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On Holomorphic Maps between Domains in $\mathbf{C}^n$.

GIORGIO PATRIZIO

Introduction.

The Kobayashi metric and distance are important tools to study holomorphic maps between complex manifolds. Whenever precise informations about them are available, it is relatively easy to get a great deal of useful results. In this paper we use them to characterize biholomorphic maps between certain classes of domains in $\mathbf{C}^n$. In Section 2 we consider the relatively elementary case of pseudoconvex, complete circular domains. Using a recent result of T. J. Barth we show that a holomorphic map between two such domains, which fixes the origin and is an isometry at the origin for the Kobayashi metric, is in fact a linear biholomorphic map (Theorem 2.5). With additional convexity assumptions on the domain we draw the same conclusion for isometries of the Kobayashi distance (Theorem 2.7). Some of the general results which are presented in Section two are given in the infinite dimensional case since they seem to belong naturally to this context. In Section 3 we take over the case of strictly convex domains in $\mathbf{C}^n$. Using the very powerful theory of Lempert [L] we can show that a holomorphic map between two of them, which is an isometry of the Kobayashi metric or distance at least at one point, is a biholomorphic map (Theorem 3.1). Analogous results are shown for maps between a circular domain and a strictly convex domain (Theorem 3.2). We also give a characterization of the unit ball in $\mathbf{C}^n$ among the strictly convex domains in terms of the Kobayashi distance and metric (Theorem 3.3). A version of our Theorem 3.1 has also been proved by I. Graham and H. Wu [GW] in the case when the first strictly convex domain is the ball. While this paper was in preparation, we were not aware of the results of Vigué [Vi2] who proves in an equivalent form part of our Theorem 3.1. We thank the referee for pointing us out the paper of Vigué.

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1. – Notations and preliminaries.

Let $U = \{z \in \mathbf{C} \colon |z| < 1\}$ be the unit disk in $\mathbf{C}$. If M is a complex (Banach) manifold, denote by $H(U, M)$ the set of all holomorphic maps from V to M. The *Kobayashi metric* of M is defined by

$$K_M(q, v) = \inf\{|u| \colon u \in T_0(U) \text{ and } df(0)(u) = v \text{ for some } f \in H(U, M) \text{ with } f(0) = p\} \tag{1.1}$$

for all $q \in M$ and $v \in T_p(M)$. Here we identify $T_0(V)$ with $\mathbf{C}$ and $|\ |$ denotes the euclidean norm. The *indicatrix* of M at $p \in M$ with respect to the Kobayashi metric is defined by

$$I_p(M) = \{v \in T_p(M) \colon K_M(p, v) < 1\}\,. \tag{1.2}$$

Let M, N be complex (Banach) manifolds and $\varphi \colon M \to N$ be a holomorphic map. We say that φ is an isometry of the Kobayashi metric (*K-isometry* for short) at $p \in M$ if for all $v \in T_p(M)$ we have $K_M(p, v) = K_N(\varphi(p), d\varphi(p)(v))$. In particular if φ is a biholomorphic map then it is a K-isometry at every point of M.

The *Kobayashi (pseudo)-distance* δ_M is defined as follows. Let $p, q \in M$. A *holomorphic chain* for p, q is a pair $\{(f_j)_{j=1,\dots,N}, (z_j)_{j=0,\dots,N}\}$ such that $f_j \in H(U, M)$, $z_j \in U$ and $p = f_1(z_0), f_1(z_1), \dots, f_i(z_j) = f_{j+1}(z_j), \dots, f_N(z_N) = q$. Then if ϱ denotes the hyperbolic distance on U,

$$\delta_M(p, q) = \inf\Big\{\sum_j \varrho(z_j, z_{j+1}) \colon \{(f_j, (z_j))\} \text{ is a holomorphic chain for } p, q\Big\}\,. \tag{1.3}$$

In fact, if D is a convex domain in $\mathbf{C}^n$, then, because of a theorem of Lempert [L], δ_D is a distance function and, if $p, q \in D$, we have

$$\delta_D(p, q) = \inf\{\varrho(z, w) \colon z, w \in U \text{ and } f(z) = p, f(w) = q \text{ for some } f \in H(U, D)\}\,. \tag{1.4}$$

Let M, N be two complex (Banach) manifold and $\varphi\colon M \to N$ be a holomorphic map. We say that φ is *δ-preserving* at $p \in M'$ if for all $q \in M$ we have

$$\delta_M(p, q) = \delta_N(\varphi(p), \varphi(q)) \,. \tag{1.5}$$

In particular if φ is a biholomorphic map, then it is δ-preserving at every point.

