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On the Local Behaviour of Solutions of Degenerate Parabolic Equations with Measurable Coefficients (*).

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0. – Introduction.

We will prove interior and boundary Hölder continuity for weak solutions of degenerate parabolic equations with principal part in divergence form, of the type

$$u_t - \operatorname{div} \boldsymbol{a}(x, t, u, {}_x u) + b(x, t, u, \nabla_x u) = 0 \qquad \text{in } \mathcal{D}'(\Omega_T) \tag{0.1}$$

where Ω is a region in $\mathbb{R}^N$, $\Omega_T \equiv \Omega \times (0, T)$, $0 < T < \infty$, and ∇_x denotes the gradient with respect only to the space variables $x \equiv (x_1, x_2, \ldots, x_N)$.

The functions $\boldsymbol{a}\colon \mathbb{R}^{2N+2} \to \mathbb{R}^N$ and $b\colon \mathbb{R}^{2N+2} \to \mathbb{R}$, are only assumed to be measurable and satisfying the structure conditions

$$[\mathrm{A}_1] \qquad \boldsymbol{a}(x, t, u, \nabla_x u) \cdot \nabla_x u \geqslant C_0 |\nabla_x u|^p - \varphi_0(x, t)\,, \qquad p \geqslant 2$$

$$[\mathrm{A}_2] \qquad |a_i(x, t, u, \nabla_x u)| \leqslant C_1 |\nabla_x u|^{p-1} + \varphi_1(x, t)\,, \qquad i = 1, 2, \ldots, N\,,$$

$$[\mathrm{A}_3] \qquad |b(x, t, u, \nabla_x u)| \leqslant C_2 |\nabla_x u|^p + \varphi_2(x, t)\,,$$

where C_i, $i = 0, 1, 2$ denote given positive constants and φ_i, $i = 0, 1, 2$ are given non-negative functions defined on Ω_T and subject to the conditions (1)

$$[\mathrm{A}_4] \qquad \varphi_0, \quad \varphi_1^{p'}, \quad \varphi_2 \in L_{\hat{q}, \hat{r}}(\Omega_T)$$

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(1) Throughout the paper the notation of [11] is employed.

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where $1/p + 1/p' = 1$ and $\hat{q}, \hat{r} \geqslant 1$ and satisfy

$$(0.2) \qquad \frac{1}{\hat{r}} + \frac{N}{p\hat{q}} = 1 - \varkappa_1 ,$$

and

$$(0.2)\,(\mathrm{i}) \qquad \hat{q} \in [1, \infty]; \qquad \hat{r} \in \left[\frac{1}{1-\varkappa_1}, \frac{p}{p(1-\varkappa_1)-1}\right]; \qquad \varkappa_1 \in \left(0, \frac{p-1}{p}\right) \quad \text{if } N = 1 ,$$

$$(0.2)\,(\mathrm{ii}) \qquad \hat{q} \in \left[\frac{N}{p(1-\varkappa_1)}, \infty\right]; \qquad \hat{r} \in \left[\frac{1}{1-\varkappa_1}, \infty\right]; \qquad \varkappa_1 \in (0, 1) \quad \text{if } N > 1,\ p \leqslant N,$$

$$(0.2)\,(\mathrm{iii}) \qquad \hat{q} \in \left[\frac{N}{p(1-\varkappa_1)}, \infty\right]; \qquad \hat{r} \in \left[\frac{1}{1-\varkappa_1}, \infty\right]; \qquad \varkappa_1 \in \left(\frac{p-N}{p}, 1\right) \quad \text{if } 1 < N < p.$$

Given the structure conditions [A_1]-[A_3], the degeneracy of (1.1) is of the same nature of

$$(0.3) \qquad u_t = \operatorname{div}\left(|\nabla_x u|^{p-2} \nabla u\right) \qquad \text{in } \mathcal{D}'(\Omega_T), \qquad p \geqslant 2 .$$

When $p = 2$, major developments, in the theory of local regularity of (0.1) have been brought about the discovery of the Harnack inequality of Moser [14, 15], for linear elliptic and parabolic equations with bounded and measurable coefficients. The Harnack inequality can be used to imply the local Hölder continuity of the solutions. The latter regularity statement had been proved previously by De Giorgi [3] in the elliptic case and Ladyzenskaja-Uralt'zeva [11] in the parabolic case.

In the case of an elliptic equation, the extension of these results from $p = 2$ to any $p > 1$ is quite direct and the theory can now be considered fairly complete [16, 17, 19].

The parabolic case is complicated by the dissymmetry of the space and time parts of the operator in (0.1), and at our knowledge no regularity results are available if p differs from 2. In particular, non-negative weak solutions of (0.1) do not in general satisfy the Harnack inequality. To see this we consider the following explicit solution of (0.3), constructed in [1].

$$(0.4) \qquad u(x,t) = \begin{cases} \dfrac{1}{R(t)^N}\left\{1 - \left(\dfrac{|x|}{R(t)}\right)^{p/(p-1)}\right\}^{(p-1)/(p-2)}, & |x| < R(t) \\ 0 , & |x| \geqslant R(t) \end{cases}$$

$$(0.5) \qquad R(t) = \left([N(p-2) + p]\frac{p}{p-2}\, t\right)^{1/(N(p-2)+p)}, \qquad t > 0 .$$

This solution exhibits a behaviour similar to the solutions of the porous medium equation; that is, it is of compact support in the space variables for all $t > 0$. Clearly for a cylinder Q intersecting the free boundary $|x| = R(t)$, the Harnack inequality fails to hold (see also Remark B section 7 of [4] p. 116). Nevertheless the solution u is $C^{1+\alpha}(\mathbf{R}^N \times [\varepsilon, T])$, $\forall 0 < \varepsilon < T < \infty$.

By a weak solution of (0.1) in Ω_T, we mean a function $u \in V_{2,p}(\Omega_T) \equiv \equiv C(0, T; L^2(\Omega)) \cap L^p(0, T; H^p(\Omega))$, satisfying

$$\int_\Omega u(x, t)\varphi(x, \tau)\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \cdot \{-u\varphi_t + a(x, t, u, \nabla_x u)\cdot\nabla_x\varphi + b(x, t, u, \nabla_x u)\varphi\}\,dx\,d\tau = 0 \tag{0.6}$$

for all $\varphi \in \mathring{W}_p^{1,0}(\Omega_T)$ such that $\varphi_t \in L_2(\Omega_T)$, and for all t_1, t_2, $0 < t_1 < t_2 \leqslant T$.

We assume throughout that

$$[\mathrm{A}_5] \qquad u \in L^\infty(\Omega_T)\,.$$

REMARK 0.1. If $[\mathrm{A}_3]$ is replaced by the more restrictive condition

$$[\mathrm{A}_3]' \qquad |b(x, t, u, \nabla_x u)| \leqslant C_2|\nabla_x u|^{p-1} + \varphi_2(x, t)\,,$$

then a local L^∞ bound for u can be calculated by a simple modification of De Giorgi-Moser techniques (see for example [11] page 102-109). The proof gives an explicit but complicated (due to the mentioned dissimetry) bound of $\|u\|_{\infty,Q}$ over a cylinder Q in terms of the norm $\|u\|_{p,Q'}$ over a larger cyolinder Q'. We have chosen to omit such calculation since they result from a variant of known techniques.

With $\partial\Omega$ we denote the boundary of Ω and set

$$S_T \equiv \partial\Omega \times (0, T]\,; \qquad \Gamma \equiv S_T \cup \bar{\Omega} \times \{0\}\,. \tag{0.7}$$

Clearly Γ is the parabolic boundary of Ω_T.

The statement that a constant γ depends only upon the data, means that γ can be calculated only in dependence of the various constants appearing in $[\mathrm{A}_1]$-$[\mathrm{A}_4]$, $\|u\|_{\infty,\Omega_T}$ and the dimension N. We can now state our main results.

I. – Interior regularity.

THEOREM 1. *Let $u \in V_{2,p}(\Omega_T) \cap L^\infty(\Omega_T)$ be a weak solution of* (0.1), *and let* $[\mathrm{A}_1]$-$[\mathrm{A}_4]$ *hold. Then $(x, t) \to u(x, t)$ is locally Hölder continuous in Ω_T*

and for every compact set $\mathcal{K} \subset \Omega_T$, *there exists a constant* γ *depending only upon the data and* $\operatorname{dist}(\mathcal{K}, \Gamma)$, *and a constant* $\alpha \in (0, 1)$ *depending only upon the data, such that*

$$|u(x_1, t_1) - u(x_2, t_2)| \leqslant \gamma\big(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/p}\big) ,$$

for every pair of points (x_1, t_1), $(x_2, t_2) \in \mathcal{K}$.

REMARK 0.2. Since our arguments are local in nature to prove Theorem 1, we do not need to have a solution in the whole Ω_T. It is sufficient to have a «local» solution; *i.e.* $u \in V^{\text{loc}}_{2,p}(\Omega_T) \cap L^\infty_{\text{loc}}(\Omega_T)$, satisfying (0.6). Also we may assume φ_0, $\varphi_1^{p'}$, $\varphi_2 \in L^{\text{loc}}_{\hat{q},\hat{r}}(\Omega_T)$.

II. – Boundary regularity.

II-(a) *Regularity at* $t = 0$.

We assume that (0.1) is associated with initial data

$$u(x, 0) = u_0(x) , \tag{0.8}$$

and on u_0 assume

[A$_6$] $\quad x \to u_0(x)$ is continuous in $\bar{\Omega}$ with modulus of continuity $\omega_0(\cdot)$.

Since we assume that $u \in C\big(0, T; L^2(\Omega)\big)$, the initial datum (0.8) is taken in the sense of (0.6) where $t_1 \geqslant 0$.

THEOREM 2. *Let* $u \in C\big(0, T; L^2(\Omega)\big) \cap L^p\big(0, T; H^p(\Omega)\big)$ *be a weak solution of* (0.1) *which takes on initial data* (0.8) *and let* [A$_1$]-[A$_6$] *hold. Then* $(x, t) \to u(x, t)$ *is continuous in* $\mathring{\Omega} \times [0, T]$, *and for every compact set* $\mathcal{K} \subset \Omega$ *there exist a function* $\varrho \to \omega(\varrho)\colon \mathbb{R}^+ \to \mathbb{R}^+$ *continuous and non-decreasing such that*

$$|u(x_1, t_1) - u(x_2, t_2)| \leqslant \omega\big(|x_1 - x_2| + |t_1 - t_2|^{1/p}\big)$$

for every pair of points (x_1, t_1), $(x_2, t_2) \in \mathcal{K} \times [0, T]$. *The function* $\omega(\cdot)$ *can be determined in terms of the data and* $\omega_0(\cdot)$.

If in particular

$$\omega_0(\varrho) = \varrho^{\bar{\sigma}} , \qquad \bar{\sigma} \in (0, 1) ,$$

then $(x, t) \to u(x, t)$ *is Hölder continuous in* $\mathring{\Omega} \times [0, T]$, *and for every compact set* $\mathcal{K} \subset \Omega$ *there exist a constant* γ *depending only upon the data and* $\operatorname{dist}(\mathcal{K}, \partial\Omega)$, *and a constant* $\sigma \in (0,1)$ *depending upon the data and* $\bar{\sigma}$ *such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \leqslant \gamma\big(|x_1 - x_2|^{\sigma} + |t_1 - t_2|^{\sigma/p}\big)$$

for every pair of points (x_1,t_1), $(x_2,t_2) \in \mathcal{K} \times [0,T]$.

REMARK 0.3. If $x \to u_0(x)$ is only known to be continuous in a open subset Ω' of Ω then the stated regularity can only be claimed in the set $\Omega' \times [0,T]$.

II-(*b*) *Regularity at* S_T *(Dirichlet data).*

The boundary $\partial\Omega$ is assumed to satisfy

[A_7] $\exists \alpha^* \in (0,1)$, $R_0 > 0$ such that $\forall x_0 \in \partial\Omega$ and every ball $B(x_0,R)$ centered at x_0, with radius $R < R_0$,

$$\operatorname{meas}[\Omega \cap B(x_0,R)] \leqslant (1-\alpha^*) \operatorname{meas} B(x_0,R) .$$

We suppose that (0.1) is associated with Dirichlet data $f(x,t)$ on S_T (taken in the sense of the traces) satisfying

[A_8] $(x,t) \to f(x,t)$ is continuous on $\bar{S}_T$ with modulus of continuity $\omega_f(\cdot)$

THEOREM 3. *Let* $u \in V_{2,p}(\Omega_T) \cap L^\infty(\Omega_T)$ *be a weak solution of* (0.1) *associated with Dirichlet data* f *on* S_T, *and assume that* [A_1]-[A_4] *and* [A_7]-[A_8] *hold. Then* $(x,t) \to u(x,t)$ *is continuous in* $\bar{\Omega} \times (0,T]$ *and* $\forall \varepsilon > 0$ *there exist a positive non-decreasing continuous function* $\varrho \to \omega_\varepsilon(\varrho)\colon \mathbf{R}^+ \to \mathbf{R}^+$ *such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \leqslant \omega_\varepsilon\big(|x_1 - x_2| + |t_1 - t_2|^{1/p}\big)$$

for every pair of points (x_1,t_2), $(x_2,t_2) \in \bar{\Omega} \times [\varepsilon,T]$. *If in particular*

$$\omega_f(\varrho) = \varrho^{\bar{\beta}} , \qquad \bar{\beta} \in (0,1) ,$$

then $(x,t) \to u(x,t)$ *is Hölder continuous in* $\bar{\Omega} \times [\varepsilon,T]$ $\forall \varepsilon > 0$, *and there exist a constant* γ *depending only upon the data and* ε, *and a constant* $\beta \in (0,1)$ *depending upon the data and* $\bar{\beta}$, *such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \leqslant \gamma\big(|x_1 - x_2|^{\beta} + |t_1 - t_2|^{\beta/p}\big)$$

for every pair of points (x_1,t_1), $(x_2,t_2) \in \bar{\Omega} \times [\varepsilon,T]$.

REMARK 0.4. If the Dirichlet data f is only known to be continuous in a open subset S' of S (open in the relative topology of S_T) then the stated regularity can only be claimed up to S'.

COROLLARY 0.1. *Consider the boundary value problem*

$$u_t - \operatorname{div} \boldsymbol{a}(x, t, u, \nabla_x u) + b(x, t, u, \nabla_x u) = 0 \quad \text{in } \Omega_T \tag{0.9}$$

$$u(x, t) = f(x, t)\,, \qquad (x, t) \in S_T \tag{0.10}$$

$$u(x, 0) = u_0(x)\,, \qquad x \in \Omega\,, \tag{0.11}$$

where $x \to u_0(x)$ *satisfies* $[A_6]$ *and* $(x, t) \to f(x, t)$ *satisfies* $[A_8]$ *and assume that* $[A_7]$ *holds. Every bounded weak solution of* (0.9)-(0.11) (*in the sense of identity* (0.6)) *is continuous in* $\bar{\Omega}_t$: *In particular if* u_0 *is Hölder continuous in* $\bar{\Omega}$ *and* f *is Hölder continuous on* $\bar{S}_T$, *then* u *is Hölder continuous in* $\bar{\Omega}_T$.

II-(c) *Regularity at* S_T (*Variational data*).

We assume here that

$[A_9]$ $\qquad \partial\Omega \quad$ is a C^1 manifold in $\mathbf{R}^{N-1}$,

and consider formally the problem

$$u_t - \operatorname{div} \boldsymbol{a}(x, t, u, \nabla_x u) + b(x, t, u, \nabla_x u) = 0 \quad \text{in } \Omega_T \tag{0.12}$$

$$\boldsymbol{a}(x, t, u, \nabla_x u) \cdot \boldsymbol{n}_{S_T}(x, t) = g(x, t, u)\,, \qquad \text{on } S_T \tag{0.13}$$

$$u(x, 0) = u_0(x)\,, \qquad x \in \Omega\,, \tag{0.14}$$

where $n_{S_T} = (n_{x_1}, n_{x_2}, \ldots, n_{x_N})$ denotes the outer unit normal to S_T. On the function $g(x, t, u)$ we assume

$[A_{10}]$ $\quad g$ is continuous over $S_T \times \mathbf{R}$ and

$$|g(x, t, u)| \leq C_3 \tag{0.15}$$

for a given non-negative constant C_3.

By a weak solution of (0.12)-(0.14) we mean a function $u \in V_{2,p}(\Omega_T)$ satisfying

$$\int_\Omega u\varphi\, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\!\!\int_\Omega \{-u\varphi_t + \boldsymbol{a}(x, t, u, \nabla_x u) \cdot \nabla_x \varphi + b(x, t, u, \nabla_x u)\varphi\}\, dx\, d\tau = \int_{t_1}^{t_2}\!\!\int_{\partial\Omega} g(x, \tau, u)\varphi\, d\sigma\, d\tau\,, \tag{0.16}$$

where $d\sigma$ denotes the H^{N-1}-measure on $\partial\Omega$, for all $\varphi \in W_p^{1,0}(\Omega_T)$ such that $\varphi_t \in L_2(\Omega_T)$, and for all t_1, t_2 satisfying $0 \leqslant t_1 < t_2 \leqslant T$.

