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Multiplicity-2 Structures on Castelnuovo Surfaces.

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0. – Introduction.

In this paper we study « nice » multiplicity-2 structures $\tilde{Y}$ on smooth surfaces $Y \subseteq \mathbf{P}_4 = \mathbf{P}_4(\mathbf{C})$. Every multiplicity-2 structures in this sense is given by a quotient $N^*_{Y/\mathbf{P}_4} \to \omega_Y(l)$ and vice versa. The existence of such a quotient for given l imposes rather strong topological conditions on Y. Under suitable conditions the non-reduced structure $\tilde{Y}$ leads to a rank-2 vector bundle E on $\mathbf{P}_4$ with a section s, such that $\tilde{Y} = \{s = 0\}$ (compare [7]).

Here we are interested in the case where E splits, in other words, where $\tilde{Y}$ is a complete intersection. We are particularly interested in the case where Y is a Castelnuovo surface. These surfaces can be characterized by the fact that, for given degree d, their geometric genus is maximal (at least if $d \geqslant 6$). If d is even, then Y is a complete intersection [3], so we only consider Castelnuovo surfaces of odd degree $d = 2b + 1$. Our main result (Theorem 13 below) is a precise description of those Castelnuovo surfaces Y which admit multiplicity-2 structures in our sense; then $\tilde{Y}$ is a complete intersection of type $(2, 2b + 1)$. Many such surfaces exist.

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1. – Multiplicity-2 structures.

Let $Y \subseteq \mathbf{P}_4$ be a smooth surface with ideal sheaf I_Y. We consider certain non-reduced strucutres $\tilde{Y}$ on Y, i.e. ideals $I_{\tilde{Y}} \subseteq I_Y$, with the following properties:

1) $\tilde{Y}$ is a locally complete intersection,

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2) $\tilde{Y}$ has multiplicity 2, i.e. for every point $P \in Y$ and a general plane E through P the local intersection multiplicity

$$i(P;\tilde{Y},E)=\dim_{\mathbf{C}}\mathcal{O}_{P/I(\tilde{Y}\cap E)}=2\ .$$

DEFINITION. A non-reduced strucutre $\tilde{Y}$ on Y with properties (1) and (2) will be called a *multiplicity*-2 *structure* on Y.

LEMMA 1. *If Y and $\tilde{Y}$ are as above then near a point $P \in Y$ there are local coordinates $x_0, \ldots, x_3$ such that $I_Y=(x_0,x_1)$ and $I_{\tilde{Y}}=(x_0,x_1^2)$.*

PROOF. Let E be a general plane through P. Then we can find local coordinates $x_0, \ldots, x_3$ such that $Y=\{x_0=x_1=0\}$ and $E=\{x_2=x_3=0\}$. Now look at the ideal $I_{\tilde{Y}}\subseteq I_Y=(x_0,x_1)$. It is generated by two functions say $I_{\tilde{Y}}=(f,g)$. We can write

$$f=x_0f_0+x_1f_1\ ,\qquad g=x_0g_0+x_1g_1\ .$$

Because of (2) it follows that at least one of the functions f_0, f_1, g_0, g_1 is a unit at P. We may assume $f_0(P)\neq 0$ and introducing $x_0f_0+x_1f_2$ as a new local coordinate we find that $I_{\tilde{Y}}$ is generated by functions of the form

$$f=x_0\ ,\qquad g=x_1g_1$$

where $g_1=g_1(x_1,x_2,x_3)$. Now $g\in I_Y$ since otherwise $\tilde{Y}$ would be generically reduced which contradicts (2). Hence we have $g=x_1^2g_2$ with $g_2=g_2(x_1,x_2,x_3)$. It again follows from (2) that g_2 is a local unit and hence we are done.

Next we observe that $I_Y^2\subseteq I_{\tilde{Y}}$ and that we have an exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_{\tilde{Y}}/I_Y^2 & \longrightarrow & I_Y/I_Y^2 & \longrightarrow & I_Y/I_{\tilde{Y}} & \longrightarrow & 0\\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & M^* & \longrightarrow & N^*_{Y/\mathbf{P}_4} & \longrightarrow & L^* & \longrightarrow & 0
\end{array}$$

which can be interpreted as a sequence of vector bundles on Y. In particular $\tilde{Y}$ defines a quotient $N^*_{Y/\mathbf{P}_4}\to L^*$. Conversely every such quotient defines a non-reduced structure $\tilde{Y}$ by setting

$$I_{\tilde{Y}}:=\ker\,(I_Y\to I_Y/I_Y^2=N^*_{Y/\mathbf{P}_4}\to L^*)\ .$$

Clearly $\tilde{Y}$ fulfills conditions (1) and (2). Hence we can state

LEMMA 2. *To define a multiplicity-2 structure $\tilde{Y}$ on Y is equivalent to defining a subbundle $L \subseteq N_{Y/\mathbf{P}_4}$.*

Since $\tilde{Y}$ is a locally complete intersection it has a dualising sheaf $\omega_{\tilde{Y}}$ which is given by

$$\omega_{\tilde{Y}} = \mathcal{E}xt^2_{\mathcal{O}_{\mathbf{P}_4}}(\mathcal{O}_{\tilde{Y}}, \omega_{\mathbf{P}_4}) = \Lambda^2 N_{\tilde{Y}/\mathbf{P}_4} \otimes \omega_{\mathbf{P}_4}.$$

From now on we assume the following additional property:

$$(3) \qquad \omega_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-l) \quad \text{for some } l \in \mathbb{Z}.$$

LEMMA 3. *If* (3) *holds then*

$$(3') \qquad L^* = \omega_Y(l).$$

PROOF. We have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_Y/I_{\tilde{Y}} & \longrightarrow & \mathcal{O}_{\tilde{Y}} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0 \\ & & \Vert & & & & & & \\ & & L^* & & & & & & \end{array}$$

Applying $\mathcal{E}xt^2_{\mathcal{O}_{\mathbf{P}_4}}(-, \omega_{\mathbf{P}_4})$ we get

$$0 \to \omega_Y \to \omega_{\tilde{Y}} \to L \otimes \omega_Y \to 0 .$$

Tensoring with $\mathcal{O}_{\mathbf{P}_4}(l)$ we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_Y(l) & \longrightarrow & \omega_{\tilde{Y}}(l) & \longrightarrow & L \otimes \omega_Y(l) & \longrightarrow & 0 \\ & & & & \Vert & & & & \\ & & & & \mathcal{O}_{\tilde{Y}} & & & & \end{array}$$

Restricting this sequence to Y the second morphism gives us an isomorphism

$$\mathcal{O}_Y = L \otimes \omega_Y(l)$$

which implies $L^* \cong \omega_Y(l)$.