Given $f \in H(U, M)$, we say that f is *extremal with respect to $p \in M$ and $v \in T_p(M)$* if $f'(0) = df(0)(1) = \lambda v$ with $\lambda \in \mathbf{R}$ and $K_M(p\ v) = \lambda^{-1}$. We say that f is *extremal with respect to $p, q \in M$* if there exist $z, w \in U$ with $f(z) = p$, $f(w) = q$ such that $\delta_M(p, q) = \varrho(z, w)$.

Throughout the paper $S = S^{2n-1}$ will denote the unit sphere in $\mathbf{C}^n$ and $\mathbf{B}$ the unit ball. Also upper indeces will denote components and lower indeces derivatives. Whenever it is not confusing, summation conventions are assumed.

2. – Complete circular domains.

Let V be a complex Banach space. We say that $G \subset V$ is a *complete circular domain* if G is a connected, bounded, open set such that if $Z \in G$ and $\lambda \in \overline{U}$, then $\lambda Z \in G$. The *Minkowski functional* $m_G = m\colon V \to \mathbf{R}_+$ associated to G is defined by

$$m(Z) = \begin{cases} 0 & \text{if } Z = 0\,, \\ \inf\{t\colon t > 0 \text{ and } Z \in tG\} & \text{if } Z \neq 0\,. \end{cases} \tag{2.1}$$

Then, for all $Z \in V - \{0\}$, we have $(m(Z))^{-1}Z \in \partial G$ and $m(Z) = 1$ if and only if $Z \in \partial G$. Moreover m has the following homogeneity property:

$$m(\lambda Z) = |\lambda| m(Z) \quad \text{for all } \lambda \in \mathbf{C} \text{ and } Z \in V\,. \tag{2.2}$$

We shall identify freely the tangent space $T_0(G)$ of G at the origin with V. Then, in particular, the indicatrix $I_0(G)$ will be a subset of V. Also we denote by K_G^0 the restriction $K_G(0, \square)$ of the Kobayashi metric to $T_0(G)$. We recall the following result.

THEOREM 2.1 (Barth [B]). *Let G be a complete circular domain in a Banach space V. Then $G \subset I_0(G)$. Moreover, if G is pseudoconvex, then $G = I_0(G)$ and, in fact, $m_G = K_G^0$.*

In [PW] the complete circular domains have been classified by considering their Minkowski functionals. Barth's theorem implies that one can also use the Kobayashi metric, even in the infinite dimensional case. More precisely we make the following observation.

PROPOSITION 2.2. *Let G, G' be two pseudoconvex, complete circular domains in a complex Banach space V. The following statements are equivalent:*

(i) *G is biholomorphic to G'.*

(ii) *There exists a linear isomorphism $A: V \to V$ such that $m_G = m_{G'} \circ A$.*

(iii) *There exists a linear isomorphism $A: V \to V$ such that $K^0_G = K^0_{G'} \circ A$.*

PROOF. Clearly (ii) $\Leftrightarrow$ (iii) because of Theorem 2.1. Assume that (ii) holds. Then, if $Z \in G$, we have $m_{G'}(A(Z)) = m_G(Z) < 1$ and hence $A(G) \subset G'$. Similarly one shows that $G' \subset A(G)$ so that $A|_G: G \to G'$ is a biholomorphic map and (i) holds. Finally if (i) is true, because of a theorem of Braun, Kaup and Upmeier [BKU], then there exists a linear isomorphism $A: V \to V$ such that $A(G) = G'$ and thus (ii) follows. q.e.d.

