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An Estimate of the Gap of the First Two Eigenvalues in the Schrödinger Operator.

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1. - Introduction.

We shall consider the following Dirichlet eigenvalue problem on a smooth bounded domain $\Omega \subset \mathbb{R}^n$,

$$(1.1) \quad \begin{cases} -\Delta u + Vu = \lambda u \\ u \equiv 0 \quad \text{on } \partial\Omega, \end{cases}$$

where V is a nonnegative function defined on $\bar{\Omega}$. As is well-known, the eigenvalues of problem (1.1) can be interpreted as the energy levels of a particle travelling under an external force field of a potential q in $\mathbb{R}^n$, where

$$q(x) = \begin{cases} V(x) & x \in \bar{\Omega} \\ +\infty & x \notin \bar{\Omega}, \end{cases}$$

and the corresponding eigenfunctions are wave functions of the Schrödinger equation $-\Delta u + qu = \lambda u$. Furthermore, the set of eigenvalues $\{\lambda_k\}$ of (1.1) are nonnegative and can be arranged in a nondecreasing order as follows,

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_m \leq \dots$$

It is a significant problem to find a lower bound for λ_1 in terms of the geometry of Ω . This subject has been studied extensively by many authors. A rather precise bound in the case $V \equiv 0$ was worked out not only for a bounded domain in $\mathbb{R}^n$ but actually valid for a general Riemannian manifold with certain curvature conditions; we refer to [4] for these recent developments. Nevertheless, very little is known about the obvious interesting question of how big the gap is between λ_2 and λ_1 . There are both physical

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and mathematical interests in finding out a lower bound for $\lambda_2 - \lambda_1$ in terms of the geometrical invariants of Ω and the given potential function V . Our main result is the following.

THEOREM (1.1). *Let Ω be a smooth convex bounded domain in $\mathbb{R}^n$ and $V: \bar{\Omega} \rightarrow \mathbb{R}$ a nonnegative convex smooth potential function.*

Suppose λ_2 and λ_1 are the first and second nonzero eigenvalues of (1.1), then the following pinching inequality holds

$$(1.2) \quad \frac{\pi^2}{4d^2} \leq \lambda_2 - \lambda_1 \leq \left(\frac{4n\pi^2}{D^2} + \frac{4(M-m)}{n} \right),$$

where d = diameter of Ω , D = the diameter of the largest inscribed ball in Ω , $M = \sup_{\bar{\Omega}} V$, and $m = \inf_{\bar{\Omega}} V$.

In the last section, we demonstrate how to make use of the main theorem here to obtain a similar theorem when $\Omega = \mathbb{R}^n$.

In Appendix B), we give a short proof of a theorem of Brascamp and Lieb on the log concavity of the first eigenfunction. A similar method of gradient estimate was used by Li and the third author in [4].

2. - A gradient estimate.

Let f_1 and f_2 be the first and second eigenfunctions of (1.1). It is a known fact that f_1 must be a positive function (a theorem of Courant [3]), and thus $u = f_2/f_1$ is a well-defined smooth function on Ω . Using the Hopf lemma and the Malgrange preparation theorem, one can actually verify that u is smooth up to the boundary $\partial\Omega$ (for a short proof of the case we need, see § 6). In this section, the following gradient estimate will be established, which is the key step to derive the lower bound for $\lambda_2 - \lambda_1$.

THEOREM 2.1. *With the same conditions stated in Theorem (1.1), we have the following estimate for the gradient of u ,*

$$|\nabla u|^2 + \lambda(\mu - u)^2 \leq \sup_{\Omega} \lambda(\mu - u)^2,$$

where $\lambda = \lambda_2 - \lambda_1$, μ is a constant not less than $\sup_{\Omega} u$.

We proceed to give the proof by dividing our argument into two propo-

sitions. In the sequel of this, we denote by $G = |\nabla u|^2 + \lambda(\mu - u)^2$, which is a smooth function on $\bar{\Omega}$ as u is.