THEOREM 4. *Let $u \in V_{2,p}(\Omega_T) \cap L^\infty(\Omega_T)$ be a weak solution of (0.12)-(0.14) in the sense of identity (0.16). Then $(x, t) \to u(x, t)$ is Hölder continuous in $\bar{\Omega}\times[\varepsilon, T]$ for all $\varepsilon > 0$, and there exist a constant γ_ε depending only upon the data and ε, and a constant $\lambda \in (0, 1)$ depending only upon the data, such that*

$$|u(x_1, t_1) - u(x_2, t_2)| \leqslant \gamma_\varepsilon\big(|x_1 - x_2|^\lambda + |t_1 - t_2|^{\lambda/p}\big)$$

for every pair of points (x_1, t_1), $(x_2, t_2) \in \bar{\Omega}\times[\varepsilon, T]$.

If in addition $x \to u_0(x)$ is Hölder continuous in $\bar{\Omega}$, then u is Hölder continuous in $\bar{\Omega}_T$ and the constant γ_ε can be taken independent of ε, whereas the Hölder exponent λ will depend also upon the Hölder exponent of u_0.

REMARK 0.5. When $p = 2$ the integrability conditions in $[A_3]$ coincide with the requirements imposed in [11], and these are known to be the best possible [10].

REMARK 0.6. If the functions $\boldsymbol{a}(x, t, u, \nabla_x u)$ and $b(x, t, u, \nabla_x u)$ are differentiable and satisfy further restrictions then one can prove that $(x, t) \to \nabla_x u$ is Hölder continuous in $\mathring{\Omega}_T$; in fact such a result holds also for systems (see [6, 7]). The point here is of course to prove the stated regularity only under the hypothesis that $\boldsymbol{a}$ and b are measurable. An extension of our results to systems, due to the generality we consider, is not expected. It is in fact false even in the elliptic case (see [8] for a survey).

REMARK 0.7. The proof presented here shows that the various Hölder constants and exponents in Theorems 1-4 are continuous functions of p. As $p \to \infty$ these estimates deteriorate, but they are «stable» as $p \to 2$.

REMARK 0.8. One of the applications of the a priori knowledge of a modulus of continuity of solutions of (0.1) is the derivation of L^∞_{loc} bounds for $|\nabla_x u|$, (see [20]).

REMARK 0.9. Existence theory for boundary value problems associated with (0.1) is based on Galerkin approximations and it is developed in [11].

REMARK 0.10. It should be noted that we have been unable to deal with the case $1 < p < 2$.

Heuristically the results will follow from the following fact. The function $(x, t) \to u(x, t)$ can be modified in a set of measure zero to yield a continuous representative out of the equivalence class $u \in V_{2,p}(\Omega_T)$ if for every

$(x_0, t_0) \in \Omega_T$ there exist a family of nested and shrinking cylinders $Q_n(x_0, t_0)$ around (x_0, t_0), such that the essential oscillation ω_n of u in $Q_n(x_0, t_0)$ tends to zero as $n \to \infty$, in a way determined by the operator in (0.1) and the data.

The key idea of the proof is to work with cylinders whose dimensions are suitably rescaled to reflect the degeneracy exhibited by the equation. This idea has been introduced in [5] and further developed in [7].

In the present situation the arguments are more complicated with respect to the ones in [5]. This is due to the fact that, unlike the solutions of porous media type equations (see [4, 7]) where the singularity occurs at only one value of the solution (say for example for $u = 0$), in our case the equation may be degenerate at any value of u.

To render the paper as self contained as possible, certain known calculations have been reproduced.

In part I we prove the interior regularity. We introduce certain classes $\mathcal{B}_p(\Omega_T, M, \gamma, r, \delta, \varkappa)$, along the lines of a similar approach of [11], and prove that local weak solutions of (0.1) belong to them.

Then we show that $\mathcal{B}_p(\Omega_T, M, \gamma, r, \delta, \varkappa)$ is embedded in $C^{\alpha,\alpha/p}_{\mathrm{loc}}(\Omega_T)$, thereby proving Theorem 1.

We prove the boundary regularity by following a similar pattern in part II. The methods of this part will rely heavily on those of part I and in fact we will limit ourselves to describe the modifications of the proof of interior regularity to achieve regularity up to the boundary.

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1. – The classes $\mathcal{B}_p(\Omega_T, M, \gamma, r, \delta, \varkappa)$, $p \geq 2$.

Let Ω be an open set in $\mathbf{R}^N$ and for $0 < T < \infty$ let $\Omega_T \equiv \Omega \times (0, T]$. If $(x_0, t_0) \in \Omega_T$ we let $B(R) \equiv \{x \in \Omega : |x - x_0| < R\}$ and

$$Q(R, \varrho) \equiv B(R) \times \{t_0 - \varrho, t_0\}, \qquad \varrho > 0 .$$

We let R, ϱ be so small that $Q(R, \varrho) \subset \Omega_T$. Denote by $(x, t) \to \zeta(x, t)$ a piecewise smooth function defined in $Q(R, \varrho)$, such that $0 \leq \zeta \leq 1$ and $\zeta(x, \cdot) = 0$ for $x \in \partial B(R)$.

For a bounded measurable function u defined in $Q(R, \varrho)$ introduce the cut functions $(u-k)^{\pm}$, $k \in \mathbb{R}$ and let $H^{\pm}$ be any number satisfying

$$\|(u-k)^{\pm}\|_{\infty, Q(R,\varrho)} \leqslant H^{\pm} \leqslant \delta \tag{1.1}$$

where δ is a given positive number.

Define also

$$\psi(H^{\pm}, (u-k)^{\pm}, \nu) = \ln^{+}\left\{\frac{H^{\pm}}{H^{\pm}-(u-k)^{\pm}+\nu}\right\}; \qquad \nu < \min\{H^{\pm}; 1\}. \tag{1.2}$$

We say that a measurable function $u\colon \Omega_T \to \mathbb{R}$ belongs to the class $\mathcal{B}_p(\Omega_T, M, \gamma, r, \delta, \varkappa)$ if

$$u \in C(0, T; L^2(\Omega)) \cap L^p(0, T; H^p(\Omega)) \tag{1.3}$$

$$\|u\|_{\infty,\Omega_T} \leqslant M \tag{1.4}$$

and if for all $Q(R, \varrho) \subset \Omega_T$ and all ζ as above, the functions $(u-k)^{\pm}$ satisfy the integral inequalities

$$\begin{aligned}\sup_{t_0-\varrho\leqslant t\leqslant t_0} \int_{B(R)} [(u-k)^{\pm}]^2 \zeta^p(x,t)\,dx + \iint_{Q(R,\varrho)} |\nabla_x (u-k)^{\pm}|^p \zeta^p\,dx\,d\tau \\ \leqslant \int_{B(R)} [(u-k)^{\pm}]^2 \zeta^p(x, t_0-\varrho)\,dx \\ + \gamma\left\{\iint_{Q(R,\varrho)} [(u-k)^{\pm}]^p |\nabla_x \zeta|^p\,dx\,d\tau + \iint_{Q(R,\varrho)} [(u-k)^{\pm}]^2 \zeta^{p-1}\zeta_t\,dx\,d\tau\right\} \\ + \gamma\left\{\int_{t_0-\varrho}^{t_0} [\operatorname{meas} A^{\pm}_{k,R}(\tau)]^{r/q}\,d\tau\right\}^{(p/r)(1+\varkappa)}.\end{aligned} \tag{1.5}$$

$$\begin{aligned}\sup_{t_0-\varrho\leqslant t\geqslant t_0} \int_{B(R)} \psi^2(H^{\pm}, (u-k)^{\pm}, \nu)\,\zeta^p(x,t)\,dx \\ \leqslant \int_{B(R)} \psi^2(H^{\pm}, (u-k)^{\pm}, \nu)\,\zeta^p(x, t_0-\varrho)\,dx \\ + \gamma \iint_{Q(R,\varrho)} \psi(H^{\pm}, (u-k)^{\pm}, \nu)\,|\psi_u(H^{\pm}, (u-k)^{\pm}, \nu)|^{2-p} |\nabla_x \zeta|^p\,dx\,d\tau \\ + \frac{\gamma}{\nu^p}\left(1 + \ln\frac{H^{\pm}}{\nu}\right)\left\{\int_{t_0-\varrho}^{t_0} [\operatorname{meas} A^{\pm}_{k,R}(\tau)]^{r/q}\,d\tau\right\}^{(p/r)(1+\varkappa)},\end{aligned} \tag{1.6}$$

where we have denoted with $A^{\pm}_{k,R}(t)$ the set

$$A^{\pm}_{k,R}(t) \equiv \{x \in B(R) \colon (u(x,t) - k)^{\pm} > 0\} .$$

The various parameters in (1.5)-(1.6) are as follows.

[A] δ and γ are arbitrary positive numbers;

[B] k is an arbitrary real number subject only to the restriction

$$\|(u-k)^{\pm}\|_{\infty, Q(R,\varrho)} \leqslant \delta; \tag{1.7}$$

[C] $\varkappa$ is an arbitrary number in $(0,1)$ and q, r are larger than one, are linked by

$$\frac{1}{r} + \frac{N}{pq} = \frac{N}{p^2}, \tag{1.8}$$

and their admissible range is

(1.8) (i) $\quad q \in (p, \infty]$, $\quad r \in [p^2, \infty)$; $\quad$ if $N = 1$

(1.8) (ii) $\quad q \in \left[p, \dfrac{Np}{N-p}\right]$, $\quad r \in [p, \infty]$; $\quad$ if $1 < p < N$

(1.8) (iii) $\quad q \in [p, \infty)$, $\quad r \in \left(\dfrac{p^2}{N}, \infty\right]$; $\quad$ if $1 < N \leqslant p$.

REMARK 1.1. These classes can be considered as an extension of the classes $\mathcal{B}_2(\Omega_T, M, \gamma, r, \delta, \varkappa)$ introduced in [11]. Besides the fact that $p \geqslant 2$ the new requirement here is the integral inequality (1.6).

They may also be viewed as a parabolic version of De Giorgi classes, fundamental in the regularity theory for quasi minima [9].

The following two facts establish the connection between local solu tions of (0.1) and the classes $\mathcal{B}_p(\Omega_T, M, \gamma, r, \delta, \varkappa)$.

PROPOSITION 1.1. *Every essentially bounded local solution of* (0.1) *belongs to* $\mathcal{B}_p(\Omega_T, M, \gamma, r, \delta, \varkappa)$.

EMBEDDING THEOREM. $\mathcal{B}_p(\Omega_T, M, \gamma, r, \delta, \varkappa)$ *is embedded in* $C^{\alpha, \alpha/p}_{\mathrm{loc}}(\Omega_T)$, *for some* $\alpha \in (0,1)$.

The proof of Theorem 1 will result by combining these two facts.

PROOF OF PROPOSITION 1.1. Introducing the Steklov averagings of $w \in V_{2,p}(\Omega_T)$,

$$w_h(x,t) = \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} w(x,s)\,ds\,, & t \in (0, T-h] \\ 0 \quad\quad , & t > T-h \end{cases}$$

$$w_{\bar h}(x,t) = \begin{cases} \dfrac{1}{h}\displaystyle\int_{t-h}^{t} w(x,s)\,ds\,, & t \in (h, T] \\ 0 \quad\quad , & t \leqslant h\,, \end{cases}$$

a standard argument (see for example [11]) implies that (0.6) can be equivalently formulated as

$$\int_{t_1}^{t_2}\int_\Omega \left\{\frac{\partial}{\partial t} u_h \varphi + [\boldsymbol{a}(x,t,u,\nabla_x u)]_{\bar h} \nabla_x\varphi + [b(x,t,u,\nabla_x u)]_{\bar h}\varphi\right\} dx\, d\tau = 0 \tag{1.9}$$

for all $\varphi \in \mathring{W}_p^{1,0}(\Omega_T)$ and $h < t_1 < t_2 < T-h$.

In (1.9) choose the test functions

$$\varphi = \pm (u_h - k)^\pm \zeta^p\,, \qquad k \in \mathbf{R}\,. \tag{1.10}$$

Estimating the various parts of (1.9) with this choice of text function we have

(i) $$\int_{B(R)}\int_{t_0-\varrho}^{t} \pm \frac{\partial}{\partial\tau} u_h (u_h - k)^\pm \zeta^p\, dx\, d\tau = \frac{1}{2}\int_{B(R)} [(u_h - k)]^{\pm 2}\zeta^p(x,t)\,dx$$
$$-\frac{1}{2}\int_{B(R)} [(u_h-k)^\pm]^2 \zeta^p(x, t_0-\varrho)\,dx - \frac{p}{2}\iint_{Q^t} [(u_h-k)^\pm]^2 \zeta^{p-1}\zeta_t\, dx\, d\tau\,.$$

where

$$Q^t \equiv B(R)\times[t_0-\varrho, t]\,, \qquad t\in(t_0-\varrho, t_0]\,.$$

Letting $h \to 0$ we obtain for all $t \in [t_0-\varrho, t_0]$

$$\iint_{Q^t} \pm \frac{\partial}{\partial\tau} u_h (u_h-k)^\pm \zeta^p\,dx\,d\tau \to \frac{1}{2}\int_{B(R)} [(u-k)^\pm]^2 \zeta^p(x,t)\,dx \tag{1.11}$$
$$-\frac{1}{2}\int_{B(R)} [(u-k)^\pm]^2\zeta^p(x,t_0-\varrho)\,dx - \frac{p}{2}\iint_{Q^t} [(u-k)^\pm]^2\zeta^{p-1}\zeta_t\,dx\,d\tau\,.$$

We estimate the remaining terms by letting $h \to 0$ first, and then using $[A_1]$-$[A_3]$.

$$\iint_{Q^t} \boldsymbol{a}(x, \tau, u, \nabla_x u)[\pm \nabla_x(u-k)^\pm \zeta^p \pm p(u-k)^\pm \zeta^{p-1} \nabla_x \zeta]\, dx\, d\tau \tag{1.12}$$

$$\geq C_0 \iint_{Q^t} |\nabla_x(u-k)^\pm|^p \zeta^p(x, \tau)\, dx\, d\tau - \iint_{Q^t} \varphi_0(x, \tau)\, \zeta^p \chi[(u-k)^\pm > 0]\, dx\, d\tau$$

$$- pC_1 \iint_{Q^t} |\nabla_x(u-k)^\pm|^{p-1}(u-k)^\pm \zeta^{p-1} |\nabla_x \zeta|\, dx\, d\tau - p \iint_{Q^t} \varphi_1 (u-k)^\pm \zeta^{p-1} |\nabla_x \zeta|\, dx\, d\tau\, .$$

Here $\chi(\Sigma)$ denotes the characteristic function of the set Σ. By Young's inequality

$$(a) \quad pC_1 \iint_{Q^t} |\nabla_x(u-k)^\pm|^{p-1}(u-k)^\pm \zeta^{p-1} |\nabla_x \zeta|\, dx\, d\tau$$
$$\leq \frac{C_0}{2} \iint_{Q^t} |\nabla_x(u-k)^\pm|^p \zeta^p\, dx\, d\tau + \gamma(C_0) \iint_{Q^t} [(u-k)^\pm]^p\, |\nabla_x \zeta|^p\, dx\, d\tau\, ,$$

and

$$(b) \quad p \iint_{Q^t} \varphi_1 (u-k)^\pm \zeta^{p-1} |\nabla_x \zeta|\, dx\, d\tau$$
$$\leq \iint_{Q^t} |(u-k)^\pm|^p |\nabla_x \zeta|^p\, dx\, d\tau + \gamma \iint_{Q} \varphi_1^{p'}\, \chi[(u-k)^\pm > 0]\, dx\, d\tau\, .$$

Combining this in (1.12) we deduce

$$\iint_{Q^t} \boldsymbol{a}(x, \tau, u, \nabla_x u)\, \nabla_x \varphi\, dx\, d\tau \geq \frac{C_0}{2} \iint_{Q^t} |\nabla_x(u-k)^\pm|^p \zeta^p\, dx\, d\tau \tag{1.13}$$
$$- \gamma \iint_{Q^t} [(u-k)^\pm]^p |\nabla_x \zeta|^p\, dx\, d\tau - \gamma \iint_{Q^t} (\varphi_0 + \varphi_1^{p'})\, \chi[(u-k)^\pm > 0]\, dx\, d\tau\, .$$

Finally

$$\iint_{Q^t} |b(x, \tau, u, \nabla_x u)(u-k)^\pm \zeta^p|\, dx\, d\tau \tag{1.14}$$
$$\leq C_2 \iint_{Q^t} |\nabla_x(u-k)^\pm|^p (u-k)^\pm \zeta^p\, dx\, d\tau + \iint_{Q^t} \varphi_2 |(u-k)^\pm| \zeta^p\, dx\, d\tau\, .$$

