which may vary with the context possibly depending on the bounded domain Ω and $\Omega' := \Omega \cap \mathbf{R}\times\{0\}$, being independent of further quantities if not otherwise indicated.

E denotes the fundamental solution of Δ^2 in $\mathbf{R}^2$, for the comprehensive formula see the appendix.

As usual, $\|w\|_p := \left(\int_\Omega |w|^p dx\right)^{1/p}$, $1 \leqslant p < \infty$, $\|w\|_\infty := \operatorname{ess\,sup}\{|w(x)|: x \in \Omega\}$ denote the norms of the Lebesgue spaces $L^p(\Omega)$ for $1 \leqslant p \leqslant \infty$. Further, $H^{m,p}(\Omega)$, $m \in \mathbf{N}$ denotes the Sobolev space of functions with distributional derivatives in $L^p(\Omega)$ up to order m provided with the norm $\|w\|_{m,p} := \sum_{i=0}^{m} \|\nabla^i w\|_p$. If the norm is taken over an open set A other that Ω, we use the notation $\|\cdot\|_{m,p,A}$ resp. $\|\cdot\|_{p,A}$. Of course, one defines the corresponding spaces over open sets in $\mathbf{R}^1$ in a similar way. The space of functions w

with $w|\Omega_0 \in H^{m,p}(\Omega_0)$ for any open set $\Omega_0 \subset\subset \Omega$ is denoted by $H^{m,p}_{\mathrm{loc}}(\Omega)$. Finally, the subspace $H^{m,p}_0(\Omega) \subset H^{m,p}(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ with respect to the $\|\cdot\|_{m,p}$-norm. Here, $C^\infty_0(\Omega)$ denotes the space of the functions of $C^\infty(\Omega)$ which have compact support in Ω.

One says that a $\varphi \in C^0(\bar{\Omega})$ satisfies a Hölder condition with exponent $\alpha \in (0,1]$ on Ω if $[\varphi]_\alpha := \sup\{|\varphi(x) - \varphi(y)|/|x-y|^\alpha : x, y \in \Omega\} < \infty$. For those functions one defines the space $C^{0,\alpha}(\bar{\Omega})$ provided with the norm $\|\varphi\|_{C^{0,\alpha}(\bar{\Omega})} := \|\varphi\|_\infty + [\varphi]_\alpha$. Further, $C^{m,\alpha}(\bar{\Omega})$ is the space with the m-derivatives in $C^{0,\alpha}(\bar{\Omega})$ provided with the norm $\|\varphi\|_{C^{m,\alpha}(\bar{\Omega})} := \|\varphi\|_{m,\infty} + [\nabla^m \varphi]_\alpha$. Let $C^{0,\alpha}(\Omega)$ resp. $C^{m,\alpha}(\Omega)$ denote the space of functions where the Hölder condition with exponent α is only locally satisfied in Ω. One also defines Hölder spaces over open sets other than Ω in a similar way.

The integration $\int$ always runs over $\mathbb{R}^1$ resp. $\mathbb{R}^2$ if not otherwise indicated. Furthermore, we recall that for a function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ which is lower semicontinuous on $\mathbb{R}^2$ and a positive Borel measure $\nu \geqslant 0$ on $\mathbb{R}^2$ with compact support the convolution $w * \nu$ taken in the distributional sense has a pointwise representative by taking the convolution integral $\int w(\cdot - y)\, d\nu(y)$ which is a well defined lower semicontinuous function on $\mathbb{R}^2$ and in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$.

By $C^\omega(\Omega)$, we denote the space of real analytic functions on Ω. Moreover, C^ω means real analytic also at other occasions. As usual, $B_\sigma(x)$ denotes the circle with radius $\sigma > 0$ and center $x \in \mathbb{R}^2$.

2. – Proof of theorem 1.1.

Our proof of theorem 1.1 rests on a level curve analysis using the alternation of the second derivative of $u|\Omega' - \Psi'$ with respect to the arc length what is the same as the x_1-variable on $\Omega' - [u|\Omega' = \Psi']$. Therefore, we have to construct a suitable harmonic extension which can not have a local accumulation of zero level curves.

As the assertion of theorem 1.1 is trival if $u|\Omega' \equiv \Psi'$, we assume that

$$u|\Omega' \not\equiv \Psi' , \qquad \textit{therefore } [u|\Omega' > \Psi'] \neq \emptyset . \tag{2.1}$$

Because of continuity, the set $[u|\Omega' = \Psi']$ is closed in Ω', so we partition $\Omega' - [u|\Omega' = \Psi']$ which is an open set in $\mathbb{R}^1$ into its connected components such that

$$\Omega' - [u|\Omega' = \Psi'] = \bigcup_{\varkappa=1}^{\varkappa^*} J_\varkappa \times \{0\} ,$$

(2.2) *where* $J_\varkappa \neq \emptyset$ *is an open interval with* $J_\varkappa \cap J_{\varkappa'}, = \emptyset$ *for* $\varkappa \neq \varkappa'$ *and* $\varkappa, \varkappa' = 1, \ldots, \varkappa^* \leqslant \infty$.

The assertion of the theorem is true if

(2.3) *the* $J_\varkappa$ *in* (2.2) *do not accumulate to points* $\xi^j \in \Omega'$ *for* $j = 1, \ldots, j^* \leqslant \infty$.

We prove (2.3) by contradiction and show that the assumption of such an accumulation point leads to

$$u|\Omega' \cap B_\sigma(\xi^j) \equiv \Psi'|\Omega' \cap B_\sigma(\xi^j) \quad \textit{for a } \sigma > 0 \tag{2.4}$$

which is not possible because of

$$J_\varkappa \times \{0\} \cap B_\sigma(\xi^j) \neq \emptyset \quad \textit{for infinitely many } \varkappa \in \mathbf{N}\,. \tag{2.5}$$

Now we assume that $\varkappa^* = \infty$ in (2.2) and that

(2.6) *there exists an accumulation point* $\xi^0 \in \Omega'$ *for the* $J_\varkappa$ *in* (2.2).

Then necessarily $\xi^0 \in [u|\Omega' = \Psi']$, and employing the ordering in $\mathbf{R}^1$ we can choose a subsequence $\{J_{\varkappa'}\}_{\varkappa'=1}^\infty$ such that either

$$J_{\varkappa'} < J_{\varkappa'+1} < \xi^0\,, \qquad \varkappa' \in \mathbf{N}\,, \tag{2.7}$$

or

$$\xi^0 < J_{\varkappa'+1} < J_{\varkappa'}\,, \qquad \varkappa' \in \mathbf{N}\,.$$

We assume (2.7) without loss of generality and write $\varkappa = \varkappa'$ for simplicity in the sequel. Instead of $u|\Omega'$ we are now considering the function

$$z' := u|\Omega' - \Psi' \in C^2(\Omega') \tag{2.8}$$

keeping in mind that the points of

$$[z' = 0] = [u|\Omega' = \Psi'] \tag{2.9}$$

are the minima of $z' \geqslant 0$ such that

$$d/ds\, z' = 0 \quad \textit{on } [z' = 0]\,. \tag{2.10}$$

Here, s denotes the arc length with respect to Ω' and can be identified with the x_1-variable on $\mathbf{R}\times\{0\}$ such that

$$(2.11)\qquad d/ds \text{ equals the partial } \partial_1 \text{ on } \Omega' .$$

Let $\varkappa\in\mathbf{N}$ be arbitrary and $J_\varkappa=(a,b)$ an interval from (2.2). Then there is $a,b\in[z'=0]$ and by (2.10) for $s\in\Omega'$

$$(2.12)\qquad 0\leqslant z'(s)=\int_a^s\int_a^t d^2/ds^2 z'\,dh\,dt=\int_b^s\int_b^t d^2/ds^2 z'\,dh\,dt .$$

So by the definition of $J_\varkappa$ from (2.12) follows that

$$(2.13)\qquad d^2/ds^2 z'|J_\varkappa\not\equiv 0 .$$

Inspecting (2.12) again, we further obtain

LEMMA 2.1 (*Alternation of d^2/ds^2z' on $\Omega'-[z'=0]$*). *For $J_\varkappa$ from* (2.2), $\varkappa\in\mathbf{N}$, *there exist $t_\varkappa^+, t_\varkappa^-\in J_\varkappa$, $t_\varkappa^+<t_\varkappa^-$, such that $d^2/ds^2z'(t_\varkappa^+)>0$, $d^2/ds^2z'(t_\varkappa^-)<0$.*

Indeed, d^2/ds^2z' must change the sign on $J_\varkappa$, otherwise z' would only be increasing resp. decreasing on $J_\varkappa$ which means $z'(b)-z'(a)\neq 0$ for $J_\varkappa=(a,b)$ by (2.13).

Now we choose domains Ω_1, Ω_2 such that

(2.14) $\xi^0\in\Omega_1\subset\subset\Omega_2\subset\subset\Omega$ *and for* $i=1,2$ $\Omega_i':=\Omega_i\cap\Omega'$ *is only one line segment.*

The crucial step now is to extend $d^2/ds^2z'|\Omega_1'$ symmetrically with respect to $\mathbf{R}\times\{0\}$ to a function which is harmonic outside of $[z'|\Omega_1'=0]$ and continuous at $[z'|\Omega_1'=0]$ in order to employ the previous lemma in a level curve analysis.

LEMMA 2.2 (*Continuous symmetric harmonic extension*). *For $\sigma>0$ sufficiently small and $\Omega_*:=\Omega_1'\times(-\sigma,\sigma)$ there exists a $\zeta\in C^0(\Omega_*)$ which is harmonic on $\Omega_*-[z'=0]\times\{0\}$ and symmetric with respect to $\mathbf{R}\times\{0\}$ such that*

$$\zeta|\Omega_1'\equiv d^2/ds^2z'|\Omega_1' .$$

PROOF. First, we show that $d^2/ds^2z'|\Omega_2'$ is the sum of a $C^\omega(\Omega_2')$-function and the restriction of a continuous logarithmic potential of a measure with support in $[z'=0]\times\{0\}$. Thus we regard the restriction of the measure μ to $\bar{\Omega}_2$

$$\mu_* := \mu|\bar{\Omega}_2 . \tag{2.15}$$

From the proof of (1.6) in [6], [7] we know that also $E * \mu_* \in C^2(\mathbf{R}^2)$, and taking (2.11) into account there is

$$d^2/ds^2(E * \mu_*)|\mathbf{R}\times\{0\} \equiv (\partial_{11}E * \mu_*)|\mathbf{R}\times\{0\} . \tag{2.16}$$

So we can employ the expression for $\partial_{11}E$, see appendix, and arrive at

$$\begin{aligned} &d^2/ds^2(E * \mu_*)|\mathbf{R}\times\{0\} \\ &\qquad \equiv (4\cdot\pi)^{-1}\cdot\big(\log|\cdot| * \mu_*\big)|\mathbf{R}\times\{0\} + (8\cdot\pi)^{-1}\cdot\mu_*(\mathbf{R})^2 \end{aligned} \tag{2.17}$$

where $(2\cdot\pi)^{-1}\cdot\log|\cdot| * \mu_* = \Delta E * \mu_* \in C^0(\mathbf{R}^2)$.

The biharmonic potential $E * \mu_*$ nearly represents the behaviour of u on Ω_2, because by virtue of $\Delta^2(u - E * \mu_*) = 0$ on Ω_2 we have $(u - E * \mu_*)|\Omega_2 \in C^\omega(\Omega_2)$, and moreover

$$w_1' := d^2/ds^2(u - E * \mu_*)|\Omega_2' \in C^\omega(\Omega_2') . \tag{2.18}$$

Setting now

$$w' := w_1' + (8\cdot\pi)^{-1}\cdot\mu_*(\mathbf{R}^2) - d^2/ds^2\,\Psi' \in C^\omega(\Omega_2') , \tag{2.19}$$

we obtain the desired splitting

$$d^2/ds^2\, z'|\Omega_2' = w' + (4\cdot\pi)^{-1}\cdot\big(\log|\cdot| * \mu_*\big)|\Omega_2' . \tag{2.20}$$

Next, we can solve the following Cauchy problem

$$\Delta w = 0 \ \textit{on}\ \Omega_1'\times(0,\sigma) , \quad w = w'|\Omega_1' , \quad \partial_2 w = 0 \ \textit{on}\ \Omega_1' , \tag{2.21}$$

by the Cauchy-Kovalevska theorem for a sufficiently small $\sigma > 0$ recalling that $\Omega_1' \subset\subset \Omega_2'$ in $\mathbf{R}^1$. The solution w can be extended symmetrically with respect to $\mathbf{R}\times\{0\}$ such that

$$\Delta w = 0 \ \textit{on}\ \Omega_1'\times(-\sigma,\sigma) =: \Omega_* \ \textit{and}\ w|\Omega_1' = w'|\Omega_1' . \tag{2.22}$$

Note, we do nothing more than performing a holomorphic extension of the power series development of w' and then taking the real part. Therefore, we notice that

(2.23) $\sigma = \sigma(w'|\Omega_1') > 0$ *can be chosen as a lower bound of the radius of convergence for the power series development of* w' *at every point of* Ω_1' .

Recalling (2.20) and that $\log|\cdot| * \mu_* \in C^0(\mathbb{R}^2)$, we get the assertion of the lemma by setting

$$\zeta := w + (4\cdot\pi)^{-1}\cdot(\log|\cdot| * \mu_*)|\Omega_* . \tag{2.24}$$

Reviewing the construction of ζ and recalling (2.13), we obviously have the

REMARK:

$$\zeta \not\equiv 0 .$$

Note that $\frac{1}{2}\cdot\Delta u - \zeta$ *is harmonic on* Ω_*.

Now we state a sequence of lemmata about the topological structure of the set $\Omega_* - [\zeta = 0]$.

LEMMA 2.3. *Let* $\Omega_* - [\zeta = 0] = \bigcup_{\nu=1}^{\nu^*} U_\nu$ *be the partition into the mutual disjoint connected open components with respect to* Ω_*. *Then* $\partial U_\nu \cap \partial\Omega_* \neq \emptyset$ *for* $\nu = 1, \ldots, \nu^* \leqslant \infty$.

PROOF. – Assuming that $\partial U_\nu \cap \partial\Omega_* = \emptyset$ we must have $\partial U_\nu \subset [\zeta = 0]$ and by the maximum principle $\zeta|U_\nu \equiv 0$. But then $\zeta \equiv 0$ by the unique continuation principle which contradicts the remark to lemma 2.2.

In view of lemma 2.1 we now select some of the U_ν. Let

(2.25) $U_\varkappa^-$, $U_\varkappa^+$ *be the components in lemma* 2.3 *with* $t_\varkappa^- \in U_\varkappa^-$, $t_\varkappa^+ \in U_\varkappa^+$ *for* $\varkappa \in \mathbb{N}$.

Obviously from the definition follows that for $\varkappa, \varkappa' \in \mathbb{N}$

$$U_\varkappa^- \cap J_\varkappa \times \{0\} \neq \emptyset , \qquad U_\varkappa^+ \cap J_\varkappa \times \{0\} \neq \emptyset , \qquad U_\varkappa^- \cap U_\varkappa^+ = \emptyset . \tag{2.26}$$

So recalling (2.6),

(2.27) ξ^0 *is an accumulation point for* $\{U_\varkappa^-\}_{\varkappa=1}^\infty$, $\{U_\varkappa^+\}_{\varkappa=1}^\infty$.

Then with the same arguments as in the proof of lemma 2.3 one shows

LEMMA 2.4. *For* $\delta_0 \in (0, \sigma)$ *and* $\varkappa \geqslant \varkappa_0 := \varkappa(\delta_0)$ *we have*

$$U_\varkappa^+ \cap \partial B_\delta(\xi^0) \neq \emptyset , \qquad \delta \in (\delta_0, \sigma) .$$

In the next lemma we show that different $t_\varkappa^+$ correspond to disjoint $U_\varkappa^+$.

LEMMA 2.5.

$$U_\varkappa^+ \cap U_{\varkappa'}^+ = \emptyset \quad \textit{for } \varkappa \neq \varkappa', \varkappa, \varkappa' \in \mathbf{N}.$$

PROOF. Assume that for $\varkappa < \varkappa'$ there is $U_\varkappa^+ \cap U_{\varkappa'}^+ \neq \emptyset$, that means $U_\varkappa^+ = U_{\varkappa'}^+$ by definition in lemma 2.3. Then we have by (2.7) and lemma 2.1 that

$$t_\varkappa^+ < t_\varkappa^- < t_{\varkappa'}^+ . \tag{2.28}$$

and can join the points $(t_\varkappa^+, 0)$ and $(t_{\varkappa'}^+, 0)$ by piecewise linear curves which are contained in $U_\varkappa^+$.

Using the symmetry with respect to $\mathbf{R}\times\{0\}$, we can find such piecewise linear Jordan curves Γ^+ with $\Gamma^+ - \{(t_\varkappa^+, 0), (t_{\varkappa'}^+, 0)\} \subset U_\varkappa^+ \cap \mathbf{R}\times\mathbf{R}_+$ and by reflection Γ^- with $\Gamma^- - \{(t_\varkappa^+, 0), (t_{\varkappa'}^+, 0)\} \subset U_\varkappa^+ \cap \mathbf{R}\times\mathbf{R}_-$. Concatenating Γ^-, Γ^+ to a closed Jordan curve $\Gamma^- \cup \Gamma^+$, we obtain a polygon $P \subset\subset \Omega_*$ with

$$\partial P = \Gamma^- \cup \Gamma^+ \subset U_\varkappa^+ . \tag{2.29}$$

By construction and (2.28), the polygon contains the line segment $(t_\varkappa^+, t_{\varkappa'}^+)\times\{0\}$ without the end points $(t_\varkappa^+, 0)$, $(t_{\varkappa'}^+, 0)$. So (2.28) yields $(t_\varkappa^-, 0) \in P$ which means $U_\varkappa^- \cap P \neq \emptyset$, moreover $U_\varkappa^- \subset P$ by (2.26), (2.29) because $U_\varkappa^-$ is connected. But this leads to

$$U_\varkappa^- \subset P \subset\subset \Omega_* \tag{2.30}$$

which is a contradiction to lemma 2.3, thus lemma 2.5 is proved.

Employing the ordering in $\mathbf{R}^1$ and recalling (2.7), with the same conclusions one gets

LEMMA 2.6:

$$U_\varkappa^+ \cap \mathbf{R}\times\{0\} < U_{\varkappa'}^+ \cap \mathbf{R}\times\{0\} \quad \textit{for } \varkappa' > \varkappa, \varkappa', \varkappa \in \mathbf{N}.$$

Indeed, otherwise we have the situation that for a $\varkappa \in \mathbf{N}$ there exists a $\varkappa'' \in \mathbf{N}$ with $\varkappa'' \neq \varkappa$ such that

$$x_1^1 < y_1 < x_1^2 \quad \textit{for } (x_1^1, 0), (x_1^2, 0) \in U_\varkappa^+ \quad \textit{and } (y_1, 0) \in U_{\varkappa''}^+ . \tag{2.31}$$

Then we can show in the same way that $U_{\varkappa''}^+ \subset\subset \Omega_*$ which is a contradiction to lemma 2.3.

The next lemma improves lemma 2.4.

Lemma 2.7. *For* $\delta_0 \in (0,\sigma)$ *and* $k \geqslant \varkappa_0 := \varkappa(\delta_0)$ *we have* $\partial U_\varkappa^+ \cap \partial B_\delta(\xi^0) \neq \emptyset$, $\delta \in (\delta_0, \sigma)$.