A simple and interesting consequence of Proposition 2.2 is the following (cfr. [PW] for a weaker statement):

COROLLARY 2.3. *Let $G \subset \mathbf{C}^n$ be a pseudoconvex, complete circular domain with C^2 boundary and let $\mathbf{B}$ be the unit ball in $\mathbf{C}^n$. The following statements are equivalent:*

(i) *G is biholomorphic to $\mathbf{B}$.*

(ii) *There exists $A \in GL(n, \mathbf{C})$ such that $m_G(Z) = K_{G'}(Z) = \|A(Z)\|$.*

(iii) *$(m_G)^2 = (K^0_G)^2: \mathbf{C}^n \to \mathbf{R}_+$ is of class C^2 at the origin.*

PROOF. Clearly (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iii). Also since $m_{\mathbf{B}} = \|\ \|$, Proposition 2.2 implies that (ii) $\Rightarrow$ (i). We shall show that (iii) $\Rightarrow$ (ii). Define $M = (m_G)^2$. If (iii) holds, then M is of class C^2 on $\mathbf{C}^n$. Moreover because of (2.2), we have for all $Z \in \mathbf{C}^n$ and $\lambda \in \mathbf{C}$

$$M(\lambda Z) = |\lambda|^2 M(Z)\,. \tag{2.3}$$

Let $Z \in \mathbf{C}^n - \{0\}$. Differentiating (2.3) for λ, we get

$$0 < M(Z) = \frac{\partial^2 M(\lambda Z)}{\partial \lambda\, \partial \bar{\lambda}} = \sum_{\mu,\nu=1}^{n} \frac{\partial^2 M(\lambda Z)}{\partial Z^\mu\, \partial \bar{Z}^\nu} Z^\mu \bar{Z}^\nu\,. \tag{2.4}$$

Taking limit in (2.4) as $\lambda \to 0$, we can conclude that M is a positive definite hermitian form and hence (ii) follows. q.e.d.

We shall use a Schwarz's lemma for holomorphic maps between circular domains. A version of it is due to Sadullaev [S] (cfr. also [FV] Theorem III.2.3 and [R] Theorem 8.1.2). We outline here a proof of the precise statement that we need using Barth's theorem.

THEOREM 2.4. *Let G, G' be complete circular domains in a complex Banach space V. Let m, m' be the respective Minkowski functionals and assume that G' is pseudoconvex. If $\varphi\colon G \to G'$ is a holomorphic map with $\varphi(0) = 0$, then*

(i) *$m'(\varphi(Z)) \leqslant m(Z)$ for all $Z \in G$ and if $m'(\varphi(Z)) = m(Z)$ for some $Z \in G$, then, for all $\lambda \in \mathbf{C}$ with $|\lambda| < (m(Z))^{-1}$, we have $m'(\varphi(\lambda Z)) = m(\lambda Z) = |\lambda| m(Z)$.*

(ii) *If $A = d\varphi(0)$ is the differential of φ at 0, then $A(G) \subset G'$.*

PROOF. Fist of all we observe that if $f \in H(U, G')$ and $f(0) = 0$, then $m(f(z)) \leqslant |z|$ for all $z \in U$. In fact, if we define $g_r\colon \overline{U} \to V$ by $g_r(z) = z^{-1} f(rz)$ for $r \in (0, 1)$, then g_r is holomorphic on U, continuous on $\overline{U}$ and $g_r(\partial U) \subset G'$. But then, since G' is pseudoconvex, by the Kontinuitätssatz, $g_r(U) \subset G'$ for all $r \in (0, 1)$. Taking limit as $r \to 1^-$, this implies $m'(f(z)) \leqslant |z|$. Let $Z \in G$. Then $Z = tc$ where $t = m(Z)$ and $c = t^{-1} Z \in \partial G$. Define $\varphi_Z \in H(U, G)$ by $\varphi_Z(z) = \varphi(zc)$. Then, as observed above, $m'(\varphi_Z(z)) \leqslant |z|$ for all $z \in U$. In particular we get

$$m'(\varphi(Z)) = m'(\varphi_Z(t)) \leqslant t = m(Z) .$$

Since G' is pseudoconvex then m' is a plurisubharmonic function (cfr. [B], Theorem 1). Thus the function h defined by $h(\lambda) = m'(\lambda^{-1}\varphi(\lambda Z))$ on the disk $U' = \{\lambda \in \mathbf{C}\colon |\lambda| < m(Z)^{-1}\}$ is subharmonic for every $Z \in G$. Since $\sup h = m(Z)$, if $m'(\varphi(Z)) = m(Z)$, then h must be constant and hence $m'(\varphi(\lambda Z)) = m(\lambda Z) = |\lambda| m(Z)$ and the proof of (i) is complete.