Now if we impose on the levels k the restriction

$$k\colon \ \|(u-k)^{\pm}\|_{\infty, Q(R,\varrho)} \leqslant \delta = \frac{1}{C_2}\frac{C_0}{4}\,, \tag{1.15}$$

we deduce from (1.14)

$$\begin{aligned} \iint\limits_{Q^t} |b(x,\tau,u,\nabla_x u)(u-k)^{\pm}\zeta^p|\,dx\,d\tau \\ \leqslant \frac{C_0}{4}\iint\limits_{Q^t} |\nabla_x(u-k)^{\pm}|^p\zeta^p\,dx\,d\tau + \iint\limits_{Q^t}\varphi_2\,\chi[(u-k)^{\pm}>0]\,dx\,d\tau\,. \end{aligned} \tag{1.16}$$

Combining these estimates and observing that $t\in[t_0-\varrho, t_0]$ is arbitrary we obtain

$$\begin{aligned} &\sup_{t_0\leqslant\varrho\leqslant t\leqslant t_0}\int\limits_{B(R)}[(u-k)^{\pm}]^2\zeta^p(x,t)\,dx \\ &\quad + \frac{C_0}{4}\iint\limits_{Q(R,\varrho)}|\nabla_x(u-k)^{\pm}]^p\zeta^p\,dx\,d\tau \leqslant \int\limits_{B(R)}[(u-k)^{\pm}]^2\zeta^p(x,t_0-\varrho)\,dx \\ &\quad + \gamma\Big\{\iint\limits_{Q(R,\varrho)}[(u-k)^{\pm}]^p|\nabla_x\zeta|^p\,dx\,d\tau + \iint\limits_{Q(R,\varrho)}[(u-k)^{\pm}]^2\zeta^{p-1}\zeta_t\,dx\,d\tau\Big\} \\ &\quad + \gamma\iint\limits_{Q(R,\varrho)}[\varphi_0+\varphi_1^{p'}+\varphi_2]\,\chi[(u-k)^{\pm}>0]\,dx\,d\tau\,. \end{aligned} \tag{1.17}$$

By Hölder's inequality

$$\begin{aligned} \iint\limits_{R,\varrho} \quad {}^{p'} + \varphi_2\}\,\chi[(u-k)^{\pm}>0]\,dx\,d\tau \leqslant \|\varphi_0+\varphi_1^{p'}+\varphi_2\|_{\hat{q},\hat{r},\Omega_T} \\ \cdot\Big\{\int\limits_{t_0-\varrho}^{t_0}[\operatorname{meas} A^{\pm}_{k,R}(\tau)]^{((q-1)/\hat{q})(\hat{r}/(r-1))}\,d\tau\Big\}^{(\hat{r}-1)/\hat{r}}. \end{aligned}$$

Set

$$q = \frac{\hat{q}}{\hat{q}-1}\,p(1+\varkappa)\,; \qquad r = \frac{\hat{r}}{\hat{r}-1}\,p(1+\varkappa)\,; \qquad \varkappa = \frac{p}{N}\varkappa_1\,, \tag{1.18}$$

From (0.2) we see that q, r satisfy (1.8), and from (0.2) (i)-(0.2) (iii) it follows that their admissible range is (1.8) (i)-(1.8) (iii).

Substituting this last estimate in (1.17) we see that u satisfies (1.5) since obviously without loss of generality we may assume that $C_0/4 = 1$.

We turn now to the proof of (1.6). For simplicity we set

$$\psi(H^{\pm}, (u-k)^{\pm}, \nu) = \psi((u-k)^{\pm}),$$

and in (1.9) select the test function

$$\varphi = [\psi^2((u_h - k)^{\pm})]' \zeta^p, \tag{1.19}$$

where $x \to \zeta^p(x)$ is a cutoff function in $B(R)$ which vanishes on $\partial B(R)$. It is apparent that $\varphi \in \mathring{W}_p^{1,0}(\Omega_T)$ and that

$$[\psi^2((u_h - k)^{\pm})]'' = 2(1+\psi)\psi'^2 \in L_\infty^{\mathrm{loc}}(\Omega_T).$$

Therefore such a φ is an admissible test function in (1.9). Estimating the various terms we have

$$I_h \equiv \iint_{Q^t} \frac{\partial}{\partial\tau} u_h [\psi^2((u_h - k)^{\pm})]' \zeta^p \, dx\, d\tau = \int_{B(R)\times\{t\}} \psi^2 \zeta^p \, dx - \int_{B(R)\times\{t_0-\varrho\}} \psi^2 \zeta^p \, dx, \tag{1.20}$$

and letting $h \to 0$ we have

$$I_h \to \int_{B(R)\times\{t\}} \psi^2(H^{\pm}, (u-k)^{\pm}, \nu)\, \zeta^p(x)\, dx - \int_{B(R)\times\{t_0-\varrho\}} \psi^2(H^{\pm}, (u-k)^{\pm}, \nu)\, \zeta^p(x)\, dx$$

for all $t \in [t_0 - \varrho, t_0]$.

In order to estimate the remaining terms we let $h \to 0$ first and then use [A$_1$]-[A$_3$].

$$\begin{aligned} J &\equiv \iint_{Q^t} \boldsymbol{a}(x, \tau, u, \nabla_x u) \nabla_x \varphi \, dx\, d\tau \\ &\geq 2C_0 \iint_{Q^t} (1+\psi)\psi'^2 |\nabla_x u|^p \zeta^p \, dx\, d\tau - 2\iint_{Q^t} (1+\psi)\psi'^2 \varphi_0(x,\zeta)\zeta^p \, dx\, d\tau \\ &\quad - pC_1 \iint_{Q^t} |\nabla_x u|^{p-1} \psi\psi' \zeta^{p-1} |\nabla_x \zeta| \, dx\, d\tau - p \iint_{Q^t} \psi\psi' \varphi_1 \zeta^{p-1} |\nabla_x \zeta| \, dx\, d\tau. \end{aligned} \tag{1.21}$$

By repeated application of the Young inequality we deduce

$$(1.22)\qquad J \geq C_0 \iint_{Q^t} (1+\psi)\,\psi'^2 |\nabla_x u|^p \zeta^p \,dx\,d\tau - 2\iint_{Q^t} (1+\psi)\,\psi'^2 \varphi_0(x,\tau)\,\zeta^p\,dx\,d\tau -$$

$$- \gamma(p)\iint_{Q^t} \psi(\psi')^{2-p} |\nabla_x \zeta|^p \,dx\,d\tau - \gamma(p)\iint_{Q^t} \psi(\psi')^2 \varphi_1^{p'} \zeta^p \,dx\,d\tau\,.$$

For the lower order terms we have

$$(1.23)\qquad \iint_{Q^t} |b(x,\tau,u,\nabla_x u)\,\psi\psi'\,\zeta^p|\,dx\,d\tau \leq$$

$$\leq C_2 \iint_{Q^t} |\nabla_x u|^p (1+\psi)\,\psi'^2 \psi'^{-1} \zeta^p \,dx\,d\tau + \iint_{Q^t} \varphi_2 \psi\psi'\,\zeta^p\,dx\,d\tau\,.$$

Next observe that $\psi'^{-1} = H^\pm - (u-k)^\pm + \nu < 2\delta$ by virtue of (1.1), and

$$\psi \leq \ln(H/\nu)\,; \qquad \psi' \leq \frac{1}{\nu}\,,$$

Therefore recalling (1.15) we have

$$(1.24)\qquad 2\iint_{Q^t} |b(x,\tau,u,\nabla_x u)\,\psi\psi'\,\zeta^p|\,dx\,d\tau \leq C_0 \iint_{Q^t} |\nabla_x u|^p (1+\psi)\,\psi'^2 \zeta^p\,dx\,d\tau$$

$$+ 2\,\frac{1}{\nu}\ln\left(\frac{H^\pm}{\nu}\right) \iint_{Q(R,\varrho)} |\varphi_2|\,\chi[(u-k)^\pm > 0]\,dx\,d\tau\,.$$

Collecting these estimates we deduce

$$(1.25)\qquad \int_{B(R)\times\{t\}} \psi^2 \zeta^p\,dx \leq \int_{B(R)\times\{t_0-\tau\}} \psi^2\zeta^p\,dx + \gamma \iint_{Q(R,\varrho)} \psi|\psi'|^{2-p} |\nabla_x \zeta|^p\,dx\,d\tau$$

$$+ \frac{\gamma}{\nu^p}\left(1 + \ln\frac{H^\pm}{\nu}\right) \iint_{Q(R,\varrho)} |\varphi_0 + \varphi_1^{p'} + \varphi_2|\,\chi[(u-k)^\pm > 0]\,dx\,d\tau$$

where we have used the fact that ν^{-1}, $\nu^{-2} < \nu^{-p}$ since $p \geq 2$. Treating the last integral as before the result follows.

The proof of the embedding theorem will be the object of sections 2-5.

PART I - INTERIOR REGULARITY

2. - Preliminaries.

Let the point (x_0, t_0) be fixed throughout, and consider the cylinder

$$Q_R^{N\varkappa} \equiv B(R) \times \{t_0 - R^{p-(N\varkappa/p)(p-2)}, t_0\}. \tag{2.1}$$

Set

$$\mu^+ = \operatorname*{ess\,sup}_{Q_{2R}^{N\varkappa}} u\,; \qquad \mu^- = \operatorname*{ess\,inf}_{Q_{2R}^{N\varkappa}} u\,,$$

and let ω be any number satisfying

$$2M \geqslant \omega \geqslant \mu^+ - \mu^- = \operatorname*{ess\,osc}_{Q_{2R}^{N\varkappa}} u\,.$$

Let s^* be a positive integer to be fixed later and set

$$\theta = \left(\frac{2^{s^*}}{\omega}\right)^{p-2}. \tag{2.2}$$

Construct the cylinder Q_R^θ given by

$$Q_R^\theta \equiv B(R) \times \{t_0 - \theta R^p, t_0\}\,. \tag{2.3}$$

If $\omega > 2^{s^*} R^{N\varkappa/p}$, then $\theta R^p < R^{p-(N\varkappa/p)(p-2)}$ and we have the inclusion

$$Q_R^\theta \subset Q_R^{N\varkappa}\,.$$

Inside Q_R^θ we consider subcylinders of the type

$$\bar{Q}_R^\eta \equiv B(R) \times \{\bar{t} - \eta R^p, \bar{t}\}\,, \qquad \eta > 0\,, \tag{2.4}$$

where $\bar{t} \leqslant t_0$ and $\bar{t} - \eta R^p \geqslant t_0 - \theta R^p$. The length of these subcylinders is determined by the choice of η

$$\eta = \left(\frac{2^{s_0}}{\omega}\right)^{p-2}, \qquad s_0 < s^*\,, \tag{2.5}$$

where s_0 is the smallest positive integer satisfying

$$\frac{2M}{2^{s_0}} \leqslant \delta . \tag{2.6}$$

The structure of the proof is based on studying separately two cases. Either we can find a cylinder of the type $\bar{Q}_R^\eta$ where u is « mostly » large, or such a subcylinder cannot be found. In both cases the conclusion is that the essential oscillation of u in a smaller cylinder around (x_0, t_0) decreases in a way that can be quantitatively measured.

We will need the following two embedding lemmas known from the literature.

LEMMA 2.1 (De Giorgi [3]). *Let $u \in W^{1,1}(B(R))$ and let l, $k \in \mathbf{R}$, $l > k$ Then*

$$(l-k)\operatorname{meas} A^+_{l,R} \leqslant \frac{CR^{N+1}}{\operatorname{meas}\{B(R)\setminus A^+_{k,R}\}} \int\limits_{A^+_{k,R}\setminus A^+_{l,R}} |\nabla u|\, dx , \tag{2.7}$$

where C depends only upon the dimension N.

REMARK 2.1. A similar lemma holds more generally for convex domains (see [11]).

For notational convenience we set

$$\begin{aligned} V_p(\Omega_T) &\equiv L^\infty(0, T; L^p(\Omega)) \cap L^p(0, T; H^p(\Omega)) , \\ \mathring{V}_p(\Omega_T) &\equiv L^\infty(0, T; L^p(\Omega)) \cap L^p(0, T; H^p_0(\Omega)) , \\ \|u\|^p_{V_p(\Omega_T)} &\equiv \operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \|u(\cdot, t)\|^p_{p,\Omega} + \|\nabla_x u\|^p_{p,\Omega_T} \end{aligned}$$

and define $\|u\|^p_{\mathring{V}_p(\Omega_T)} = \|u\|^p_{V_p(\Omega_T)}$.

LEMMA 2.2. *Let $u \in \mathring{V}_p(\Omega_T)$, then*

$$\|u\|_{q,r,\Omega_T} \leqslant C\|u\|_{\mathring{V}_p(\Omega_T)}$$

where C does not depend on u nor on Ω_T, and where q, r are subject to the conditions (1.8)-(1.8) (iii).

The proof results from a straightforward adaptation of the arguments of [11] page 74, carried for $p = 2$.

From Lemma 2.2 we deduce two corollaries.

COROLLARY 2.3. *Let* $u \in \overset{\circ}{V}_p(\Omega_T)$, *then*

$$\|u\|_{p((N+p)/p),\Omega_T} \leqslant C\|u\|_{V_p(\Omega_T)}.$$

COROLLARY 2.4. *Let* $u \in \overset{\circ}{V}_p(\Omega_T)$, *then*

$$\|u\|_{p,\Omega_T}^p \leqslant C(\operatorname{meas}[u \neq 0] \cap \Omega_T)^{p/(N+p)}\|u\|_{V_p(\Omega_T)}^p.$$

REMARK 2.2. These Corollaries still hold if $u \in V_p(\Omega_T)$ and does not necessarily vanish on $\partial\Omega$.

In such a case C depends on Ω_T via

$$C = \tilde{C}\left\{1 + \left(\frac{T^{N/p}}{\operatorname{meas}\Omega}\right)^{1/N+p}\right\}.$$

With C we will denote a generic non negative constant depending only upon the various parameters in the classes $\mathcal{B}_p(\Omega_T, M, \gamma, r, \delta, \omega)$ and independent of R, ω, s^*. For a measurable set Σ we write also $\operatorname{meas}\Sigma \equiv |\Sigma|$.

3. – The first alternative.

LEMMA 3.1. *There exists a number* $\alpha_0 \in (0,1)$ *independent of* ω, R, s^* *such that if for some subcylinder* $\bar{Q}_R^\eta$

$$\left|(x,t) \in \bar{Q}_R^\eta\colon u(x,t) < \mu^- + \frac{\omega}{2^{s_0}}\right| \leqslant \alpha_0 |\bar{Q}_R^\eta|,$$

then either

$$R^{N\varkappa/p} \geqslant \left(\frac{\omega}{2^{s^*}}\right)^{1+((1+\varkappa)/r-1/p)^+(p-2)}, \tag{3.1}$$

or

$$u(x,t) \geqslant \mu^- + \frac{\omega}{2^{s_0+1}}, \quad \forall (x,t) \in \bar{Q}_{R/2}^\eta. \tag{3.2}$$

PROOF. We assume that (3.1) is violated so that $Q_R^\theta \subset Q_R^{N\varkappa}$ and fix a cylinder $\bar{Q}_R^\eta$ for which the assumption of the lemma holds. Let

$$R_n = \frac{R}{2} + \frac{R}{2^n}; \quad \bar{R}_n = \frac{R_n + R_{n+1}}{2} = \frac{R}{2} + \frac{3R}{2^{n+2}}, \qquad n = 1, 2, \ldots.$$

We will write (1.5) over the pair of culinders $\bar{Q}^{\eta}_{\bar{R}_n}$ and $\bar{Q}^{\eta}_{R_n}$, by choosing the function ζ so that $\zeta(x,t)=1$ for $(x,t)\in\bar{Q}^{\eta}_{\bar{R}_n}$ and vanishing for $t=\bar{t}-\eta R^p_n$. In this case

$$|\nabla_x\zeta|\leqslant\frac{2^{n+1}}{R}\,;\qquad 0\leqslant\zeta_t\leqslant C\,\frac{2^{n+1}}{\eta R^p}=C\,\frac{2^{n+1}}{R^p}\,(\omega/2^{s_0})^{p-2}\,.$$

As for levels k we take

$$k_n=\frac{\omega}{2^{s_0+1}}+\frac{\omega}{2^{s_0+n}}\,,\qquad n=1,2,\dots\,.$$

In this setting (1.5) can be rewritten as

$$\text{(3.3)}\qquad \sup_{\bar{t}-\eta\bar{R}^p_n\leqslant t\leqslant\bar{t}}\|(u-k_n)^-\|^2_{2,B(\bar{R}_n)}(t)+\|\nabla_x(u-k_n)^-\|^p_{p,\bar{Q}_{\bar{R}_n}}$$
$$\leqslant C\,\frac{2^{np}}{R^p}\left\{\iint_{\bar{Q}^{\eta}_{R_n}}[(u-k_n)^-]^p\,dx\,d\tau+(\omega/2^{s_0})^{p-2}\iint_{\bar{Q}^{\eta}_{R_n}}[(u-k_n)^-]^2\,dx\,d\tau\right\}$$
$$+\;C\left(\int_{\bar{t}-\eta R^p_n}^{\bar{t}}|A^-_{k_n,R_n}(\tau)|^{r/q}\,d\tau\right)^{(p/r)(1+\varkappa)}.$$