Proof. Fix $\delta_0 \in (0,\sigma)$ and $\delta \in (\delta_0, \sigma)$. By lemma 2.4 for $\varkappa \geqslant \varkappa_0$ there is $U_\varkappa^+ \cap \partial B_\delta(\xi^0) \neq \emptyset$. Assume $\partial U_\varkappa^+ \cap \partial B_\delta(\xi^0) = \emptyset$ for a $\varkappa \geqslant \varkappa_0$. Then $\partial B_\delta(\xi^0) \subset U_\varkappa^+$ because any two points of $\partial B_\delta(\xi^0)$ can be joined by a connected segment of $\partial B_\delta(\xi^0)$ which does not intersect $\partial U_\varkappa^+$. On account of (2.27) consider now a $U_{\varkappa'}$ for $\varkappa'$ sufficiently large with $U_{\varkappa'}^+ \cap B_\delta(\xi^0) \neq \emptyset$ and $\varkappa' \neq \varkappa$ which means $U_{\varkappa'}^+ \cap U_\varkappa^+ = \emptyset$ by lemma 2.5. But then $U_{\varkappa'}^+ \cap \partial B_\delta(\xi^0) = \emptyset$ so that $U_{\varkappa'}^+ \subset\subset \Omega_*$ because $U_{\varkappa'}^+$ is connected which is a contradiction to lemma 2.3, thus lemma 2.7 is proved. A last important property of $\{U_\varkappa^+\}_{\varkappa=1}^\infty$ we show is

Lemma 2.8:

$$\partial U_\varkappa^+ \cap \partial U_{\varkappa'}^+ \cap \Omega_* = \emptyset \qquad \textit{for } \varkappa \neq \varkappa',\ \varkappa, \varkappa' \in \mathbf{N}\,.$$

Proof. We proceed similar as in the proof of lemma 2.5. Recalling lemma 2.6 we only have to consider the points of $\Omega_* - \mathbf{R}\times\{0\}$. Assume that for $\varkappa < \varkappa'$ there exists a

$$x^0 \in \partial U_\varkappa^+ \cap \partial U_{\varkappa'}^+ \cap \Omega_* \cap \mathbf{R}\times\mathbf{R}_+\,. \tag{2.32}$$

Then we can join $(t_\varkappa^+, 0)$ and x^0 by a Jordan curve $\omega^{+,1}$ with $\omega^{+,1} - \{(t_{\varkappa'}^+, 0)\} \subset U_\varkappa^+ \cap \mathbf{R}\times\mathbf{R}_+$. For the sake of completeness we give a construction.

Indeed, for $x^0 \in \partial U_\varkappa^+$ we can choose a sequence $\{x^\nu\}_{\nu=1}^\infty$,

$$x^\nu \in U_\varkappa^+ \cap \partial B_{1/\nu}(x^0)\,, \quad \nu \in \mathbf{N}\,, \tag{2.33}$$

such that for $\nu \in \mathbf{N}$ the points x^ν and $x^{\nu+1}$ are in the closure of one and the same connected component of $U^\nu := U_\varkappa^+ - \overline{B_{1/(\nu+1)}(x^0)}$ and can be joined by a piecewise linear Jordan curve ω_ν with $\omega_\nu - \{x^{\nu+1}\} \subset U^\nu - \{x^1, \dots, x^{\nu-1}\}$ in a way that for every $k \in \mathbf{N}$ there exists a $\nu_k \in \mathbf{N}$ such that $\omega_\nu \cap U^k = \emptyset$, $\nu \geqslant \nu_k$. This is possible since $U_\varkappa^+ - \overline{U}^\nu$ possesses only a finite number of connected components (*). Further, we can arrange that $\big(\omega_\nu - \{x^\nu, x^{\nu+1}\}\big) \cap \big(\omega_{\nu'} - \{x^{\nu'}, x^{\nu'+1}\}\big) = \emptyset$, $\nu \neq \nu'$, ν, $\nu' \in \mathbf{N}$. Concatenating the ω_ν to

$$\tilde\omega_\nu := \omega_1 \cup \dots \cup \omega_\nu\,, \qquad \nu \in \mathbf{N}\,,$$

we obtain a piecewise linear Jordan curve $\tilde\omega_\nu \subset U_\nu \cap \{x^{\nu+1}\}$ joining x^1 and $x^{\nu+1}$ where

$$\textit{for } k \in \mathbf{N}\colon\ \ \tilde\omega_\nu \cap U^k = \tilde\omega_{\nu_k} \cap U^k \qquad \textit{for } \nu \geqslant \nu_k\,,\quad \nu \in \mathbf{N}\,. \tag{2.34}$$

So we can define a curve $\tilde\omega$ joining $(t_\varkappa^+, 0)$ and x^0 in a way that $\tilde\omega(-1)$

(*) See Added in proof.

$= (t_\varkappa^+, 0)$, $\tilde{\omega}(0) = x^1$, and $\tilde{\omega}$ is parametrized on the interval $[1 - 1/\nu, 1 - 1/(\nu + 1)]$ as ω_ν, therefore as $\tilde{\omega}_\nu$ on $[0, 1 - 1/(\nu + 1)]$, so that by continuous extension $\tilde{\omega}(1) := x^0$ can be defined and $\tilde{\omega}$ is parametrized on the full interval $[-1, 1]$. Further, we can arrange $\tilde{\omega}|[-1, 0]$ in such a way that $\tilde{\omega}|[-1, 0]$ is a Jordan curve joining $(t_\varkappa^+, 0)$ and x^1 such that $\tilde{\omega}|[-1, 0] \cap \tilde{\omega}|[0, 1] = \{x^1\}$. As by construction this parametrization is a bijective continuous map of the compact space $[-1, 1]$ onto the Hausdorff space trace $(\tilde{\omega})$, it is moreover a homeomorphism, thus $\tilde{\omega}$ is a Jordan curve. On account of $x^0 \in \mathbf{R}\times\mathbf{R}_+$ we can carry out this construction in a way that besides $\tilde{\omega} - \{(t_\varkappa^+, 0)\} \subset \mathbf{R}\times\mathbf{R}_+$, and hence obtain $\omega^{+,1}$. Further, in the same way we can join x^0 and $(t_{\varkappa'}^+, 0)$ by a Jordan curve $\omega^{+,2}$ with $\omega^{+,2} - \{(t_{\varkappa'}^+, 0)\} \subset U_{\varkappa'}^+ \cap \mathbf{R}\times\mathbf{R}_+$.

Concatenating yields the Jordan curve $\Gamma^+ := \omega^{+,1} \cup \omega^{+,2}$ joining $(t_\varkappa^+, 0)$ and $(t_{\varkappa'}^+, 0)$ where

$$\Gamma^+ - \{(t_\varkappa^+, 0), (t_{\varkappa'}^+, 0)\} \subset \left(U_\varkappa^+ \subset U_{\varkappa'}^+ \cup \{x^0\}\right) \cap \mathbf{R}\times\mathbf{R}_+ . \tag{2.35}$$

Let $\breve{x}^0 = (\breve{x}_1^0, \breve{x}_2^0) = (x_1^0, -x_2^0) \in \partial U_\varkappa^+ \cap \partial U_{\varkappa'}^+ \cap \Omega_* \cap \mathbf{R}\times\mathbf{R}_-$ be the reflected point of $x^0 = (x_1^0, x_2^0)$. By the symmetry with respect to $\mathbf{R}\times\{0\}$ we also have a Jordan curve Γ^- again joining $(t_\varkappa^+, 0)$ and $(t_{\varkappa'}^+, 0)$ where

$$\Gamma^- - \{(t_\varkappa^+, 0), (t_{\varkappa'}^+, 0)\} \subset \left(U_\varkappa^+ \cup U_{\varkappa'}^+ \cup \{\breve{x}^0\}\right) \cap \mathbf{R}\times\mathbf{R}_- . \tag{2.36}$$

Concatenating Γ^- and Γ^+ results in a closed Jordan curve

$$\Gamma := \Gamma^- \cup \Gamma^+ \subset U_\varkappa^+ \cup U_{\varkappa'}^+ \cup \{x^0\} \cup \{\breve{x}^0\} . \tag{2.37}$$

By Jordan's curve theorem, Γ divides the plane $\mathbf{R}^2$ into two disjoint connected open components Ω_Γ and $\mathbf{R}^2 - \bar{\Omega}_\Gamma$ where Ω_Γ is bounded with $\partial\Omega_\Gamma = \Gamma \subset \Omega_*$. So there is $\Omega_\Gamma \cap \Omega_* \neq \emptyset$, and because Ω_Γ is connected, we even have

$$\Omega_\Gamma \subset\subset \Omega_* . \tag{2.38}$$

Now we recall that $(t_\varkappa^+, 0)$, $(t_{\varkappa'}^+, 0) \in \partial\Omega_\Gamma$, and moreover for the joining line segment $(t_\varkappa^+, t_{\varkappa'}^+) \times \{0\} \subset \Omega_\Gamma$ by construction. So (2.28) yields $(t_\varkappa^-, 0) \in \Omega_\Gamma$ what means $U_\varkappa^- \cap \Omega_\Gamma \neq \emptyset$, and recalling that on account of (2.26) the inclusion (2.37) implies $\partial\Omega_\Gamma \cap U_\varkappa^- = \emptyset$ we even have

$$U_\varkappa^- \subset \Omega_\Gamma \subset\subset \Omega_* \tag{2.39}$$

because $U_\varkappa^-$ is connected. But (2.39) is a contradiction to lemma 2.3 so that the assumption (2.32) is not possible. Taking the symmetry with respect to $\mathbb{R}\times\{0\}$ into account, the assertion of lemma 2.8 is proved.

Now we have shown all the properties of $\{U_\varkappa^+\}_{\varkappa=1}^\infty$ which we need and continue the studying of the level set $[\zeta = 0]$. Recalling that $\partial U_\varkappa^+ \subset [\zeta = 0]$, $\varkappa \in \mathbb{N}$, and taking into account the symmetry with respect to $\mathbb{R}\times\{0\}$, we join lemma 2.7, 2.8 and obtain the following crucial result where we for simplicity write $\partial B_\delta^+(\zeta^0) = \partial B_\delta(\xi^0) \cap \mathbb{R}\times\mathbb{R}_+$. There is

LEMMA 2.9:

$$\#\big(\partial B_\delta^+(\zeta^0) \cap [\zeta = 0]\big) = \infty \quad \textit{for } \delta \in (0, \sigma)\,.$$

By virtue of lemma 2.9 we are in the position to show that the restrictions $\zeta|\partial B_\delta(\xi^0)$ vanish for sufficiently many δ.

LEMMA 2.10. *There is $\zeta|\partial B_\delta(\xi^0) \equiv 0$ for $\delta \in \Lambda \subset (0, \sigma)$ where Λ is dense in $(0, \sigma)$.*

PROOF. After a translation of the coordinate system we can assume that $\xi^0 = (0, 0)$. Considering the restriction of ζ on $\Omega_* \cap \mathbb{R}\times\{0\} = \Omega_1' \times\{0\}$ and writing $\zeta' := \zeta|\Omega_1'\times\{0\}$, we define Λ where $\operatorname{int}_{\mathbb{R}^1}$ denotes the interior and $\partial_{\mathbb{R}^1}$ the boundary with respect to the $\mathbb{R}^1$-topology as

$$\begin{aligned}(2.40)\qquad \Lambda := &\big\{\delta \in (0, \sigma)\colon -\delta, \delta \in [\zeta' \neq 0] \cup \operatorname{int}_{\mathbb{R}^1}\big([\zeta' = 0]\big)\big\}\\ = &\,(0, \sigma) - \big(\partial_{\mathbb{R}^1}[\zeta' \neq 0] \cup \partial_{\mathbb{R}^1}[\zeta'(-\,\cdot) \neq 0]\big)\,.\end{aligned}$$

Since $\operatorname{int}_{\mathbb{R}^1}(\partial Q) = \emptyset$ for an open set $Q \subset \mathbb{R}^1$, Λ is dense in $(0, \sigma)$. Now we fix an arbitrary $\delta \in \Lambda$ and consider the restriction $\zeta|\partial B_\delta^+(\xi^0)$ which is a real analytic function in one variable with respect to the arc length of $\partial B_\delta(\xi^0)$. By lemma 2.9 we have for the zeros Z of $\zeta|\partial B_\delta(\zeta^0)$

$$(2.41)\qquad \# Z := \#\big[\,\zeta|\partial B_\delta(\xi^0) = 0\,\big] = \infty\,,$$

and because $\partial B_\delta(\xi^0)$ is compact, there must exist at least one accumulation point x_0 of Z. In case that

$$(2.42)\qquad Z \textit{ has an accumulation point } x^0 \in \partial B_\delta^+(\xi^0)\,,$$

the identity principle for analytic functions implies that $\zeta|\partial B_\delta^+(\xi^0) \equiv 0$, moreover $\zeta|\partial B_\delta(\zeta^0) \equiv 0$ by the symmetry with respect to $\mathbb{R}\times\{0\}$ and the continuity of ζ. In case that (2.42) is not true, then

$$(2.43)\qquad Z \textit{ has an accumulation point } x^0 \in \partial B_\delta(\xi^0) \cap \mathbf{R}\times\{0\}\ .$$

By the continuity of ζ and the definition of Λ in (2.40) we then have

$$(2.44)\qquad x^0 \in \mathrm{int}_{\mathbf{R}^1}\big([\zeta'=0]\big)\times\{0\}\ .$$

For a $\varepsilon>0$ sufficiently small there is $\zeta|B_\varepsilon(x^0)\cap\mathbf{R}\times\{0\}\equiv 0$ and we extend the harmonic function $\zeta|B_\varepsilon(x^0)\cap\mathbf{R}\times\mathbf{R}_+$ by the Schwarz reflection principle to a function ζ_{ext} which is harmonic on $B_\varepsilon(x^0)$ where $\zeta_{\mathrm{ext}}:=-\zeta(-\cdot)$ on $B_\varepsilon(x^0)\cap\mathbf{R}\times\mathbf{R}_-$. Hence, we also get an real analytic continuation $(\zeta|\partial B_\delta^+)_{\mathrm{con}}$ of $\zeta|\partial B_\delta^+(\zeta^0)$ at least on the larger arc segment of $\partial B_\delta(\zeta^0)$

$$(2.45)\qquad \partial B^+_{\delta,\varepsilon}(\xi^0):=\partial B^+_\delta(\xi^0)\cup\big(\partial B_\delta(\xi^0)\cap B_\varepsilon(x^0)\big)\ .$$

Recalling the symmetry with respect to $\mathbf{R}\times\{0\}$ we set

$$(2.46)\qquad \begin{aligned}(\zeta|\partial B^+_\delta)_{\mathrm{con}} &:= \zeta|\partial B^+_\delta(\xi^0) \quad \textit{on } \partial B^+_\delta(\xi^0)\\ &:= 0 \quad \textit{at } x^0\\ &:= -\zeta|\partial B^-_\delta(\xi^0) \quad \textit{on } \partial B^-_\delta(\xi^0)\end{aligned}$$

where $\partial B^-_\delta(\xi^0):=\partial B_\delta(\xi^0)\cap\mathbf{R}\times\mathbf{R}_-$. Then $(\zeta|\partial B^+_\delta)_{\mathrm{con}}$ is a real analytic function even on $\partial B^+_{\delta,\varepsilon}(\xi^0)\cup\partial B^-_\delta(\xi^0)$. Since x^0 is an interior point of $\partial B^+_{\delta,\varepsilon}(\xi^0)$ with respect to the $\mathbf{R}^1$-topology, again the identity principle for analytic functions implies that $(\zeta|\partial B^+_\delta)_{\mathrm{con}}\equiv 0$ what yields $\zeta|\partial_\delta(\xi^0)\equiv 0$ as before. Hence, with the Λ defined in (2.40) we obtain the assertion of lemma 2.10.

By lemma 2.10 we immediately arrive at the conclusion of the proof of theorem 1.1. Indeed, by the continuity of ζ lemma 2.10 leads to $\zeta|B_\sigma(\xi^0)\equiv 0$, and moreover to

$$(2.47)\qquad \zeta\equiv 0$$

by the unique continuation principle for harmonic functions. But (2.47) is a contradiction to the remark on lemma 2.2 as well as to (2.5), thus the assumption (2.6) is not possible. Therefore, the assertion (2.3) is valid which is equivalent to the assertion of theorem 1.1, and the proof is complete now.

Note, the method of proof only is based on studying the one-sided approach of the connected components of the coincidence set to an assumed limit point. Hence the idea arises to prove the same under weaker assumptions on Ψ'.

It is important in view of applications that we are able to obtain the assertion of theorem 1.1 also under piecewise analyticity assumptions on Ψ'. Therefore, let again

$$f \in C^\omega(\Omega) \tag{2.48}$$

and assume that

$$\Psi' \in C^2(\Omega') \textit{ is piecewise analytic on } \Omega' . \tag{2.49}$$

The expression « piecewise analytic on Ω' » means that

(2.50) *for arbitrary* $x^0 = (x_1^0, 0) \in \Omega'$ *there exist* $\Psi_1', \Psi_2' \in C^\omega(x_1^0 - \alpha, x_1^0 + \alpha)$ *with* $\alpha = \alpha(x^0) > 0$ *such that* $\Psi' \equiv \Psi_1'$ *on* $(x_1^0 - \alpha, x_1^0]$, $\Psi' \equiv \Psi_2'$ *on* $[x_1^0, x_1^0 + \alpha)$.

Then we show

THEOREM 1.1′. *Assume* (2.48), (2.49) *and let* u *solve* (1.2). *Then the assertion of theorem* 1.1 *is valid.*

REMARK. The hypothesis (2.49) in particular includes the cubic spline functions given on Ω'.

PROOF OF THEOREM 1.1′. We repeat the proof of theorem 1.1 and indicate the necessary changes. Furthermore, we take this opportunity in order to restate the basic steps on which all conclusions base in a more general way. Therefore the method of proof is applicable also on other occasions. First, again we exclude a trivial case by assuming (2.1). Then we perform the partition in (2.2), so that proving theorem 1.1′ means proving (2.3) by contradiction.

Thus we again assume the existence of an accumulation point $\xi^0 = (\xi_1^0, 0) \in \Omega'$ of the in-between intervals of the coincidence set in (2.6), and further without loss of generality the left-handed approach (2.7). In fact, for our conclusions it is sufficient to study the accumulating in-between intervals only on one side of ξ^0 totally neglecting the intervals on the other side. Since Ψ' is piecewise analytic, we have to choose a suitable small neighborhood of ξ^0 in order to proceed. Writing

$$I^\alpha := (\xi_1^0 - \alpha, \xi_1^0 + \alpha) , \quad I_-^\alpha := (\xi_1^0 - \alpha, \xi_1^0) , \quad I_+^\alpha := (\xi_1^0, \xi_1^0 + \alpha) , \tag{2.51}$$

by (2.49) resp. (2.50) we have for a $\alpha = \alpha(\xi_0) > 0$ that

$$\Psi' \equiv \Psi_1' \text{ on } I_-^\alpha, \quad \Psi' \equiv \Psi_2' \text{ on } I_+^\alpha \quad \text{for } \Psi_1', \quad \Psi_2' \in C^\omega(I^\alpha)\,. \tag{2.52}$$

In case of

$$\Psi_1' \equiv \Psi_2'\,, \tag{2.53}$$

the proof of theorem 1.1 applies without any changes apart from choosing Ω_2 such that $\Omega_2' \subset I^\alpha \times \{0\}$, and we obtain that assumption (2.6) is not possible. Cconsidering the opposite case when

$$\Psi_1' \not\equiv \Psi_2'\,, \tag{2.54}$$

according to (2.7) Ψ_1' replaces Ψ' in our further considerations. Shrinking I^α with a new $\alpha > 0$, we can achieve that $d^2/ds^2(\Psi_1' - \Psi')$ keeps a sign on I_+^α. Indeed, since the zeros of not identically vanishing $C^\omega(I^\alpha)$-functions do not posses accumulation points in I^α, recalling (2.52), (2.54) we have on account of $d^2/ds^2(\Psi_1' - \Psi_2') \not\equiv 0$, $\in C^\omega(I^\alpha)$ for a $\alpha > 0$ sufficiently small that either

$$d^2/ds^2(\Psi_1' - \Psi') = d^2/ds^2(\Psi_1' - \Psi_2') > 0 \quad \text{on } I_+^\alpha\,, \tag{2.55}$$

or

$$d^2/ds^2(\Psi_1' - \Psi') = d^2/ds^2(\Psi_1' - \Psi_2') < 0 \quad \text{on } I_+^\alpha\,, \tag{2.56}$$

and assume (2.56) without loss of generality. Hence, we now regard instead of (2.8)

$$z' := u|I^\alpha \times \{0\} - \Psi_1' \in C^2(I^\alpha)\,, \tag{2.57}$$

and notice that from now on

(2.58) $[z' = 0] \cap \bar{I}_-^\alpha$ *plays the role of* (2.9) *in all the coming assertions.*

But we also have to study z' on I_+^α. Therefore, define

$$[u|\Omega' = \Psi']^* := [u|\Omega' = \Psi'] - \{\text{isolated points of } [u|\Omega' = \Psi']\}\,. \tag{2.59}$$

Then by the continuity of the second derivative and the assumption (2.56) there is

$$
(2.60)\qquad \begin{aligned}[u|\Omega' = \Psi']^* \cap I_+^\alpha \subset [d^2/ds^2(u|\Omega' - \Psi') = 0] \cap I_+^\alpha \\ \subset [z' > 0] \cap I_+^\alpha .\end{aligned}
$$

From the first inclusion in (3.3) which bases on (3.2) and from (2.60) we infer the following one

$$
(2.61)\qquad \operatorname{supp}(\mu) \cap I_+^\alpha \subset [u|\Omega' = \Psi']^* \cap I_+^\alpha \subset [z' > 0] \cap I_+^\alpha .
$$

Taking (2.58) into consideration, one can show an appropriate version of lemma 2.1 with the same arguments.