Part (ii) follows immediately from Barth's theorem because under the hypothesis we have

$$A(G) = A(I_0(G)) \subset I_0(G') = G' . \qquad \text{q.e.d.}$$

Using the above results we can now give the following characterization of the biholomorphic maps between complete circular domains.

THEOREM 2.5. *Let G, G' be pseudoconvex, complete circular domains in $\mathbf{C}^n$ and $\varphi\colon G \to G'$ be a holomorphic map with $\varphi(0) = 0$. If φ is a K-isometry at 0, then φ is a linear biholomorphic map of G into G'.*

PROOF. Let $A = d\varphi(0)\colon \mathbf{C}^n \to \mathbf{C}^n$ be the differential of φ at 0. By hypothesis $K^0_{G'} = K^0_G \circ A$. Thus A is non singular and, by proposition 2.2, G is biholomorphic to G' and, in fact, $G' = A(G)$. But then $\psi = \varphi \circ A^{-1}\colon G \to G$ is a holomorphic map such that $\psi(0) = 0$ and $d\psi(0) = Id$. By Cartan's uniqueness theorem, we conclude that $\psi = Id$ and hence $\varphi = A = d\varphi(0)$.

q.e.d.

Since biholomorphic maps are K-isometry at every point, one gets at once the following

COROLLARY 2.6. *Let G, G' be pseudoconvex, complete circular domains in $\mathbf{C}^n$ and $\varphi\colon G \to G'$ be a holomorphic map.*

(i) *If G is homogeneous and $Z \in \varphi^{-1}(0)$ exists such that is a K-isometry at Z, then φ is a biholomorphic map.*

(ii) *If G' is homogeneous and φ is a K-isometry at $0 \in G$, then φ is a biholomorphic map.*

In particular if both G, G' are homogeneous, then φ is biholomorphic if and only if φ is a K-isometry at some point $Z \in G$.

REMARK. It is known that every bounded symmetric domain can be realized as a bounded, pseudoconvex, complete circular domain (even in the infinite dimensional case, cfr. [Vi1]). Thus the above result implies in particular that a holomorphic map between two bounded symmetric domains in $\mathbf{C}^n$ is biholomorphic if and only if it is a K-isometry at one point.

We now restrict our considerations to more special domains. First we need some terminology. Let V be a complex Banach space and $K \subset V$. A point $p \in K$ is called a *complex extreme point* of K if $q = 0$ is the only vector in V such that $\{p + \lambda q\colon \lambda \in U\} \subset K$. Let D be a bounded, convex domain in V. We say that D is *E-convex* if every point $p \in \partial D$ is a complex extreme point of N. It is known (cfr. [V]) that if G is a E-convex, complete circular domain in a complex Banach space V and $Z \in G$, then the only extremal maps $f \in H(U, G)$, for the Kobayashi distance or metric, such that $f(0) = 0$ are of the type $f = f_c$ with $f_c(z) = zc$ for some $c \in \partial G$. This fact together with Barth's theorem implies the following formula for all $Z \in G$:

$$\delta_G(0, Z) = \frac{1}{2}\log\frac{1 + m(Z)}{1 - m(Z)} = \frac{1}{2}\log\frac{1 + K_G^0(Z)}{1 - K_G^0(Z)}\,. \tag{2.5}$$

THEOREM 2.7. *Let G, G' be E-convex, complete circular domains in $\mathbf{C}^n$ and let m, m' be the respective Minkowski functionals. If $\varphi\colon G \to G'$ is a holomorphic map with $\varphi(0) = 0$, then the following statements are equivalent:*

(i) *φ is a linear biholomorphic map.*

(ii) $\delta_G(0, Z) = \delta_{G'}(0, \varphi(Z))$ *for all* $Z \in G$.

(iii) *For some* $r \in (0, 1)$, *if* $m(Z) = r$, *then* $m'(\varphi(Z)) = r$.