REMARKS:

(i) The converse implication $(3)' \Rightarrow (3)$ is more difficult. It holds for $l \geqslant 0$ and if $H^1(\omega_Y(l)) = 0$ (see [7]). The latter is automatically satisfied for $l > 0$ by Kodaira's vanishing theorem.

(ii) If there exists a quotient $N^*_{Y/\mathbf{P}_4} \to \omega_Y(l)$, then $c_2(N_{Y/\mathbf{P}_4} \otimes \omega_Y(l)) = 0$. This is equivalent to

$$d^2 + d(l^2 + 5l) + (3l + 5)HK + 2K^2 = 0$$

where d is the degree of Y.

(iii) There are only a few surfaces which admit a quotient $N^*_{Y/\mathbf{P}_4} \to \omega_Y(l)$ for $l \geqslant 0$. They are the complete intersections of type (a, b) with $2a = b < 5$, the cubic ruled surface and the quintic elliptic scroll (see [7]).

2. - Locally free resolutions.

Let $Y \subseteq \mathbf{P}_4$ be a smooth surface and assume that its ideal sheaf I_Y has a locally free resolution

$$0 \to E_1 \to E_0 \to I_Y \to 0 . \tag{4}$$

Dualising this sequence and tensoring it with $\mathcal{O}_{\mathbf{P}_4}(l-5)$ we get a resolution for the twisted canonical bundle $\omega_Y(l)$ which reads as follows

$$0 \to \mathcal{O}_{\mathbf{P}_4}(l-5) \to E_0^*(l-5) \to E_1^*(l-5) \to \omega_Y(l) \to 0 . \tag{5}$$

We are interested in epimorphisms $I_Y \to \omega_Y(l)$. Every such epimorphism defines a quotient $N^*_{Y/\mathbf{P}_4} \to \omega_Y(l)$.

LEMMA 4. *If there is an epimorphism $\Gamma\colon E_0 \to E_1^*(l-5)$ such that the diagram*

$$\begin{array}{ccc} E_1 & \xrightarrow{\ s\ } & E_0 \\ \big\downarrow{\scriptstyle \Gamma^*(l-5)} & & \big\downarrow{\scriptstyle \Gamma} \\ E_0^*(l-5) & \xrightarrow{s^*(l-5)} & E_1^*(l-5) \end{array}$$

commutes, then Γ induces an epimorphism $\gamma\colon I_Y \to \omega_Y(l)$.

PROOF. Let $c_1 := c_1(E_1)$ and $r := \operatorname{rank} E_1$. From (4) and (5) we get the following « standard diagram »:

$$
(6)\qquad
\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^4}(2c_1 + r(5-l)) & \xrightarrow{g} & I_{\tilde{Y}} & \longrightarrow & I_{\tilde{Y}/X} \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & E_1 & \xrightarrow{s} & E_0 & \longrightarrow & I_Y & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \Gamma^*(l-5)} & & \downarrow{\scriptstyle \Gamma} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^4}(l-5) & \longrightarrow & E_0^*(l-5) & \xrightarrow{s^*(l-5)} & E_1^*(l-5) & \longrightarrow & \omega_Y(l) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbf{P}^4}(-2c_1 + (r+1)(l-5)) & & 0 & & 0 & & \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}
$$

Here X is the hypersurface defined by the equation g.

REMARK. Assume that H is defined by (4) and that there is an epimorphism $\gamma: I_Y \to \omega_Y(l)$. Let $F := \operatorname{Im}(s^*(l-5))$. If

$$h^1(E_0^* \otimes F) = h^1(E_1^*(l-5)) = 0$$

then γ can be lifted to give a commutative diagram

$$
\begin{array}{ccc}
E_1 & \xrightarrow{s} & E_0 \\
\downarrow{\scriptstyle \Gamma'} & & \downarrow{\scriptstyle \Gamma} \\
E_0^*(l-5) & \xrightarrow{s^*(l-5)} & E_1^*(l-5)
\end{array}
$$

such that γ is induced by Γ. Note that if Γ is generically surjective then $\ker \Gamma \subseteq \ker \gamma$ is invertible. This follows from [4, Prop. 1.1 and Prop. 1.9].

We now want to consider surfaces with a special resolution, namely

$$(7)\qquad 0 \longrightarrow \mathcal{O}_{\mathbf{P}_4}^r \xrightarrow{s(b+ar)} \mathcal{O}_{\mathbf{P}_4}(a)^r \oplus \mathcal{O}_{\mathbf{P}_4}(b) \longrightarrow I_Y(b + ar) \longrightarrow 0$$

where $r \geqslant 1$ and $1 \leqslant a \leqslant b$. If $r = 1$ then Y is a complete intersection of type (a, b). If $r > 1$ then Y is in liaison with a surface Y' defined by a resolution

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}_4}^{r-1} \xrightarrow{s'(ar)} \mathcal{O}_{\mathbf{P}_4}(a)^r \longrightarrow I_{Y'}(ar) \longrightarrow 0 .$$

The union $Y \cup Y'$ is a complete intersection of type $(ar, b + a(r+1))$. (See [11]).

The map $s(b + ar)\colon \mathcal{O}_{\mathbf{P}_4}^r \to \mathcal{O}_{\mathbf{P}_4}(a)^r \oplus \mathcal{O}_{\mathbf{P}_4}(b)$ is given by an $(r+1) \times r$ matrix

$$s(b + ar) = \begin{pmatrix} & A & \\ f_1 & \dots & f_r \end{pmatrix}$$

where A is an $r \times r$ matrix with entries $a_{ij} \in H^0(\mathcal{O}_{\mathbf{P}_4}(a))$ and $f_i \in H^0(\mathcal{O}_{\mathbf{P}_4}(b))$.