LEMMA 2.1′. (*Alternation of* $d^2/ds^2 z'$ *on* $I_-^\alpha - [z' = 0]$). *For* $J_\varkappa \subset I_-^\alpha$ *from* (2.2), $\varkappa \in \mathbf{N}$, *there exist* $t_\varkappa^+, t_\varkappa^- \in J_\varkappa$, $t_\varkappa^+ < t_\varkappa^-$, *such that* $d^2/ds^2 z'(t_\varkappa^+) > 0$, $d^2/ds^2 z'(t_\varkappa^-) < 0$.

In order to proceed in the proof, in (2.14) we choose Ω_2 in such a way that

$$
(2.62)\qquad \Omega_2' := \Omega_2 \cap \Omega' \subset I^\alpha \times \{0\} .
$$

Taking now (2.56), (2.61), (2.62) into account, the construction leading to lemma 2.2. now gives the weaker version

LEMMA 2.2′. *For* $\sigma > 0$ *sufficiently small and* $\Omega_* := \Omega_1' \times (-\sigma, \sigma)$ *there exists a* $\zeta \in C^0(\Omega_*)$ *which is harmonic on* $\Omega_* - ([\zeta = 0] \cap \bar{I}_-^\alpha \times \{0\} \cup [\zeta > 0] \cap I_+^\alpha \times \{0\})$ *and symmetric with respect to* $\mathbf{R} \times \{0\}$ *such that* $\zeta|\Omega_1' \equiv d^2/ds^2 z'|\Omega_1'$.

REMARK:

$$
\zeta \not\equiv 0 .
$$

Since lemma 2.2′ has a weaker assertion than lemma 2.2, in fact it is only a one-sided version, we have to be more carefully. In view of possible extensions of the method of proof, we study the present situation carefully. Therefore, we state the conditions on which the proof of a suitable version of lemma 2.3 bases in a more general way.

First, in order to proceed we need the following condition on the behaviour of ζ on $\Omega_* - I_+^\alpha \times \{0\}$ controlling the left-handed approach to ξ^0.

CONDITION i). *ζ is subharmonic on* $[\zeta > 0] - I_+^\alpha \times \{0\}$ *and superharmonic on* $[\zeta < 0] - I_+^\alpha \times \{0\}$.

In our present case condition i) is satisfied by lemma 2.2′ independently of the assumption (2.56). In addition to the former condition we need one further condition on ζ which also controls the right-handed behaviour of ζ with respect to ξ^0 on $I_+^\alpha \times \{0\}$. We have the choice of one of the two following conditions.

CONDITION ii). *ζ is subharmonic on $[\zeta > 0]$.*

CONDITION iii). *ζ is superharmonic on $[\zeta < 0]$.*

In our present case condition iii) is satisfied by lemma 2.2′ under the assumption (2.56). In the case when assumption (2.55) is valid, we have instead of (2.61)

$$\operatorname{supp}(\mu) \cap I_+^\alpha \subset [z' < 0] \cap I_+^\alpha\,, \tag{2.63}$$

and therefore in lemma 2.2′ that

$$\zeta \text{ is harmonic on } \Omega_* - \big([\zeta = 0] \cap \bar{I}_-^\alpha \times \{0\} \cup [\zeta < 0] \cap I_+^\alpha \times \{0\}\big) \tag{2.64}$$

which implies that condition ii) is satisfied.

It is noteworthy to remark that in our present situation we can use a property which we have not mentioned before and which is closely connected with the case of interior unilateral constraints. Looking at the definition of ζ in (2.24), we see that ζ is the sum of a function which is harmonic on Ω_* and a negative logarithmic potential of a positive measure. Thus,

$$\zeta \text{ is subharmonic on full } \Omega_*\,, \tag{2.65}$$

which means that condition (ii) is satisfied independently of (2.61) or (2.63). But basing the proof on (2.55), (2.56) allows much more generality.

Instead of the maximum principle for harmonic functions leading lemma 2.2 to lemma 2.3, we now use the maximum principle for subharmonic functions asserting that a subharmonic function is dominated by its boundary values, and further the minimum principle for superharmonic functions stating the dual assertion. Hence, we infer from condition i) and one of the conditions ii), iii) the following more general version of lemma 2.3.

LEMMA 2.3′. *Let $\Omega_* - [\zeta = 0] = \bigcup_{\nu=1}^{\nu^*} U_\nu$ be the partition into the mutual disjoint open connected components with respect to Ω_*. Then we have $\partial U_\nu \cap \partial \Omega_* \neq \emptyset$ for all U_ν satisfying $\partial U_\nu \cap I_+^\alpha \times \{0\} = \emptyset$, and moreover for* *all* $U_\nu \subset [\zeta > 0]$ *if condition* ii) *is valid, for all* $U_\nu \subset [\zeta < 0]$ *if condition* iii) *is valid.*

Comparing this result with lemma 2.3, lemma 2.3′ gives actually less information about the U_ν in the case under consider when only one of the two conditions ii), iii) is valid. Fortunately, the proof of theorem 1.1 does not use all the U_ν from lemma 2.3′. Again we select the $U_\varkappa^-$, $U_\varkappa^+$ as in (2.25), and for those (2.26), (2.27) is valid. In order to follow the lines of the proof of theorem 1.1, it is necessary and sufficient to show the assertion in lemma 2.3′ only for all the $U_\varkappa^-$, $U_\varkappa^+$. Therefore, we prove the following replacement for lemma 2.3.

LEMMA 2.3″. *Let condition* i) *and one of the conditions* ii) *or* iii) *be valid. Then* $\partial U_\varkappa^- \cap \partial\Omega_* \neq \emptyset$, $\partial U_\varkappa^+ \cap \partial\Omega_* \neq \emptyset$ *for* $\varkappa \in \mathbf{N}$.

PROOF. First, we

(2.66) *assume that conditions* i) *and* ii) *are valid*.

Then the assertion for the $U_\varkappa^+$ is already contained in the assertion of lemma 2.3′. Again by lemma 2.3′, concerning the $U_\varkappa^-$ we only have to show that

$$\partial U_\varkappa^- \cap I_+^\alpha \times \{0\} = \emptyset\,, \qquad \varkappa \in \mathbf{N}\,. \tag{2.67}$$

In view of (2.7) and (2.26) recalling that $J_\varkappa \subset I_-^\alpha$, $\varkappa \in \mathbf{N}$, employing the ordering in $\mathbf{R}^1$ the assertion (2.67) immediately follows from

$$U_\varkappa^- \cap \mathbf{R}\times\{0\} < U_{\varkappa+1}^+ \cap \mathbf{R}\times\{0\}\,, \qquad \varkappa \in \mathbf{N}\,, \tag{2.68}$$

which is similar to lemma 2.6. Thus, the proof is like the one of lemma 2.6. Assuming that (2.68) is not valid for a $\varkappa' \in \mathbf{N}$, one shows in a way which is familiar from the proof of lemma 2.5 using the symmetry with respect to $\mathbf{R}\times\{0\}$, that then necessarily $U_{\varkappa'+1}^+ \subset\subset \Omega_*$ which is a contradiction to lemma 2.3′ under (2.66). Therewith (2.68) is shown and also the assertion of lemma 2.23″ in case of (2.66). In the other case we

(2.69) *assume that conditions* i) *and* iii) *are valid*.

Reversing the roles of $U_\varkappa^-$, $U_\varkappa^+$, we can conclude in the same way. Indeed, by lemma 2.3′ we only have to show that

$$\partial U_\varkappa^+ \cap I_+^\alpha \times \{0\} = \emptyset\,, \qquad \varkappa \in \mathbf{N}\,. \tag{2.70}$$

But (2.70) follows from an inequality of type (2.68) which can be verified using that now $\partial U_\varkappa^- \cap \partial \Omega_* \neq \emptyset$, $\varkappa \in \mathbf{N}$, by lemma 2.3′ under (2.69). Thus the assertion of lemma 2.3″ is also shown in case of (2.69) and therefore the proof of lemma 2.3″ complete.

From now on we can continue the proof of theorem 1.1 up to the end relying on lemma 2.3″ instead of lemma 2.3. Hence, we obtain that the existence of an accumulation point ξ^0 in (2.6) is not possible and confirm assertion (2.3). Thus the proof of theorem 1.1′ is complete.

Reviewing the proof of theorem 1.1′, we notice that the continuity of the second order derivatives of u is crucially needed, but can be dropped at some isolated points provided that the second order derivatives remains bounded in a neighborhood. Therefore, we reduce the assumption (2.49) to

$$\Psi' \in C^1(\Omega') \textit{ is piecewise analytic on } \Omega' . \tag{2.71}$$

Then we show

THEOREM 1.1″. *Assume* (2.71), (2.48) *and let u solve* (1.2). *Then the assertion of theorem* 1.1 *is valid.*

PROOF OF THEOREM 1.1″. We repeat the proof of theorem 1.1′ step by step where we have to take into account that we can not expect $u \in C^2(\Omega)$ any any longer. From (2.71) in particular follows that

$$\Psi' \in C^2\big(\Omega' - \bigcup_{i=1}^{i_1} \{x^i\}\big) \cap H^{2,\infty}_{\mathrm{loc}}(\Omega') \tag{2.72}$$

where $\{x^i\}_{i=1}^{i_1}$, $i_1 \in \mathbf{N} \cup \{\infty\}$, has no accumulation point in Ω. Since the regularity proofs in [6], [7] work locally, by virtue of (2.72) we obtain that

$$u \in C^2\Big(\Omega - \bigcup_{i=1}^{i_1} \{x^i\}\Big) \cap H^{2,\infty}_{\mathrm{loc}}(\Omega) . \tag{2.73}$$

Now we begin with the conclusions of the proof of theorem 1.1′ and obtain a suitable small interval I^α in (2.52). In the case (2.53), nothing new happens and we carry on with the lines of the proof of theorem 1.1′ up to the end. In the case (2.54) we have to take into account the possibility that

$$I^\alpha \times \{0\} \cap \bigcup_{i=1}^{i_1} \{x^i\} = \{\xi^0\} . \tag{2.74}$$

Then lemma 2.1′ is again valid without a change. Recalling (2.73), the construction in lemma 2.2 yields

$$
\begin{aligned}
&E * \mu_* \in C^2(\mathbb{R}^2 - \{\xi^0\}) \cap H^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^2)\,, \\
&\log|\cdot| * \mu_* \in C^0(\mathbb{R}^2 - \{\xi^0\}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^2)\,,
\end{aligned}
\tag{2.75}
$$

thus we have to replace $\zeta \in C^0(\Omega_*)$ by the weaker property $\zeta \in C^0(\Omega_* - \{\xi^0\}) \cap L^\infty_{\mathrm{loc}}(\Omega_*)$ in the assertion of lemma 2.2′ in our present situation. Consequently, in lemma 2.3′ we consider the partition

$$
\Omega_* - ([\zeta = 0] \cup \{\xi^0\}) = \bigcup_{\nu=1}^{\nu^*} U_\nu
\tag{2.76}
$$

into the mutual disjoint connected open components with respect to $\Omega_* - \{\xi^0\}$. One proves the same assertion of lemma 2.3′ by virtue of the generalized maximum principle for bounded subharmonic functions and the related minimum principle for superharmonic functions. This generalized maximum principle for bounded subharmonic functions states that such a function is majorized by the upper quasi-everywhere bounds for its boundary values which is at least valid in bounded domains, see theorem 3.5 in [4].

Then again one selects $U_\varkappa^-$, $U_\varkappa^+$ in (2.76) associated to the $t_\varkappa^-$, $t_\varkappa^+$ from lemma 2.1′ for $\varkappa \in \mathbf{N}$ as in (2.25) and proves the crucial lemma 2.3″. Relying on lemma 2.3″ we can continue the proof of theorem 1.1 resp. 1.1′ up to the end. Thus we confirm the assertion (2.3) and the proof of theorem 1.1″ is complete.

3. – Proof of theorem 1.2.

In order to prove the regularity of the solution u of (1.2) under the assumption (1.5), we have to recall that due to the convention in § 1 there is

$$
\Delta^2 u = 0 \ \ on\ \ \Omega - \operatorname{supp}(\mu)\,, \qquad and\ \ u \in C^\omega(\Omega - \operatorname{supp}(\mu))\,,
\tag{3.1}
$$

and so it remains to study the behaviour of u at supp (μ). For single points there is

$$
\mu(\{x\}) = 0\,, \qquad x \in \Omega\,,
\tag{3.2}
$$

which follows on account of the regularity (1.6) from local representations of Δu in [6], [7] as a logarithmic potential of a local restriction of μ plus a smooth remainder. In case of $\operatorname{supp}(\mu) \subset\subset \Omega$ we obtain (3.2) from the fact that then $\Delta E * \mu \in L^{\infty}_{\mathrm{loc}}(\mathbf{R}^2)$. So by (1.4), (3.2) we arrive at

$$\begin{aligned} \operatorname{supp}(\mu) \subset [u|\Omega' = \Psi'] - \{\textit{isolated points of } [u|\Omega' = \Psi']\} \\ = \bigcup_{i=1}^{i^*} [a^i, b^i] \times \{0\} \qquad \textit{with } a^i < b^i \end{aligned} \tag{3.3}$$

for $i = 1, \ldots, i^* \leqslant \infty$. Thank to theorem 1.1 the connected components of $[u|\Omega' = \Psi']$ are locally finite. Therefore, we only have to study the solution u on a $\mathbf{R}^2$-neighborhood U of an arbitrary $[a^i, b^i] \times \{0\}$, $i = 1, \ldots, i^*$. As the assertion of the theorem becomes trivial by (3.1) in the case of $\operatorname{supp}(\mu) = \emptyset$, we assume that $\operatorname{supp}(\mu) \neq \emptyset$ and chose an interval

$$I := [a, b] \times \{0\} \textit{ from } (3.3) \textit{ arbitrary but fixed} \tag{3.4}$$

and show that

$$u|U(I) \in C^{2,\frac{1}{2}}(U(I)) \tag{3.5}$$

where $U(I)$ is a suitable $\mathbf{R}^2$-neighborhood of I.

In order to study the behaviour of $\Delta u|\Omega'$ on a $\mathbf{R}^1$-neighborhood of I, we can use the function ζ from lemma 2.2 Let $\Omega_* = \Omega_1' \times (-\sigma, \sigma)$ in lemma 2.2 such that

$$I \subset\subset \Omega_*, \qquad \Omega_* \cap ([u|\Omega' = \Psi'] - I) = \emptyset. \tag{3.6}$$

Of course we have

$$\zeta|[a, b] \times \{0\} \equiv 0, \tag{3.7}$$

so that using the symmetry of ζ with respect to $\mathbf{R} \times \{0\}$, by the Schwarz reflection principle follows

LEMMA 3.1:

$$\begin{aligned} \zeta_{\mathrm{ext}} \textit{ defined by } \zeta_{\mathrm{ext}} &:= \zeta \quad \textit{on } [a, b] \times [0, \sigma] \\ &:= -\zeta \quad \textit{on } [a, b] \times [-\sigma, 0] \end{aligned}$$

is harmonic on $(a, b) \times (-\sigma, \sigma)$.

COROLLARY. *Let* $V_\varrho := B_\varrho((a, 0)) \cup B_\varrho((b, 0))$ *for* $\varrho > 0$. *Then* $\zeta|\Omega_* - \overline{V}_\varrho \in C^{0,1}(\Omega_* - \overline{V}_\varrho)$.

In order to prove any degree of regularity for $\zeta|\Omega_1'\times\{0\}$, we only have to study the behaviour of ζ at $(a, 0)$, $(b, 0)$. Assuming that

$$(a, 0) = (0, 0) \in R^2 , \tag{3.8}$$

we study by the use of suitable comparison functions how $\zeta|\Omega_1'\times\{0\}$ attains its boundary value at 0. Although this is well-known in the theory of the Dirichlet problem for the Laplace equation, we present the details in short to refer to later in our work. Furst, we introduce polar coordinates $(r, \theta) \in \mathbf{R}_+\times[0, 2\cdot\pi)$ where $[\theta = 0] = \mathbf{R}_+\times\{0\}$, and for the point $x \in R^2$ we have in polar coordinates $r = |x|$ which is the radius, $\theta = \sphericalangle(\mathbf{R}_+\times\{0\}, 0^{\to}x)$ which is the angle of the vector $0^{\to}x$ with the positive x_1-axis. Then we introduce the comparison function

$$G(r, \theta) := r^{\frac{1}{2}}\cdot\cos\left(\tfrac{1}{2}\cdot(\theta - \pi)\right) \geqslant 0 \tag{3.9}$$

which is harmonic on $\mathbf{R}^2 - \mathbf{R}_+\times\{0\}$ and $G(r, 0) = 0$ for $r \in \mathbf{R}_+$. Now we choose $\lambda > 0$ such that

$$B_\lambda := B_\lambda((0, 0)) \subset\subset \Omega_* \quad \textit{and } \lambda \leqslant (l - a)/3 . \tag{3.10}$$

Further, we set

$$M := 2^{\frac{1}{2}}\cdot\lambda^{-\frac{1}{2}}\cdot\left\{\|\zeta\|_{\infty,\partial B_\lambda} + 2\cdot\lambda\cdot\|d\zeta|\partial B_\lambda/ds\|_{\infty,\partial B_\lambda}\right\} \tag{3.11}$$

where s denotes the arc length with respect to ∂B_λ. Then

LEMMA 3.2:

$$-M\cdot G \leqslant \zeta \leqslant M\cdot G \quad \textit{on } B_\lambda .$$

PROOF. As $\zeta, G \in C^0(\bar{B}_\lambda)$ are harmonic on $B_\lambda - [0, \lambda)\times\{0\}$ and $\zeta = G = 0$ on $[0, \lambda]\times\{0\}$, the assertion follows by the maximum principle from

$$-M\cdot G \leqslant \zeta \leqslant M\cdot G \quad \textit{on } \partial B_\lambda . \tag{3.12}$$

In order to prove (3.12) we take into account that

$$\begin{aligned} G(\lambda, \theta) &\geqslant 2^{-\frac{1}{2}}\cdot\lambda^{\frac{1}{2}} , && \theta \in [\pi/2, 3/2\cdot\pi] \\ &\geqslant \theta/2\cdot 2^{-\frac{1}{2}}\cdot\lambda^{\frac{1}{2}} , && \theta \in [0, \pi/2] \cup [3/2\cdot\pi, 2\cdot\pi] \end{aligned} \tag{3.13}$$

where the latter inequality follows from $G(\lambda, 0) = 0$ estimating $\partial G/\partial\theta|\partial B_\lambda$

from below. Recalling for $\zeta|\partial B_\lambda \in C^{0,1}(\partial B_\lambda) = H^{1,\infty}(\partial B_\lambda)$ that there is $\partial\zeta/\partial\theta|\partial B_\lambda/ds = \lambda\cdot d\zeta/\partial B/ds$ and using that $\zeta(\lambda, 0) = 0$, we obtain that

$$(3.14) \qquad |\zeta(\lambda,\theta)| \leqslant \theta\cdot\lambda\cdot d\zeta|\partial B_\lambda/ds\,, \qquad \theta\in[0, 2\cdot\pi)\,,$$

thus by the choice of M the assertion of the lemma follows.