PROOF. Since biholomorphic maps are δ-preserving, (i) $\Rightarrow$ (ii). Also (ii) $\Rightarrow$ (iii) because of (2.5). Again because of (2.5) and of (i) of Theorem 2.4. we have that (iii) $\Rightarrow$ (ii). Only (ii) $\Rightarrow$ (i) remains to be shown. Given any $c \in \partial G$, the map $f_c \in H(U, G)$ defined by $f_c(z) = zc$ is an extremal map of G. If (ii) holds, then the map $g = \varphi \circ f_c \in H(U, G')$ is an extremal map of G' and $g(0) = 0$. Thus, as remarked above, g must be linear and hence

$$\varphi(zc) = g(z) = dg(0, z) = d\varphi(0, zc) .$$

Since this holds for all $c \in \partial G$, φ is linear. Also since δ_G is a distance, if $\varphi(Z) = 0$, then $0 = \delta_{G'}(0, \varphi(Z)) = \delta_G(0, Z)$ and hence $Z = 0$ i.e. φ is injective and therefore $\varphi \in GL(n, \mathbf{C})$. Because of (2.5) then $K_{G'} = K_G^0 \circ \varphi = K_G^0 \circ d\varphi(0)$ i.e. φ is K-isometric at 0 and the claim follows from Theorem 2.5.
q.e.d.

As with Corollary 2.6. since biholomorphic maps are δ-preserving, one shows at once the following consequence of the above theorem.

COROLLARY 2.9. *Let G, G' be E-convex, complete circular domains in* $\mathbf{C}^n$ *and let* $\varphi: G \to G'$ *be a holomorphic map.*

(i) *If G is homogeneous and there exists* $Z \in G$ *such that* $\varphi(Z) = 0$ *and* $\delta_G(Z, W) = \delta_{G'}(0, \varphi(W))$ *for all* $W \in G$, *then φ is biholomorphic.*

(ii) *If G' is homogeneous and* $\delta_G(0, W) = \delta_{G'}(\varphi(0), \varphi(W))$ *for all* $W \in G$, *then φ is biholomorphic.*

In particular, if both G and G' are homogeneous, φ is biholomorphic if and only if there exists $Z \in G$ such that $\delta_G(Z, W) = \delta_{G'}(\varphi(Z), \varphi(W))$ for all $W \in G$.

3. – Strictly convex domains.

We say that $D \subset \mathbf{C}^n$ is a *strictly convex domain* if it is an open, bounded, connected set and there exists a defining function $r: \mathbf{C}^n \to \mathbf{R}$ for it of class C^∞ and such that the real Hessian of r is everywhere positive definite. For such domains Lempert has shown the existence and the unicity of the extremal maps for the Kobayashi metric and distance. Here we shall recall a few notions that will be used below (cfr. [L] for proofs).

Let D be a strictly convex domain in $\mathbf{C}^n$. There exists a C^∞, proper, surjective map

$$F = F_D\colon D \times \overline{U} \times S \to \overline{D} \tag{3.1}$$

with the following properties:

(3.2) For every $p \in D$, $b \in S$, the map $F(p, \square, b)\colon U \to D$ is holomorphic with $F(p, 0, b) = p$ and $F'(p, 0, b) = \|F'(p, 0, b)\|\, b$, and it is the unique extremal map with respect to p and b (here and below we identify $T_p(D)$ with $\mathbf{C}^n$) and $K_D(p, b) = (\|F'(p, 0, b)\|)^{-1}$.

(3.3) For every $z, w \in U$ and $p \in D$, the map $F(p, \square, b)\colon U \to D$ is the unique extremal map with respect to $F(p, z, b)$ and $F(p, w, b)$.

(3.4) If $\lambda \in \partial U$ then $F(p, z, \lambda b) = F(p, \lambda z, b)$ for all $(p, z, b) \in D \times \overline{U} \times S$.

(3.5) If $b_j \in S$ for $j = 1, 2$ and $L_j = F(p, \square, b_j)(\overline{U})$ for some $p \in D$, then either $L_1 \cap L_2 = \{p\}$ or $L_1 = L_2$ and there exists $\mu \in \partial U$ such that $b_1 = \mu b_2$. In fact, for all $p \in D$, $F(p, z, b) = F(p, w, c)$ if and only if $|z| = |w|$ and $c = \mu b$ for some $\mu \in \partial U$.