PROPOSITION 5. *If A is symmetric then there exists a multiplicity-2 structure $\tilde{Y}$ on Y such that $\tilde{Y}$ is a complete intersection of type $(ar, 2b + a(r-1))$.*

PROOF. Let $l := 5 - 2b - a(2r-1)$. Since A is symmetric we get a commutative diagram:

$$(6')\qquad \begin{array}{ccccccccc}
 & & & & & & 0 & & 0 \\
 & & & & & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & & & \mathcal{O}_{\mathbf{P}^4}(-ar) & \xrightarrow{s} & I_{\tilde{Y}} \to Q \to \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
 & 0 \to & \mathcal{O}_{\mathbf{P}^4}(-b-ar)^r & \xrightarrow{g} & & & \mathcal{O}_{\mathbf{P}^4}(-b-a(r-1))^r \oplus \mathcal{O}_{\mathbf{P}^4}(-ar) & \longrightarrow & I_Y \longrightarrow 0 \\
 & & \downarrow{\scriptstyle \Gamma^*(l-5)} & & & & \downarrow{\scriptstyle \Gamma} & & \downarrow \\
0 \to \mathcal{O}_{\mathbf{P}^4}(-2b-a(2r-1)) & \longrightarrow & \mathcal{O}_{\mathbf{P}^4}(-b-ar)^r \oplus \mathcal{O}_{\mathbf{P}^4}(-2b-a(r-1)) & & \xrightarrow{s^*(l-5)} & & \mathcal{O}_{\mathbf{P}^4}(-b-a(r-1))^r & \longrightarrow & \omega_Y(l) \longrightarrow 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
 & & \mathcal{O}_{\mathbf{P}^4}(-2b-a(r-1)) & & & & 0 & & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Here Γ is the projection map. By diagram chasing one sees that

$$Q = \mathcal{O}_{(g)_0}(-2b - a(r-1))$$

where $g = \det(A)$. Hence $\tilde{Y}$ is a complete intersection of $\det A$ with a hypersurface of degree $2b + a(r-1)$.

REMARK. Since $H^1(\omega_Y(l)) = 0$ for all l it follows from [7], that there exists a vector bundle E together with a section $s \in H^0(E)$ such that $\tilde{Y} = \{s = 0\}$. Using the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}_4} \xrightarrow{s} E \longrightarrow I_{\tilde{Y}}(2b + a(2r-1)) \longrightarrow 0$$

and the section $g \in H^0(I_{\tilde{Y}}(ar))$ one finds a section $t\colon \mathcal{O}_{\mathbf{P}_4} \to E(-2b-a(r-1))$. Since $c_2(E(-2b-a(r-1))) = 0$ this defines a subbundle $\mathcal{O}_{\mathbf{P}_4}(2b + a(r-1)) \subseteq E$ which must necessarily split off.

We want to give explicit equations for the complete intersection $\tilde{Y}$ (Compare [13], [14]).

PROPOSITION 6. *The complete intersection $\tilde{Y}$ is given by the equations*

$$g = \det A\,, \qquad h = \det \tilde{A}$$

where

$$\tilde{A} = \begin{pmatrix} & A & & f_1 \\ & & & \vdots \\ & & & f_r \\ f_1 & \cdots & f_r & 0 \end{pmatrix}$$

PROOF. We first want to show the equality of sets:

$$|(g)_0 \cup (h)_0| = Y\,.$$

The surface Y is the set of all points $x \in \mathbf{P}_4$ where

$$\operatorname{rank}\begin{pmatrix} A \\ f_1 \cdots f_r \end{pmatrix} < r\,. \tag{8}$$

Since A is symmetric it is at any given point equivalent to a diagonal matrix. We can, therefore, write

$$\tilde{A} = \begin{pmatrix} 1 & & & & & f_1 \\ & \ddots & & & & \cdot \\ & & 1 & & & \cdot \\ & & & 0 & & \cdot \\ & & & & \ddots & \cdot \\ & & & & 0 & f_r \\ f_1 & \cdots & \cdots & \cdots & f_r & 0 \end{pmatrix}$$

From this description it is obvious that (8) is equivalent to $g = h = 0$.

We have already seen that $g \in H^0(I_{\tilde{Y}}(ar))$. Next we want to show that $h \in H^0\big(I_{\tilde{Y}}(2b + a(r-1))\big)$. To see this note that the map

$$\beta\colon \mathcal{O}_{\mathbf{P}^4}(b)^r \oplus \mathcal{O}_{\mathbf{P}^4}(2b-a) \to I_Y(2b + a(r-1))$$

is given by

$$(\det A_1, \quad -\det A_2, \ldots, \quad \pm \det A_r, \quad \pm \det A)$$

where A_i is the $r \times r$ matrix which one gets from the matrix

$$\begin{pmatrix} & A & \\ f_1 & \ldots & f_r \end{pmatrix}$$

by deleting the i-th row. Hence

$$h = \det \tilde{A} = \sum_{i=1}^{r} (-1)^{i+1} f_i \det A_i = \beta(f_1 \ldots f_r, 0)\,.$$

Since

$$(f_1, \ldots, f_r) = s^*(l - 5 + 2b + a(r-1))\,(0, \ldots, 0, 1)$$

it follows that $h \in H^0\big(I_{\tilde{Y}}(2b + a(r-1))\big)$. Hence g and h define a complete intersection $\tilde{\tilde{Y}}$ of degree $ar(2b + a(r-1))$ with $\tilde{Y} \subseteq \tilde{\tilde{Y}}$. Since both varieties have the same degree it follows that $\tilde{Y} = \tilde{\tilde{Y}}$.

3. – Castelnuovo surfaces.

We now consider surfaces with a special kind of resolution i.e. we consider resolutions of type (7) with $r = 2$, $q = 1$:

$$(9) \qquad 0 \longrightarrow \mathcal{O}_{\mathbf{P}^4} \xrightarrow{s(b+2)} \mathcal{O}_{\mathbf{P}^4}(1)^2 \oplus \mathcal{O}_{\mathbf{P}^4}(b) \longrightarrow I_Y(b+2) \longrightarrow 0\,.$$

LEMMA 7. *The numerical invariants of Y are*

$$d = 2b + 1\,, \qquad \pi = 2\binom{b}{2}, \qquad p_g = 2\binom{b}{3}, \qquad q = 0\,,$$

$$K^2 = 2b^3 + 3(-3b^2 + 2b + 3)\,, \qquad HK = 2b^2 - 4b - 3\,,$$

$$c_2 = 2b^3 - 3b^2 + 2b + 3\,.$$

PROOF. This is a straightforward calculation using the resolution (9) and its dual.