The previous lemma enables us to prove the following estimate for the finite difference-quotient $\delta_h^{\frac{1}{2}}$ of ζ in the x_1-direction for $h > 0$ where for a function φ

$$(3.15) \qquad \delta_h^{\frac{1}{2}}(\varphi)(x_1, x_2) := \big(\varphi(x_1 + h, x_2) - \varphi(x_1, x_2)\big)/h^{\frac{1}{2}}\,.$$

LEMMA 3.3:

$$\|\delta_h^{\frac{1}{2}}(\zeta)\|_{\infty, B_{\lambda\cdot 5/12}} \leqslant \lambda/12\cdot\|\zeta\|_{C^{0,\frac{1}{2}}(B_{\lambda/2}-B_{\lambda/3})} + M$$

for h with $0 < h \leqslant \lambda/12$.

COROLLARY:

$$\zeta|\Omega_1'\times\{0\}\cap B_{\lambda\cdot 5/12} \in C^{0,\frac{1}{2}}[-\lambda\cdot 5/12, \lambda\cdot 5/12]\,.$$

Of course the behaviour of ζ at $(b, 0)$ is the same as at $(a, 0)$. Recalling the definition of ζ in lemma 2.2, we then have

$$(3.16) \qquad (\Delta E * \mu_*)|\Omega_1'\times\{0\}\in C^{0,\frac{1}{2}}(\Omega_1')$$

and because of $(u - E * \mu_*)|\Omega_2 \in C^\infty(\Omega_2)$

$$(3.17) \qquad (\Delta u)|\Omega_1'\times\{0\}\in C^{0,\frac{1}{2}}(\Omega_1')\,.$$

Further by (3.1) and the inclusion in (1.4), $\Delta u|\Omega_*^+$ solves on $\Omega_*^+ := \Omega_* \cap \mathbb{R}\times\mathbb{R}_+ = \Omega_1'\times(0,\sigma)$ a Dirichlet problem for the Laplace equation with $C^{0,\frac{1}{2}}$-boundary data given on $\Omega_1'\times\{0\}$. Recalling (3.4) we can now choose a fixed $\varrho > 0$ such that the larger line segment $I^{2\varrho} := [a - 2\cdot\varrho, b + 2\cdot\varrho] \times\{0\} \subset\subset \Omega_*$ and define the domain $V_*^+ \subset \Omega_*^+$ with $I^\varrho := [a-\varrho, b+\varrho] \times\{0\}\subset\partial V_*^+$ by

$$(3.18) \qquad V_*^+ := \{x\in\Omega_*^+ : \operatorname{dist}(x, I^\varrho) \leqslant \operatorname{dist}(x, \partial\Omega_*^+ - I^\varrho)\}\,.$$

Then on account of (3.17) we show that

$$(3.19) \qquad \Delta u|V_*^+ \in C^{0,\frac{1}{2}}(\overline{V}_*^+)\,.$$

But this is the well-known problem of establishing the Hölder continuity of the solution up to a portion of the boundary where Hölder continuous Dirichlet data is given. We summarize this standard procedure in short. Again employing the comparison function G from (3.9), one proves in the first step that on account of (3.17), $\Delta u|\Omega_*^+$ satisfies a Hölder condition at the boundary points $x^0 \in I^\varrho$, that is

$$\begin{aligned} &H(x^0) := \sup\{|\Delta u(x^0) - \Delta u(x)|/|x^0 - x|^{\frac{1}{2}}: x \in \Omega_*^+\} < \infty \\ &\textit{with } \|H(\cdot)\|_{\infty, I^\varrho} \leqslant C\cdot\left(\|\Delta u\|_{\infty,\Omega_*^+} + \|\Delta u\|_{C^{0,\frac{1}{2}}(\bar{I}^{2\varrho})}\right) \end{aligned} \tag{3.20}$$

where $C = C(\varrho, I^{2\varrho}, \Omega_*^+) > 0$. Now let $x, y \in V_*^+$ with $x \neq y$ be arbitrary and estimate $|\Delta u(x) - \Delta u(y)|/|x-y|^{\frac{1}{2}}$. Then if

$$\tfrac{1}{2}\cdot \operatorname{dist}(x, I^\varrho) = \tfrac{1}{2}\cdot|x^0 - x| \leqslant |x - y| \quad \textit{for a } x^0 \in I^\varrho\,, \tag{3.21}$$

we obtain by the triangle inequality using the Hölder condition in x^0 that

$$|\Delta u(x) - \Delta u(y)|/|x-y|^{\frac{1}{2}} \leqslant 6\cdot H(x^0)\,. \tag{3.22}$$

Otherwise, there is

$$|x - y| \leqslant \tfrac{1}{2}\cdot \operatorname{dist}(x, I^\varrho) = \tfrac{1}{2}\cdot|x^0 - x| \quad \textit{for a } x^0 \in I^\varrho\,, \tag{3.23}$$

and again we have to employ the comparison function G from (3.9). Taking now the polar coordinates in such a way that $[\theta = 0] = \{0\}\times\mathbf{R}_+$, we have

$$\Delta u|\Omega_*^+ < \Delta u(x^0) + 2^{\frac{1}{2}}\cdot H(x^0)\cdot G(\cdot - x^0) \quad \textit{on } \Omega_*^+\,. \tag{3.24}$$

By virtue of $G \in C^{0,\frac{1}{2}}_{\mathrm{loc}}(\mathbf{R}^2)$ and (3.20) it suffices to consider $v := \Delta u(x^0) + 2^{\frac{1}{2}}\cdot H(x^0)\cdot G(\cdot - x^0) - \Delta u|\Omega_*^+$ on Ω_*^+ with $\Delta v = 0$ in Ω_*^+ satisfying the inequalites

$$0 < v < 3\cdot H(x^0)\cdot|\operatorname{dist}(x^0, \cdot)|^{\frac{1}{2}} \quad \textit{on } \Omega_*^+\,. \tag{3.25}$$

Writing $R := |x^0 - x|$ and $r := |x - y|$ there is $r/R \leqslant \frac{1}{2}$ by (3.23). Applying the Harnack inequalities to the positive harmonic function v on $B_R(x) \subset \Omega_*^+$, we obtain by (3.25)

$$\begin{aligned} |\Delta v(x) - \Delta v(y)|/r^{\frac{1}{2}} &\leqslant \left(\frac{R+r}{R-r} - \frac{R-r}{R+r}\right)\cdot 3\cdot H(x_0)\cdot\frac{R^{\frac{1}{2}}}{r^{\frac{1}{2}}} = \\ &= 3\cdot H(x^0)\cdot 2\cdot\frac{r^{\frac{1}{2}}}{R^{\frac{1}{2}}}\cdot(1 - r^2/R^2)^{-1} \leqslant 12\cdot H(x_0)\,. \end{aligned} \tag{3.26}$$

Thus (3.19) is shown where by (3.22), (3.26) there is the estimate $\|\Delta u\|_{C^{0,\frac{1}{2}}(\overline{V})^+_*} \leqslant C\cdot\|H(\cdot)\|_{\infty,I^\varrho}$ where $C > 0$ only depends on diam (Ω^+_*). By the symmetry with respect to $\mathbf{R}\times\{0\}$, we also have $\Delta u|V^-_* \in C^{0,\frac{1}{2}}(\overline{V}^-_*)$ where $V^-_* := \{(x_1, -x_2)\colon x \in V^+_*\}$. Further, by construction $U_*(I) := \operatorname{int}(\overline{V^-_* \cup V^+_*})$ is a $\mathbf{R}^2$-neighborhood of I, and because $U_*(I)$ is convex, one shows immediately that

$$\Delta u|U_*(I) \in C^{0,\frac{1}{2}}(\overline{U_*(I)})\,. \tag{3.27}$$

From (3.27) we obtain (3.5) by the regularity results for the Poisson equation, see for example [3], so the proof of theorem 1.2 is complete now.

In the following theorem we prove the optimal $C^{2,\frac{1}{2}}$-regularity of the solution u under the more general assumption that $\Psi' \in C^2(\Omega')$ is piecewise analytic.

THEOREM 1.2′. *Under the assumptions* (2.48), (2.49) *the solution u of* (1.2) *has the optimal regularity* $u \in C^{2,\frac{1}{2}}_{\mathrm{loc}}(\Omega)$.

REMARK. See remark ii), iii) on theorem 1.2.

PROOF OF THEOREM 1.2′. Replacing theorem 1.1 by theorem 1.1′ in our considerations, we repeat the proof of theorem 1.2 with the necessary modifications. Again choosing an interval $I = [a, b]\times\{0\}$ in (3.4), we have to show (3.5) which follows from

$$\Delta u|(a-\eta, b+\eta)\times\{0\} \in C^{0,\frac{1}{2}}(a-\eta, b+\eta)\,, \qquad \eta > 0 \tag{3.28}$$

sufficiently small chosen, in the same way we deduce (3.5) from (3.17) in the proof of theorem 1.2. In order to get (3.28), we first prove for one endpoint $(a, 0)$ of I that for $\lambda > 0$ sufficiently small

$$\Delta u|\Omega'\times\{0\} \cap B_{\lambda/3}((a, 0)) \in C^{0,\frac{1}{2}}[a-\lambda/3, a+\lambda/3]\,. \tag{3.29}$$

Due to the piecewise analyticity of Ψ' we have to regard the two possibilities in (2.50) separately.

In the case when for a $\alpha > 0$

$$\Psi'|(a-\alpha, a+\alpha)\times\{0\} \in C^\omega(a-\alpha, a+\alpha)\,, \tag{3.30}$$

we can proceed as in the proof of theorem 1.2. Therefore, we choose a $\lambda \in (0, \alpha/3)$ sufficiently small such that

$$(a-2\cdot\lambda, a+2\cdot\lambda)\times\{0\} \cap ([u|\Omega' = \Psi'] - I) = \emptyset\,. \tag{3.31}$$

Then choosing Ω_1, Ω_2 in § 2 such that

$$(3.32) \qquad \Omega_1' = (a - 2\cdot\lambda, a + 2\cdot\lambda)\times\{0\}\,, \qquad \Omega_2' \subset (a-\alpha, a+\alpha)\times\{0\}\,,$$

we again use a function ζ constructed on $\Omega_* = \Omega_1'\times(-\sigma, \sigma)$ in lemma 2.2 for a $\sigma > 0$ sufficiently small. For this ζ there are appropriate versions of lemma 3.1, 3.2, 3.3 valid. Hence recalling the definition of ζ in lemma 2.2, we have $(\Delta E * \mu_*)|\Omega'\times\{0\} \cap B_{\lambda/3}((a, 0)) \in C^{0,\frac{1}{2}}[a - \lambda/3, a + \lambda/3]$, and moreover obtain (3.29) on account of $(u - E * \mu_*)|\Omega_2 \in C^\omega[\Omega_2)$. In the opposite case when for a $\alpha > 0$

$$(3.33) \qquad \Psi' \equiv \Psi_1' \ \textit{on}\ (a-\alpha, a]\times\{0\}\,, \qquad \Psi' \equiv \Psi_2' \ \textit{on}\ [a, a+\alpha)\times\{0\}$$
$$\textit{for}\ \Psi_1', \Psi_2' \in C^\omega(a-\alpha, a+\alpha)\,, \quad \Psi_1' \neq \Psi_2'\,,$$

we again chose a $\lambda \in (0, \alpha/3)$ sufficiently small such that (3.31) is valid, and Ω_1, Ω_2 in § 2 satisfying (3.32). As a consequence of (3.31), (3.32), we have the inclusion

$$(3.34) \qquad I \cap \Omega_1' \subset [\, d^2/ds^2(u|\Omega' - \Psi_2') = 0\,] \cap \Omega_1'$$

remembering that $(a, 0)$ is the left end point of I. Thus employing Ψ_2', we construct a ζ on $\Omega_* = \Omega_1'\times(-\sigma, \sigma)$ in lemma 2.2′ for a $\sigma > 0$ sufficiently small which is harmonic on $\Omega_* - I$. Thank to (3.34) we can then proceed as before and also obtain (3.29) in the case (3.33). Obviously for the other endpoint $(b, 0)$ of I by the same arguments one gets analogue to (3.29) for a $\lambda > 0$ sufficiently small that

$$(3.35) \qquad \Delta u|\Omega'\times\{0\} \cap B_{\lambda/3}((b, 0)) \in C^{0,\frac{1}{2}}[b - \lambda/3, b + \lambda/3]\,.$$

Finally, in order to get (3.28) we study the regularity of $\Delta u|\Omega'$ on I. From the assumption that $\Psi' \in C^2(\Omega')$ is piecewise analytic on Ω', see (2.49), (2.50), immediately follows that

$$(3.36) \qquad \Psi' \in C^{2,1}(\Omega')\,.$$

Choosing Ω_1, Ω_2 in § 2 such that (3.6) is satisfied, then formula (2.17) is valid, and by (2.18), (3.36) we now have in (2.19)

$$(3.37) \qquad w' \in C^{2,1}(\Omega_2')\,.$$

Since $z'|I \equiv 0$ where we recall the definition of z' in (2.8), equation (2.20) reads as

$$(3.38)\qquad 0 = w'|I + \tfrac{1}{2}\cdot \Delta u|I - \tfrac{1}{2}\cdot(\Delta u - \Delta E * \mu_*)|I$$

where $(u - E * \mu_*)|\Omega_2 \in C^\omega(\Omega_2)$. Hence, from (3.38) follows by (3.37) that

$$(3.38')\qquad \Delta u|I \in C^{2,1}[a, b]\,.$$

Putting (3.29), (3.35), (3.38′) together, we obtain (3.28), and the proof of theorem 1.2′ is complete.

The question of what regularity of the solution u of (1.2) one can expect at all is discussed by means of an example in § 1. Closing § 3 we show the stated properties of the u_ε in this example where $-\Delta u_\varepsilon = -\Delta E * \nu_\varepsilon$ is the logarithmic capacitary potential with respect to the line segment L'_ε in $\mathbb{R}^2$ of length $2\cdot\varepsilon$, see (1.12), (1.13), (1.14).

Continuing our notation in § 3, let for $\varepsilon \in (0, \frac{1}{4}]$

$$(3.39)\qquad L'_\varepsilon = I = [a, b]\times\{0\} = [0, 2\cdot\varepsilon]\times\{0\}\,.$$

Then by the use of comparison functions one obtains as in the first part of this chapter that $\Delta u_\varepsilon \in C^{0,\frac{1}{2}}(\bar\Omega)$ and moreover (1.20). Further, the comparison function G from (3.9) can also be used to prove the second part of (1.19). Indeed, because $1 - (-\Delta E * \nu_\varepsilon)$, G are harmonic on $\mathbb{R}^2 - L'_\varepsilon$, $\equiv 0$ on L'_ε, and by the fact that $(-\Delta E * \nu_\varepsilon)|\partial B_N(0) \to -\infty$ uniformly $(N \to \infty)$, there exists a $N \in \mathbb{N}$ and $A_N \in \mathbb{R}_+$ such that

$$(3.40)\qquad 1 - (-\Delta E * \nu_\varepsilon) \geqslant A_N \cdot G \geqslant 0 \quad \textit{on}\ \partial B_N(0)\,.$$

Then we have for $h > 0$, $\beta > \frac{1}{2}$ at the point $(0, 0)$

$$(3.41)\qquad \big(1 - (-\Delta E * \nu_\varepsilon)(-h, 0) - 1 + (-\Delta E * \nu_\varepsilon)(0, 0)\big)\cdot h^{-\beta} =$$

$$= \big(1 - (-\Delta E * \nu_\varepsilon)(-h, 0)\big)\cdot h^{-\beta} \geqslant A_N \cdot G(-h, 0)\cdot h^{-\beta} = A_N \cdot h^{\frac{1}{2}-\beta} \to \infty \quad (h \to 0),$$

thus (1.19) is complete shown.

In order to study the u_ε for $\varepsilon \to 0$, let again $L'_\varepsilon = [-\varepsilon, \varepsilon]\times\{0\}$ as in § 1. First, we derive the relation (1.18) what is

LEMMA 3.4:

$$\nu_\varepsilon = q_\varepsilon \cdot \nu_{\frac{1}{4}}\big((4\cdot\varepsilon)^{-1}\cdot .\big)\,, \qquad \varepsilon \in (0, \tfrac{1}{4}]\,.$$

PROOF. Setting $\nu_{\frac{1}{4},\varepsilon} := \nu_{\frac{1}{4}}((4\cdot\varepsilon)^{-1}\cdot .)$ and writing for simplicity $\tilde{x} = 4 \cdot\varepsilon\cdot x$, $\tilde{y} = 4\cdot\varepsilon\cdot y$, there is

$$(3.42)\qquad \begin{aligned} -\log|.| * \nu_{\frac{1}{4}}(x) &= -\int \log|x-y|\,d\nu_{\frac{1}{4}}(y) \\ &= -\int \log|(4\cdot\varepsilon)^{-1}\cdot(\tilde{x}-\tilde{y})|d\nu_{\frac{1}{4},\varepsilon}(\tilde{y}) = -\int \log|\tilde{x}-\tilde{y}|d\nu_{\frac{1}{4},\varepsilon}(\tilde{y}) + \\ &+ \int \log|4\cdot\varepsilon|d\nu_{\frac{1}{4}}(y) = -\log|.| * \nu_{\frac{1}{4},\varepsilon}(\tilde{x}) + \log|4\cdot\varepsilon|\cdot\nu_{\frac{1}{4}}(\mathbf{R}^2)\,. \end{aligned}$$

Since $x \in L'_{\frac{1}{4}}$ if and only if $\tilde{x} \in L'_{\varepsilon}$, there is on account of (3.42) and the property (1.14) of $\nu_{\frac{1}{4}}$

$$(3.43)\qquad -(2\cdot\pi)^{-1}\cdot\log|.| * \nu_{\frac{1}{4},\varepsilon} + (2\cdot\pi)^{-1}\cdot\log|4\cdot\varepsilon|\cdot\nu_{\frac{1}{4}}(\mathbf{R}^2) \equiv 1 \qquad \textit{on } L'_{\varepsilon}$$

so that

$$(3.44)\qquad q_\varepsilon\cdot(-\Delta E * \nu_{\frac{1}{4},\varepsilon}) \equiv 1 \qquad \textit{on } L'_{\varepsilon}\,.$$

As $q_\varepsilon\cdot\nu_{\frac{1}{4},\varepsilon} \geqslant 0$ is a positive measure with $\operatorname{supp}(q_\varepsilon\cdot\nu_{\frac{1}{4},\varepsilon}) \subset L'_\varepsilon$, $q_\varepsilon\cdot\nu_{\frac{1}{4},\varepsilon}$ has the properties (1.13), (1.14) as ν_ε, thus by the uniqueness of the capacitary measure the assertion of the lemma follows.