For strictly convex domains we have the following analogue of Theorem 2.5 and 2.7:

THEOREM 3.1. *Let D, D' be strictly convex domains in $\mathbf{C}^m$ and $\varphi\colon D \to D'$ be a holomorphic map. Then the following statements are equivalent*:

(i) *φ is a K-isometry at one point $p \in D$.*

(ii) *φ is δ-preserving at one point $p \in D$.*

(iii) *φ is a biholomorphic map.*

PROOF. It is clear that (iii) implies both (i) and (ii). Let $F_D, F_{D'}$ be the maps introduced in (3.1) relative to D, D' respectively. Assume that (ii) holds. If $q = \varphi(p)$ and $b \in S$ is any given vector, then the map $f\colon U \to D'$ defined by $f(z) = \varphi(F_D(p, z, b))$ is extremal with respect to $q = f(0)$ and $f(w)$ for all $w \in U$ and also (cfr. [L], Theorem 2) it is extremal with respect to $q = f(0)$ and $f'(0) = A(F'_D(p, 0, b)) = \|F'_D(p, 0, b)\| A(b)$ where $A = d\varphi(p)$. But then it is also extremal with respect to q and $A(b)/\|A(b)\|$ and therefore, because of the unicity of the extremal maps we have $\varphi(F_D(p, z, b)) = F_{D'}(q, z, A(b)/\|A(b)\|)$. Differentiating this equality with respect to z and setting $z = 0$, one gets:

$$\frac{A(b)}{K_{D'}(q, A(b))} = \left\| F'_{D'}\left(q, 0, \frac{A(b)}{\|A(b)\|}\right)\right\| \frac{A(b)}{\|A(b)\|}$$
$$= F'_{D'}\left(q, 0, \frac{A(b)}{\|A(b)\|}\right)$$
$$= A(F'_D(p, 0, b))$$
$$= \|F'_D(p, 0, b)\| A(b)$$
$$= \frac{A(b)}{K_D(p, b)}$$

and therefore $K_{D'}(q, A(b)) = K_D(p, b)$. Since b was arbitrary it follows that φ is a K-isometry at q i.e. we have shown that (ii) implies (i). It remains only to prove that (i) implies (iii). Assume that (i) holds and let $q = \varphi(p)$, $A = d\varphi(p)$. Since φ is a K-isometry at p, then for all $b \in S$ the map $f\colon U \to D'$ defined by $f(z) = \varphi(F_D(p, z, b))$ is extremal with respect to $f(0) = q$ and $f'(0) = A(F'_D(p, 0, b)) = \|F'(p, 0, b)\| A(b)$. Thus f is also extremal with respect to p and $A(b)/\|A(b)\|$ and therefore, by the unicity of extremal maps we get for all $z \in U$

$$\varphi(F_D(p, z, b)) = F_{D'}\left(q, z, \frac{A(b)}{\|A(b)\|}\right). \tag{3.6}$$

From (3.6) one gets immediately that φ is surjective. In fact if $S \in D'$, then there exists $x \in U$ and $c \in S$ such that $X = F_{D'}(q, x, c)$. But then $X = \varphi(F_D(p, x, A^{-1}(c)/\|A^{-1}(c)\|))$. Let $Z, W \in D$ and assume $\varphi(Z) = \varphi(W)$. Then there exist $z, w \in U$ and $b, c \in S$ such that $Z = F_D(p, z, b)$ and $W = F_D(p, w, c)$. Then

$$F_{D'}\left(q, z, \frac{A(b)}{\|A(b)\|}\right) = \varphi(F_D(p, z, b)) = \varphi(F_D(p, w, c)) = F_{D'}\left(q, w, \frac{A(c)}{\|A(c)\|}\right).$$

Thus $|z| = |w|$ and there exists $\lambda \in \partial U$ such that

$$\frac{A(c)}{\|A(c)\|} = \lambda \frac{A(b)}{\|A(b)\|}.$$

But then, since $\|b\| = \|c\| = 1$, we have $\|A(b)\| = \|A(c)\|$ and $c = \lambda b$. Thus $Z = F_D(p, z, b) = F_D(p, w, c) = X$. It follows that φ is also injective and therefore bijective and hence (iii) holds. q.e.d.

REMARKS. As mentioned in the introduction, the part (i) $\Leftrightarrow$ (iii) of the above theorem is contained in Vigué's paper [Vi2]. In fact, he shows that a holomorphic map between two convex domains (not necessarily strictly convex) is biholomorphic if and only if it is an isometry at one point for the Caratheodory metric. Since for convex domains the Kobayashi and Caratheodory metric coincide (cfr. [L2] and [RW]), our statement (i) $\Leftrightarrow$ (iii) follows at once.