The choice of levels k_n is justified since

$$\|(u-k_n)^-\|_{\infty,\bar{Q}^{\eta}_{R_n}}\leqslant\frac{\omega}{2^{s_0}}\leqslant\delta\,.$$

We estimate the various terms in (3.3) as follows. First

$$\iint_{\bar{Q}^{\eta}_{R_n}}[(u-k_n)^-]^p\,dx\,d\tau+(\omega/2^{s_0})^{p-2}\iint_{\bar{Q}^{\eta}_{R_n}}[(u-k_n)^-]^2\,dx\,d\tau$$
$$\leqslant\left(\frac{\omega}{2^{s_0}}\right)^p\int_{\bar{t}-\eta R^p_n}^{\bar{t}}|A^-_{k_n,R_n}(\tau)|\,d\tau$$

Next for all $t\in[\bar{t}-\eta\bar{R}^p_n,\bar{t}]$

$$\|(u-k_n)^-\|^2_{2,B(\bar{R}_n)}(t)\geqslant\left(\frac{2^{s_0}}{\omega}\right)^{p-2}\|(u-k_n)^-\|^p_{p,B(\bar{R}_n)}(t)=\eta\,\|(u-k_n)^-\|^p_{p,B(\bar{R}_n)}(t)\,.$$

Using these remarks in (3.3) and dividing by η

$$(3.4)\qquad \sup_{\bar t-\eta\bar R_n^p\leqslant t\leqslant\bar t}\|(u-k_n)^-\|^p_{p,B(\bar R_n)}(t)+\frac{1}{\eta}\|\nabla_x(u-k_n)^-\|^p_{p,\bar Q^\eta_{\bar R_n}}$$
$$\leqslant C\frac{2^{np}}{R^p}\left(\frac{\omega}{2^{s_0}}\right)^p\left\{\frac{1}{\eta}\int_{\bar t-\eta R_n^p}^{\bar t}|A^-_{k_n,R_n}(\tau)|\,d\tau\right\}$$
$$+\,C\left(\frac{1}{\eta}\int_{\bar t-\eta R_n^p}^{\bar t}|A^-_{k_n,R_n}(\tau)|^{r/q}d\tau\right)^{(p/r)(1+\varkappa)}\eta^{(p/r)(1+\varkappa)-1}.$$

The change of variable $z=(t-\bar t)/\eta$, transforms $\bar Q_{R_n}$ and $\bar Q_{\bar R_n}$ respectively into

$$Q_n\equiv B(R_n)\times\{-R_n^p,0\}\,;\qquad \bar Q_n\equiv B(\bar R_n)\times\{-\bar R_n^p,0\}\,.$$

Setting also $v(x,z)=u(x,\bar t+\eta z)$, inequalities (3.4) can be written more concisely as

$$(3.5)\qquad \|(v-k_n)^-\|^p_{V_p(\bar Q_n)}\leqslant C\frac{2^{np}}{R^p}\left(\frac{\omega}{2^{s_0}}\right)^p|A_n|$$
$$+\,C\left(\int_{-R_n^p}^{0}|A_n(z)|^{r/q}dz\right)^{(p/r)(1+\varkappa)}\eta^{(p/r)(1+\varkappa)-1},$$

where we have set

$$A_n(z)\equiv\{x\in B(R_n)\colon v(x,z)<k_n\}\,;\qquad |A_n|=\int_{-R_n^p}^{0}|A_n(z)|dz\,.$$

Let $x\to\varphi_n(x)$ be a piecewise smooth cutoff function in $B(\bar R_n)$ which equals one on $B(R_{n+1})$ and such that $|\nabla_x\varphi_n|\leqslant 2^{n+2}/R$. Then $(v-k_n)^-\varphi_n\in\in\mathring{V}_p(\bar Q_n)$ and by Corollary 2.4.

$$(3.6)\qquad \|(v-k_n)^-\|^p_{p,Q_{n+1}}\leqslant\|(v-k_n)^-\varphi_n\|^p_{p,\bar Q_n}\leqslant C|A_n|^{p/(N+p)}\|(v-k_n)^-\varphi_n\|^p_{\mathring{V}_p(\bar Q_n)}$$
$$\leqslant C|A_n|^{p/(N+p)}\left\{\|(v-k_n)^-\|^p_{V_p(\bar Q_n)}+\frac{2^{p(n+2)}}{R^p}\|(v-k_n)^-\|^p_{p,Q_n}\right\}.$$

Since

$$\|(v-k_n)^-\|^p_{p,Q_{n+1}}\geqslant|k_n-k_{n+1}|^p\,|A_{n+1}|\geqslant(\omega/2^{s_0})^p2^{-p(n+1)}|A_{n+1}|$$

we deduce from (3.6) by making use of (3.5)

$$(3.7)\qquad |A_{n+1}| \leqslant C4^{np}\left\{\frac{|A_n|^{1+p/(N+p)}}{R^p} + |A_n|^{p/N+p}\left(\int_{-R_n^p}^{0}|A_n(z)|^{r/q}\,dz\right)^{(p/r)(1+\varkappa)} \cdot\left(\frac{2^{s_0}}{\omega}\right)^p \eta^{(p/r)(1+\varkappa)-1}\right\}.$$

We set

$$Y_n = \frac{|A_n|}{|Q_n|}; \qquad Z_n = \frac{1}{|B(R_n)|}\left(\int_{-R_n^p}^{0}|A_n(z)|^{r/q}dz\right)^{p/r}.$$

Then from (3.7) in dimensionless form we have

$$(3.8)\qquad Y_{n+1} \leqslant C4^{np}\{Y_n^{1+p/(N+p)} + Y_n^{p/(N+p)}Z_n^{1+\varkappa}\},$$

where we have used the inequality

$$\left(\frac{2^{s_0}}{\omega}\right)^p \eta^{(p/r)(1+\varkappa)-1}R^{N\varkappa} \leqslant 1$$

which follows from the definition (2.5) of η and the fact that we have assumed that (3.1) is violated.

On the other hand, by the embedding lemma 2.2

$$Z_{n+1}(k_n - k_{n+1})^p \leqslant |B(R_{n+1})|^{-1}\|(v-k_n)^-\|^p_{q,r,Q_{n+1}} \leqslant |B(R_{n+1})|^{-1}\|(v-k_n)^-\varphi_n\|^p_{q,r,\bar{Q}_n}$$
$$\leqslant CR^{-N}\left\{\|(v-k_n)^-\|^p_{V_p(\bar{Q}_n)} + \frac{2^{p(n+2)}}{R^p}\|(v-k_n)^-\|^p_{p,Q_n}\right\}.$$

Then using (3.5)

$$(3.9)\qquad Z_{n+1} \leqslant C4^{np}\{Y_n + Z_n^{1+\varkappa}\}.$$

From lemma 5.7 of [11] page 96, Y_n, $Z_n \to 0$ as $n \to \infty$, provided

$$Y_1 \leqslant \lambda_0; \qquad Z_1^{1+\varkappa} \leqslant \lambda_0$$

where λ_0 is a small constant depending upon C, p, $\varkappa$ only and independent of R, ω.

Therefore the lemma is proved if we choose α_0 sufficiently small depending only upon λ_0.

We suppose that the assumptions of lemma 3.1 are verified for some subcylinder $\bar{Q}_R^\eta$ and construct the cylinder

$$Q \equiv B\left(\frac{R}{2}\right)\times\left\{\bar{t}-\eta\left(\frac{R}{2}\right)^p, t_0\right\}.$$

The length of such cylinder is at least $\eta(R/2)^p$ and at most $\eta(R/2)^p + (\theta-\eta)R^p \leqslant \theta R^p$, so that setting for simplicity $\varrho = R/4$ we may write

$$Q \equiv Q_{2\varrho}^{\tilde{\theta}} \equiv B(2\varrho)\times\{t_0-\tilde{\theta}(2\varrho)^p, t_0\}$$

where

$$\tilde{\theta} = 2^p\left(\frac{2^{\bar{s}}}{\omega}\right)^{p-2}, \qquad s_0 \leqslant \bar{s} \leqslant s^*. \tag{3.10}$$

LEMMA 3.2. *Assume that*

$$H^- = \left\|\left(u-\left(\mu^- + \frac{\omega}{2^{s_0+1}}\right)\right)^-\right\|_{\infty, Q} > \frac{\omega}{2^{s_0+2}}.$$

Then for every $\alpha_1 \in (0,1)$, *there exists a positive integer* $s_1 = s_1(\alpha_1, \gamma, \varkappa, \delta, r)$ *independent of* ω *and* R *such that either*

$$R^{N\varkappa/p} \geqslant \left(\frac{\omega}{2^{s_1}}\right)^{1+((1+\varkappa)/r-1/p)^+(p-2)}, \tag{3.11}$$

or

$$\text{meas}\left\{x\in B(\varrho) : u(x,t) < \mu^- + \frac{\omega}{2^{s_1}}\right\} < \alpha_1|B(\varrho)| \tag{3.12}$$

for all $t\in[t_0-\tilde{\theta}(R/2)^p, t_0]$.

PROOF. We will employ inequality (1.6) over the cylinder $Q_{2\varrho}^{\tilde{\theta}}$. As a cutoff function $(x,t)\to\zeta(x,t)$ we take a function independent of t, such that $\zeta(x)=1$ on $B(\varrho)$ and $|\nabla\zeta|\leqslant\varrho^{-1}$. We observe that for $t = t_0-\tilde{\theta}(2\varrho)^p = \bar{t}-\eta(R/2)^p$, by lemma 3.1 $u > \mu^- + \omega/2^{s_0+1}$ and therefore $\forall n \geqslant 1$

$$\psi\left(H^-, \left(u-\left(\mu^-+\frac{\omega}{2^{s_0+1}}\right)\right)^-, \frac{\omega}{2^{s_0+n}}\right)(t_0-\tilde{\theta}(2\varrho)^p) = 0 .$$

Next for $\nu = \omega/2^{s_0+n}$, from the definition (1.2) of $\psi(\cdot)$ we have, since $H^- \leqslant \omega/2^{s_0+1}$

$$\psi\left(H^-, \left(u-\left(\mu^-+\frac{\omega}{2^{s_0+1}}\right)\right)^-, \frac{\omega}{2^{s_0+n}}\right) \leqslant (n-1)\ln 2 .$$

Moreover a quick calculation gives

$$\left|\psi_u\left(H^{\pm},\left(u-\left(\mu^-+\frac{\omega}{2^{s_0+1}}\right)\right)^-,\frac{\omega}{2^{s_0+n}}\right)\right|^{2-p}\leqslant\left(\frac{\omega}{2^{s_0}}\right)^{p-2}.$$

Using these remarks in (1.6) we have

$$\int_{B(\varrho)\times\{t\}}\psi^2\left(H^-,\left(u-\left(\mu^-+\frac{\omega}{2^{s_0+1}}\right)\right)^-,\frac{\omega}{2^{s_0+n}}\right)dx \tag{3.12}$$
$$\leqslant\frac{C}{\varrho^p}(n-1)\left(\frac{\omega}{2^{s_0}}\right)^{p-2}|Q_{2\varrho}^{\tilde\theta}|+Cn\left(\frac{2^{s_0+n}}{\omega}\right)^p\tilde\theta^{(p/r)(1+\varkappa)}R^{N\varkappa}|B(\varrho)|\,.$$

Let n be a positive integer to be selected and set $s_1=s_0+n$. Then recalling (3.10) if (3.11) is violated the right hand side of (3.12) is bounded above by

$$C(s^*)\,n|B(\varrho)|\,.$$

We bound the integral on the left hand side of (3.12) from below by extending the integration to the smaller set

$$\left\{x\in B(\varrho):u(x,t)<\mu^-+\frac{\omega}{2^{s_0+n}}\right\}.$$

On such set, since $H^->\omega/2^{s_0+2}$ we have

$$\psi^2\left(H^-,\left(u-\left(\mu^-+\frac{\omega}{2^{s_0+1}}\right)\right)^-,\frac{\omega}{2^{s_0+n}}\right)\geqslant\ln^2\left(\frac{\omega/2^{s_0+2}}{\omega/2^{s_0+n-1}}\right)=(n-3)^2\ln^2 2\,.$$

Therefore for all $t\in[t_0-\tilde\theta(R/2)^p,\,t_0]$

$$|A^-_{\mu^-+\omega/2^{s_0+n},\,p}(t)|\leqslant C(s^*)\frac{n}{(n-3)^2}|B(\varrho)|\,. \tag{3.12}$$

To prove the lemma we have only to choose n so large that $\big(C(s^*)\,n\big)/(n-3)^2\leqslant\alpha_1$.

REMARK 3.1. The number $s_1=s_1(\alpha)$ claimed by lemma 3.2 depends upon k, $\varkappa$, δ, r and s^*. The number s^* is not fixed as yet. It will be fixed later independent of ω, R and therefore we can say that s_1 is independent of ω and R.

Without loss of generality we may assume that $s_1 > s^*$.

LEMMA 3.3. *Suppose the assumptions of lemma* 3.1 *hold and assume that* $H^- > \omega/2^{s_0+2}$. *Then there exist an integer* $s > s^*$ *independent of* ω, R *such that either*

$$R^{N\varkappa/p} \geqslant \left(\frac{\omega}{2^s}\right)^{1+((1+\varkappa)/r-1/p)^+(p-2)} \tag{3.13}$$

or

$$u(x,t) > \mu^- + \frac{\omega}{2^s}, \qquad \forall (x,t) \in B\left(\frac{R}{8}\right) \times \left\{t_0 - \eta\left(\frac{R}{8}\right)^p, t_0\right\}. \tag{3.14}$$

PROOF. Let

$$\varrho_n = \frac{\varrho}{2} + \frac{\varrho}{2^n}, \qquad \bar{\varrho}_n = \frac{\varrho_n + \varrho_{n+1}}{2^q} = \frac{\varrho}{2} + \frac{3\varrho}{2^{n+2}}, \qquad n = 1, 2, \dots,$$

and consider the cylinders

$$D_n^{\tilde{\theta}} \equiv B(\varrho_n) \times \left\{t_0 - \tilde{\theta}\left(\frac{R}{2}\right)^p, t_0\right\}$$

$$\bar{D}_n^{\tilde{\theta}} \equiv B(\bar{\varrho}_n) \times \left\{t_0 - \tilde{\theta}\left(\frac{R}{2}\right)^p, t_0\right\}.$$

We observe that these cylinders decrease in the space variables but their length is unchanged with respect to n. This is due to the fact that lemma 3.1 gives information on the level $t_0 - \tilde{\theta}(R/2)^p = 1 - \eta(R/2)^p$, and such information we want to exploit.

We assume (3.13) is violated and write (1.5) over the pair of cylinders $\bar{D}_n^{\tilde{\theta}}$ and $D_n^{\tilde{\theta}}$ as follows.

We choose a cutoff function ζ independent of t such that $\zeta \equiv 1$ on $B(\bar{\varrho}_n)$ and $|\nabla_x \zeta| \leqslant 2^{n+2}/\varrho$. Then the term involving ζ_t in (1.5) is eliminated. As for level k we choose

$$k_n = \mu^- + \frac{\omega}{2^{s_1+1}} + \frac{\omega}{2^{s_1+n}}, \qquad n = 1, 2, \dots,$$

where s_1 is the number claimed by lemma 3.2.