Thank of lemma 3.4 we are now able to establish the lower estimate (1.21) for $\|\Delta u_\varepsilon\|_{C^{0,\alpha}(\overline{B}_\varepsilon)}$ for $\varepsilon \to 0$. Therefore, let $[\varphi]_{\alpha,B_\sigma} := \sup\{|\varphi(x) - \varphi(y)| / |x-y|^\alpha \colon x, y \in B_\sigma\}$ denote the α-Hölder coefficient of $\varphi \in C^{0,\alpha}(\overline{B}_\sigma)$ for $\alpha \in (0, \frac{1}{2}]$ where $B_\sigma = B_\sigma(0)$ with $\sigma > 0$. Then there trivially holds

$$(3.45)\qquad [\varphi((4\cdot\varepsilon)^{-1}\cdot .)]_{\alpha,B_{4\cdot\varepsilon\cdot\sigma}} = (4\cdot\varepsilon)^{-\alpha}\cdot[\varphi]_{\alpha,B_\sigma}\,,$$

so that on account of lemma 3.4 and (3.42)

$$(3.46)\qquad \begin{aligned} [-\Delta E * \nu_\varepsilon]_{\alpha,B_\varepsilon} &= q_\varepsilon\cdot[-\Delta E * \nu_{\frac{1}{4},\varepsilon}]_{\alpha,B_\varepsilon} \\ &= q_\varepsilon\cdot(4\cdot\varepsilon)^{-\alpha}\cdot[-\Delta E * \nu_{\frac{1}{4}}]_{\alpha,B_{\frac{1}{4}}}\,. \end{aligned}$$

Recalling that $-\Delta E * \nu_{\frac{1}{4}} \not\equiv$ const., the last factor on the right-handed side in (3.46) does not vanish, thus (1.21) follows.

In order to obtain (1.22) one first has to recall that

$$(3.47)\qquad E,\, \partial_i E \in C^0(\mathbf{R}^2) \textit{ and } \partial_{12}E,\, (\partial_{ii} - \tfrac{1}{2}\cdot\Delta)E \in C^0(\mathbf{R}^2 - \{0\}) \cap L^\infty(\mathbf{R}^2)\,, \qquad i = 1, 2\,,$$

and to use that from lemma 3.4 in particular follows that

$$\nu_\varepsilon(\mathbf{R}^2) \leqslant C\cdot 1/\log\frac{1}{\varepsilon} \quad \textit{where } \operatorname{supp}(\nu_\varepsilon)\subset B_\varepsilon \tag{3.48}$$

and $C > 0$. That gives

LEMMA 3.5:

$$\|u_\varepsilon\|_{1,\infty,B_1},\ \|\partial_{12}u_\varepsilon\|_{\infty,B_1},\ \|\partial_{ii}u_\varepsilon - \tfrac{1}{2}\cdot\Delta u_\varepsilon\|_{\infty,B_1}\to 0 \qquad (\varepsilon\to 0)\,,\quad i=1,2\,.$$

Now employing (3.48) again and estimating Δu_ε on ∂B_1 from below, by by the maximum principle we get recalling (1.14)

$$\|\Delta u_\varepsilon\|_{\infty,B_1}\leqslant C\,,\qquad \varepsilon\in(0,\tfrac{1}{4}] \tag{3.49}$$

where $C > 0$. Further, by the splitting $\partial_{ii}u_\varepsilon = \frac{1}{2}\cdot\Delta u_\varepsilon + (\partial_{ii} - \frac{1}{2}\cdot\Delta)u_\varepsilon$ and lemma 3.5 we obtain the remaining properties of the u_ε in (1.22).

Finally, by the fact $E\in C^\infty(\mathbf{R}^2 - \{0\})$ also the assertion (1.24) follows from (3.48).

4. – Proof of theorem 1.3.

First, we prove the following theorem which refers to the remark i) of theorem 1.3 and gives some information about the behaviour of the measure μ on a nondegenerated connected component of $[u|\Omega' = \Psi']$. For isolated points of $[u|\Omega' = \Psi']$ compare (3.3).

THEOREM 4.1. *Assume* (1.5) *and let* u *solve* (1.2). *Then*

$$\operatorname{supp}(\mu) = [u|\Omega' = \Psi'] - \{\textit{isolated points of } [u|\Omega'=\Psi']\} = \bigcup_{i=1}^{i^*}[a^i, b^i]\times\{0\}$$

where $a^i < b^i$ *and* $[a^i, b^i]\cap[a^{i'}, b^{i'}] = \emptyset$ *for* $i, i' = 1, \ldots, i^* \leqslant \infty$ *with* $i\neq i'$, *and there exists a* $\mathbf{R}^1$*-Lebesgue density* $\Theta'_i \in C^\omega(a^i, b^i)\cap L^1(a^i, b^i)$ *such that* $\Theta'_i \geqslant 0$,

$$\mu|[a^i, b^i]\times\{0\} = \Theta'_i\cdot dx_1|[a^i, b^i]\,,\qquad \#[\Theta'_i = 0] < \infty\,.$$

REMARK. $dx_1|[a^i, b^i]$ denotes the Lebesgue measure dx_1 in $\mathbf{R}^1$ restricted to $[a^i, b^i]$.

The assertion of theorem 4.1 follows from a sequence of lemmata. Recalling (1.4), theorem 1.1 and (3.3), we only have the consider μ in the neighborhood of an arbitrary nondegenerated component and can proceed similar as in § 3. As the theorem becomes trivial in the case of supp $(\mu) = \emptyset$, we assume that supp $(\mu) \neq \emptyset$ and choose

(4.1) $I := [a, b] \times \{0\}$ *from* (3.3) *arbitrarily but fixed and* ζ, Ω_* *as in* § 3, *see* (3.6), (3.7).

Now we consider the behaviour of the connected components U_ν of $\Omega_* - [\zeta = 0]$ in lemma 2.3.

LEMMA 4.1. *Let* $U_\nu^+, U_{\nu'}^+ \subset [\zeta > 0]$ *be connected components with* $\partial U_i^+ \cap [\zeta = 0] \cap \Omega' \neq \emptyset$, $i = \nu, \nu'$. *Then*

$$\partial U_\nu^+ \cap \partial U_{\nu'}^+ \cap \Omega_* - \Omega' = \emptyset \quad \textit{for } \nu \neq \nu' .$$

A similar assertion holds for the connected components $U_\nu^-, U_{\nu'}^- \subset [\zeta < 0]$.

PROOF. Assuming $\nu \neq \nu'$ we proceed similar as in the proof of lemma 2.8. So we assume that there exists a

$$x^0 \in \partial U_\nu^+ \cap \partial U_{\nu'}^+ \cap \Omega_* \cap \mathbf{R} \times \mathbf{R}_+ . \tag{4.2}$$

Now we have to consider two different cases. First, let

$$\partial U_\nu^+ \cap \partial U_{\nu'}^+ \cap \Omega_* \cap \Omega' = \emptyset \tag{4.3}$$

and choose

$$x^\nu \in \partial U_\nu^+ \cap \Omega_* \cap \Omega' , \qquad x^{\nu'} \in \partial U_{\nu'}^+ \cap \Omega_* \cap \Omega' . \tag{4.4}$$

Then one can join $x^0, \breve{x}^0 := (x_1^0, -x_2^0)$, $x^\nu, x^{\nu'}$ by a closed Jordan curve Γ as indicated in the proof of lemma 2.8 where

$$\Gamma \subset U_\nu^+ \cup U_{\nu'}^+ \cup \{x^0\} \cup \{\breve{x}^0\} \cup \{x^\nu\} \cup \{x^{\nu'}\} . \tag{4.5}$$

By Jordan's curve theorem, Γ divides the plane $\mathbf{R}^2$ into two disjoint connected open components Ω_Γ and $\mathbf{R}^2 - \overline{\Omega_\Gamma}$ where Ω_Γ is bounded with $\partial\Omega_\Gamma = \Gamma$ and then by (4.5)

$$\Omega_\Gamma \subset\subset \Omega_* . \tag{4.6}$$

Concluding further in a way we used in § 2, by lemma 2.3 we have on account of (4.2), (4.4), (4.5) that

$$\Omega_\Gamma \cap U_j = \emptyset \quad \textit{for all } U_j \neq U_\nu^+, U_{\nu'}^+ \textit{ in lemma } 2.3 . \tag{4.7}$$

But (4.7) and (4.6) mean that

$$\Omega_\Gamma \subset (\overline{U_\nu^+} \cup \overline{U_{\nu'}^+}) \cap \Omega_* \subset [\zeta \geqslant 0] . \tag{4.8}$$

By the maximum resp. minimum principle then follows that

$$\Omega_\Gamma \cap (\partial U_\nu^+ \cup \partial U_{\nu'}^+) = [\zeta = 0] \cap \Omega_\Gamma \subset [\zeta = 0] \cap \Omega' . \tag{4.9}$$

Further, by the symmetry with respect to $\mathbf{R} \times \{0\}$ and the construction of Γ there is $\Omega_\Gamma^+ := \Omega_\Gamma \cap \mathbf{R} \times \mathbf{R}_+$ a connected open set with $\Omega_\Gamma^+ \neq \emptyset$ and

$$\Omega_\Gamma^+ \cap U_\nu^+ \neq \emptyset , \quad \Omega_\Gamma^+ \cap U_{\nu'}^+ \neq \emptyset . \tag{4.10}$$

But putting (4.8), (4.9) and (4.10) together yields

$$\Omega_\Gamma^+ \cap U_\nu^+ = \Omega_\Gamma^+ \cap U_{\nu'}^+ \tag{4.11}$$

which is a contradiction to $\nu \neq \nu'$, so the assumption (4.2) is not possible in case of (4.3).

Now we consider the opposite case when

$$\partial U_\nu^+ \cap \partial U_{\nu'}^+ \cap \Omega_* \cap \Omega' \neq \emptyset \tag{4.12}$$

and choose

$$x^\nu \in \partial U_\nu^+ \cap \partial U_{\nu'}^+ \cap \Omega_* \cap \Omega' . \tag{4.13}$$

Then we can proceed as in the case (4.3) in order to obtain a contradiction to (4.2). We can join x^0 and x^ν by a closed Jordan curve Γ where

$$\Gamma \subset \big((U_\nu^+ \cup U_{\nu'}^+) \cap \mathbf{R} \times \mathbf{R}_+\big) \cup \{x^0\} \cup \{x^\nu\} \tag{4.14}$$

by the method of construction which is stated in the proof of lemma 2.8. Now the bounded domain Ω_Γ with $\partial\Omega_\Gamma = \Gamma$ plays the role of the Ω_Γ^+ of the previous case (4.3) and again we arrive at a contradiction to (4.2).

So the assumption (4.2) is not possible, and respecting the symmetry with respect to $\mathbf{R} \times \{0\}$, the proof of lemma 4.1 is complete now.

We need lemma 4.1 essentially for the next lemma.

LEMMA 4.2:

$$\#\{U_\nu \textit{ in lemma } 2.3\colon \partial U_\nu \cap [\zeta = 0] \cap \Omega' \neq \emptyset\} < \infty$$

PROOF. By construction, there is

$$(4.15)\qquad [\zeta = 0] \cap \Omega' = [a, b] \times \{0\} \subset\subset \Omega_* \textit{ and}$$
$$\zeta > 0 \textit{ on } \Omega_* \cap \Omega' - [a, b] \times \{0\}\,.$$

So we can choose $\tilde{a}, \tilde{b}, \lambda \in \mathbf{R}$ with $\tilde{a} < a < b < \tilde{b}$, $\lambda > 0$ such that

$$(4.16)\qquad B_\lambda((\tilde{a}, 0)),\, B_\lambda((\tilde{b}, 0)) \subset [\zeta > 0]\,.$$

Assuming the contrary to the assertion of lemma 4.2, by lemma 4.1, 2.3 and the symmetry with respect to $\mathbf{R} \times \{0\}$ we must have for the connected components U_ν of $\Omega_* - [\zeta = 0]$ that

(4.17) $\partial U_\nu \cap (\tilde{a}, \tilde{b}) \times \{x_2\} \neq \emptyset$ *for infinitely many* $\nu \in \mathbf{N}$ *and arbitrary* $x_2 \in (0, \lambda)$.

But using the identity principle for analytic functions in one variable for $\zeta|(\tilde{a}, \tilde{b}) \times \{x_2\}$ with $x_2 \in (0, \lambda)$ fixed, (4.17) leads to $\zeta|(\tilde{a}, \tilde{b}) \times (0, \lambda) \equiv 0$, and by the unique continuation principle to

$$(4.18)\qquad \zeta \equiv 0$$

which is a contradiction to $[\zeta = 0] \cap \Omega' \subset\subset \Omega_*$, thus lemma 4.2 is proved.

LEMMA 4.3. *Let* U_ν *be from lemma* 2.3 *with* $\partial U_\nu \cap [\zeta = 0] \cap \Omega' \neq \emptyset$. *Then* $\partial U_\nu \cap [\zeta = 0] \cap \Omega'$ *is connected.*

PROOF. Assume for a U_ν with $V'_\nu := \partial U_\nu \cap [\zeta = 0] \cap \Omega' \neq \emptyset$ that V'_ν has at least two connected components, that means that there exist $\alpha, \beta, \gamma \in \mathbf{R}$ with $\alpha < \beta < \gamma$ such that

$$(4.19)\qquad (\alpha, 0), (\gamma, 0) \in V'_\nu \textit{ and } (\beta, 0) \notin V'_\nu\,.$$

Let $x^0 \in U_\nu \cap \mathbf{R} \times \mathbf{R}_+$ be arbitrary chosen but fixed, then we can join $(\alpha, 0)$, x^0, $(\gamma, 0)$ by a Jordan curve ω with the property $\omega - \{(\alpha, 0), (\gamma, 0)\} \subset U_\nu \cap \mathbf{R} \times \mathbf{R}_+$ because U_ν is connected using the symmetry with respect to $\mathbf{R} \times \{0\}$, for the details of the construction see the proof of lemma 2.8. Taking into account that $(\alpha, \gamma)\times\{0\} \subset [\zeta = 0]$, we concatenate ω and the line segment $(\alpha, \gamma)\times\{0\}$ in order to obtain the

(4.20) *closed Jordan curve* $\Gamma \subset (U_\nu \cap \mathbf{R}\times\mathbf{R}_+) \cup ([\zeta = 0] \cap \Omega')$ *with* $x^0 \in U_\nu \cap \Gamma \neq \emptyset$ *and* $\Gamma \cap \Omega' = (\alpha, \gamma)\times\{0\}$,

so that by Jordan's curve theorem there exists a simply connected bounded domain

$$\Omega_\Gamma \subset\subset \Omega_* \quad \text{with } \partial\Omega_\Gamma = \Gamma . \tag{4.21}$$

Recalling that $(\beta, 0) \notin \partial U_\nu$, there must exist a $U_{\nu'}$ with $\nu \neq \nu'$ such that

$$(\beta, 0) \in \partial U_{\nu'} . \tag{4.22}$$

Further, using (4.20) and the symmetry with respect to $\mathbf{R}\times\{0\}$, we have

$$\Omega_\Gamma \cap U_{\nu'} \neq \emptyset . \tag{4.23}$$

But because of $U_{\nu'} \cap \partial\Omega_\Gamma = U_{\nu'} \cap \Gamma = \emptyset$ by (4.20) and the fact that $U_{\nu'}$ is connected, (4.21), (4.23) lead to

$$U_{\nu'} \subset \Omega_\Gamma \subset\subset \Omega_* \tag{4.24}$$

which is a contradiction to lemma 2.3, thus lemma 4.3 is shown.

We need lemma 4.3 for the following result.

LEMMA 4.4. *Let* $U_\nu^- \subset [\zeta < 0]$ *be from lemma* 2.3. *Then for the number of elements there is* $\#(\partial U_\nu^- \cap [\zeta = 0] \cap \Omega') \leqslant 1$.

PROOF. Let $U_\nu^- \subset [\zeta < 0]$ be from lemma 2.3 with $V'_\nu := \partial U_\nu^- \cap [\zeta = 0] \cap \Omega' \neq \emptyset$ and $\check{U}_\nu^- := \{(x_1, -x_2): (x_1, x_2) \in U_\nu^-\} \subset [\zeta < 0]$ be the reflected component which also appears in lemma 2.3 by the symmetry with respect to $\mathbf{R}\times\{0\}$. We then have

$$\zeta \leqslant 0 \quad on \quad W_\nu^- := \operatorname{int}_{\mathbf{R}^2}(\operatorname{cl}_{\mathbf{R}^2}(U_\nu^- \cup \check{U}_\nu^-)) , \tag{4.25}$$

and by the symmetry with respect to $\mathbf{R}\times\{0\}$ there is

$$\operatorname{int}_{\mathbf{R}^1}(V'_\nu)\times\{0\} \subset W_\nu^- \tag{4.26}$$

where $\operatorname{int}_{\mathbf{R}^1}$ denotes the interior with respect to the $\mathbf{R}^1$-topology. But by the contruction of ζ in (2.24) as a sum of a function which is harmonic on Ω_* and a negative logarithmic potential of a positive measure, ζ is subharmonic on Ω_*, that is $-\Delta\zeta \leqslant 0$ on Ω_* in the distributional sense. So by the maximum principle for subharmonic functions, (4.25) leads to

$$\zeta < 0 \ \text{on} \ W_\nu^- . \tag{4.27}$$

Hence, by (4.26) there is $\operatorname{int}_{\mathbf{R}^1}(V'_\nu) = \emptyset$, and by lemma 4.3 the assertion of lemma 4.4 follows.

From lemma 3.1 we derive the following result.

LEMMA 4.5. *$\zeta|(a+\varepsilon, b-\varepsilon)\times(0, \sigma/2) \in C^1([a+\varepsilon, b-\varepsilon]\times[0, \sigma/2])$ for arbitrary $\varepsilon > 0$, and for the right-handed derivative ∂_2^+ on $[\zeta = 0] \cap \Omega'$ there is $\partial_2^+\zeta|(a, b)\times\{0\} \in C^\omega(a, b)$.*

Studying $\partial_2^+\zeta|[\zeta = 0] \cap \Omega'$ further we get

LEMMA 4.6. *Let $U_\nu^+ \subset [\zeta > 0]$ be from lemma 2.3. Then there is $\partial_2^+\zeta > 0$ on $\operatorname{int}_{\mathbf{R}^1}\big(\partial U_\nu^+ \cap [\zeta = 0] \cap (\mathbf{R}\times\{0\})\big)$ where $\operatorname{int}_{\mathbf{R}^1}$ denotes the interior with respect to the $\mathbf{R}^1$-topology.*

PROOF. Assume that $\operatorname{int}_{\mathbf{R}^1}\big(\partial U_\nu^+ \cap [\zeta = 0] \cap (\mathbf{R}\times\{0\})\big) \neq \emptyset$, which is a nondegenerated line segment. Then the assertion of the lemma follows by the Hopf boundary point lemma which is well-known from the proof of the strong maximum principle for harmonic functions.

With lemma 4.2, 4.4 we then obtain

COROLLARY. *$\partial_2^+\zeta > 0$ on $(a, b)\times\{0\} - \{x^\nu\}_{\nu=2}^{i_0-1}$ for a $i_0 \in \mathbf{N}$.*

Now we show that $2\cdot\partial_2^+\zeta|\Omega'_1$ is a Lebesgue density of μ on I.

LEMMA 4.7.

$$\partial_2^+\zeta|[a, b]\times\{0\}\cdot dx_1|[a, b] = \tfrac{1}{2}\cdot\mu|[a, b]\times\{0\} ,$$

where dx_1 denotes the Lebesgue measure in $\mathbf{R}^1$.