Given a strictly convex domain $D \subset \mathbf{C}^n$ and $p \in D$, using (3.2), one has the following description of the indicatrix of D at p:

$$\begin{aligned} I_p(D) &= \{v \in T_p(D)\colon K_D(p, v) < 1\} \\ &= \{zb \in \mathbf{C}^n\colon z \in \mathbf{C},\ b \in S \text{ and } |z| K_D(p, b) < 1\} \qquad (3.7) \\ &= \{zb \in \mathbf{C}^n\colon z \in \mathbf{C},\ b \in S \text{ and } |z| < \|F_D'(p, 0, b)\|\} \end{aligned}$$

and thus we have also

$$\partial I_p(D) = \{zb \in \mathbf{C}^n\colon z \in \mathbf{C},\ b \in S \text{ and } |z| = \|F_D'(p, 0, b)\|\} . \qquad (3.8)$$

Because of (3.4) and (3.5) a map $h\colon \overline{I_p(D)} \to \overline{D}$ is well defined by $h(zc) = F_D(p, z, c/\|c\|)$ where $z \in \overline{U}$ and $c \in \partial I_p(D)$. The map h, which we called in [P] the *circular representation* of the domain D at p, has the following properties (cfr. [P] for proofs):

(3.9) h is a homeomorphism.

(3.10) $h\colon \overline{I_p(D)} - \{0\} \to \overline{D} - \{p\}$ is a diffeomorphism.

(3.11) For all complex line $L \subset \mathbf{C}^n$, the restriction $h|_{L \cap I_p(D)}$ is holomorphic.

(3.12) h is biholomorphic if and only if it is of class C^∞ at 0.

Using this map h we can show the following:

THEOREM 3.2. *Let $D \subset \mathbf{C}^n$ be a strictly convex domain and $G \subset \mathbf{C}^n$ be a pseudoconvex, complete circular domain.*

(i) *If $\varphi\colon G \to D$ is a holomorphic map and a K-isometry at 0 then φ is a biholomorphic map.*

(ii) *If $\psi\colon D \to G$ is a holomorphic map with $\psi(p) = 0$, ψ is a K-isometry at p and G is an E-convex domain, then ψ is a biholomorphic map.*

PROOF. (i) If φ is a K-isometry at 0 and $A = d\varphi(0)$ then $A \in GL(n, \mathbf{C})$ and, since by Barth's theorem $G = I_D(G)$, we have $A(G) = I_p(D)$. If we show that $\varphi = h \circ A$, then it follows that φ is a holomorphic bijective map and hence a biholomorphic map. Let $Z \in G$ be any point. Then $z \in U$ and $c \in \partial G$ exist with $Z = zc$. Define $f: U \to D$ by $f(z) = \varphi(zc)$. Since φ is a K-isometry at 0 then f is extremal with respect to $p = f(0)$ and $f'(0) = A(c)$. Thus f is also extremal with respect to p and $A(c)/\|A(c)\| \in S$. But then we have

$$\varphi(Z) = f(z) = F_D\left(p, z, \frac{A(c)}{\|A(c)\|}\right) = h(zA(c)) = h(A(Z))$$

because of the unicity of extremal maps for strictly convex domains and the claim is proved.

(ii) Again by Barth's theorem, since ψ is a K-isometry at 0, if $B = d\psi(p)$, we have $B(I_p(D)) = G$. If we show that $\psi \circ h = B$ then it will follow that $\psi = B \circ h^{-1}$ is a bijective holomorphic map and therefore biholomorphic. Let $Z \in I_p(D)$. Then $Z = zc$ for some $z \in U$ and $c \in \partial I_p(D)$. Since ψ is a K-isometry at p and the unique extremal maps $f: U \to G$ with $f(0) = 0$ are of the type $f(z) = zf'(0)$, we have

$$\psi(h(Z)) = \psi\left(F_D\left(p, z, \frac{c}{\|c\|}\right)\right) = z\, B\left(F'_D\left(p, 0, \frac{c}{\|v\|}\right)\right)$$
$$= z \left\|F'_D\left(p, 0, \frac{c}{\|c\|}\right)\right\| \frac{B(c)}{\|c\|} = z\, \frac{\|c\|}{K_D(p, c)}\, \frac{B(c)}{\|c\|} = zB(c) = B(Z).$$

which proves the claim. q.e.d.