By lemma 3.1,

$$u > \mu^- + \frac{\omega}{2^{s_1}} \qquad \text{for} \qquad t = t_0 - \tilde{\theta}\left(\frac{R}{2}\right)^p = \bar{t} - \eta\left(\frac{R}{2}\right)^p,$$

and therefore we have

$$\int_{B(\varrho_n)} [(u-k_n)^-]^2 \zeta^p \left(x, t_0 - \tilde{\theta}\left(\frac{R}{2}\right)^p\right) dx = 0.$$

From (1.5) with the indicated choices we deduce

$$\sup_{t_0-\tilde{\theta}(R/2)^p \leqslant t \leqslant t_0} \|(u-k_n)^-\|^2_{2,B(\bar{\varrho}_n)}(t) + \|\nabla_x (u-k_n)^-\|^p_{p,\bar{D}_n^{\tilde{\theta}}} \leqslant C\frac{2^{np}}{\varrho^p}\|(u-k_n)^-\|^p_{p,D_n^{\tilde{\theta}}} + C\left(\int_{t_0-\tilde{\theta}(R/2)^p} |A_{\bar{k}_n,\varrho}(\tau)|^{r/q}\,d\tau\right)^{(p/r)(1+\varkappa)}. \tag{3.15}$$

For all $t \in [t_0 - \tilde{\theta}(R/2)^p, t_0]$

$$\|(u-k_n)^-\|^2_{2,B(\bar{\varrho}_n)}(t) \geqslant \left(\frac{2^{s_1}}{\omega}\right)^{p-2} \|(u-k_n)^-\|^p_{p,B(\bar{\varrho}_n)}(t) \geqslant \tilde{\theta}\,\|(u-k_n)^-\|^p_{p,B(\bar{\varrho}_n)}(t)\,.$$

We carry this estimate below, divide by $\tilde{\theta}$ and make the change of variables

$$z = \frac{t-t_0}{\tilde{\theta}}\,.$$

The cylinders $D_n^{\tilde{\theta}}$ and $\bar{D}_n^{\tilde{\theta}}$ are transformed into

$$D_n = B(\varrho_n) \times \left\{-\left(\frac{R}{2}\right)^p, 0\right\}; \qquad \bar{D}_n = B(\bar{\varrho}_n) \times \left\{-\left(\frac{R}{2}\right)^p, 0\right\}.$$

Setting also $v(x,z) = u(x, t_0 + \tilde{\theta} z)$, inequalities (3.15) can be rewritten as

$$\|(v-k_n)^-\|^p_{V_p(\bar{D}_n)} \leqslant \frac{C2^{np}}{\varrho^p}\|(v-k_n)^-\|^p_{p,D_n} + C\left(\int_{-(R/2)^p}^{0} |A_n(z)|^{r/q}\,dz\right)^{(p/r)(1+\varkappa)} \left(\frac{2^{s_1}}{\omega}\right)^{p[(1+\varkappa)/r-1/p]^+(p-2)}, \tag{3.16}$$

with the obvious definition of $A_n(z)$.

Using (3.16) we may repeat an iteration process in all analogous to lemma 3.1 and conclude that there exist $\lambda_1 > 0$ independent of R and ω, such that if

$$\operatorname{meas}\left\{(x,z) \in D_1 : v(x,z) < \mu^- + \frac{\omega}{2^{s_1}}\right\} \leqslant \lambda_1 |D_1|\,, \tag{3.17}$$

then either

$$R^{N\varkappa/p} \geqslant \left(\frac{\omega}{2^{s_1+1}}\right)^{1+((1+\varkappa)/r-1/p)^+(p-2)}$$

or

$$v(x,z) \geqslant \mu^- + \frac{\omega}{2^{s_1+1}}, \qquad \forall (x,t) \in D_\infty \equiv B\left(\frac{\varrho}{2}\right) \times \left\{-\left(\frac{R}{2}\right)^p, 0\right\}.$$

Scaling back to the cylinder $D^{\tilde{\theta}}_{\varrho_1}$ and choosing $\alpha_1 = \lambda_1$ in lemma 3.2 we see that we can choose $s_1 = s_1(\lambda_1, s^*)$ so that (3.17) holds. This proves the lemma.

We summarize the results obtained so far.

PROPOSITION 3.1. *There exist $\alpha_0 \in (0,1)$ and a positive integer s independent of ω, R, such that if for some cylinder of the form $\bar{Q}^\eta_R$ with η given by* (2.5)

$$\text{meas}\left\{(x,t) \in \bar{Q}^\eta_R \colon u(x,t) < \mu^- + \frac{\omega}{2^{s_0}}\right\} < \alpha_0 |\bar{Q}^\eta_R| \tag{3.18}$$

then either

$$\omega \leqslant 2^s R^{(N\varkappa/p)\xi}; \qquad \xi = \left[1 + \left(\frac{1+\varkappa}{r} - \frac{1}{p}\right)^+ (p-2)\right]^{-1} \tag{3.19}$$

or

$$\underset{Q^\eta_{R/8}}{\text{ess osc}}\, u \leqslant \omega\left(1 - \frac{1}{2^s}\right). \tag{3.20}$$

PROOF. If a cylinder satisfying (3.18) exists, then by lemma 3.1 and lemma 3.2, the set where $u < \mu^- + \omega/2^s$, relatively to $B(R/4) \times \{t_0 - \tilde{\theta}(R/2)^p, t_0\}$, can be made arbitrarily small provided

$$H^- = \left\| \left(u - \left(\mu^- + \frac{\omega}{2^{s_0+1}}\right)\right)^- \right\|_{\infty, B(R/4) \times \}t_0 - \tilde{\theta}(R/2)^p, t_0\}} \geqslant \frac{\omega}{2^{s_0+2}}. \tag{3.21}$$

Then by lemma 3.3

$$u(x,t) > \mu^- + \frac{\omega}{2^{s_1+1}}, \qquad \forall (x,t) \in B\left(\frac{R}{8}\right) \times \left\{t_0 - \tilde{\theta}\left(\frac{R}{8}\right)^p, t_0\right\}. \tag{3.22}$$

Since $\eta \leqslant \tilde{\theta} \leqslant \theta$ from (3.22) we also have

$$\underset{Q^\eta_{R/8}}{\text{ess inf}}\, u > \mu^- + \frac{\omega}{2^s}; \qquad s = s_1 + 1$$

and hence

$$\operatorname*{ess\,osc}_{Q^{\eta}_{R/8}} u = \operatorname*{ess\,sup}_{Q^{\eta}_{R/8}} u - \operatorname*{ess\,inf}_{Q^{\eta}_{R/8}} u \leqslant \mu^{+} - \mu^{-} - \frac{\omega}{2^{s}} = \omega\left(1 - \frac{1}{2^{s}}\right).$$

On the other hand if (3.12) is violated, since obviously $H^{-} \leqslant \omega/2^{s_0+1}$, we have

$$\operatorname*{ess\,inf}_{Q^{\eta}_{R/8}} u \geqslant \mu^{-} + \frac{\omega}{2^{s_0+1}} - \frac{\omega}{2^{s_0+2}} \geqslant \mu^{-} + \frac{\omega}{2^{s}},$$

from which the conclusion follows.

REMARK 3.2. The various constants in (3.18)-(3.20) are independent of ω and R. The number s_1 depends upon s^* as shown by lemma 3.2. The number s^* will be fixed later independent of ω, R.

4. – The second alternative.

We assume in this section that the assumptions of lemma 3.1 are violated, *i.e.* for every subcylinder $\bar{Q}^{\eta}_{R}$

$$\text{meas}\left\{(x, t) \in \bar{Q}^{\eta}_{R} \colon u(x, t) < \mu^{-} + \frac{\omega}{2^{s_0}}\right\} > \alpha_0 |\bar{Q}^{\eta}_{R}| . \tag{4.1}$$

Since if $s_0 \geqslant 2$ we obviously have

$$\mu^{+} - \frac{\omega}{2^{s_0}} \geqslant \mu^{-} + \frac{\omega}{2^{s_0}}$$

we will rewrite (4.1) as

$$\text{meas}\left\{(x, t) \in \bar{Q}^{\eta}_{R} \colon u(x, t) > \mu^{+} - \frac{\omega}{2^{s_0}}\right\} < (1 - \alpha_0) |\bar{Q}^{\eta}_{R}| . \tag{4.2}$$

valid for all cylinders $\bar{Q}^{\eta}_{R} \subset Q^{\theta}_{R}$. The parameters θ and η are those fixed in (2.2) and (2.5). In this section we will determine the value of s^*.

LEMMA 4.1. *Let $\bar{Q}^{\eta}_{R} \subset Q^{\theta}_{R}$ be fixed and let* (4.2) *hold. Then there exist*

$$t^* \in \left[\bar{t} - \eta R^{p},\ \bar{t} - \frac{\alpha_0}{2}\, \eta R^{p}\right]$$

such that

$$|A^+_{\mu^+-\omega/2^{s_0},R}(t^*)|\leq\left(\frac{1-\alpha_0}{1-\alpha_0/2}\right)|B(R)|\,.$$

PROOF. If not, for a.e. $t\in[\bar t-\eta R^p,\ \bar t-(\alpha_0/2)\eta R^p]$

$$|A^+_{\mu^+-\omega/2^{s_0},R}(t)|>\left(\frac{1-\alpha_0}{1-\alpha_0/2}\right)|B(R)|$$

and

$$\operatorname{meas}\left\{(x,t)\in\bar Q^\eta_R\colon u(x,t)>\mu^+-\frac{\omega}{2^{s_0}}\right\}\geq\int\limits_{\bar t-\eta R^p}^{\bar t-(\alpha_0/2)\eta R^p}|A^+_{\mu^+-\omega/2^{s_0},R}(\tau)|\,d\tau>(1-\alpha_0)|\bar Q^\eta_R|$$

contradicting (4.2).

As before we let

$$H^+=\left\|\left(u-\left(\mu^+-\frac{\omega}{2^{s_0}}\right)\right)^+\right\|_{\infty,Q^\theta_R}\leq\frac{\omega}{2^{s_0}}\,.$$

LEMMA 4.2. *Let $\bar Q_R\subset Q_R$ be fixed, and assume that $H^+>\omega/2^{s_0+1}$. There exist a positive integer m independent of ω and R, such that either*

$$R^{N\varkappa/p}\geq\left(\frac{\omega}{2^{s_0+m}}\right)^{1+((1+\varkappa)/2-1/p)^+(p-2)} \tag{4.3}$$

or

$$|A^+_{\mu^+-\omega/2^{s_0+m},R}(t)|\leq\left[1-\left(\frac{\alpha_0}{2}\right)^2\right]|B(R)| \tag{4.4}$$

for all $t\in[\bar t-(\alpha_0/2)\eta R^p,\ \bar t]$.

PROOF. We will employ inequality (1.6) over the cylinders

$$Q^*_R\equiv B(R)\times[t^*,\bar t]\,;\qquad Q^*_{R-\sigma R}\equiv B(R-\sigma R)\times[t^*,t]\,.$$

Here t^* is the number claimed in lemma 4.1 and $\sigma\in(0,1)$ is arbitrary. We take also

$$k=\frac{\omega}{2^{s_0}}\,;\qquad \nu=\frac{\omega}{2^{s_0+m}}$$

where m has to be chosen. The cutoff function ζ will be independent of t and such that $\zeta\equiv1$ on $B(R-\sigma R)$ and $|\nabla\zeta|\leq(\sigma R)^{-1}$.

With these choices (1.6) can now be written for all $t \in [t^*, \bar{t}]$ as

$$
\begin{aligned}
(4.5)\qquad \int\limits_{B(R-\sigma R)\times\{t\}} &\psi^2\left(H^+, \left(u-\left(\mu^+-\frac{\omega}{2^{s_0}}\right)\right)^+, \frac{\omega}{2^{s_0+m}}\right)dx \\
&\leqslant \int\limits_{B(R)\times\{t^*\}} \psi^2\left(H^+, \left(u-\left(\mu^+-\frac{\omega}{2^{s_0}}\right)\right)^+, \frac{\omega}{2^{s_0+m}}\right)dx \\
&+\frac{C}{(\sigma R)_a}\int\limits_{t^*}^{\bar{t}}\int\limits_{B(R)}\psi^2\left(H^+, \left(u-\left(\mu^+-\frac{\omega}{2^{s_0}}\right)\right)^+, \frac{\omega}{2^{s_0+m}}\right) \\
&\times\left|\psi_u\left(H^+, \left(u-\left(\mu^+-\frac{\omega}{2^{s_0}}\right)\right)^+, \frac{\omega}{2^{s_0+m}}\right)\right|^{2-p} dx\,d\tau \\
&+C\left(\frac{2^{s_0+m}}{\omega}\right)^p \ln\frac{H^+2^{s_0+m}}{\omega}\left(\int\limits_{t^*}^{\bar{t}}|A^+_{\mu^+-\omega/2^{s_0},R}(\tau)|^{r/q}\,d\tau\right)^{(p/r)(1+\varkappa)}.
\end{aligned}
$$

The various terms in (4.5) are estimated as follows. First we observe that

$$
\psi\left(H^+, \left(u-\left(\mu^+-\frac{\omega}{2^{s_0}}\right)\right)^+, \frac{\omega}{2^{s_0+m}}\right)\leqslant m\ln 2
$$

$$
\left|\psi_u\left(H^+, \left(u-\left(\mu^+-\frac{\omega}{2^{s_0}}\right)\right)^+, \frac{\omega}{2^{s_0+m}}\right)\right|^{2-p}\leqslant\left(\frac{\omega}{2^{s_0}}\right)^{p-2}
$$

$$
\ln\frac{H^+2^{s_0+m}}{\omega}\leqslant m\ln 2\,.
$$

Next from the definition (1.2) of ψ we see that $\psi = 0$ on the set $[u < \mu^+ - \omega/2^{s_0}]$. Therefore by using lemma 4.1 the first integral on the right hand side of (4.5) is estimated above by

$$
\int\limits_{B(R)\times\{t^*\}}\psi^2\left(H^+, \left(u-\left(\mu^--\frac{\omega}{2^{s_0}}\right)\right)^+, \frac{\omega}{2^{s_0+m}}\right)dx\leqslant m^2\ln^2 2\left(\frac{1-\alpha_0}{1-\alpha_0/2}\right)|B(R)|\,.
$$

Since $\bar{t}-t^*\leqslant\eta R^p=(2^{s_0}/\omega)^{p-2}R^p$, the second integral is estimated by

$$
\frac{C}{\sigma^p}m|B(R)|\,.
$$

Finally for the last term we have the estimate

$$Cm|B(R)|\left(\frac{2^{s_0+m}}{\omega}\right)^{p[1+((1+\varkappa)/r-1/p)^+(p-2)]}R^{N\varkappa}\,.$$

If (4.3) does not hold, this last term is majorized by

$$Cm|B(R)|\,.$$

Putting together these remarks, from (4.5) we have for all $t\in[t^*,t]$

$$\int\limits_{B(R-\sigma R)\times\{t\}}\psi^2\left(H^+,\left(u-\left(\mu^+-\frac{\omega}{2^{s_0}}\right)\right)^+,\frac{\omega}{2^{s_0+m}}\right)dx \tag{4.6}$$
$$\leqslant m^2\ln^2 2\left(\frac{1-\alpha_0}{1-\alpha_0/2}\right)|B(R)|+\frac{C}{\sigma^p}m|B(R)|\,.$$

We estimate the left hand side of (4.6) below by integrating over the smaller set

$$B(R-\sigma R)\cap\left[u>\mu^+-\frac{\omega}{2^{s_0+m}}\right](t)\,.$$

Then on such a set, since $H^+>\omega/2^{s_0+1}$ we have

$$\psi\left(H^+,\left(u-\left(\mu^+-\frac{\omega}{2^{s_0}}\right)\right)^+,\frac{\omega}{2^{s_0+m}}\right)\geqslant\ln\left(\frac{\omega/2^{s_0+1}}{\omega/2^{s_0+m-1}}\right)=(m-2)\ln 2\,.$$

Carrying this estimate in (4.6)

$$(m-2)^2\ln^2 2|A^+_{\mu^+-\omega/2^{s_0+m},R-R\sigma}(t)|\leqslant m^2\ln^2 2\left(\frac{1-\alpha_0}{1-\alpha_0/2}\right)|B(R)|+\frac{C}{\sigma^p}m|B(R)|\,,$$

and dividing by $(m-2)^2\ln^2 2$

$$|A^+_{\mu^+-\omega/2^{s_0+m},R-\sigma R}(t)|\leqslant\left(\frac{m}{m-2}\right)^2\left(\frac{1-\alpha_0}{1-\alpha_0/2}\right)|B(R)|+\frac{C}{\sigma^p}m^{-1}|B(R)|\,. \tag{4.7}$$

On the other hand

$$|A^+_{\mu^+-\omega/2^{s_0+m},R}(t)|\leqslant|A^+_{\mu^+-\omega/2^{s_0+m},R-\sigma R}(t)|+|B(R)\diagdown B(R-\sigma R)|$$
$$\leqslant|A^+_{\mu^+-\omega/2^{s_0+m},R-\sigma R}(t)|+N\sigma|B(R)|\,.$$

Combining this with (4.7)

$$|A^+_{\mu^+-\omega/2^{s_0+m},R}(t)| \leq \left[\left(\frac{m}{m-2}\right)^2\left(\frac{1-\alpha_0}{1-\alpha_0/2}\right)+\frac{C}{\sigma^p m}+N\sigma\right]|B(R)| \tag{4.8}$$

for all $t\in[t^*,\bar{t}]$.

Choose σ so small that $\sigma N\leq(3/8)\alpha_0^2$ and m so large that

$$\left(\frac{m}{m-2}\right)^2\leq(1-\alpha_0/2)(1+\alpha_0);\ \frac{C}{\sigma^p m}\leq\frac{3}{8}\alpha_0^2\,.$$

Then for such a choice of m

$$|A^+_{\mu^+-\omega/2^{s_0+m},R}(t)|\leq\left[1-\left(\frac{\alpha_0}{2}\right)^2\right]|B(R)|\,. \tag{4.9}$$

REMARK 4.1. Since α_0 is independent of ω, R, the number m is independent of ω, R. The number s^* which determines the length of Q^θ_R is still to be chosen. We will choose it later subject to the condition $s^*>s_0+m$.