PROOF. Recalling that in (2.24) $w \in C^\omega((a, b)\times(-\sigma, \sigma))$ is symmetric with respect to $\mathbf{R}\times\{0\}$ which implies $\partial_2 w|(a, b)\times\{0\} \equiv 0$, we have for $\delta > 0$ fixed

$$\begin{aligned} &\partial_2^+\zeta|(a, b)\times\{0\} = \tfrac{1}{2}\cdot(\partial_2^+\Delta E * \mu_*)|(a, b)\times\{0\} ,\\ &(\Delta E * \mu_*)|(a+\delta, b-\delta)\times(0, \infty) \in C^1([a+\delta, b-\delta]\times[0, \infty)) . \end{aligned} \tag{4.28}$$

First, we mollify $(\Delta E * \mu_*)(.,.) \in C^0(\mathbf{R}^2)$ with respect to the x_1-variable. Therefore, we introduce the usual mollifier $\omega_\varepsilon := \varepsilon^{-1}\cdot\omega(./\varepsilon)$, $\varepsilon > 0$, where

$$\omega := \left(\int_{-1}^{1} \exp\left(1/(x_1^2-1)\right) dx_1\right)^{-1} \cdot \exp\left(1/(x_1^2-1)\right) \qquad \text{on } (-1,1), \tag{4.29}$$
$$:= 0 \qquad \text{on } \mathbf{R}-(-1,1),$$

with the well-known properties of mollification and convergence in function spaces resp. in $\mathcal{M}(\mathbf{R}^1)$ which is the space of the Borel measures in $\mathbf{R}^1$. Denoting by $*_{(1)}$ the convolution in $\mathbf{R}^1$, we have for a fixed $x_2 \in \mathbf{R}$, $\varepsilon > 0$, by the usual rules for convolutions recalling that supp (μ_*) is bounded with supp $(\mu_*) \subset \mathbf{R}\times\{0\}$ that

$$\begin{aligned}(4.30)\qquad &(\Delta E * \mu_*)(\cdot, x_2) *_{(1)}\omega_\varepsilon \\ &= \left(\int \Delta E\left(\sqrt{(\cdot - y_1)^2 + (x_2-0)^2}\right) d\mu_*(y_1)\right) *_{(1)}\omega_\varepsilon \\ &= \left(\Delta E\left(\sqrt{(\cdot - y_1)^2 + x_2^2}\right) *_{(1)}\mu_*\right) *_{(1)}\omega_\varepsilon = \Delta E\left(\sqrt{(\cdot)^2 + x_2^2}\right) *_{(1)}\mu_{*_\varepsilon} \\ &= \int \Delta E\left(\sqrt{(\cdot - y_1)^2 + (x_2-0)^2}\right) d\mu_{*_\varepsilon}(y_1) = (\Delta E * \mu_{*_\varepsilon})(\cdot, x_2)\,.\end{aligned}$$

Here we identify the measure μ_* which is supported on $\mathbf{R}\times\{0\}$ with its restriction on $\mathbf{R}$ and set

$$\mu_{*_\varepsilon} := \mu_* *_{(1)} \omega_\varepsilon \in C_0^\infty(\mathbf{R})\,, \qquad \varepsilon > 0\,, \tag{4.31}$$

considering μ_{*_ε} again also as a measure supported on $\mathbf{R}\times\{0\}$ in an obvious way. Further, writing out the formula for $\partial_2\Delta E$ there is for $x = (x_1, x_2)$ $\in \mathbf{R}\times\mathbf{R}_+$

$$\partial_2\Delta E(x) = (2\cdot\pi)^{-1}\cdot x_2/(x_1^2 + x_2^2) = \mathcal{P}_{x_2}(x_1) \tag{4.32}$$

where $\mathcal{P}(.)$ denotes the Poisson kernel associated to the upper half-space $\mathbf{R}\times\mathbf{R}_+$. So $\partial_2\Delta E * \mu_{*_\varepsilon}$ can on $\mathbf{R}\times\mathbf{R}_+$ be regarded as a Poisson integral of the $C_0^\infty(\mathbf{R})$-function μ_{*_ε}, and taking into account that we mollify only with respect to the x_1-variable there is for $x_2 \in \mathbf{R}_+$

$$\left((\partial_2\Delta E * \mu_*) *_{(1)} \omega_\varepsilon\right)(., x_2) = (\partial_2\Delta E * \mu_{*_\varepsilon})(., x_2) = \mathcal{P}_{x_2}(.) *_{(1)} \mu_{*_\varepsilon}\,. \tag{4.33}$$

From the well-known properties of the Poisson integral of a $C_0^0(\mathbf{R})$-function, see the book of Stein and Weiss [9] for example, for fixed $\varepsilon > 0$ follows

$$\|\mathcal{P}_{x_2}(.) *_{(1)} \mu_{*_\varepsilon} - \mu_{*_\varepsilon}\|_{\infty,\mathbf{R}} \to 0 \qquad (x_2 \to 0+) . \tag{4.34}$$

Recalling (4.28) we obtain by the mean value theorem for $\delta > 0$ fixed that

$$(\partial_2 \Delta E * \mu_*)(., x_2) \to (\partial_2^+ \Delta E * \mu_*) \text{ in } C^0[a+\delta, b-\delta] \qquad (x_2 \to 0+) . \tag{4.35}$$

Putting (4.33), (4.34), (4.35) together yields

$$(\partial_2^+ \Delta E * \mu_*) *_{(1)} \omega_\varepsilon = \mu_{*_\varepsilon} \text{ on } (a + 2\cdot\varepsilon, b - 2\cdot\varepsilon) \times \{0\} . \tag{4.36}$$

Letting $\varepsilon \to 0$ we have $\mu_{*_\varepsilon} \cdot dx_1 \xrightarrow{\cdot} \mu_*$ vaguely in $\mathcal{M}(\mathbf{R}^1)$ $(\varepsilon \to 0)$, further $(\partial_2^+ \Delta E * \mu_*) *_{(1)} \omega_\varepsilon \to \partial_2^+ \Delta E * \mu_*$ in $C^0[a+\delta, b-\delta]$ $(\varepsilon \to 0)$ for $\delta > 0$ fixed, and recalling (4.28), (3.2), (3.6) we arrive at

$$\begin{aligned} \mu|[a,b]\times\{0\} = \mu_* &= \partial_2^+ \Delta E * \mu_* \cdot dx_1|[a,b] \\ &= 2\cdot\partial_2^+ \zeta|(a,b)\times\{0\}\cdot dx_1|[a,b] \end{aligned} \tag{4.37}$$

which is the assertion of the lemma.

By lemma 4.5 and the corollary of lemma 4.6 then follows

COROLLARY: $\Theta' := 2\cdot\partial_2^+\zeta|(a,b)\times\{0\} \in L^1(a,b) \cap C^\omega(a,b)$ *where*

$$\operatorname{supp}(\mu|I) = \operatorname{supp}(\Theta') = I .$$

Now we have all the arguments for the

PROOF OF THEOREM 4.1. Considering μ on a neighborhood of I we get the assertion of the theorem from lemma 4.7 with its corollary and the corollary of lemma 4.6 because the choice of I in (4.1) is arbitrary.

In the next theorem we show that for the problem after suitable small perturbations the number of the components of the coincidence set is uniformly bounded.

THEOREM 4.2. *Let u resp. $\breve{u}$ be the solution of* (1.2) *associated to g, Ψ' resp. $\breve{g}, \breve{\Psi}'$ under the assumptions* (1.5), (1.7) *and $\delta > 0$ given. Then there exists $\varepsilon_0 = \varepsilon_0(g, \Psi', \delta) > 0$ and $k = k(g, \Psi', \delta) \in \mathbf{N}$ such that if*

$$\|g - \breve{g}\|_{2,2} + \sum_{i=0}^{\infty} \|d^i(\Psi' - \breve{\Psi}')/ds^i\|_{\infty,\Omega'} \cdot \delta^i/i! \leq \varepsilon_0 ,$$

then $\#\{\breve{I} : \breve{I}$ is connected component of $[\breve{u}|\Omega' = \breve{\Psi}']\} \leq k$.

For the proof of the previous theorem we need several steps which are stated in lemmata. Note, that in the sequel

(4.38) $\varepsilon > 0$ *refers to the assumption in theorem* 1.3 *and is supposed to be sufficiently small.*

First, we consider the stability of the solution u in the $H^{2,2}$-norm. By Sobolev's theorem, there is $u,\ g \in C^0(\bar{\Omega})$ recalling that $\partial\Omega$ is $C^{0,1}$ due to our assumption in (1.7), and because $u - g \in \overset{\circ}{H}{}^{2,2}_0(\Omega)$ and (1.7)

$$u|\partial\Omega' = g|\partial\Omega' > \Psi'|\partial\Omega' \tag{4.39}$$

which yields $d_0 := \operatorname{dist}\big([u|\Omega' = \Psi'],\ \partial\Omega\big) > 0$. Let $\chi \in C_0^\infty(\Omega)$ be a suitable cut-off function with

$$0 \leqslant \chi \leqslant 1 \quad \textit{and}\ \chi \equiv 1\ \textit{on}\ U_{d_0/3}\big([u|\Omega' = \Psi']\big)\,. \tag{4.40}$$

Then for $\varepsilon > 0$ sufficiently small

$$\begin{aligned} u_{\check{\Psi}',\check{g}} := u + \chi\cdot\|\Psi' - \check{\Psi}'\|_{\infty,\Omega'} + (1-\chi)\cdot(\check{g} - g) \\ \in \check{\mathbf{K}}_{\check{\Psi}'} \subset \check{g} + \overset{\circ}{H}{}^{2,2}_0(\Omega)\,. \end{aligned} \tag{4.41}$$

Recalling that $\check{f} \equiv 0$ due to the convention in § 1, we set $u_{\check{\Psi}',\check{g}}$ into (1.3) associated to $\check{g}$, $\check{\Psi}'$ and obtain by a standard estimate using a Poincaré type inequality

$$\|\check{u}\|_{2,2} \leqslant C_\chi\cdot\big(\|u\|_{2,2} + \|g\|_{2,2} + \|\check{g}\|_{2,2} + \|\Psi' - \check{\Psi}'\|_{\infty,\Omega'}\big) \tag{4.42}$$

where $C_\chi > 0$. From the obvious inequality

$$\Big(\operatorname{dist}\big([\check{u}|\Omega' = \check{\Psi}'],\ \partial\Omega\big)\Big)^{\frac12}\cdot\|\check{u}|\Omega' - \check{\Psi}'\|_{C^{0,\frac12}(\overline{\Omega'})} \geqslant \inf\,(\check{g}|\partial\Omega' - \check{\Psi}'|\partial\Omega') \tag{4.43}$$

we infer by a Sobolev type inequality that

$$\begin{aligned} &\operatorname{dist}\big([\check{u}|\Omega' = \check{\Psi}'],\ \partial\Omega\big) \\ &\qquad \geqslant C_\chi\cdot\Big(\inf\,(\check{g}|\partial\Omega' - \check{\Psi}'|\partial\Omega')/\big(\|\check{u}\|_{2,2} + \|\check{\Psi}'\|_{C^{0,\frac12}(\overline{\Omega'})}\big)\Big)^2 \end{aligned} \tag{4.44}$$

where $C_\chi > 0$. Note, that by assumption (1.7) there can not be $u \equiv 0$, $\Psi' \equiv 0$ at the same time. Hence by (4.42), for $\varepsilon > 0$ sufficiently small there is $\operatorname{dist}\left([\check{u}|\Omega' = \check{\Psi}'], \partial\Omega\right) \geq d_0/3$, so that like (4.41)

$$(4.45) \qquad \check{u}_{\Psi',g} := \check{u} + \chi \cdot \|\Psi' - \check{\Psi}'\|_{\infty,\Omega'} + (1-\chi)\cdot(g - \check{g}) \in \mathbf{K}_{\Psi'} \subset g + \overset{\circ}{H}{}^{2,2}_0(\Omega)\,.$$

Setting now $u_{\check{\Psi}',\check{g}}$ into (1.2) associated to $\check{\Psi}'$, $\check{g}$ as well as $\check{u}_{\Psi',g}$ into (1.2) associated to Ψ', g, we get for $\varepsilon > 0$ sufficiently small by a standard estimate using a Poincaré type inequality

LEMMA 4.8:

$$\|u - \check{u}\|_{2,2} \leq A(\varepsilon) \quad \textit{where } A(\varepsilon) \to 0 \ (\varepsilon \to 0)\,.$$

COROLLARY.

$$[\check{u}|\Omega' = \check{\Psi}'] \subset U_{A(\varepsilon)}\left([u|\Omega' = \Psi']\right), \quad A(\varepsilon) \to 0 \ (\varepsilon \to 0)\,.$$

Thank to the corollary and theorem 1.1, it is sufficient to verify the assertion of theorem 4.2 locally in a neighborhood of a connected component of $[u|\Omega' = \Psi']$. So in order to proceed, we again consider I, Ω_* as in (4.1).

Now we choose a suitable cut-off function $\chi_* \in C_0^\infty(\Omega_*)$ with

$$(4.46) \qquad 0 \leq \chi_* \leq 1 \quad \textit{and } \chi_* \equiv 1 \textit{ on } U(I)$$

where $U(I)$ is an open $\mathbf{R}^2$-neighborhood of $I = [u|\Omega' = \Psi'] \cap \Omega_*$ recalling that $I \subset\subset \Omega_*$ by (4.1). Then using the inclusion in (1.4) we have $\mu_*(\mathbf{R}^2) = \chi_* \cdot \mu(\mathbf{R}^2) = \int \Delta\chi_* \cdot \Delta u\, dx$ and by the corollary of lemma 4.8 a similar expression for $\check{\mu}_*(\mathbf{R}^2)$ provided that $\varepsilon > 0$ is sufficiently small. So we can infer from lemma 4.8 that

LEMMA 4.9.

$$|\mu_*(\mathbf{R}^2) - \check{\mu}_*(\mathbf{R}^2)| \leq A(\varepsilon) \to 0 \qquad (\varepsilon \to 0)\,.$$

By virtue of the representation

$$E * (\mu_* - \check{\mu}_*) = E * \left(\chi_* \cdot (\mu - \check{\mu})\right) = \int \chi_*(y) \cdot E(. - y)\, d(\mu - \check{\mu})(y)$$

and recalling that $E \in H^{2,2}(\mathbf{R}^2)$, we obtain by integration by parts on account of lemma 4.8

LEMMA 4.10:

$$\|E * \mu_* - E * \check{\mu}_*\|_{\infty,\Omega} \leq A(\varepsilon) \quad \textit{where } A(\varepsilon) \to 0 \ (\varepsilon \to 0)\,.$$

After a possible shrinking of Ω_* we can assume that there exists a domain Ω_2 such that $\Omega_* \subset\subset \Omega_2 \subset\subset \Omega$ and further dist $\big(\Omega_2, [u|\Omega' = \Psi'] - I\big) > 0$ which yields $[u|\Omega' = \Psi'] \cap \Omega_2 = I = [u|\Omega' = \Psi'] \cap \Omega_*$. So by the inclusion in (1.4) then there is $\mu|\Omega_2 = \mu|\Omega_* = \mu_*$, therefore

$$(4.47) \qquad \Delta^2(u - E * \mu_*) = 0 \quad \text{on } \Omega_2 ,$$

and the same for $\check{u} - E * \check{\mu}_*$ on account of the corollary of lemma (4.8) provided that $\varepsilon > 0$ is sufficiently small. Then we get the following estimate for the power series development of the perturbation of $u - E * \mu_*$ on Ω_1' recalling that due to our choice of Ω_2 there is dist $(\Omega_1', \partial\Omega_2) > 0$.

Lemma 4.11. *For δ with $0 < \delta \leqslant \operatorname{dist}(\Omega_1', \partial\Omega_2)/3$ there is*

$$\sum_{i=0}^{\infty} \|d^i(u - E * \mu_* - \check{u} + E * \check{\mu}_*)|\Omega'/ds^i\|_{\infty,\Omega_1'} \cdot \delta^i/i! \leqslant A(\varepsilon) \to 0 \qquad (\varepsilon \to 0) .$$

Proof. For the $\mathbf{R}^2$-neighborhood $U_{2\delta}(\Omega_1') \subset\subset \Omega_2$ of Ω_1' we get by an interior estimate for biharmonic functions on account of (4.47) using lemma 4.8, 4.10

$$(4.48) \qquad \|u - E * \mu_* - (\check{u} - E * \check{\mu}_*)\|_{2,\infty,U_{2\delta}(\Omega\ )} \leqslant A(\varepsilon) \to 0 \qquad (\varepsilon \to 0) .$$

Now we employ the following representation formula for a biharmonic function v on $B_r = B_r(0)$, $r > 0$, that is for a $v \in C^1(\bar{B}_r)$ with $\Delta^2 v = 0$ on B_r, see Schulze-Wildenhain [8],

$$(4.49) \qquad v(x) = \frac{(r^2 - |x|^2)^2}{2\cdot r^2\cdot\pi}\cdot \int_{\partial B_r} \frac{-(\partial_{\mathcal{N}} v|\partial B_r)(y)}{|x-y|^2}\, dy +$$
$$+ \frac{(r^2 - |x|^2)^2}{2\cdot r^3\cdot\pi}\cdot \int_{\partial B_r} \frac{2\cdot(r^2 - x\cdot y)\cdot(v|\partial B_r)(y)}{|x-y|^4}\, dy \qquad \text{for } x \in B_r = B_r(0) .$$

Considering $v := u - E * \mu_* - (\check{u} - E * \check{\mu}_*)$ on $B_{2\delta}(x^0)$ for $x^0 \in \Omega_1'$ in (4.49), we obtain after differentiating the formula which obviously is invariable under translations the following estimate for the power series development

$$(4.50) \qquad \sum_{i=0}^{\infty} \|d^i(u - E * \mu_* - \check{u} + E * \check{\mu}_*)|\Omega'/ds^i\|_{\infty,\Omega_1'} \cdot \delta^i/i! \leqslant$$
$$\leqslant C\cdot \|u - E * \mu_* - \check{u} + E * \check{\mu}_*\|_{1,\infty,U_{2\delta}(\Omega_1')}$$

where $C = C(\Omega_1', \Omega_2, \delta) > 0$. Putting (4.48) and (4.50) together, the assertion of lemma 4.11 follows.

We can assume for the sequel that $0 < \delta \leqslant \operatorname{dist}(\Omega_1', \partial\Omega_2)/3$ and $\Omega_* \subset U_\delta(\Omega_1')$ after choosing σ such that $0 < \sigma \leqslant \delta$. Then having a suitable majorant for estimating the power series development of $w - \check{w}$ now by the assumption concerning $\Psi' - \check{\Psi}'$ and lemma 4.9, 4.11 where w resp. $\check{w}$ is from (2.19), (2.22) recalling (2.23), we immediately obtain that

$$\|w - \check{w}\|_{\infty,\Omega_*} \leqslant A(\varepsilon) \to 0 \qquad (\varepsilon \to 0)\,. \tag{4.51}$$

Further, we infer from (4.48) using interior estimates for harmonic functions and the corollary of lemma 4.8 in association with the inclusion in (1.4) that

LEMMA 4.12. *$\|\Delta E * \mu_* - \Delta E * \check{\mu}_*\|_{2,\Omega_*} \leqslant A(\varepsilon) \to 0$ $(\varepsilon \to 0)$, and $\Delta E * \mu_* - \Delta E * \check{\mu}_* \to 0$ uniformly in $\varepsilon > 0$ on compact subsets of $U_\delta(\Omega_*) - I$ $(\varepsilon \to 0)$ where $U_\delta(\Omega_*)$ is the δ-neighborhood of Ω_*.*

Considering the definition of ζ resp. $\check{\zeta}$ in (2.24), then (4.51) and lemma 4.12 yield

LEMMA 4.13. *$\|\zeta - \check{\zeta}\|_{2,\Omega_*} \leqslant A(\varepsilon) \to 0$ $(\varepsilon \to 0)$, and $\zeta - \check{\zeta} \to 0$ uniformly in $\varepsilon > 0$ on compact subsets of $\Omega_* - I$ $(\varepsilon \to 0)$.*

Now let $\Omega_1' - [\check{u}|\Omega' = \check{\Psi}'] = \bigcup_{\varkappa=1}^{\check{\varkappa}^*} \check{J}_\varkappa$ be the decomposition into mutually disjoint open intervals as in (2.2) where $\check{\varkappa}^* \in \mathbf{N}$ by theorem 1.1. Recalling lemma 2.1 and (2.25), we consider the associated $\check{U}_\varkappa^+$ and show an uniform upper bound for $\check{\varkappa}^*$.