Our last theorem is a characterization of the ball $\mathbf{B}$ in $\mathbf{C}^n$ among the strictly convex domains which has the same flavor of the previous results although the proof relies on different techniques. First we need to introduce one more notion. Let $D \subset \mathbf{C}^n$ be a strictly convex domain and $p \in D$ be any point. Because of (3.4) and (3.5) an exhaustion $\tau = \tau_p: \overline{D} \to [0, 1]$, called the *Lempert exhaustion* at p, is well defined by

$$\tau(F_D(p, z, b)) = |z|^2 \tag{3.13}$$

for all $z \in \overline{U}$ and $b \in S$ where F_D is the map defined in (3.1). Because of (3.3) one has immediately that for any $q \in D$.

$$\tau(q) = \left(\frac{\exp 2\delta_D(p, q) - 1}{\exp 2\delta_D(p, q) + 1}\right)^2. \tag{3.14}$$

In addition the function τ has the following properties (cfr. [L] and [P]):

(3.15) τ is continuous and proper, of class C^∞ on $\bar{D}-\{p\}$ with $\tau(p)=0$, $0<\tau(q)<1$ if $q\in D-\{p\}$ and $\tau\equiv 1$ on ∂D.

(3.16) τ is strictly plurisubharmonic and $\log\tau$ is plurisubharmonic on $D-\{p\}$.

(3.17) If $u=\log\tau$ then $\det(u_{\mu\bar{\nu}})\equiv 0$ on $D-\{p\}$.

Using Stoll's characterization of the unit ball ([SPM]) we can prove the following:

THEOREM 3.3. *Let $D\subset\mathbf{C}^n$ be a strictly convex domain.*

(i) *If there exists $p\in D$ such that the squared Kobayashi distance from p $\delta_D^2(p,\square)\colon D\to[0,\infty)$ is a function of class C^∞, then D is biholomorphic to $\mathbf{B}$.*

(ii) *If there exists $p\in D$ such that the Kobayashi metric is a smooth hermitian metric in a neighborhood of p, then D is biholomorphic to $\mathbf{B}$.*

PROOF. First observe that if (ii) holds then (i) follows too. In fact for strictly convex domains $\delta_D^2(p,\square)\in C^\infty(D-\{p\})$. On the other hand under the assumption of (ii) it follows that $\delta_D^2(p,\square)$, which is the squared integrated distance of the Kobayashi metric, is smooth in a neighborhood of p and therefore the assumption of (i) are verified.

To prove (i), first observe that, if $\tau=\tau_p$ is the Lempert exhaustion at p, we have $\tau=(\operatorname{tg}h\delta_D(p,\square))^2$ because of (3.14) and therefore τ is an analytic function of $\delta_D^2(p,\square)$ and thus τ is of class C^∞ on D. Moreover τ is strictly plurisubharmonic also at p. In fact, if $b\in S$, then, using (3.13), we have

$$\begin{aligned}(3.14)\qquad \tau_{\mu\bar{\nu}}(p)b^\mu b^{\bar{\nu}} &= \frac{1}{\|F_D'(p,0,b)\|^2}\tau_{\mu\bar{\nu}}(p)F_D'^{\mu}(p,0,b)\overline{F_D'^{\nu}(p,0,b)}\\ &=\lim_{z\to 0}\frac{1}{\|F_D'(p,z,b)\|^2}\tau_{\mu\bar{\nu}}(F_D(p,z,b))F_D'^{\mu}(p,z,b)\overline{F_D'^{\nu}(p,z,b)}\\ &=\lim_{z\to 0}\frac{1}{\|F_D'(p,z,b)\|^2}\frac{\partial^2\tau}{\partial z\,\partial\bar{z}}(F_D(p,z,b))\\ &=\lim_{z\to 0}\frac{1}{\|F_D'(p,z,b)\|}\frac{\partial^2|z|^2}{\partial z\,\partial\bar{z}}=\frac{1}{\|F_D'(p,0,b)\|^2}>0\,.\end{aligned}$$

This together with (3.15), (3.16) and (3.17) shows that τ is a strictly parabolic exhaustion for D of radius 1 and thus by Stoll's theorem D is biholomorphic to p (cfr. [SPM]). q.e.d.

REMARKS. Part (ii) of the above theorem is a version, in the case of strictly convex domains, of a theorem of C. Stanton [St]. In fact, in light of the fact that for strictly convex domains the Kobayashi and Caratheodory metrics agree (cfr. [L2] and [RW]), it could be derived directly from Stanton's theorem.

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