We will set

$$s_2=s_0+m\,.$$

The arguments of lemma 4.1 and 4.2 are carried under the assumptions that (4.2) holds, and we know that (4.2) holds for *every* cylinder of the form $\bar{Q}^\eta_R\subset\bar{Q}^\theta_R$.

Since $s^*>s_0$ we have $\forall p\geq2$

$$\left(1-\frac{\alpha_0}{2}\right)2^{s_0(p-2)}<\frac{1}{1+\alpha_0/2}\,2^{s^*(p-2)}\,. \tag{4.10}$$

COROLLARY 4.3. *Assume that* $H^+>\omega/2^{s_0+1}$. *Then either*

$$R^{N\varkappa/p}\geq\left(\frac{\omega}{2^{s_2}}\right)^{1+((1+\varkappa)/r-1/p)^+(p-2)}, \tag{4.11}$$

or

$$|A^+_{\mu^+-\omega/2^{s_0+m},R}(t)|\leq\left[1-\left(\frac{\alpha_0}{2}\right)^2\right]|B(R)| \tag{4.12}$$

for all $t\in[t_0-(\alpha_0/3)\theta R^p, t_0]$.

PROOF. Every cylinder of the type $\bar{Q}^\eta_R$ satisfies (4.2) and lemma 4.2 holds for every such cylinder. Therefore the conclusion of the lemma holds for all t satisfying

$$t \in \left[t_0 - \left(\theta - \left(1 - \frac{\alpha_0}{2}\right)\eta\right)R^p, t_0\right].$$

Because of (4.10) and the definition of θ and η

$$\theta - \left(1 - \frac{\alpha_0}{2}\right)\eta > \left(\frac{\alpha_0}{\alpha_0 + 2}\right)\theta > \frac{\alpha_0}{3}\theta$$

and the Corollary follows.

From now on we will focus on the cylinders

$$Q_R^\theta(\alpha_0) \equiv B(R) \times \left\{t_0 - \frac{\alpha_0}{3}\theta R^p, t_0\right\}.$$

LEMMA 4.4. *Assume that* (4.12) *holds. Then for every* $\beta_0 \in (0, 1)$ *there exists a number* s^* (*which determines the length of* Q_R^θ), *independent of* ω *and* R *such that either*

$$R^{N\varkappa/p} \geqslant \left(\frac{\omega}{2^{s^*}}\right)^{1+((1+\varkappa)/r-1/p)^+(p-2)}, \tag{4.13}$$

or

$$\operatorname{meas}\left\{(x, t) \in Q_R^\theta(\alpha_0) : u(x, t) > \mu^+ - \frac{\omega}{2^{s^*}}\right\} \leqslant \beta_0 |Q_R^\theta(\alpha_0)|. \tag{4.14}$$

PROOF. We write inequalities (1.5) over the cylinders $Q_R^\theta(\alpha_0)$ and $Q_{2R}^\theta(\alpha_0)$ as follows. We choose a cutoff function ζ such that $\zeta \equiv 1$ on $Q_R^\theta(\alpha_0)$ and $\zeta(x, t_0 - (\alpha_0/3)\theta(2R)^p) = 0$, $0 \leqslant \zeta_t \leqslant C(\alpha_0\theta R^p)^{-1}$, $|\nabla_x \zeta| \leqslant 2R^{-1}$.

As for the levels k we take $k = \mu^+ - \omega/2^n$ where $s^* \geqslant n \geqslant s_2$ and s_2 is the number claimed by Corollary 4.3.

Neglecting the first term on the right hand side of (1.5) and using the indicated choices we have

$$\begin{aligned} &\iint_{Q_R^\theta(\alpha_0)} \left|\nabla_x\left(u - \left(\mu^+ - \frac{\omega}{2^n}\right)\right)^+\right|^p dx\,d\tau \\ &\leqslant \frac{C}{R^p} \iint_{Q_{2R}^\theta(\alpha_0)} \left[\left(u - \left(\mu^+ - \frac{\omega}{2^n}\right)\right)^+\right]^p dx\,d\tau + \frac{C}{\alpha_0\theta R^p} \iint_{Q_{2R}^\theta(\alpha_0)} \left[\left(u - \left(\mu^+ - \frac{\omega}{2^n}\right)\right)^+\right]^2 dx\,d\tau \\ &+ C\left(\int_{t_0-(\alpha_0/3)\theta(2R)^p}^{t_0} |A^+_{\mu^+-\omega/2^n, R}(t)|^{r/q}\,dt\right)^{(p/r)(1+\varkappa)}. \end{aligned} \tag{4.15}$$

We estimate the right hand side of (4.15) as follows

(i) $$\frac{C}{R^p}\iint_{Q^{\theta}_{2R}(\alpha_0)}\left[\left(u-\left(\mu^+-\frac{\omega}{2^n}\right)\right)^+\right]^p dx\,d\tau \leqslant \frac{C}{R^p}\left(\frac{\omega}{2^n}\right)^p |Q^{\theta}_R(\alpha_0)|\,.$$

(ii) Recalling the definition of θ

$$\frac{C}{\alpha_0\theta R^p}\iint_{Q^{\theta}_{2R}(\alpha_0)}\left[\left(u-\left(\mu^+-\frac{\omega}{2^n}\right)\right)^+\right]^q dx\,d\tau \leqslant \frac{C}{R^p}\left(\frac{\omega}{2^n}\right)^2\left(\frac{\omega}{2^{s^*}}\right)^{p-2}|Q^{\theta}_R(\alpha_0)|\,.$$

(iii) $$\left(\int_{t_0-(\alpha_0/3)\theta(2R)^p} |A^+_{\mu^+-\omega/2^n,R}(t)|^{r/q}\,dt\right)^{(p/r)(1+\varkappa)} \leqslant \frac{C}{R^p}\,|Q^{\theta}_R(\alpha_0)|\,R^{N\varkappa}\,\theta^{p((1+\varkappa)/r-1/p)^+}.$$

Carrying these estimates in (4.15)

$$\int_{Q^{\theta}_R(\alpha_0)}\left|\nabla_x\left(u-\left(\mu^+-\frac{\omega}{2^n}\right)\right)^+\right|^p dx\,d\tau \tag{4.16}$$
$$\leqslant \frac{C}{R^p}\left\{\left(\frac{\omega}{2^n}\right)^p+\left(\frac{\omega}{2^n}\right)^2\left(\frac{\omega}{2^{s^*}}\right)^{p-2}+\theta^{p((1+\varkappa)/r-1/p)^+}R^{N\varkappa}\right\}|Q^{\theta}_R(\alpha_0)|\,.$$

Next we use lemma 2.1 over $B(R)$ for all the levels $t\in[t_0-(\alpha_0/3)\theta R^p, t_0]$. As for levels l, k we take

$$l=\mu^+-\frac{\omega}{2^n}\,;\qquad k=\mu^+-\frac{\omega}{2^{n-1}}\,.$$

Notice that for all $t\in[t_0-(\alpha_0/3)\theta R^p, t_0]$ by virtue of Corollary 4.3 we have

$$\operatorname{meas}\{B(R)\setminus A^+_{\mu^+-\omega/2^n,R}(t)\}\geqslant\left(\frac{\alpha_0}{2}\right)^2|B(R)|\,.$$

Therefore (2.7) in this setting gives $\forall t\in[t_0-(\alpha_0/3)\theta R^p, t_0]$

$$\left(\frac{\omega}{2^n}\right)|A^+_{\mu^+-\omega/2^n,R}(t)|\leqslant CR\int_{A^+_{k,R}(t)\setminus A^+_{l,R}(t)}|\nabla_x u|\,dx\,. \tag{4.17}$$

We majorize the right hand side of (4.17) by

$$\int_{A^+_{k,R}(t)\setminus A^+_{l,R}(t)}|\nabla_x u|\leqslant\left(\int_{B(R)}|\nabla_x(u-k)^+|^p\right)^{1/p}|A^+_{k,R}(t)\setminus A^+_{l,R}(t)|^{(p-1)/p}.$$

Integrating (4.17) over $[t_0-(\alpha_0/3)\theta R^p, t_0]$ and setting

$$A_n=\int_{t_0-(\alpha_0/3)\theta R^p}^{t_0}|A^+_{\mu^+-\omega/2^n,R}(\tau)|d\tau\,,$$

we have for all $n\geqslant s_2+1$

$$(4.18)\qquad \left(\frac{\omega}{2^n}\right)A_n\leqslant CR\left(\iint_{Q_R^\theta(\alpha_0)}\left|\nabla_x\left(u-\left(\mu^+-\frac{\omega}{2^{n-1}}\right)\right)^+\right|^p dx\,d\tau\right)^{1/p}[A_{n-1}-A_n]^{(p-1)/p}.$$

We take the $p/(p-1)$ power, estimate the integral on the right hand side by using (4.16) and divide by $(\omega/2^n)^{p/(p-1)}$ to obtain

$$A_n^{p/(p-1)}\leqslant C\left\{1+2^{(p-2)(n-s^*)}+R^{N\varkappa}\left(\frac{2^{s^*}}{\omega}\right)^{p[1+((1+\varkappa)/r-1/p)^+(p-2)]}\right\}^{1/(p-1)}$$
$$\cdot|Q_R^\theta(\alpha_0)|^{1/(p-1)}[A_{n-1}-A_n]\,.$$

Since $s^*\geqslant n>s_2$, if also (4.13) is violated, the quantity in brackets is bounded independent of ω, R, s^* and we deduce

$$(4.19)_n\qquad A_n^{p/(p-1)}\leqslant C|Q_R^\theta(\alpha_0)|^{1/(p-1)}[(_{n-1}-A_n]\,.$$

These inequalities are valid for all $n>s_2$ and $n\leqslant s^*$. We add $(4.19)_n$ for $n=s_2+1,\ s_2+2,\dots,s^*$.

The right hand side can be majorized with a convergent series and therefore we obtain

$$(4.20)\qquad (s^*-s_2-1)\,A_{s^*}^{p/(p-1)}\leqslant C|Q_R^\theta(\alpha_0)|^{p/(p-1)},$$

and

$$A_{s^*}\leqslant\frac{C}{[s^*-(s_2+1)]^{(p-1)/p}}|Q_R^\theta(\alpha_0)|\,.$$

To prove the lemma we take s^* so large that

$$(4.21)\qquad \frac{C}{[s^*-s_2-1]^{(p-1)/p}}\leqslant\beta_0\,.$$

Notice that if β_0 is independent of ω and R, also s^* is independent of ω and R.

REMARK 4.2. The process described by lemma 4.4 has a double meaning. On one hand, given β_0, determines a level $k = \mu^+ - \omega/2^{s^*}$ and on the other hand (recalling the definition (2.2) of θ) determines the cylinder Q_R^θ. That is, given $\beta_0 \in (0, 1)$, the measure of the set where $u > \mu^+ - \omega/2^{s^*}$ can be made smaller than β_0 only on a particular cylinder $Q_R^\theta(\alpha_0)$ related to the level $\mu^+ - \omega/2^{s^*}$.

LEMMA 4.5. *Suppose the conclusion of Corollary* 4.3 *holds. Then* s^* *can be chosen so that either*

$$R^{N\varkappa/p} \geq \left(\frac{\omega}{2^{s^*+1}}\right)^{1+((1+\varkappa)/r-1/p)^+(p-2)} \tag{4.22}$$

or

$$u(x, t) \leq \mu^+ - \frac{\omega}{2^{s^*+1}}, \qquad \forall (x, t) \in Q^\theta_{R/2}(\alpha_0)\,. \tag{4.23}$$

PROOF. Set

$$R_n = \frac{R}{2} + \frac{R}{2^n}; \qquad \bar{R}_n = \frac{R_{n+1} + R_n}{2} = \frac{R}{2} + \frac{3R}{2^{n+2}}, \qquad n = 1, 2, \ldots .$$

We will write inequalities (1.5) over the pair of cylinders $Q^\theta_{R_n}(\alpha_0)$ and $Q^\theta_{\bar{R}_n}(\alpha_0)$. The cutoff function ζ will be taken so that $\zeta \equiv 1$ on $Q^\theta_{\bar{R}_n}(\alpha_0)$, $\zeta(x, t_0 - (\alpha_0/3)\,\theta R_n^p) = 0$ and

$$|\nabla_x \zeta| \leq 2^{n+2}/R\,; \qquad 0 \leq \zeta_t \leq C 2^n/\theta R^p\,.$$

The levels k are taken to be

$$k_n = \mu^+ - \frac{\omega}{2^{s^*+1}} - \frac{\omega}{2^{s^*+n}}, \qquad n = 1, 2, \ldots .$$

In this setting (1.5) can be written as

$$\begin{aligned}\sup_{t_0-(\alpha_0/3)\theta\bar{R}_n^p \leq t \leq t_0} &\|(u - k_n)^+\|^2_{2,B(\bar{R}_n)}(t) + \|\nabla_x(u - k_n)^+\|^p_{p,Q^\theta_{\bar{R}_n}(\alpha_0)} \\ &\leq C\frac{2^{pn}}{R^p}\left\{\|(u - k_n)^+\|^p_{p,Q^\theta_{R_n}(\alpha_0)} + \theta^{-1}\|(u - k_n)^+\|^2_{2,Q^\theta_{R_n}(\alpha_0)}\right\} \\ &\quad + C\left(\int_{t_0-(\alpha_0/3)\theta R_n^p}^{t_0} |A^+_{k_n,R_n}(\tau)|^{r/q}\, d\tau\right)^{(p/r)(1+\varkappa)}.\end{aligned} \tag{4.24}$$

We estimate the various terms in (4.24), recalling the definition of θ as follows. First for all $t\in[t_0-(\alpha_0/3)\theta\bar{R}_n^p, t_0]$

$$\|(u-k_n)^+\|^2_{2,B(\bar{R}_n)}(t)\geqslant\theta\|(u-k_n)^+\|^p_{p,\bar{B}(R_n)}(t)\,.$$

Next

$$\left\{\|(u-k_n)^+\|^p_{p,Q^\theta_{R_n}(\alpha_0)}+\theta^{-1}\|(u-k_n)^+\|^2_{2,Q^\theta_{R_n}(\alpha_0)}\right\}\leqslant 2\left(\frac{\omega}{2^{s^*}}\right)^p\int\limits_{t_0-(\alpha_0/3)\theta R_n^p}^{t_0}|A^+_{k_n,R_n}(\tau)|d\tau\,.$$

Then from (4.24) dividing by θ

$$\sup_{t_0-(\alpha_0/3)\theta\bar{R}_n^p\leqslant t\leqslant t_0}\|(u-k_n)^+\|^p_{p,B(\bar{R}_n)}(t)+\theta^{-1}\|\nabla_x(u-k_n)^+\|^p_{p,Q^\theta_{\bar{R}_n}(\alpha_0)}\leqslant C\frac{2^{np}}{R^p}\left(\frac{\omega}{2^{s^*}}\right)^p\frac{1}{\theta}\int\limits_{t_0-(\alpha_0/3)\theta R_n^p}^{t_0}|A^+_{k_n,R_n}(t)|\,dt$$
$$+\,C\left(\frac{1}{\theta}\int\limits_{t_0-(\alpha_0/3)\theta R_n^p}^{t_0}|A^+_{k_n,R_n}(t)|^{r/q}\,dt\right)^{(p/r)(1+\varkappa)}\theta^{p((1+\varkappa)/r-1/p)^+}.\tag{4.25}$$

The change of variable $z=3(t-t_0)/\alpha_0\theta$, transforms $Q^\theta_{R_n}(\alpha_0)$ and $Q^\theta_{\bar{R}_n}(\alpha_0)$ respectively into

$$Q_n\equiv B(R_n)\times\{-R_n^p, t_0\}\,;\qquad \bar{Q}_n\equiv B(\bar{R}_n)\times\{-\bar{R}_n^p, 0\}\,.$$

We also set $v(x,z)=u(x, t_0+(\alpha_0/3)\theta z\}$ and,

$$A_n(z)\equiv\{x\in B(R_n)\colon v(x,z)>k_n\}\qquad A_n\equiv\int\limits_{-R_n^p}^{0}|A_n(z)|dz\,.$$

Then (4.25) can be rewritten more concisely as

$$\|(v-k_n)^+\|^p_{V_p(\bar{Q}_n)}\leqslant\frac{C}{R^p}2^{np}\left(\frac{\omega}{2^{s^*}}\right)^pA_n+C\left(\int\limits_{-R_n^p}^{0}|A_n(z)|^{r/q}\,dz\right)^{(p/r)(1+\varkappa)}\theta^{p((1+\varkappa)/p-1/p)^+}.\tag{4.26}$$