In [3] Harris investigated so called *Castelnuovo varieties*. These are non-degenerate irreducible varieties $V_d^k \subset \mathbb{P}_n$ of dimension k and degree d with $d \geqslant k(n-k)+2$ whose geometric genus p_g is maximal with respect to all varieties of this type. For surfaces in $\mathbb{P}_4$ he showed that

$$p_g^{\max} = 2\binom{M}{3} + \binom{M}{2}\varepsilon$$

where

$$M = \left[\frac{d-1}{2}\right], \qquad \varepsilon = d - 1 - 2M .$$

Here $[x]$ denotes the greatest integer less than or equal to x. Harris showed that every *Castelnuovo surface* in $\mathbb{P}_4$ of even degree $2b \geqslant 6$ is the complete intersection of a hyperquadric with a hypersurface of degree b. Moreover every Castelnuovo surface of odd degree $2b+1 \geqslant 6$ is together with a plane a complete intersection of a hyperquadric and a hypersurface of degree $b+1$.

PROPOSITION 8. *The Castelnuovo surfaces of odd degree* $\geqslant 6$ *are just the surfaces defined by a resolution of type* (9).

PROOF. If Y is defined by (9) its geometric genus is $p_g = 2\binom{b}{3} = p_g^{\max}$. If Y is a Castelnuovo surface then there is a plane E such that $Y \cup E$ is a complete intersection of type $(2, b+1)$. The plane E has the resolution

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-2) \to \mathcal{O}_{\mathbb{P}^4}(-1)^2 \to I_E \to 0 .$$

Hence it follows from [10, Cor. 1.7] that Y has a resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-b-2)^2 \to \mathcal{O}_{\mathbb{P}^4}(-b-1)^2 \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \to I_Y \to 0$$

which gives the desired result.

We call every surface Y with a resolution of type (9) a Castelnuovo surface.

REMARK. Okonek proved in [8], [9] that

(i) The only Castelnuovo surface of degree 3 is the cubic ruled surface, i.e. $\mathbb{P}_2$ blown up in a point x_0 and embedded by the linear system $|2l - x_0|$.

(ii) For $d=5$ the surface Y is a $\mathbf{P}_2$ blown up in 8 points i.e. $= \tilde{\mathbf{P}}_2(x_0, \ldots, x_7)$ embedded by $\left|4l - 2x_0 - \sum_{i=1}^{7} x_i\right|$.

(iii) Every Castelnuovo surface of degree 7 (where $p_g^{\max} = 2$) is an elliptic surface over $\mathbf{P}_1$ with Kodaira dimension $\varkappa = 1$.

Let us now return to the resolution (9). The map $s(b+2)$ is given by a matrix

$$\begin{pmatrix} & A & \\ f_1 & & f_2 \end{pmatrix}$$

where the entries a_{ij} of the 2×2 matrix A are linear forms and where $f_i \in H^0(\mathcal{O}_{\mathbf{P}_4}(b))$. In particular Y is contained in the hyperquadric $Q_Y = \{\det A = 0\}$. If the degree of Y is at least 5 then this is the only hyperquadric through Y. For the cubic ruled surface the f_i are also linear forms and Y is contained in a net of quadrics.

DEFINITION. A Castelnuovo surface Y is called *symmetric* if I_Y has a resolution (9) with symmetric matrix A.

PROPOSITION 9. *Y is symmetric if and only if it is contained in a corank 2 hyperquadric Q_Y. This hyperquadric is unique.*

PROOF. Clearly if A is symmetric then $Q_Y = \{\det A = 0\}$ has corank 2. Now assume that $Q_Y = \{\det A = 0\}$ has corank 2. Then there are coordinates x_i on $\mathbf{P}_4$ such that A is equivalent to

$$A = \begin{pmatrix} x_0 & l \\ x_1 & x_2 \end{pmatrix}$$

where $l = l(x_0, x_1, x_2)$ is a linear form. By elementary transformations A is equivalent to

$$A' = \begin{pmatrix} x'_0 & x_1 \\ x_1 & x'_2 \end{pmatrix}.$$

The uniqueness is clear for $d \geqslant 5$. Every ruled cubic surface is projectively equivalent to the surface defined by the matrix

$$\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Hence the net of quadrics is spanned by

$$x_0x_2 - x_1^2 = 0\,, \qquad x_0x_4 - x_1x_3 = 0\,, \qquad x_1x_4 - x_2x_3 = 0$$

and Q_Y is the only corank 2 quadric in this net.

Our next purpose is to show that there are many smooth symmetric Castelnuovo surfaces of given degree $d = 2b + 1$. This will follow from:

PROPOSITION 10. *Let Y be the Castelnuovo surface defined by*

$$\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ f_1 & f_2 \end{pmatrix}$$

where $f_1, f_2 \in H^0(\mathcal{O}_{\mathbf{P}_4}(b))$ depend only on x_3 and x_4. Then Y is smooth if the complete intersection $(f_1)_0 \cap (f_2)_0$ is smooth and does not intersect the line $L_0 = \{x_0 = x_1 = x_2 = 0\}$.

PROOF. Y is defined by the equations

$$x_0x_2 - x_1^2\,, \qquad x_0f_2 - x_1f_1\,, \qquad x_1f_2 - f_1x_2\,.$$

We put $\partial_i f_j := \partial f_j/\partial x_i$: Since f_1 and f_2 only depend on x_3 and x_4 the Jacobian matrix is

$$J = \begin{pmatrix} x_2 & -2x_1 & x_0 & 0 & 0 \\ f_2 & -f_1 & 0 & x_0\partial_3f_2 - x_1\partial_3f_1 & x_0\partial_4f_2 - x_1\partial_4f_1 \\ 0 & f_2 & -f_1 & x_1\partial_3f_2 - x_2\partial_3f_1 & x_1\partial_4f_2 - x_2\partial_4f_1 \end{pmatrix}.$$

Y is smooth if and only if rank $J \geqslant 2$ for all points $x \in Y$. For a point $x \in Y$ we have rank $J \leqslant 1$ only in two cases, namely when

$$x_0 = x_1 = x_2 = f_1 = f_2 = 0$$

or when $(x_0, x_1, x_2) \neq 0$ and

$$f_1 = f_2 = 0 \quad \text{and} \quad \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}\begin{pmatrix} \operatorname{grad} f_2 \\ -\operatorname{grad} f_1 \end{pmatrix} = 0\,.$$

Since here the matrix $\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}$ has rank 1, this implies that $\operatorname{grad} f_1$ and $\operatorname{grad} f_2$ are linearly dependent and $(f_1)_0 \cap (f_2)_0$ is singular at x.