LEMMA 4.14. *There exists $k_0 = k_0(\Omega_1', g, \Psi', \delta) \in \mathbf{N}$ such that*

$$\check{\varkappa}_* := \#\{\check{J}_\varkappa : \check{J}_\varkappa \text{ is connected component of } \Omega' - [\check{u}|\Omega' = \check{\Psi}']\} \leqslant k_0\,.$$

PROOF. On account of (4.15) we have for a λ sufficiently small with $0 < \lambda < \sigma/2$

$$[\zeta = 0] \cap \Omega_1' \times [-\lambda, \lambda] \subset\subset \Omega_*\,. \tag{4.52}$$

Then we conclude by the identity principle for analytic functions in one variable that

$$\#[\zeta|\Omega_1' \times \{\lambda\} = 0] < \infty\,, \tag{4.53}$$

because the assumption $\zeta|\Omega_1' \times \{\lambda\} \equiv 0$ leads to a contradiction to (4.52).

Now we have to find an upper bound for $\#[\check{\zeta}|\Omega_1'\times\{\lambda\}=0]$. From (4.25) we know that $r_0 := \min\big(\lambda, \operatorname{dist}([\zeta|\Omega_1'\times\{\lambda\}=0], \partial\Omega_*)\big) > 0$. In the same way we obtained (4.50) from the representation formula (4.49), by virtue of the Poinsson integral formula related to the disc of radius $\frac{2}{3}r_0$ we get an estimate of type (4.50) for the power series deevelopment of $\zeta-\check{\zeta}$ on $U_{r_0/3} := U_{r_0/3}([\zeta|\Omega_1'\times\{\lambda\}=0])$ in terms of $\|\zeta-\check{\zeta}\|_{\infty,U_{\frac{2}{3}r_0}}$. Further, the radius of convergence for the power series development of $\zeta|\Omega_1'\times\{\lambda\}$ resp. $\check{\zeta}|\Omega_1'\times\{\lambda\}$ on U_{r_0} must be at least $r_0/3$. Hence, plugging in complex variables we get the holomorphic extension at least on $U_{r_0/3}$ and can control the holomorphic extension of $(\zeta-\check{\zeta})|\Omega_1'\times\{\lambda\}$ uniformly on $U_{r_0/3}$ in terms of $\varepsilon>0$ using lemma 4.13. Recalling the fact that zeros of holomorphic functions which do not vanish identically are isolated, we obtain by virtue of Rouché's theorem the existence of a $k_0 = k_0(\Omega_1', g, \Psi', \delta)\in\mathbf{N}$ such that

$$(4.54)\qquad \#[\check{\zeta}|\Omega_1'\times\{\lambda\}=0]\leqslant k_0 \quad \textit{for } \|\zeta-\check{\zeta}\|_{\infty,U_{\frac{2}{3}r_0}}>0 \textit{ sufficiently small}$$

where k_0 of course depends on (4.53).

But on account of (2.25) and lemma 2.8, this is also an estimate for $\check{\varkappa}^*$ which is the number of open intervals in Ω_1' between the components of the coincidence set $[\check{u}|\Omega'=\check{\Psi}']\cap\Omega_*$,

$$(4.55)\qquad \check{\varkappa}^*\leqslant\#\Big(\bigcup_{\varkappa=1}^{\check{\varkappa}^*}\partial\check{U}_\varkappa^+\cap(\Omega_1'\times\{\lambda\})\Big)\leqslant\#[\zeta|\Omega_1'\times\{\lambda\}=0].$$

thus lemma 4.14 is proved.

REMARK. *In the case when* $\{x^0\}$ *is an isolated point of the coincidence set* $[u|\Omega'=\Psi']$ *and* $\Omega_*=\Omega_1'\times(-\sigma,\sigma)$ *for a* $\sigma>0$ *such that* $\Omega_*\cap[u|\Omega'=\Psi']=\{x^0\}$, *lemma* 4.14 *is also true.*

Considering the case which is indicated in the remark, we have $\mu_*\equiv\mu|\Omega_*\equiv0$ by the inclusion in (1.4) and (3.2). Then performing the construction of ζ there is $\zeta=w$ now and the assertion of lemma 4.9, 4.10, 4.11, 4.12, 4.13 trivially holds again and so the proof of lemma 4.14 does.

Now we are able to give the

PROOF OF THEOREM 4.2. By theorem 1.1 and lemma 4.14 with the remark, there exists an open $\mathbf{R}^2$-neighborhood V_0 of the coincidence set $[u|\Omega'=\Psi']$ and an upper bound k for the number of connected components of $[\check{u}|\Omega'=\check{\Psi}']\cap V_0$, and the corollary of lemma 4.8 assures that $[\check{u}|\Omega'=\check{\Psi}']\cap V_0=[\check{u}|\Omega'=\check{\Psi}']$ for $\varepsilon>0$ sufficiently small. Thus theorem 4.2 is proved.

For the proof of theorem 1.3 we need the following additional arguments. Recalling that $\mu \not\equiv 0$ according to our assumptions and using the simple estimate $(\tau\cdot(\mu-\check{\mu})(\mathbf{R}^2)) \leqslant C_\tau \cdot \|u-\check{u}\|_{2,2}$ for $\tau \in C_0^\infty(\Omega)$, we obtain recalling lemma 4.8

LEMMA 4.15. *For* $x^0 \in \operatorname{supp}(\mu)$ *and* $\tau \in C_0^\infty(\Omega)$ *with* $0 \leqslant \tau \leqslant 1$, $\tau(x^0) = 1$ *there is* $\operatorname{supp}(\tau\cdot\check{\mu}) \neq \emptyset$ *for* $\varepsilon > 0$ *small enough.*

Again, let us consider I, Ω_* as in (4.1) in order to proceed. Now we have by theorem 4.1 a Lebesgue density Θ' such that $\mu_* = \mu|I = \Theta'\cdot dx_1|I$ where for the zeros of Θ' and the endpoints $(a, 0)$, $(b, 0)$ of I

$$(4.56)\qquad \begin{aligned}&[\Theta' = 0] \cup \{(a, 0), (b, 0)\} = \{x^1, \ldots, x^{i_0}\} \textit{ where } i_0 \in \mathbf{N}\,,\\ &x_1^1 := a < x_1^2 < \ldots < x_1^{i_0-1} < x_1^{i_0} := b\,,\end{aligned}$$

and of course $x_2^i = 0$ for $i = 1, \ldots, i_0$. We see in the following lemmata what important consequences the zeros of Θ' have in view of the stability of the coincidence set.

LEMMA 4.16. *For* $\eta > 0$ *and* $0 < \varepsilon \leqslant \varepsilon_0$ *where* $\varepsilon_0 = \varepsilon_0(\eta) > 0$ *there exist intervals* $\check{L}^{i,\eta} := [\check{\alpha}^{i,\eta}, \check{\beta}^{i,\eta}] \times \{0\} \subset [\check{u}|\Omega' = \check{\Psi}']$ *with length*

$$\check{\beta}^{i,\eta} - \check{\alpha}^{i,\eta} = \varrho := k^{-1}\cdot\eta/2\,, \qquad i = 1, \ldots, 2\cdot i_0 - 2\,,$$

such that

$$\check{L}^{1,\eta} \subset (x_1^1, x_1^1 + \eta) \times \{0\}\,, \qquad \check{L}^{2,\eta} \subset (x_1^2 - \eta, x_1^2) \times \{0\}\,,$$
$$\check{L}^{3,\eta} \subset (x_1^2, x_1^2 + \eta) \times \{0\}, \ldots, \check{L}^{2\cdot i_0 - 2,\eta} \subset (x_1^{i_0} - \eta, x_1^{i_0}) \times \{0\}\,.$$

REMARK. *The k which appears in the assertion of lemma 4.16 is the same as in the assertion of theorem 4.2.*

PROOF. Without loss of generality we prove the existence of a $\check{L}^{i,\eta} \subset (x_1^1, x_1^1 + \eta) \times \{0\} \cap [\check{u}|\Omega' = \check{\Psi}']$, the method of proof also works in the other cases. First, we perform a partition of the interval $(x_1^1, x_1^1 + \eta)$ such that

$$(4.57)\qquad \begin{aligned}s^0 := x_1^1 < s^1 := x_1^1 + \eta\cdot 1/k < \ldots < s^i := x_1^1 + \eta\cdot i/k&\\ < \ldots < s^k := x_1^1 + \eta\cdot k/k = x_1^1 + \eta&\end{aligned}$$

where $s^i - s^{i-1} = 2\cdot\varrho$ for $i = 1, \ldots, k$.
By lemma 4.15 there is for $\varepsilon > 0$ sufficiently small

$$(4.58)\qquad (s^i - \varrho/2, s^i + \varrho/2) \times \{0\} \cap \operatorname{supp}(\check{\mu}) \neq \emptyset\,, \quad i = 1, \ldots, k\,,$$

But then according to theorem 4.2 for $\varepsilon > 0$ small enough, supp $(\check{\mu})$ has no more than k connected components, so that

$$\check{L}^{1,\eta} := (s^{i'} - \varrho/2, s^{i'} + \varrho/2) \times \{0\} \subset \operatorname{supp}(\check{\mu}) \subset [\check{u}|\Omega' = \check{\Psi}'] \tag{4.59}$$

for at least one $i' \in \{0, \ldots, k-1\}$, because otherwise we would have the impossible situation when supp $(\check{\mu})$ has at least $k+1$ connected components. Since we can get the other $\check{L}^{i,\eta}$ in a similar way, lemma 4.16 is shown.

By virtue of $\zeta - \check{\zeta} = 0$ on $\check{L}^{i,\eta}$ we can apply a reflection argument, see lemma 3.1, in order to obtain by standard interior estimates for harmonic functions using lemma 4.13

Lemma 4.17. *Define for $\check{L}^{i,\eta} = [\check{\alpha}^{i,\eta}, \check{\beta}^{i,\eta}] \times \{0\}$ and $i = 1, \ldots, 2 \cdot i_0 - 2$ the interval $\check{L}^{i,\eta}_{in} := [\check{\alpha}^{i,\eta} + \varrho/4, \check{\beta}^{i,\eta} - \varrho/4]$. Then we have $\|\zeta - \check{\zeta}\|_{1,\infty,\check{L}^{i,\eta}_{in} \times [0,\sigma/2]} \leqslant C_{\varrho,\sigma} \cdot A(\varepsilon)$, $A(\varepsilon) \to 0 (\varepsilon \to 0)$, where $C_{\varrho,\sigma} > 0$ only depends on $\varrho, \sigma > 0$.*

By theorem 4.1, in particular by the corollary of lemma 4.6, there exist $\gamma = \gamma(\varrho) > 0$, $\lambda = \lambda(\varrho) \in (0, \sigma/2)$ such that

$$\partial_2 \zeta \geqslant \gamma > 0 \;\; on \;\; [x_1^i + \varrho/4, x_1^{i+1} - \varrho/4] \times [0, \lambda], \quad i = 1, \ldots, i_0 - 1, \tag{4.60}$$

which in addition implies

$$\zeta \geqslant \gamma \cdot \lambda > 0 \;\; on \;\; [x_1^i + \varrho/4, x_1^{i+1} - \varrho/4] \times \{\lambda\}, \quad i = 1, \ldots, i_0 - 1. \tag{4.61}$$

Therefore, we have

Corollary. *For $\varepsilon > 0$ sufficiently small and $i = 1, \ldots, 2 \cdot i_0 - 2$ there is $\partial_2 \check{\zeta} \geqslant \gamma/2$ on $\check{L}^{i,\eta}_{in} \times [0, \lambda]$.*

Remark. *Note, that*

$$x_1^i + \varrho/4 \leqslant \check{L}^{2\cdot i-1,\eta}_{in} < \check{L}^{2\cdot i,\eta}_{in} \leqslant x_1^{i+1} - \varrho/4,$$

$$L^{2\cdot i-1,\eta}_{in} \times \{0\}, \quad L^{2\cdot i,\eta}_{in} \times \{0\} \subset [\check{u}|\Omega' = \check{\Psi}'] \subset [\zeta = 0], \qquad i = 1, \ldots, i_0 - 1.$$

By the previous corollary and (4.61) in association with lemma 4.13, we arrive at

Lemma 4.18. *For $\varepsilon > 0$ sufficiently small there is*

$$\check{\zeta} > 0 \;\; on \;\; \check{L}^{i,\eta}_{in} \times (0, \lambda], \quad i = 1, \ldots, 2 \cdot i_0 - 2,$$

$$\check{\zeta} > 0 \;\; on \;\; [x_1^i + \varrho/4, x_1^{i+1} - \varrho/4] \times \{\lambda\}, \quad i = 1, \ldots, i_0 - 1.$$

Recalling the above remark we are now in the position to show the crucial property of the $\check{L}_{in}^{i,\eta}\times\{0\}$.

LEMMA 4.19. *For $\varepsilon > 0$ sufficiently small and $i = 1, \ldots, i_0 - 1$ the intervals $\check{L}_{in}^{2\cdot i-1,\eta}\times\{0\}$, $\check{L}_{in}^{2\cdot i,\eta}\times\{0\}$ are contained in the same connected component of $[\check{u}|\Omega' = \check{\Psi}']$.*

PROOF. Let us assume the contrary to the assertion of the lemma. Then for a $i' \in \{1, \ldots, i_0 - 1\}$ there exists a nondegenerated connected component $\check{J}_{\varkappa'} \subset \Omega' - [\check{u}|\Omega' = \check{\Psi}']$ in (2.2) such that employing the ordering in $\mathbb{R}^1$

$$\check{L}_{in}^{2\cdot i'-1,\eta} < \check{J}_{\varkappa'} < \check{L}_{in}^{2\cdot i',\eta} . \tag{4.62}$$

Now by lemma 2.1 and (2.25) there exists a component $\check{U}_{\varkappa'}^{-} \subset [\check{\zeta} < 0]$ in the partition in lemma 2.3 such that recalling (2.26) $\check{U}_{\varkappa'}^{-} \cap \check{J}_{\varkappa'}\times\{0\} \neq \emptyset$. So by (4.62) recalling the above remark there is

$$\check{U}_{\varkappa'}^{-} \cap [x_1^{i'} + \varrho/4, x_1^{i'+1} - \varrho/4]\times[-\lambda, \lambda] \neq \emptyset . \tag{4.63}$$

But by lemma 4.18 and the symmetry with respect to $\mathbb{R}\times\{0\}$ we also have

$$\check{U}_{\varkappa'}^{-} \cap \partial([x_1^{i'} + \varrho/4, x_1^{i'+1} - \varrho/4]\times[-\lambda, \lambda]) = \emptyset . \tag{4.64}$$

Since $\check{U}_{\varkappa'}^{-}$ is connected, by (4.63), (4.64) there must hold the inclusion

$$\check{U}_{\varkappa'}^{-} \subset [x_1^{i'} + \varrho/4, x_1^{i'+1} - \varrho/4]\times[-\lambda, \lambda] \subset\subset \Omega_* \tag{4.65}$$

which is a contradiction to lemma 2.3, thus lemma 4.19 is shown.

Recalling the definition of the $\check{L}_{in}^{i,\eta}$, there is

$$\check{L}_{in}^{2\cdot i-1,\eta} \subset (x_1^i, x_1^i + \eta) , \quad \check{L}_{in}^{2\cdot i,\eta} \subset (x_1^{i+1} - \eta, x_1^{i+1}) \quad \textit{for } i = 1, \ldots, i_0 - 1 , \tag{4.66}$$

so that $\check{L}_{in}^{2\cdot i-1,\eta} < (x_1^i + \eta, x^{i+1} - \eta) < \check{L}_{in}^{2\cdot i,\eta}$, and we obtain

COROLLARY: *For $\varepsilon > 0$ sufficiently small and $i = 1, \ldots, i_0 - 1$ there is $[x_1^i + \eta, x_1^{i+1} - \eta]\times\{0\} \subset [\check{u}|\Omega' = \check{\Psi}']$.*

Tracing back now the definition and existence of the $\check{L}^{i,\eta}$ in lemma 4.16, by the previous corollary we arrive at the following final result.

LEMMA 4.20. *For* $\eta > 0$ *and* $0 < \varepsilon \leqslant \varepsilon_0$ *where* $\varepsilon_0 = \varepsilon_0(\eta) > 0$ *there is* $I - \bigcup_{i=1}^{i_0} (x_1^i - \eta, x_1^i + \eta) \times \{0\} \subset [\check{u} | \Omega' = \check{\Psi}']$.

Now we have all the arguments for the

PROOF OF THEOREM 1.3. Thank of lemma 4.20 we have for $\varepsilon > 0$ arbitrary small that $\varepsilon \in (0, \varepsilon_0(\eta)]$ for a $\eta > 0$ such that for $\varepsilon \to 0$ one can choose corresponding $\eta \to 0$ for which the inclusion in lemma 4.20 is valid, thus theorem 1.3 is shown.

5. - Proof of theorem 1.4.

First, we consider the assertion i) of the theorem which means that we study the stability of the solution u of the variational inequality (1.2) under the assumptions (1.5), (1.7) in a neighborhood of a nondegenerated connected component I of $[u | \Omega' = \Psi']$ using the notation of § 3, § 4. Let

(5.1) $I := [a, b] \times \{0\}$ *from* (3.3) *arbitrarily but fixed and* ζ, Ω_* *as in* § 3, *see* (3.6), (3.7).

Further, as in (4.38) we propose for the sequel that

(5.2) $\varepsilon > 0$ *refers to the assumption in theorem* 1.4 *and is supposed to be sufficiently small.*

Writing the corollary of lemma 4.8 and the lemma 4.20 in a different fashion, we have

$$\begin{aligned}
&\textit{for } \eta = \eta(\varepsilon) > 0 \textit{ with } \eta(\varepsilon) \to 0 \ (\varepsilon \to 0) \\
&[\check{\zeta} | \Omega_1' = 0] \subset [\zeta | \Omega_1' = 0] \cup \bigcup_{i=1}^{i_0} (x_1^i - \eta, x_1^i + \eta) \times \{0\} , \\
&[\zeta | \Omega_1' = 0] \subset [\check{\zeta} | \Omega_1' = 0] \cup \bigcup_{i=1}^{i_0} (x_1^i - \eta, x_1^i + \eta) \times \{0\} .
\end{aligned} \tag{5.3}$$

Employing a reflection argument, see lemma 3.1, we then obtain by (5.3) analogue to lemma 4.17

LEMMA 5.1. *For* $q := \min \{|x^{i-1} - x^i| : i = 1, \ldots, i_0 - 1\}/4 > 0$

$$\|\zeta - \check{\zeta}\|_{1,\infty,[x_1^i + q, x_1^{i+1} - q] \times [-\sigma/2, \sigma/2]} \leqslant C_{q,\sigma} \cdot A(\varepsilon) , \qquad A(\varepsilon) \to 0 \qquad (\varepsilon \to 0) ,$$

where $i = 1, \ldots, i_0 - 1$ *and* $C_{q,\sigma} > 0$ *only depends on* $q, \sigma > 0$.

Now again using (5.3) one can show the following estimate for a comparison function $M \cdot G$ of the type which is constructed in § 3 with the G from (3.9) and a suitable chosen $M > 0$ in (3.11) analogue to lemma 3.2 using lemma 4.13, 5.1.