Let $x\to\zeta_n(x)$ be a cutoff function in $B(\bar{R}_n)$ which equals one on $B(R_{n+1})$

and $|\nabla_x \zeta_n| \leqslant 2^{n+2}/R$. Then $(v-k_n)^+\zeta_n \in \mathring{V}_p(\bar{Q}_n)$ and by Corollary 2.4

$$\|(v-k_n)^+\|^p_{p,Q_{n+1}} \leqslant \|(v-k_n)^+\zeta_n\|^p_{p,\bar{Q}_n} \leqslant CA_n^{p/(N+p)}\|(v-k_n)^+\zeta_n\|^p_{\mathring{V}_p(\bar{Q}_n)} \tag{4.27}$$

$$\leqslant CA_n^{p/(N+p)}\left(\|(v-k_n)^+\|^p_{V_p(\bar{Q}_n)} + \frac{2^{p(n+2)}}{R^p}\|(v-k_n)^+\|^p_{p,Q_n}\right).$$

Using (4.26) we find

$$\|(v-k_n)^+\|^p_{p,Q_{n+1}} \leqslant C\frac{2^{np}}{R^p}\left(\frac{\omega}{2^{s^*}}\right)^p A_n^{1+p/(N+p)} \tag{4.28}$$

$$+ CA_n^{p/(N+p)}\left(\int_{-R_n^p}^0 |A_n(z)|^{r/q}dz\right)^{(p/r)(1+\varkappa)}\theta^{p((1+\varkappa)/r-1/p)^+}.$$

Since

$$\|(v-k_n)^+\|^p_{p,Q_{n+1}} \geqslant \frac{1}{2}\left(\frac{\omega}{2^{s^*}}\right)^p A_{n+1},$$

recalling the definition of θ, from (4.28) we obtain

$$A_{n+1} \leqslant C\frac{4^{pn}}{R^p}A_n^{1+p/(N+p)} + C4^{pn}A_n^{p(p/p)}\left(\int_{-R_n^p}^0 [A_n(z)]^{r/q}dz\right)^{(p/r)(1+\varkappa)} \tag{4.29}$$

$$\cdot\left(\frac{2^{s^*}}{\omega}\right)^{p[1+((1+\varkappa)/r-1/p)^+(p-2)]}.$$

Set

$$Y_n = \frac{A_n}{|Q_n|}; \qquad Z_n = \frac{1}{|B(R)_n|}\left(\int_{-R_n^p}^0 |A_n(z)|^{r/q}dz\right)^{p/r}.$$

Then proceeding as in the proof of lemma 3.1, if (4.22) is violated we have the recursion inequalities

$$Y_{n+1} \leqslant C4^{pn}\{Y_n^{1+p/(N+p)} + Y^{p/(N+p)}Z_n^{1+\varkappa}\} \qquad Z_{n+1} \leqslant C4^{pn}\{Y_n + Z_n^{1+\varkappa}\}.$$

It follows from these, with the aid of lemma 5.7 of [11] page 96 that Y_n, $Z_n \to 0$ as $n \to \infty$ if

$$Y_1 \leqslant \beta_0; \qquad Z_1^{1+\varkappa} \leqslant \beta_0$$

where

$$\beta_0 = \min\left\{(4C)^{-(N+p)/p}4^{-(N+p)/d}; \qquad (4C)^{-(1+\varkappa)/\varkappa}4^{-p/\varkappa d}\right\} \tag{4.30}$$

$$d = \min\left\{\frac{p}{N+p}; \frac{\varkappa}{1+\varkappa}\right\}.$$

Therefore to prove the lemma we choose β_0 according to (4.30) and then s^* so large that (4.21) is verified for this choice of β_0.

Arguing as in Proposition 3.1 we can now summarize the results of this section.

PROPOSITION 4.1. *There exists a positive integer s^* independent of ω, R such that if* (4.2) *holds for every cylinder $\bar{Q}_R^\eta \subset Q_R^\theta$, $\theta = (2^{s^*}/\omega)^{p-2}$, then either*

$$\omega < 2^{s^*+1} R^{(N\varkappa/p)\xi}, \qquad \xi^{-1} = 1 + \left(\frac{1+\varkappa}{r} - \frac{1}{p}\right)^+ (p-2) \tag{4.31}$$

or

$$\operatorname*{ess\,osc}_{Q^\theta_{R/2}(\alpha_0)} u \leqslant \omega\left(1 - \frac{1}{2^{s^*+1}}\right), \tag{4.32}$$

where α_0 is the number claimed by Proposition 3.1 *and*

$$Q^\theta_{R/2}(\alpha_0) \equiv B(R/2) \times \left\{t_0 - \frac{\alpha_0}{3}\,\theta\left(\frac{R}{2}\right)^p,\, t_0\right\}.$$

5. – Proof of the embedding theorem.

First we remark that the proof presented only uses the fact that the essential oscillation of u in Q_R^θ, $\theta = (2^{s^*}/\omega)^{p-2}$ is less than ω. Since this is not a priori guaranteed we used the device of introducing the cylinder $Q_R^{N\varkappa}$ (see (3.1)) to claim that if Q_R^θ is not included in $Q_R^{N\varkappa}$

$$\operatorname*{ess\,osc}_{Q_R^{N\varkappa}} u \leqslant \omega \leqslant 2^{s^*} R^{N\varkappa/p}$$

Keeping this in mind we now iterate the process described, over a sequence of nested and shrinking cylinders.

Let $\bar{s} = \max\{s; s^*+1\}$ where s is the number claimed by Proposition 3.1, and set

$$\eta_0 = 1 - \frac{1}{2^{\bar{s}}}; \qquad \delta_0 = \frac{N\varkappa}{p}\,\xi; \qquad C_0 = 2^{\bar{s}}.$$

All these numbers are independent of ω, R. Setting

$$2^{\sigma_0(p-2)} = \min\left\{2^{s_0(p-2)}; \frac{\alpha_0}{3}\, 2^{s^*(p-2)}\right\},$$

both Proposition 3.1 and 4.1 can be combined by stating that in either case we have the following alternative. Either

$$\omega \leq C_0 R^{\delta_0} \tag{5.1}$$

or

$$\operatorname*{ess\,osc}_{Q^{\sigma_0}} u \leq \omega\eta_0 = \omega_1 \tag{5.2}$$

where

$$Q^{\sigma_0} \equiv B\left(\frac{R}{8}\right) \times \left\{t_0 - \left(\frac{2^{\sigma_0}}{\omega}\right)^{p-2}\left(\frac{R}{8}\right)^p, t_0\right\}.$$

Obviously (5.2) remains valid if we take the essential oscillation of u over a cylinder contained in Q^{σ_0}.

We set $R_0 = 2R$, and

$$R_1 = \frac{R_0}{[2^{s^*-\sigma_0}/\eta_0]^{(p-2)/p4^2}} = \frac{1}{C_1} R_0 .$$

Then the cylinder

$$Q_{R_1}^{\theta_1} \equiv B(R_1) \times \{t_0 - \theta_1 R_1^p, t_0\} ; \qquad \theta_1 = \left(\frac{2^{s^*}}{\omega}\right)^{p-2}$$

is contained in Q^{σ_0} and we have

$$\operatorname*{ess\,osc}_{Q_{R_1}^{\theta_1}} u \leq \omega_1 = \gamma_0 .$$

Therefore the process can be continued starting from the cylinder $Q_{R_1}^{\theta_1}$. By iteration we define sequences

$$R_0 = 2R ; \qquad \omega_0 = \operatorname*{ess\,osc}_{Q_{R_0}^{N\varkappa}} u ; \qquad \theta_0 = \left(\frac{2^{s^*}}{\omega_0}\right)^{p-2}$$

$$R_n = \frac{1}{C_1^n} R_0 ; \qquad \omega_n = \eta_0 \omega_{n-1} ; \qquad \theta_n = \left(\frac{2^{s^*}}{\omega_n}\right)^{p-2}$$

and the cylinders $Q_{R_n}^{\theta_n} \equiv B(R_n) \times \{t_0 - \theta_n R_n^p, t_0\}$.

For them the following iteration holds.

Either

$$\omega_n \leq C_0 R_n^{\delta_0} \tag{5.3}$$

or

$$\operatorname*{ess\,osc}_{Q_{R_n}^{\theta_n}} u \leq \omega_n = \eta_0 \omega_{n-1} . \tag{5.4}$$

The theorem is now a straightforward consequence of lemma 5.8 of [11] page 96.

Part II - BOUNDARY REGULARITY

We say that a function $u: \Omega_T \to \mathbf{R}$ belongs to the class $\mathcal{B}_p(\Omega_T \cap \Gamma, M, \gamma, r, \delta, \varkappa)$ if u satisfies all the requirements listed in section 1, with the only difference that the cylinders $Q(R, \varrho)$ may intersect Γ, and the various integrals in (1.5)-(1.6) are extended over $Q(R, \varrho) \cap \Omega_T$ and $B(R) \cap \Omega$. We impose an extra requirement.

The cutoff function $(x, t) \to \zeta(x, t)$ vanishes on $\partial B(R)$, or on the parabolic boundary of $Q(R, \varrho)$, but it does not vanish on Γ. Because of this, a function u belongs to $\mathcal{B}_p(\Omega_T \cap \Gamma, M, \gamma, r, \delta, \varkappa)$ if (1.5)-(1.6) hold for all the levels k for which

$$(u - k)^{\pm} \zeta^p = 0 \qquad \text{on } \Gamma .$$

Given such a requirement it is immediate to see, by following the same arguments of section 1, that a weak solution u of (0.1) defined in $\bar{\Omega}_T$, belongs to $\mathcal{B}_p(\Omega_T \cap \Gamma, M, \gamma, r, \delta, \varkappa)$.

The proof of regularity up to the boundary is based again on inequalities (1.5)-(1.6). In fact it is much simpler since we may simplify such inequalities by making use of the information coming from the boundary data.

6. - Proof of Theorem 2.

Let $x_0 \in \mathring{\Omega}$ be fixed and let $R > 0$ be so small that $B(R) \subset \Omega$. We consider also the cylinder

$$Q(R) \equiv B(R) \times [0, R^p] .$$

As before we set

$$\mu^+ = \operatorname*{ess\,sup}_{Q(R)} u ; \qquad \mu^- = \operatorname*{ess\,inf}_{Q(R)} u ; \qquad \omega = \operatorname*{ess\,osc}_{Q(R)} u .$$

If the initial datum u_0 satisfies $[A_6]$ we set

$$\mu_0^+ = \operatorname*{ess\,sup}_{B(R)} u_0 ; \qquad \mu_0^- = \operatorname*{ess\,inf}_{B(R)} u_0 ; \qquad \omega_0 = \operatorname*{ess\,osc}_{B(R)} u_0 .$$

Let s_0 be the smallest positive integer satisfying

$$\frac{2M}{2^{s_0}} \leqslant \delta \,, \tag{6.1}$$

where δ is the number introduced in (1.15). We consider the following two cases.

Case 1. The inequalities

$$\mu^+ - \frac{\omega}{2^{s_0}} < \mu_0^+ ; \qquad \mu^- + \frac{\omega}{2^{s_0}} > \mu_0^- \tag{6.2}$$

both hold, or

Case 2. At least one of (6.2) is violated.

In case 1, subtracting the second inequality from the first we obtain

$$\operatorname*{ess\,osc}_{Q(R)} u \leqslant 2 \operatorname*{ess\,osc}_{B(R)} u_0 \,. \tag{6.3}$$

To examine Case 2, suppose for example that the second of (6.2) is violated. Then

$$\left(u - \left(\mu^- + \frac{\omega}{2^{s_0}}\right)\right)^-(x, 0) = 0 \,, \qquad \forall x \in B(R) \,, \qquad \forall s \geqslant s_0 \,. \tag{6.4}$$

Let $x \to \zeta(x)$ be a smooth cutoff function in $B(R)$ which equals one on $B(R - \sigma R)$, $\sigma \in (0, 1)$ and such that $|\nabla_x \zeta| \leqslant (\sigma R)^{-1}$. Then, proceeding as in section 1 and making us of (6.4) we deduce that the following two inequalities hold.

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant R^p} \left\| \left(u - \left(\mu^- + \frac{\omega}{2^s}\right)\right)^- \right\|^2_{2, B(R - \sigma R)} \\ &+ \int_0^{R^p} \int_{B(R - \sigma R)} \left| \nabla_x \left(u - \left(\mu^- + \frac{\omega}{2^s}\right)\right)^- \right|^p dx\, d\tau \leqslant \frac{\gamma}{(\sigma R)^p} \iint_{Q(R)} \left| \left(u - \left(\mu^- + \frac{\omega}{2^s}\right)\right)^- \right|^p dx\, d\tau \\ &+ \gamma \left(\int_0^{R^p} |A^-_{\mu^- + \omega/2^s, R}(\tau)|^{r/q}\, d\tau \right)^{(p/r)(1 + \varkappa)} . \end{aligned} \tag{6.5}$$

$$(6.6)\qquad \sup_{0\leqslant t\leqslant R^p}\int_{B(R-\sigma R)\times\{t\}}\psi^2\left(H^-,\left(u-\left(\mu^-+\frac{\omega}{2^s}\right)\right)^-,\nu\right)dx$$
$$\leqslant\frac{\gamma}{(\sigma R)^p}\iint_{Q(R)}\psi\left(H^-,\left(u-\left(\mu^-+\frac{\omega}{2^s}\right)\right)^-,\nu\right)$$
$$\times\left|\psi_u\left(H^-,\left(u-\left(\mu^-+\frac{\omega}{2^s}\right)\right)^-,\nu\right)\right|^{2-p}dx\,d\tau$$
$$+\frac{\gamma}{\nu^p}\left(\ln\frac{H^-}{\nu}\right)\left(\int_0^{R^p}|A^-_{\mu^-+\omega/2^s,R}(\tau)|^{r/q}d\tau\right)^{(p/r)(1+\varkappa)}.$$

These inequalities hold (in view of (6.4)) for all $s\geqslant s_0$.

The proof can now be completed as follows. First, using (6.6) and the procedure of lemma 3.2, given any $\alpha_1\in(0,1)$ we can find a positive integer s_1 such that either

$$(6.7)\qquad R^{(N\varkappa/p)\xi}\geqslant\omega/2^{s_1+1};\qquad \xi^{-1}=1+\left(\frac{1+\varkappa}{r}-\frac{1}{p}\right)^+(p-2)$$

or

$$(6.8)\qquad \left|(x,t)\in B(R/2)\times[0,R^p]:\ u(x,t)<\mu^-+\frac{\omega}{2^{s_1}}\right|\leqslant\alpha_1|B(R/2)\times[0,R^p]|.$$

Second, using (6.5) and the procedure of lemma 3.3 we deduce that either (6.7) holds or

$$(6.9)\qquad u(x,t)>\mu^-+\frac{\omega}{2^{s_1+1}},\qquad \forall(x,t)\in B(R/4)\times[0,R^p].$$

These facts are much easier to establish than the corresponding ones in the quoted lemmas. In particular in establishing (6.9) no shrinking occurs in the t-direction. This is due to (6.4), and the relatively simple form of (6.5)-(6.6).

Combining these remarks and recalling the definition of $Q(R)$ we deduce that

$$(6.10)\qquad \operatorname*{ess\,osc}_{Q(R/4)}u\leqslant\max\left\{\eta\operatorname*{ess\,osc}_{Q(R)}u;\ CR^{(N\varkappa/p)\xi};\ \operatorname*{ess\,osc}_{B(R)}u_0\right\},$$

where

$$\eta=1-\frac{1}{2^{s_1+1}};\qquad C=2^{s_1+1};\qquad \xi^{-1}=1+\left(\frac{1+\varkappa}{r}-\frac{1}{p}\right)^+(p-2).$$

Since this estimate can be reproduced over a sequence of cylinders $Q(R/4^n)$, $n=1,2,\ldots$, with the same constants η, C, ξ, standard arguments imply Theorem 2.

7. – Proof of Theorem 3.

Let $(x_0, t_0) \in S_T$ be fixed and consider the cylinder

$$Q_{2R}^{\varepsilon} \equiv B(2R) \times \{t_0 - (2R)^{p-\varepsilon}, t_0\}$$

where

$$\varepsilon = \frac{N\varkappa}{p}\xi(p-2); \qquad \xi^{-1} = 1 + \left(\frac{1+\varkappa}{r} - \frac{1}{p}\right)^{+}(p-2).$$

We let R be so small that $t_0 - (2R)^{p-\varepsilon} > 0$ and define

$$\mu^+ = \operatorname*{ess\,sup}_{Q_{2R}^{\varepsilon}\cap\Omega_T} u; \qquad \mu^- = \operatorname*{ess\,inf}_{Q_{2R}^{\varepsilon}\cap\Omega_T} u; \qquad \omega = \mu^+ - \mu^- = \operatorname*{ess\,osc}_{Q_{2R}^{\varepsilon}\cap\Omega_T} u. \tag{7.1}$$

If the boundary datum f satisfies $[A_8]$ we let

$$\mu_f^+ = \operatorname*{ess\,sup}_{Q_{2R}\cap S_T} f; \qquad \mu_f^- = \operatorname*{ess\,inf}_{Q_{2R}\cap S_T} f; \qquad \omega_f = \operatorname*{ess\,osc}_{Q_{2R}\cap S_T} f. \tag{7.2}$$

Define also the cylinders

$$Q_{\varrho}^{\theta} \equiv B(\varrho) \times \{t_0 - \theta\varrho^p, t_0\}, \qquad 0 < \varrho \leqslant 2R$$

$$Q_{\varrho}^{\theta}(\sigma_1, \sigma_2) \equiv B(\varrho - \sigma_1\varrho) \times \{t_0 - \theta(1-\sigma_2)\varrho^p, t_0\}$$

where $\sigma_i \in (0,1)$, $i=1,2$ and

$$\theta = \left(\frac{2^{s^*}}{\omega}\right)^{p-2} \tag{7.3}$$

and s^* is a large positive integer to be chosen.