4. – The main theorem.

Dualising the resolution (9) we get the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^4}(-b-2) \to \mathcal{O}_{\mathbf{P}^4}(-1)^2 \oplus \mathcal{O}_{\mathbf{P}^4}(-b) \to \mathcal{O}_{\mathbf{P}^4} \to \omega_Y(3-b) \to 0 .$$

This shows that $\omega_Y(3-b)$ is generated by 2 sections and hence every Castelnuovo surface Y admits a fibration

$$\varphi := \Phi_{|K+(3-b)H|}\, Y \to \mathbf{P}_1 .$$

By construction the class of the fibre F is

$$F \sim K + (3-b)H .$$

LEMMA 11. *The following two conditions are equivalent*

(i) *There exists a line L_0 on Y with $L_0^2 = 1 - 2b$.*

(ii) *There is a line L_0 on Y which is a b-section of φ.*

PROOF. Let $L_0 \subseteq Y$ be a line. Since $H \cdot L_0 = 1$ the condition $F \cdot L_0 = b$ is equivalent to $K \cdot L_0 = 2b - 3$. But by the adjunction formula this is equivalent to $L_0^2 = 1 - 2b$.

Our main aim is to characterise those Castelnuovo surfaces Y which possess a multiplicity-2 structure $\tilde{Y}$, such that $\tilde{Y}$ is a complete intersection.

We start with

PROPOSITION 12. *Let Y be a smooth Castelnuovo surface of odd degree $2b+1$. If Y has a multiplicity-2 structure $\tilde{Y}$ with induced canonical bundle $\omega_{\tilde{Y}}$ then this structure is given by a quotient $N^*_{Y/\mathbf{P}_4} \to \omega_Y(2-2b)$. In this case $\tilde{Y}$ is a complete intersection of type $(2, 2b+1)$. The hyperquadric through $\tilde{Y}$ is unique and is singular along a line $L_0 \subset Y$.*

PROOF. By lemmas 2 and 3 every multiplicity-2 structure with induced canonical bundle comes from a quotient $N^*_{Y/\mathbf{P}_4} \to \omega_Y(l)$. The integer l must fulfill the quadratic equation

$$d^2 + d(l^2 + 5l) + HK(3l + 5) + 2K^2 = 0 .$$

Using lemma 7 this equation becomes

$$l^2(2b+1) + l(6b^2 - 2b - 4) + 4(b^3 - b^2 - b + 1) = 0 .$$

There are two solutions

$$l_- = 2 - 2b\,, \qquad l_+ = \frac{2 - 2b^2}{1 + 2b}\,.$$

It is easy to check that $l_+ \notin \mathbb{Z}$ unless $b = 1$ in which case $l_- = l_+ = 0$.

One can now use the remark after lemma 4 to construct a diagram similar to (6′). The only difference is that $\Gamma^*(l - 5)$ has to be replaced by some arbitrary map Γ'. Nevertheless it follows from this diagram that $\tilde{Y}$ is a complete intersection of type $(2, 2b + 1)$. In particular $\tilde{Y}$ is contained in a hyperquadric. This is clearly unique if $d \geqslant 5$. For the case $d = 3$ see [7]. We now have to show that Q_Y has corank 2. Again we can restrict ourselves to the case $d \geqslant 5$. Let us assume that $\operatorname{corank} Q_Y \leqslant 1$. Let $C = Y \cap H$ be a general hyperplane section. Its genus is $b(b - 1) > 0$ if $b \geqslant 2$. The curve C lies on the smooth quadric $Q_H = Q_Y \cap H$. On the other hand we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_H^* & \longrightarrow & N_{C/H}^* & \longrightarrow & \omega_Y(2-2b)|C & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & L_H^* & & \end{array}$$

We claim that this sequence splits which gives a contradiction to [5, theorem 1]. To show the splitting it is enough to see that

$$h^1(M_H^* \otimes L_H) = h^0(M_H \otimes L_H^* \otimes \omega_C) = 0\,.$$

But this follows from

$$\begin{aligned} \deg\,(M_H \otimes L_H^* \otimes \omega_C) &= \deg M_H - \deg L_H + 2g(C) - 2 \\ &= \deg N_{C/H}^* - 2\deg L_H + 2g(C) - 2 = -2\,. \end{aligned}$$

Hence we have seen that $\operatorname{corank} Q_Y = 2$. Let L_0 be the singular line. Then L_0 must lie on Y, otherwise projection from a general point of L_0 would immediately give a contradiction to the fact that the degree of Y is odd.

Now we are ready to prove the main result of this paper.

THEOREM 13. *Let $Y \subseteq \mathbf{P}_4$ be a smooth Castelnuovo surface of degree $2b + 1$. Then the following conditions are equivalent:*

(i) *Y is symmetric.*

(ii) *Y is contained in a corank 2 quadric Q_Y.*

(iii) *There is a line* $L_0 \subseteq Y$ *which is a* b*-section of the fibration* $\varphi\colon Y \to \mathbf{P}_1$.

(iv) Y *contains a projective line* L_0 *with self-intersection* $L_0^2 = 1 - 2b$.

(v) *There exists a multiplicity-*2 *structure* $\tilde{Y}$ *on* Y *such that* $\tilde{Y}$ *is a complete intersection of type* $(2, 2b+1)$.

PROOF. (i)$\Leftrightarrow$(ii) is proposition 9; (i) $\Rightarrow$(v) follows from proposition 5 and (v) $\Rightarrow$(ii) is proposition 12. (iii)$\Leftrightarrow$(iv) is nothing but lemma 11.