LEMMA 5.2. *With* $M \leqslant A(\varepsilon)$ *where* $A(\varepsilon) \to 0$ $(\varepsilon \to 0)$ *we have*

$$|\zeta - \check{\zeta}| \leqslant M \cdot G(. - x^i - (\eta, 0)) \text{ on } B_{2a}(x^i + (\eta, 0)), \quad i = 1, \dots, i_0 - 1,$$

$$|\zeta - \check{\zeta}| \leqslant M \cdot G(. - x^i + (\eta, 0)) \text{ on } B_{2a}(x^i - (\eta, 0)), \quad i = 2, \dots, i_0.$$

COROLLARY:

$$\sum_{i=1}^{i_0} \|\zeta - \check{\zeta}\|_{\infty, [x_1^i - \eta, x_1^i + \eta] \times \{0\}} \leqslant A(\varepsilon) \cdot \eta^{\frac{1}{2}}.$$

Recalling that $\zeta - \check{\zeta}$ is harmonic on $\Omega_* - \big([\zeta|\Omega_1' = 0] \cup [\check{\zeta}|\Omega_1' = 0]\big)$, we get on account of (5.3) by the maximum principle using lemma 4.13 and the corollary of lemma 5.2 the following improvement of lemma 4.13.

LEMMA 5.3. *For every domain* $\Omega_{**} \subset\subset \Omega_*$ *there is*

$$\|\zeta - \check{\zeta}\|_{\infty, \Omega_{**}} \leqslant A(\varepsilon), \text{ where } A(\varepsilon) \to 0 \ (\varepsilon \to 0).$$

Considering the definition of ζ resp. $\check{\zeta}$ in (2.24), (4.51) then yields using lemma 4.12 for an estimate on a neighborhood of $\partial\Omega_*$

COROLLARY.

$$\|\Delta E * \mu_* - \Delta E * \check{\mu}_*\|_{\infty, \Omega_*} \leqslant A(\varepsilon) \to 0 \quad (\varepsilon \to 0).$$

The previous corollary enables us to establish an uniform estimate for $\partial_{22}(E * \mu_* - E * \check{\mu}_*)$ on Ω_1' by virtue of a property of the corresponding derivative of the biharmonic fundamental solution E in $\mathbb{R}^2$. On account of the expression of $\partial_{22}E$, see the appendix, there is

$$(5.4) \qquad (\partial_{22}E)|\mathbb{R} \times \{0\} \equiv (4 \cdot \pi)^{-1} \cdot (\log|.| - \tfrac{1}{2})|\mathbb{R} \times \{0\}$$

which because of $\operatorname{supp}(\mu_*) \subset \mathbb{R} \times \{0\}$ implies that

$$(5.5) \qquad (\partial_{22}E * \mu_*)|\mathbb{R} \times \{0\} \equiv (\tfrac{1}{2} \cdot \Delta E * \mu_*)|\mathbb{R} \times \{0\} - (8 \cdot \pi)^{-1} \cdot \mu_*(\mathbb{R}^2).$$

So we infer from the corollary of lemma 5.3 and from lemma 4.9 for the traces on Ω_1'

$$\|(\partial_{22}E * \mu_* - \partial_{22}E * \check{\mu}_*)|\Omega_1'\|_{\infty,\Omega_1'} \leqslant A(\varepsilon) \to 0 \qquad (\varepsilon \to 0)\,. \tag{5.6}$$

Further, employing the symmetry of $E * \mu$ with respect to $\mathbf{R}\times\{0\}$, there is

$$(\partial_2 E * \mu_*)|\mathbf{R}\times\{0\} \equiv 0\,, \qquad (\partial_{12}E * \mu_*)|\mathbf{R}\times\{0\} \equiv 0 \tag{5.7}$$

where of course the same is true for the corresponding derivatives of $E * \check{\mu}_*$. So we arrive at

LEMMA 5.4. *For a $A(\cdot)$ with $A(\varepsilon) \to 0$ $(\varepsilon \to 0)$ there is*

$$\|(\partial_2 E * \mu_* - \partial_2 E * \check{\mu}_*)|\Omega_1'\|_{\infty,\Omega_1'} + \|(\partial_2 \nabla E * \mu_* - \partial_2 \nabla E * \check{\mu}_*)|\Omega_1'\|_{\infty,\Omega_1'} \leqslant A(\varepsilon)\,.$$

The previous lemma is crucial in order to obtain uniform bounds for $\partial_2\nabla E * \mu_* - \partial_2 \nabla E * \check{\mu}_*$ on Ω_*. But first we need an estimate on $\Omega_* - U(I)$. Let $\lambda := \operatorname{dist}(I, \partial\Omega_*)/3 > 0$ and choose a suitable cut-off function $\tau \in C_0^\infty \cdot (U_{2\lambda}(I))$ such that $\tau \equiv 1$ on $U_\lambda(I)$ where $U_\lambda(I)$ resp. $U_{2\lambda}(I)$ is the λ- resp. $2\cdot\lambda - \mathbf{R}^2$-neighborhood of $I = [\zeta|\Omega_1' = 0] = [u|\Omega' = \Psi'] \cap \Omega_*$. Then

$$\tau\cdot\partial_i\,(x - .)\,, \quad \tau\cdot\partial_{ij}E(x - .) \in C_0^\infty(\Omega_*)\,, \qquad i, j = 1, 2\,,$$

$$\textit{where } \tau\cdot\partial_i E(x - .) \equiv \partial_i E(x - .)\,, \qquad \tau\cdot\partial_{ij}E(x - .) \equiv \partial_{ij}E(x - .)$$

$$\textit{on } U_\lambda(I),\ \textit{for every fixed } x \in \Omega_* - U_{2\lambda}(I)\,. \tag{5.8}$$

Remembering how to get lemma 4.10, by integration by parts one shows on account of (5.8) using lemma 4.8

LEMMA 5.5:

$$\|E * \mu_* - E * \check{\mu}_*\|_{2,\infty,\Omega_* - U_{2\lambda}(I)} \leqslant A(\varepsilon) \to 0 \qquad (\varepsilon \to 0)\,.$$

Now we are in the position to extend the estimate (5.6) totally on Ω_*.

LEMMA 5.6. *For a $A(.)$ with $A(\varepsilon) \to 0$ $(\varepsilon \to 0)$ there is*

$$\|\partial_{22}E * \mu_2 - \partial_{22}E * \check{\mu}_*\|_{\infty,\Omega_*} + \|\partial_{12}E * \mu_* - \partial_{12}E * \check{\mu}_*\|_{\infty,\Omega_*} \leqslant A(\varepsilon)\,.$$

PROOF. Our estimate relies on lemma 5.4, 5.5 by the following maximum estimate for biharmonic functions in $\mathbf{R}^2$, see for example the book [8] for our stated version.

(5.9) *Let $\hat{\Omega} \subset \mathbf{R}^2$ be a bounded domain such that $\partial\hat{\Omega}$ is a closed C^2-Jordan curve. Then there exists a $c_{\hat{\Omega}} > 0$ such that if $\varphi \in C^4(\hat{\Omega}) \cap C^1(\bar{\hat{\Omega}})$ satisfies the biharmonic equation $\Delta^2\varphi = 0$ on $\hat{\Omega}$, then there is $\|\nabla\varphi\|_{\infty,\hat{\Omega}} \leqslant c_{\hat{\Omega}} \cdot (\|\varphi|\partial\hat{\Omega}\|_{1,\infty,\partial\hat{\Omega}} + \|\partial_{\mathcal{N},\hat{\Omega}}\varphi\|_{\infty,\partial\hat{\Omega}})$.*

Here, $\partial_{\mathcal{N},\hat{\Omega}}$ denotes the normal derivative relative to $\hat{\Omega}$. Choosing now suitable domains with C^2-boundaries which are contained in $\Omega_* \cap \mathbf{R}\times\mathbf{R}_+ = \Omega_1' \times (0, \sigma)$ resp. $\Omega_* \cap \mathbf{R}\times\mathbf{R}_- = \Omega_1' \times (-\sigma, 0)$ in such a way that the boundary in $U_{2\lambda}(I)$ coincides with $\Omega_1' \cap U_{2\lambda}(I)$, we obtain the assertion of lemma 5.6 by (5.9) using lemma 5.4, 5.5.

By the identity $\partial_{11} = \Delta - \partial_{22}$ and the corollary of lemma 5.3 we even have

$$\|\nabla^2 E * \mu_* - \nabla^2 E * \check{\mu}_*\|_{\infty,\Omega_*} \leqslant A(\varepsilon) \to 0 \quad (\varepsilon \to 0)\,, \tag{5.10}$$

and representing the first derivatives as line integrals of the second order derivatives also using lemma 5.4,

COROLLARY:

$$\|E * \mu_* - E * \check{\mu}_*\|_{2,\infty,\Omega_*} \leqslant A(\varepsilon) \to 0 \quad (\varepsilon \to 0)\,.$$

Recalling that $\Omega_* \subset U_\delta(\Omega_1')$ due to our assumptions, by virtue of the previous corollary and (4.48) we arrive at our final estimate.

LEMMA 5.7. $\|u - \check{u}\|_{2,\infty,\Omega_*} \leqslant A(\varepsilon)$ *where* $A(\varepsilon) \to 0$ $(\varepsilon \to 0)$.

Further, we get in addition the following L^2-bound on Ω_* for the differences of the third derivatives.

LEMMA 5.8. $\|\nabla^3 u - \nabla^3 \check{u}\|_{2,\Omega_*} \leqslant A(\varepsilon)$ *where* $A(\varepsilon) \to 0$ $(\varepsilon \to 0)$.

PROOF. Setting $v := \Delta u - \Delta\check{u}$ we have in the distributional sense that $\Delta(\Delta u - \Delta\check{u}) = \mu - \check{\mu} = \mu_* - \check{\mu}_*$ on a neighborhood $U(\Omega_*)$ of Ω_*. Using mollification we obtain by Green's theorem that

$$\int_{\Omega_*} |\nabla v|^2\, dx = \int v\, d(\mu_* - \check{\mu}_*) - \int_{\partial\Omega_*} v \cdot \partial_{\mathcal{N},\Omega_*} w\, ds \tag{5.11}$$

where $\partial_{\mathcal{N},\Omega_*}$ denotes the normal derivative relative to Ω_* and ds the arc length of $\partial\Omega_*$. By lemma 5.7, 4.9 and the conclusions leading to lemma 4.9

we have

$$\int v\,d(\mu_* - \breve{\mu}_*) \leqslant \|v\|_{\infty,\Omega_*}\cdot(\mu(\mathbf{R}^2) + \breve{\mu}(\mathbf{R}^2)) \tag{5.12}$$
$$\leqslant \|v\|_{\infty,\Omega_*}\cdot(2\cdot\mu(\mathbf{R}^2) + 1)$$
$$\leqslant \|v\|_{\infty,\Omega_*}\cdot C\cdot(\|u\|_{2,2} + 1)$$
$$\leqslant A(\varepsilon) \to 0 \qquad (\varepsilon \to 0)\,.$$

As $\Delta^2(u - \breve{u}) = 0$ on $U(\partial\Omega_*)$ suitable small, we get by an interior estimate of biharmonic functions using lemma 4.8

$$\|\partial_{\mathcal{N},\Omega_*} v\|_{\infty,\partial\Omega_*} \leqslant A(\varepsilon) \to 0 \qquad (\varepsilon \to 0)\,, \tag{5.13}$$

so that further by lemma 5.7

$$\int_{\partial\Omega_*} v\cdot\partial_{\mathcal{N},\Omega_*} v\,ds \leqslant A(\varepsilon) \to 0 \qquad (\varepsilon \to 0)\,. \tag{5.14}$$

Putting (5.12), (5.13) into (5.11) we have

$$\|\nabla\Delta u - \nabla\Delta\breve{u}\|_{2,\Omega_*} \leqslant A(\varepsilon) \to 0 \qquad (\varepsilon \to 0)\,, \tag{5.15}$$

and the assertion of lemma 5.8 immediately follows by Gårdings inequality using lemma 5.7.

Proof of theorem 1.4 *i*). Let U be an open $\mathbf{R}^2$-neighborhood of I with $U \cap \big([u|\Omega' = \Psi'] - I\big) = \emptyset$. By the corollary of lemma 4.8 we then have for $\varepsilon > 0$ sufficiently small that $[\breve{u}|\Omega' = \breve{\Psi}'] \subset U_\eta\big([u|\Omega' = \Psi']\big)$ with $\eta = \eta(\varepsilon) > 0$, $\eta \to 0$ $(\varepsilon \to 0)$, denoting by $U_\eta\big([u|\Omega' = \Psi']\big)$ the $\eta - \mathbf{R}^2$-neighborhood of $[u|\Omega' = \Psi']$. Now one obtains by an interior estimate for biharmonic functions and lemma 4.8

$$\|u - \breve{u}\|_{3,\infty,U-\Omega_*} \leqslant A(\varepsilon) \to 0 \qquad (\varepsilon \to 0)\,. \tag{5.16}$$

Putting (5.16), lemma 5.7, 5.8 together, we finally obtain the assertion i).

Proof of theorem 1.4 ii). Now let $\{x^0\}$ be a connected component of $[u|\Omega' = \Psi']$ of at least third order contact what means

$$d^2(u|\Omega' - \Psi')(x_1^0) = 0 \qquad \textit{where } x_0 = (x_1^0, 0)\,. \tag{5.17}$$

Choosing Ω_1' in such a way that $\Omega_1' \subset\subset \Omega$ and

$$[u|\Omega_1' - \Psi'|\Omega_1' = 0] = \{x_1^0\}, \tag{5.18}$$

we proceed as in the previous case by constructing ζ on a $\Omega_* = \Omega_1' \times (-\sigma, \sigma)$ as in § 2, § 3. Noting that now $\mu_* \equiv \mu|\Omega_* \equiv 0$ on account of (5.18), (3.2) and the inclusion in (1.4), we have $\zeta \equiv w$ by (2.24) which is harmonic on Ω_*. So all the lemmata which do not rest on the requirement of $[\zeta|\Omega_1' = 0]$ being a nondegenerated line segment are still valid. All we have to do is to replace lemma 5.3 by the following. Recalling that $\zeta - \check{\zeta}$ is harmonic on $\Omega_* - [\check{u}|\Omega' = \check{\Psi}']$, by the maximum principle we obtain using lemma 4.13

$$\|\zeta - \check{\zeta}\|_{\infty,\Omega_*} \leqslant A(\varepsilon) + \|\zeta\|_{\infty,[\check{u}|\Omega' = \check{\Psi}'] \cap \Omega_1'} = A'(\varepsilon) \tag{5.19}$$

$$\textit{where } A(\varepsilon)\,,\quad A'(\varepsilon) \to 0 \quad (\varepsilon \to 0)\,.$$

By virtue of (5.19) we can proceed as before showing appropriate versions of lemma 5.4, 5.5, 5.6, 5.7, 5.8 and complete the proof.

6. – Appendix: The formula of the fundamental solution of Δ^2 in $\mathbf{R}^2$.

Let E be the fundamental solution of the biharmonic operator Δ^2 in $\mathbf{R}^2$ what means $\Delta^2 E = \delta$ in the distributional sense where δ denotes the Dirac measure. We give the detailed formula for E and its most important derivatives. Note, that ΔE is the fundamental solution of the Laplacian Δ and its partial derivative $\partial_2 \Delta E$ is the Poisson kernel $\mathcal{T}$. associated with the upper half plane $\mathbf{R} \times \mathbf{R}_+$.

So for $x = (x_1, x_2) \in \mathbf{R}^2$, $x \neq 0$, there is

$$\begin{aligned}
E(x) &= (8\cdot\pi)^{-1}\cdot|x|^2\cdot(\log|x| - 1)\,,\\
\partial_i E(x) &= (4\cdot\pi)^{-1}\cdot(x_i\cdot\log|x| - \tfrac{1}{2}\cdot x_i)\,, \qquad i = 1, 2\,,\\
\partial_{ij} E(x) &= (4\cdot\pi)^{-1}\cdot x_i\cdot x_j/|x|^2\,, \qquad i \neq j\,,\quad i, j = 1, 2\,,\\
\partial_{ii} E(x) &= (4\cdot\pi)^{-1}\cdot(\log|x| + x_i^2/|x|^2 - \tfrac{1}{2})\,, \qquad i = 1, 2\,,\\
&= \tfrac{1}{2}\cdot\Delta E(x) + (4\cdot\pi)^{-1}\cdot(x_i^2/|x|^2 - \tfrac{1}{2})\,, \qquad i = 1, 2\,,\\
\Delta E(x) &= (2\cdot\pi)^{-1}\cdot\log|x|\,,\\
\partial_2 \Delta E(x) &= (2\cdot\pi)^{-1}\cdot x_2/(x_1^2 + x_2^2) = \mathcal{T}_{x_2}(x_1)\,, \qquad x_2 > 0\,.
\end{aligned}$$

Added in proof.

i) In the proof of lemma 2.8, 4.1 we use the following fact. *Let $x^0 \in \partial U_\varkappa^+$ and ζ harmonic on $B_{\varrho_0}(x^0)$ for a $\varrho_0 > 0$, then the intersection $U_\varkappa^+ \cap B_\varrho(x^0)$,*

$\varrho \in (0, \varrho_0)$, *posseses only a finite number of connected components.* Indeed, let us assume the contrary which means that for a $\varrho \in (0, \varrho_0)$ we have the decomposition of $U_\varkappa^+ \cap B_\varrho(x^0) = \bigcup_{m=1}^{\infty} W_m$ into the mutual disjoint connected components. By the maximum principle, $(\overline{W}_m - [\zeta = 0]) \cap \partial B_\varrho(x^0) \neq \emptyset$, $m = 1, \ldots, \infty$. Then also $[\zeta|\partial B_\varrho(x^0) > 0]$ posseses an infinite number of mutual disjoint connected components which by the identity principle for real analytic functions in one real variable leads to $\zeta|\partial B_\varrho(x^0) \equiv 0$ and moreover to $\zeta|B_\varrho(x^0) \equiv 0$ by the maximum principle. But this is a contradiction to $x^0 \in \partial U_\varkappa^+$, thus the assertion is true. In the situation of lemma 4.1 the last conclusions had to be modified performing a reflection to continue $\zeta|B_\varrho(x_0)$ analytically at the points of $[\zeta = 0] \cap \Omega'$. Again, one derives a contradiction.

ii) Using similar ideas but more elaborated techniques, it is possible to extend all our results to the according two-sided problem where in (1.2) $\mathbf{K}_{\Psi'}$ is replaced by $\mathbf{K}_{\Psi',T'} := \{v \in g + \overset{\circ}{H}{}^{2,2}(\Omega) : \Psi' \leqslant v \leqslant T' \textit{ on } \Omega'\}$. We assume that $\partial\Omega \in C^{0,1}$, Ψ', $T' \in C^0(\overline{\Omega}') \cap C^2(\Omega')$ are piecewise analytic on Ω' and $\Psi' < g < T'$ on $\partial\Omega'$, $f \in L^p(\Omega) \cap C^\omega(\Omega)$. Obviously, $[\Psi' = T'] \neq \emptyset$ is allowed. Then

THEOREM. *Let u denote the solution of* (1.2) *associated to* $\mathbf{K}_{\Psi',T'} \neq \emptyset$. *Then* i) $[u = \Psi'] \cup [u = T'] = \bigcup_{j=1}^{j^0} [a^j, b^j] \times \{0\}$, $j^0 \in \mathbf{N}$. ii) $u \in C^{2,\frac{1}{2}}_{\mathrm{loc}}(\Omega) \cap H^{3,2}_{\mathrm{loc}}(\Omega)$. iii) *If in addition Ψ', T' are piecewise analytic with $C^\omega(\Omega')$-pieces, $\Delta^2 u - f$ is a signed measure on Ω.*

Now we consider an extension of the stability results in § 4, § 5 to two-sided problems. Since globally analytic obstacles Ψ', T' $C^\omega(\Omega')$ are to limited to work with because $[\Psi' = T']$ is only a set of single points, we are interested in results for piecewise analytic obstacles. But then it is not possible to study the case when the perturbations of lower and upper obstacle are independent of each other. Fortunately, if we fix the obstacles Ψ', T' and study only perturbations of the boundary value g in the $H^{2,2}(\Omega)$-norm, under the assumption of theorem iii) then the assertion of theorem 1.3, 1.4 is valid.

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