If $\theta \geqslant (2R)^{-\varepsilon}$ we have

$$\omega \leqslant 2^{s^*}(2R)^{\varepsilon/(p-2)} = 2^{s^*}(2R)^{(N\varkappa/p)\xi}. \tag{7.4}$$

If (7.4) does not hold, then $\theta < (2R)^{-\varepsilon}$ and

$$Q_{2R}^{\theta} \subset Q_{2R}^{\varepsilon}.$$

We will assume that such inclusion holds, in what follows.

Defining s_0 as in (6.1), we may also assume that at least one of the two inequalities

$$\mu^+ - \frac{\omega}{2^{s_0}} \leqslant \mu_f^+ ; \qquad \mu^- + \frac{\omega}{2^{s_0}} \geqslant \mu_f^- \tag{7.5}$$

does not hold. In fact if both are satisfied we have

$$\operatorname*{ess\,osc}_{Q^\varepsilon_{2R}} u \leqslant 2 \operatorname*{ess\,osc}_{Q^\varepsilon_{2R}\cap S_T} f . \tag{7.6}$$

Let us assume that for example the first of (7.5) is violated. Then $\forall s \geqslant s_0$

$$\left(u - \left(\mu^+ - \frac{\omega}{2^s}\right)\right)^+ = 0 \qquad \text{on} \qquad S_T \cap Q^\varepsilon_{2R} . \tag{7.7}$$

Proceedings as in section 1 and using (7.7) we see that the following inequalities are valid $\forall s \geqslant s_0$ and $\forall 0 < \varrho \leqslant 2R$.

$$\begin{aligned}
&\sup_{t_0 - (1-\sigma_2)\theta\varrho^p \leqslant t \leqslant t_0} \left\| \left(u - \left(\mu^+ - \frac{\omega}{2^s}\right)\right)^+ \right\|^2_{2, B(\varrho - \sigma_1\varrho\ \cap \Omega} (t) \\
&+ \left\| \nabla_x \left(u - \left(\mu^+ - \frac{\omega}{2^s}\right)\right)^+ \right\|^p_{p, Q^\theta_\varrho(\sigma_1, \sigma_2)\cap\Omega_T} \leqslant \frac{\gamma}{(\sigma_1\varrho)^p} \left\| \left(u - \left(\mu^+ - \frac{\omega}{2^s}\right)\right)^+ \right\|^p_{p, Q^\theta_\varrho \cap \Omega_T} \\
&+ \frac{\gamma}{\sigma_2\theta\varrho^p} \left\| u - \left(\mu^+ - \left(\frac{\omega}{2^s}\right)\right)^+ \right\|^2_{2, Q_\varrho \cap \Omega_T} + \gamma \left(\int_{t_0 - \theta\varrho^p}^{t_0} |A_{\mu^+ - \omega/2^s, \varrho}(\tau)|^{r/q} d\tau \right)^{(p/r)(1+\varkappa)} .
\end{aligned} \tag{7.8}$$

Since $(u - (\mu^+ - \omega/2^s))^+$ vanishes on $S_T \cap Q^\theta_{2R}$, we may extend $(u - (\mu^+ - \omega/2^s))^+$ with zero outside Q^θ_ϱ, $0 < \varrho \leqslant 2R$, and therefore the domains of integration in (7.8) may be considered to be $Q^\theta_\varrho(\sigma_1, \sigma_2)$, Q^θ_ϱ, $B(\varrho - \sigma_1\varrho)$.

By virtue of assumption [A_7], for all $t \in [t_0 - \theta\varrho^p, t_0]$

$$\left| x \in B(\varrho) \colon u(x, t) > \mu^+ - \frac{\omega}{2^s} \right| < (1 - \alpha_*) |B(\varrho)| , \qquad \forall \varrho \leqslant 2R .$$

Consequently the assumptions of lemma 4.4 are verified, and given $\beta_0 \in (0, 1)$ we may find $s^* \in \mathbf{N}$ such that either

$$\frac{\omega}{2^{s^*+1}} \leqslant R^{(N\varkappa/p)\xi} \quad \xi^{-1} = 1 + \left(\frac{1+\varkappa}{r} - \frac{1}{p}\right)^+ (p-2) , \tag{7.9}$$

or

$$\left|(x, t) \in Q_R^\theta\colon u(x, t) > \mu^+ - \frac{\omega}{2^{s^*}}\right| < \beta_0 |Q_R^\theta| . \tag{7.10}$$

REMARK. The choice of s^* will determine also the size of the cylinder Q_R^θ (see (7.3)). As shown in lemma 4.4 such a choice can be made a priori, independent of ω and R.

Finally by the method of lemma 4.5, and using inequalities (7.8), we conclude that either (7.9) holds or

$$u(x, t) \leqslant \mu^+ - \frac{\omega}{2^{s^*+1}}, \qquad \forall (x, t) \in Q_{R/2}^\theta .$$

Combining the various alternatives presented, we have

$$\operatorname*{ess\,osc}_{Q_{R/2}^\theta \cap \Omega_T} u \leqslant \max\left\{\eta \operatorname*{ess\,osc}_{Q_{2R}^\varepsilon} u;\ C R^{(N\varkappa/p)\xi};\ \operatorname*{ess\,osc}_{Q_{2R}^\varepsilon \cap S_T} f\right\} \tag{7.11}$$

where

$$\eta = 1 - \frac{1}{2^{s^*+1}}; \qquad C = 2^{s^*+1}; \qquad \xi^{-1} = 1 + \left(\frac{1+\varkappa}{r} - \frac{1}{p}\right)^+ (p-2) .$$

Interation of (7.11) yields Theorem 3.

REMARK. The proof of Corollary 0.1 follows from the previous arguments except for proving regularity at points $(x_0, 0) \in \partial\Omega \times \{0\}$. The latter case can be demonstrated by a straightforward adaptation of the previous methods.

8. – Proof of Theorem 4.

The proof is essentially the same as for the interior regularity and it is based on the arguments of sections 2-5, except that rather than working with cylinders of the type $Q(R, \varrho) \equiv B(R) \times \{t_0 - \varrho, t_0\}$ we will be working with cylinders $C(R, \varrho) \equiv B(R) \cap \Omega \times \{t_0 - \varrho, t_0\}$.

First we indicate how to derive inequalities analogous to (1.5)-(1.6).

Let $x_0 \in \partial\Omega$ be fixed and consider the portion of the boundary $\partial\Omega$ given by

$$S_0(R) \equiv \partial\Omega \cap \{|x - x_0| < R\} .$$

Since $\partial\Omega$ is of class C^1 and our arguments are local in nature, we may assume, without loss of generality that $S_0(R)$ lies on the hyperplane $x_N = 0$ and that for example

$$B(R) \cap \Omega \subset \{x_N > 0\} .$$

If $(x_0, t_0) \in S_T$, consider the cylinder

$$C(R, \varrho) \equiv \{B(R) \cap \Omega\} \times \{t_0 - \varrho, t_0\} , \tag{8.1}$$

where $\varrho > 0$ is so small that $t_0 - \varrho > 0$.

Let $(x, t) \to \zeta(x, t)$ be a piecewise smooth function defined in $Q(R, \varrho)$ such that $0 \leqslant \zeta \leqslant 1$ and $\zeta(x, \cdot) = 0$ for $x \in \partial B(R)$. We observe that ζ vanishes on the lateral boundary of $Q(R, \varrho)$ and not on the lateral boundary of $C(R, \varrho)$. We write (0.16) is terms of the Steklov averaging and take test functions of the type

$$\pm (u_h - k)^{\pm} \zeta^p$$

where $k \in \mathbf{R}$ satisfies the restriction

$$\|(u - k)^{\pm}\|_{\infty, C(R, \varrho)} \leqslant \delta$$

and δ is defined in (1.15). Performing exactly the same calculations and limiting processes described in the proof of Proposition 1.1, we arrive at inequality (1.5), with the domains of integrations being now $B(R) \cap \Omega$ and $C(R, \varrho)$, and with, on the right hand side the extra boundary integral

$$\Lambda = \int_{t_0 - \varrho}^{t_0} \int_{S_0(R)} \pm g(x, \tau, u)(u - k)^{\pm} \zeta^p \, d\sigma \, d\tau .$$

This last integral is estimated by making use of assumption $[A_{10}]$ and the fact that $u \in L^\infty(\Omega_T)$ as follows.

$$\Lambda \leqslant \Big| \iint_{C(R,\varrho)} \operatorname{div} [(u - k)^{\pm} \zeta^p] \, dx \, d\tau \Big|$$

$$\leqslant \gamma \iint_{C(R,\varrho)} \{(u - k)^{\pm} (\zeta^p + (p - 1) \zeta^{p-1} |\nabla_x \zeta|) + |\nabla_x (u - k)^{\pm}| \zeta^p\} \, dx \, d\tau .$$

By Young's inequality, $\forall \varepsilon > 0$

$$\Lambda \leqslant \varepsilon \iint_{C(R,\varrho)} |\nabla_x (u - k)^{\pm}|^p \zeta^p \, dx \, d\tau +$$

$$+ \gamma \iint_{C(R,\varrho)} |(u - k)^{\pm}|^p |\nabla_x \zeta|^p \, dx \, d\tau + \gamma(\varepsilon) \int_{t_0 - \varrho}^{t_0} [\operatorname{meas} A^{\pm}_{k,R}(\tau)] \, d\tau .$$

where

$$A^{\pm}_{k,R}(t) \equiv \{x \in B(R) \cap \Omega \colon (u-k)^{\pm}(x,t) > 0\} .$$

Combining these estimates, we see that the following inequalities are valid

$$\begin{aligned}(8.2)\qquad &\sup_{t_0-\varrho\leqslant t\leqslant t_0}\int_{B(R)\cap\Omega}[(u-k)^{\pm}]^2\zeta^p(x,t)\,dx+\iint_{C(R,\varrho)}|\nabla_x(u-k)^{\pm}|^p\zeta^p\,dx\,d\tau\\ &\leqslant\int_{B(R)\cap\Omega}[(u-k)^{\pm}]^2\zeta^p(x,t_0-\varrho)\,dx+\gamma\iint_{C(R,\varrho)}[(u-k)^{\pm}]^p|\nabla_x\zeta|^p\,dx\,d\tau\\ &+\iint_{C(R,\varrho)}[(u-k)^{\pm}]^2\zeta^{p-1}\zeta_t\,dx\,d\tau+\gamma\left(\int_{t_0-\varrho}^{t_0}[\operatorname{meas}A^{\pm}_{k,R}(\tau)]^{r/q}\,d\tau\right)^{(p/r)(1+\varkappa)}\\ &+\gamma\int_{t_0-\varrho}^{t_0}[\operatorname{meas}A^{\pm}_{k,R}(\tau)]\,d\tau\,.\end{aligned}$$

In order to derive an inequality similar to (1.6) we proceed as in the proof of Proposition 1.1 and in addition we treat the boundary integral

$$\int_{t_0-\varrho}^{t_0}\int_{S_0(R)}g(x,\tau,u)\,\psi\psi'\,\zeta^p\,d\sigma\,d\tau\,,$$

by transforming it into an interior integral over $C(R,\varrho)$ as indicated above. As a result we obtain the inequalities

$$\begin{aligned}(8.3)\qquad &\sup_{t_0-\tau\leqslant t\leqslant t_0}\int_{B(R)\cap\Omega}\psi^2(H^{\pm},(u-k)^{\pm},\nu)\,\zeta^p(x,t)\,dx\\ &\leqslant\int_{B(R)\cap\Omega}\psi^2(H^{\pm},(u-k)^{\pm},\nu)\,\zeta^p(x,t_0-\varrho)\,dx\\ &+\gamma\iint_{C(R,\varrho)}\psi(H^{\pm},(u-k)^{\pm},\nu)\,|\psi_u(H^{\pm},(u-k)^{\pm},\nu)|^{2-p}|\nabla_x\zeta|^p\,dx\,dt\\ &+\frac{\gamma}{\nu^p}\left(\ln\frac{H^{\pm}}{\nu}\right)\left\{\left(\int_{t_0-\varrho}^{t_0}[\operatorname{meas}A^{\pm}_{k,R}(\tau)]^{r/q}\,d\tau\right)^{(p/r)(1+\varkappa)}+\int_{t_0-\varrho}^{t_0}[\operatorname{meas}A^{\pm}_{k,R}(\tau)]\,d\tau\right\}.\end{aligned}$$

With these inequalities at hand, the proof can now be completed exactly, step by step, as in the proof of interior regularity. The only significant modification regards the proof of the recursion inequalities (3.8)-(3.9) in lemma 3.1 (and similar inequalities in lemmas 3.3 and 4.5). For these we used the embedding of Corollary 2.4 valid for functions $u \in \mathring{V}_p\big(C(R, \varrho)\big)$. In our case $(u-k)^{\pm}\zeta^p$ does not vanish on the lateral boundary of $C(R, \varrho)$ and therefore we must use (2.8) with the constant C given by (2.9). We observe however that for domains of the type $\{B(R) \cap \Omega\} \times \{-R^p, 0\}$, the constant in (2.9) is independent of R.

Finally the last modification occurs in the use of De Giorgi's inequality (2.7) (employed in lemma 4.4).

Now such inequality holds for convex domains (see Remark 2.1) and therefore (2.7) holds with $B(R)$ replaced by $B(R) \cap \Omega \equiv B(R) \cap \{x_N > 0\}$. The remainder of the proof stays unchanged.

REFERENCES

[1] N. D. Alikakos - R. Rostamian, *Gradient estimates for degenerate diffusion equations II*, Proc. Royal Soc. Edinburgh A, **91** (1982), pp. 335-346.

[2] D. G. Aronson - J. Serrin, *Local behaviour of solutions of quasilinear parabolic equations*, Arch. Rational Mech. Anal., **25** (1967), pp. 81-123.

[3] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Acc. Sci. Torino, Cl. Sci. Fis. Mat. Natur., (3) **3** (1957), pp. 25-43.

[4] E. DiBenedetto, *Continuity of weak solutions to a general porous medium equation*, Indiana Univ. Math. J., **32**, no. 1 (1983), pp. 83-117.

[5] E. DiBenedetto, *A boundary modulus of continuity for a class of singular parabolic equations*, J. Differential Equations, **63**, no. 3 (1986), pp. 418-447.

[6] E. DiBenedetto - A. Friedman, *Regularity of solutions of nonlinear degenerate parabolic systems*, J. Reine Angew. Math., **349** (1984), pp. 83-128.

[7] E. DiBenedetto - A. Friedman, *Hölder estimates for nonlinear degenerate parabolic systems*, J. Reine Math., **357** (1985), pp. 1-22.

[8] M. Giaquinta, *Multiple integrals in the calculus variations and nonlinear elliptic systems*, Annals of Mathematics Studies, Princeton, New Jersey (1983).

[9] M. Giaquinta - E. Giusti, *On the regularity of the minima of variational integrals*, Acta Math., **148** (1982), pp. 31-46.

[10] S. N. Kružkov, *Results concerning the nature of the continuity of solutions of parabolic equations and some of their applications*, Mat. Zametki, **6** (1969), pp. 97-108 (Russian).

[11] O. A. Ladyzhenskaya - N. A. Solonnikov - N. N. Ural'tzeva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monographs, **23**, Amer. Math. Soc., Providence, R.I. (1968).

[12] O. A. LADYZHENSKAYA - N. N. URAL'ZEVA, *Linear and quasilinear elliptic equations*, Acadmic Press, New York (1968).

[13] J. MOSER, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math., **13** (1960), pp. 457-468.

[14] J. MOSER, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math., **14** (1961), pp. 577-591.

[15] J. MOSER, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math., **17** (1964), pp. 101-134.

[16] J. SERRIN, *Local behaviour of solutions of quasilinear elliptic equations*, Acta Math., **111** (1964), pp. 302-347.

[17] N. S. TRUDINGER, *On Harnack type inequalities and their applications to quasilinear elliptic partial differential equations*, Comm. Pure Appl. Math., **20** (1967), pp. 721-747.

[18] N. S. TRUDINGER, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math., **21** (1968), pp. 205-226.

[19] K. UHLENBECK, *Regularity for a class of nonlinear elliptic systems*, Acta Math., **138** (1977), pp. 219-240.

[20] G. M. LIEBERMAN, *Interior gradient bounds for nonuniformly parabolic equations*, Indiana Univ. Math. J., **32**, no. 4 (1983), pp. 679-701.

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