(ii) $\Rightarrow$(iii). Assume that Y is contained in the corank 2 quadric Q_Y. The singular line L_0 must necessarily lie on Y. The quadric Q_Y defines a section in $N^*_{Y/\mathbf{P}_4}(2)$ which vanishes along L_0. Hence we get a diagram

$$\begin{array}{ccccccccc} & & & & \mathcal{O}_Y(L_0) & & & & \\ & & & \swarrow & \downarrow{\scriptstyle q} & & & & \\ 0 & \longrightarrow & M^*(2) & \longrightarrow & N^*_{Y/\mathbf{P}_4}(2) & \longrightarrow & \omega_Y(4-2b) & \longrightarrow & 0\,. \end{array}$$

Since (ii)$\Leftrightarrow$(v) it follows that $Q_Y \in H^0(I_{\tilde{Y}}(2))$ hence q factors through $M^*(2)$. Since q is injective outside L_0 there must be an integer $k \geq 1$ such that

$$\mathcal{O}_Y(kL_0) \cong M^*(2)\,.$$

This implies

$$kL_0 \sim (-5H - K + 4H) - \big(K + (4-2b)H\big)\,, \qquad kL_0 \sim (2b-5)H - 2K\,.$$

Since

$$\big((2b-5)H - 2K\big)\cdot H = 1$$

it follows that $k = 1$ and hence

$$L_0 \sim (2b-5)\cdot H - 2K\,.$$

From this it is straightforward to compute

$$F \cdot L_0 = b\,.$$

(iii) $\Rightarrow$(ii). We assume that there is a line $L_0 \subseteq Y$ which is a b-section of the fibration $\varphi\colon Y \to \mathbf{P}_1$. Since

$$H \cdot F = \big(K + (3-b)H\big)\cdot H = b$$

the fibres F are curves of degree b which intersect the line L_0 in b points. This implies (look at all hyperplanes through L_0) that each fibre F is contained in a unique plane E through L_0. In this way we get an injective map

$$\psi\colon \mathbf{P}_1 \to \mathbf{P}_2^{L_0} = \{\text{planes } E \supseteq L_0\} .$$

Pulling back the universal bundle we get a threefold W which is a $\mathbf{P}_2$-bundle over $\mathbf{P}_1$ and a map from W onto a threefold V which contains Y. Moreover there is a surface $\bar{Y} \subseteq \tilde{V}$ which is mapped isomorphically onto Y. Let $\tilde{W} \subseteq \tilde{V}$ be the inverse image of L_0 in $\tilde{V}$. The fibres of $\tilde{W} \to L_0$ are all isomorphic to a rational curve R. Since $\tilde{V}$ is a $\mathbf{P}_2$-bundle over $\mathbf{P}_1$ we have

$$\operatorname{Pic} \tilde{V} = \mathbb{Z}H \oplus \mathbb{Z}\mathbf{P}_2 .$$

Hence the class of $\bar{Y}$ in $\tilde{V}$ is of the form

$$\bar{Y} \sim m\cdot H + n\cdot \mathbf{P}_2 .$$

Since the intersection of $\bar{Y}$ with each plane $\mathbf{P}_2$ is a curve of degree b we find $m = b$. Moreover, since $\bar{Y}$ meets each curve R transversally in one point we find $n = 1$, i.e.

$$\bar{Y} \sim bH + \mathbf{P}_2 .$$

Then

$$2b + 1 = \bar{Y}H^2 = (bH + \mathbf{P}_2)H^2 = bH^3 + 1 .$$

This implies $H^3 = 2$ and V must be a quadric. Clearly V is singular along L_0.

5. – Castelnuovo surfaces of degree 5.

According to Okonek [8] every Castelnuovo surface of degree 5 is a $\mathbf{P}_2$ blown up in 8 points:

$$Y = \tilde{\mathbf{P}}_2(x_0, \ldots, x_7)$$

embedded by the linear system $\left|4l - 2x_0 - \sum_{i=1}^{7} x_i\right|$. Let $E_0, \ldots, E_7$ be the exceptional curves on Y. Then E_0 is a conic, whereas $E_1, \ldots, E_7$ are lines.

PROPOSITION 14. *The Castelnuovo surface Y is symmetric if and onļy if the points $x_1, ,.., x_7$ lie on a smooth conic C.*

PROOF. We first note that if such a C exists it must necessarily be smooth. Otherwise at least 4 of the points $x_1, \ldots, x_7$ would lie on a line L and $H \cdot L \leqslant 0$. It then follows from

$$\left(4l - 2E_0 - \sum_{i=1}^{7} E_i\right)\left(2l - \sum_{i=1}^{7} E_i\right) = 1$$

that C does not pass through x_0 and that it is mapped to a line $L_0 \subseteq Y$. Since $L_0^2 = -3$ the surface Y is symmetric by theorem 13.

Now assume that Y is symmetric. The singular line L_0 of Q_Y lies on Y. It intersects the lines $E_1, \ldots, E_7$ as can be seen by projecting from a general point of L_0. This also implies $L_0 \neq E_i$ and $L_0 \cdot E_i = 1$ for $i = 1, \ldots, 7$. Let E_0 be the exceptional conic. Since $L_0 \cdot E_0 \leqslant 2$ and $H \cdot L_0 = 1$ there are two possibilities:

$$L_0 \sim 3l - 2E_0 - \sum_{i=1}^{7} E_i \quad \text{or} \quad L \sim 2l - \sum_{i=1}^{7} E_i .$$

In the first case $L_0^2 = -2$ whereas in the second case $L_0^2 = -3$. We know, however, from the proof of theorem 13 that $L_0^2 = -3$ and hence we are done.

REMARK. The number of moduli for Castelnuovo surfaces of degree 5 is

$$2 \# \text{ points blown up} - \dim PGL(3, \mathbf{C}) = 16 - 8 = 8 .$$

The condition that $x_1, \ldots, x_7$ lie on a conic is 2-codimensional hence the symmetric Castelnuovo surfaces depend on 6 moduli.

6. – Castelnuovo manifolds.

We call a codimension 2 manifold $Y \subset \mathbf{P}_{n+2}$ a *Castelnuovo manifold* of dimension n if Y has a resolution of type (9), i.e.

$$0 \to \mathcal{O}^2_{\mathbf{P}_{n+2}} \to \mathcal{O}_{\mathbf{P}_{n+2}}(1)^2 \oplus \mathcal{O}_{\mathbf{P}_{n+2}}(b) \to I_Y(b+2) \to 0 .$$

Here we want to point out the following remarkable fact.

PROPOSITION 15. *The only Castelnuovo manifold Y of dimension $n \geqslant 3$ which admits a multiplicity-2 structure $\tilde{Y}$ such that $\tilde{Y}$ is a complete intersection is $\mathbf{P}_n$ embedded linearly.*

PROOF. It is enough to prove this for Castelnuovo 3-folds $Y \subset \mathbf{P}^3$. Just as in lemma 2 we see that every multiplicity-2 structure comes from a subbundle $L \subseteq N_{Y/\mathbf{P}_5}$. If Y admits a multiplicity-2 structure $\tilde{Y}$ which is a complete intersection, then $\tilde{Y}$ must be the intersection of a hyperquadric Q with a hypersurface of degree $2b+1$. The quadric Q must be of corank 3 and the singular plane V of Q must be contained in Y. This can be seen by taking hyperplane sections and applying proposition 12 and theorem 13. Our claim now follows from

LEMMA 16. *If $Y \subseteq \mathbf{P}_5$ is a smooth threefold such that*

(i) *Y contains a plane V*

(ii) *There exists a subbundle $L \subseteq N_{Y/\mathbf{P}_5}|V$ then Y is $\mathbf{P}_3$ embedded linearly.*

PROOF. Let $N_{V/Y} = \mathcal{O}_V(a)$. From the sequence

$$0 \to N_{V/Y} \to N_{V/\mathbf{P}_5} \to N_{Y/\mathbf{P}_5}|V \to 0$$

we find

$$c_1(N_{Y/\mathbf{P}_5}|V) = 3 - a\,, \quad c_2(N_{Y/\mathbf{P}_5}|V) = a^2 - 3a + 3\,.$$

Now suppose $N_{Y/\mathbf{P}_5}|V$ has a 1-subbundle $\mathcal{O}_V(b)$. Then

$$c_2\big((N_{Y/\mathbf{P}_5}|V)(-b)\big) = b^2 - b(3-a) + (a^2 - 3a + 3) = 0$$

and looking at this as a quadratic equation for b, this implies

$$(3-a)^2 - 4(a^2 - 3a + 3) \geqslant 0$$

which implies $a = 1$. Since $H^1(\mathcal{O}_Y) = 0$ by Barth's theorem $([1, \text{Th. III}])$ we see that $|V|$ is a linear system of planes on Y of (projective) dimension 3. Now choose two different points $x, y \in Y$. There is (at least) a 1-dimensional linear subsystem $|V|^0 \subseteq |V|$ of planes which contain the line L spanned by x and y. Let $V_1, V_2 \in |V|^0$ be two different planes containing L. They span a space $\mathbf{P}_3$. By construction $\mathbf{P}_3$ is tangent to Y along L. Hence all planes in $|V|^0$ are contained in this $\mathbf{P}_3$, i.e. their union equals this space. Hence $\mathbf{P}_3 \subseteq Y$ and we are done.

7. – A remark on normal bundles.

In this section we want to say a few words about the normal bundle of Castelnuovo and Bordiga surfaces. We first consider a Castelnuovo surface $Y \subseteq \mathbf{P}_4$ of odd degree.

When we speak of stability, we always mean stability with respect to the hyperplane section H.

PROPOSITION 17. *Let $Y \subseteq \mathbf{P}_4$ be a smooth Castelnuovo surface of odd degree d. Then the following holds:*

(i) *If Y is the cubic ruled surface then its normal bundle $N_{Y/\mathbf{P}_4}$ is semi-stable but not stable.*

(ii) *If $d \geqslant 5$ then the normal bundle $N_{Y/\mathbf{P}_4}$ is properly unstable.*

(iii) *The normal bundle of Y is always indecomposable.*

PROOF. (i) If Y is the cubic ruled surface we have an epimorphism $N^*_{Y/\mathbf{P}_4} \to \omega_Y$. Since

$$c_1(N_{Y/\mathbf{P}_4} \otimes \omega_Y)\cdot H = (5H + 3K)\cdot H = 0$$

it follows that $N_{Y/\mathbf{P}_4}$ cannot be stable. On the other hand the generic hyperplane section C of Y is a rational normal curve of degree 3. Since $N_{C/\mathbf{P}_3} = \mathcal{O}_{\mathbf{P}_1}(5) \oplus \mathcal{O}_{\mathbf{P}_1}(5)$ is semi-stable, it follows that $N_{Y/\mathbf{P}_4}$ must be semi-stable too.

(ii) Every Castelnuovo surface lies in a quadric, i.e. there is a section $0 \neq s \in H^0(N^*_{Y/\mathbf{P}_4}(2))$. Since

$$c_1(N^*_{Y/\mathbf{P}_4}(2))\cdot H = -(H + K)\cdot H = 2 - 2\pi > 0$$

for $d \geqslant 5$ the normal bundle $N_{Y/\mathbf{P}_4}$ is properly unstable.

(iii) If Y is not symmetric then the generic hyperplane section $C = Y \cap H$ is a smooth curve lying on a smooth quadric Q. Since C is neither rational nor a hypersurface section of Q if follows from [5, Theorem 1] that $N_{C/\mathbf{P}_3}$ and hence also $N_{Y/\mathbf{P}_4}$ is indecomposable. Now let Y be symmetric and consider the sequence

$$0 \to M^* \to N^*_{Y/\mathbf{P}_4} \to \omega_Y(2 - 2b) \to 0 \tag{10}$$

We claim that $N_{Y/\mathbf{P}_4}$ splits if and only if (10) splits. If Y is the cubic ruled surface this follows from looking at the rulings of Y. Let us now assume $d \geqslant 5$. For every smooth hypersurface section C we saw in the proof of proposition 12 that

$$N^*_{C/H} = M^*|C \oplus \omega_Y(2-2b)|C$$

and this is the only way $N^*_{C/H}$ can decompose. Hence if $N^*_{Y/\mathbf{P}_4} = L_1 \oplus L_2$ we can assume that $L_1|C \cong M^*|C$ and $L_2|C \cong \omega_Y(2-2b)|C$. Since $q(Y) = 0 \neq \pi$ we can apply a result of A. Weil, (compare [12, prop. 0.9]) to conclude that $L_1 \cong M^*$ and $L_2 \cong \omega_Y(2-2b)$ and we are done. Hence it remains to show that (10) does not split. For this purpose we restrict (10) to the line L_0 with $L_0^2 = 1-2b$. Then (10) becomes

$$0 \to \mathcal{O}_{L_0}(-2b-1) \to N^*_{Y/\mathbf{P}_4}|L_0 \to \mathcal{O}_{L_0}(-1) \to 0 . \tag{11}$$

If this sequence splits then

$$N_{Y/\mathbf{P}_4}|L_0 = \mathcal{O}_{L_0}(1+2b) \oplus \mathcal{O}_{L_0}(1) .$$

In particular we have a quotient $N_{Y/\mathbf{P}_4}(-1)|L_0 \to \mathcal{O}_{L_0}$ and we can argue as in [6] to conclude that there is a hyperplane H which contains all the tangent planes of Y along L_0. But this cannot be, since these tangent planes form the corank 2 quadric Q which contains Y.

Let us now turn to Bordiga surfaces [8]. These are rational surfaces $Y \subseteq \mathbf{P}_4$ of degree 6. They can be constructed by blowing up $\mathbf{P}_2$ in 10 points

$$Y = \tilde{\mathbf{P}}_2(x_1, \dots, x_{10})$$

and embedding this surface with the linear system

$$\left|4l - \sum_{i=1}^{10} x_i\right| .$$

These surfaces have a resolution

$$0 \to \mathcal{O}^3_{\mathbf{P}_4} \to \mathcal{O}_{\mathbf{P}_4}(1)^4 \to I_Y(4) \to 0 .$$

One checks easily that

$$K \cdot H = -2 , \quad K^2 = -1 .$$

LEMMA 18. *If Y has a multiplicity-2 structure $\tilde{Y}$ which is a complete intersection, then $\tilde{Y}$ is given by a quotient $N^*_{Y/\mathbf{P}_4} \to \omega_Y(-2)$. In this case $\tilde{Y}$ is a complete intersection of a cubic and a quartic hypersurface.*

PROOF. Every multiplicity-2 structure $\tilde{Y}$ which is a complete intersection is given by a quotient $N^*_{Y/\mathbf{P}_4} \to \omega_Y(l)$. The condition

$$c_2(N_{Y/\mathbf{P}_4} \otimes \omega_Y(l)) = 0$$

reads

$$36 + 6(l^2 + 5l) - 2(3l + 5) - 2 = 0$$

or equivalently

$$(l + 2)^2 = 0 .$$

Hence $l = -2$. On the other hand if we have a quotient $N^*_{Y/\mathbf{P}_4} \to \omega_Y(-2)$ it follows from the remark after lemma 4 and the proof of proposition 5 that $\tilde{Y}$ is a complete intersection of type (3, 4).

PROPOSITION 19. *Let $Y \subseteq \mathbf{P}_4$ be a smooth Bordiga surface. Then the following conditions are equivalent:*

(i) *There exists a multiplicity-*2 *structure $\tilde{Y}$ on Y such that $\tilde{Y}$ is a complete intersection of type* (3, 4).

(ii) *There exists a quotient* $N^*_{Y/\mathbf{P}_4} \to \omega_Y(-2)$.

(iii) $N^*_{Y/\mathbf{P}_4}$ *is not stable.*

PROOF. The equivalence (i)$\Leftrightarrow$(ii) is lemma 18. (ii) $\Rightarrow$(iii) follows since

$$c_1(N_{Y/\mathbf{P}_4} \otimes \omega_Y(-2)) \cdot H = (H + 3K) \cdot H = 0 .$$

To prove (iii) $\Rightarrow$ (ii) we look at the normal bundle $N_{C/\mathbf{P}_3}$ of smooth hyperplane sections $C = Y \cap H$ of Y. C is a curve of degree 6 and genus 3. The normal bundle of such curves was investigated thoroughly by Ellia in [2]. Now assume that $N_{Y/\mathbf{P}_4}$ is unstable. Then there is a map $N^*_{Y/\mathbf{P}_4} \to L$ to a line bundle L which is surjective outside a finite number of points such that

$$c_1(N_{Y/\mathbf{P}_4} \otimes L) \cdot H \leqslant 0 .$$

If we restrict this map to a generic hyperplane section we get a quotient $N^*_{C/H} \to L|C$ which makes $N_{C/H}$ unstable. By [2, prop. 7] this implies that $L|C = \omega_C(-3) = \omega_Y(-2)|C$. Again we can use Weil's result [12, prop. 0.9] to conclude that $L = \omega_Y(-2)$. Since $c_2(N_{Y/\mathbf{P}_4} \otimes \omega_Y(-2)) = 0$ it follows that the map $N^*_{Y/\mathbf{P}_4} \to \omega_Y(-2)$ must indeed be surjective everywhere and we are done.

REMARK. Since $N_{C/H}$ is always semi-stable [2], it follows that $N_{Y/\mathbf{P}_4}$ must be semi-stable too.

We want to conclude with the following

COROLLARY 20. *The normal bundle of a Bordiga surface* $Y \subseteq \mathbf{P}_4$ *is indecomposable.*

PROOF. Assume that $N_{Y/\mathbf{P}_4}$ splits. Then the same is true for all hyperplane sections $C = Y \cup H$. If, however, C is smooth and $N_{C/H}$ is decomposable then $N^*_{C/H} = \omega_C(-3) \oplus \omega_C(-3)$ by 2, prop. 8]. Using once more Weil's result it follows that $N_{Y/\mathbf{P}_4} = \omega_Y(-2) \oplus \omega_Y(-2)$. But this is a contradiction, since

$$c_2(N^*_{Y/\mathbf{P}_4}) = 36 \neq 31 = c_2\big(\omega_Y(-2) \oplus \omega_Y(-2)\big) .$$

REMARK. We don't know if there exist smooth Bordiga surfaces with the properties of Prop. 18.

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