PROPOSITION 2.2. *With the same conditions in Theorem (1.2), if G attains its maximum in an interior point of Ω , we have the following inequality*

$$G \leq \sup_{\Omega} \lambda(\mu - u)^2.$$

PROOF. By direct computation, we have

$$(2.1) \quad G_i = \sum_{j=1}^n 2u_j u_{ji} - 2\lambda(\mu - u)u_i$$

$$(2.2) \quad \Delta G = \sum_{i=1}^n G_{ii} = 2 \sum_{i,j=1}^n u_{ij}^2 + \sum_{i,j=1}^n 2u_j u_{jii} + 2\lambda \sum_{i=1}^n u_i^2 - 2\lambda(\mu - u) \left(\sum_{i=1}^n u_{ii} \right).$$

It is by straightforward computation that

$$(2.3) \quad \Delta u = -\lambda u - 2(\nabla u \cdot \nabla \log f_1)$$

($\log f_1$ is well-defined since $f_1 > 0$ on Ω).

We substitute (2.3) into (2.2) and obtain

$$(2.4) \quad \Delta G = \left\{ 2 \sum_{i,j=1}^n (u_{ij})^2 + 2\lambda^2 u(\mu - u) \right\} \\ + \{ 4\lambda(\mu - u)(\nabla u \cdot \nabla \log f_1) \} - \{ 4(\nabla u) \cdot [\nabla \cdot (\nabla u \cdot \nabla \log f_1)] \}.$$

Suppose G attains its maximum in an interior point $p \in \Omega$. If $(\nabla u)(p) \neq 0$, then we can choose a coordinate such that $u_1(p) \neq 0$, $u_i(p) = 0$ for $2 \leq i \leq n$. Furthermore, since $\nabla G(p) = 0$, one easily deduces from (2.1) that relative to the above coordinate the following is true

$$(2.5) \quad u_{11}(p) = \lambda(\mu - u)(p)$$

$$(2.6) \quad u_{1i}(p) = 0, \quad 2 \leq i \leq n.$$

Putting (2.5) and (2.6) into (2.4), we find a simplification for $\Delta G(p)$ with respect to this particular coordinate system,

$$(2.7) \quad \Delta G(p) = \left\{ 2 \sum_{i,j=1}^n u_{ij}^2 + 2\lambda^2(\mu - u)u \right\} - \{ 4u_1^2(\log f_1)_{11} \} \leq 0.$$

Since both V and Ω are convex by assumption, according to a result of Brascamp and Lieb [1], $\log f_1$ is concave, in particular $(\log f_1)_{11}(p) \leq 0$. Consequently, the second term of (2.7), namely $-4u_1^2(\log f_1)_{11}$, is nonnegative. Therefore, we have

$$(2.8) \quad \left\{ \sum_{i,j=1}^n u_{ij}^2 + \lambda^2(\mu - u)u \right\}(p) \leq 0.$$

Furthermore, $u_{ij}^2(p) \geq 0 \ \forall i, j$ implies

$$\{u_{11}^2 + \lambda^2(\mu - u)u\}(p) \leq 0.$$

Again from (2.5), this leads to

$$(2.9) \quad \mu(\mu - u(p)) \leq 0.$$

We can assume that $\sup_{\Omega} u$ is positive. On the other hand, $\sup_{\Omega} u$ is greater than $u(p)$ as $u_1(p) \neq 0$. If $\mu \geq \sup_{\Omega} u > 0$, it gives rise to a contradiction of (2.9).

Our argument above shows that $\nabla u(p) = 0$ and establishes the inequality $G \leq \sup_{\Omega} \lambda(\mu - u)^2$ as desired.

PROPOSITION 2.3. *Let us assume equation (1.1) satisfying all the conditions in Theorem (2.1). If G attains its maximum on $\partial\Omega$, then we have the same estimate*

$$G \leq \sup_{\Omega} \lambda(\mu - u)^2.$$

REMARK. We recall a differential geometric description of convexity here which will be used later. Suppose $H = (h_{\alpha\beta})_{2 \leq \alpha, \beta \leq n}$ is the second fundamental form of $\partial\Omega$ relative to a unit normal of $\partial\Omega$ pointing outward to Ω . It is known that $\partial\Omega$ is convex iff H is positive definite.

PROOF OF PROPOSITION 2.3. Suppose G attains its maximum on $\partial\Omega$ at a point p . We can choose an orthonormal frame $\{l_1, l_2, \dots, l_n\}$ around p such that l_1 is perpendicular to $\partial\Omega$ and pointing outward. We also use the notation $\partial/\partial x_1$ to denote the restriction of l_1 on $\partial\Omega$, that is the normal unit vector field along $\partial\Omega$.

A simple computation shows

$$(2.10) \quad \begin{aligned} \frac{\partial G}{\partial x_1}(p) &= 2 \sum_{i=1}^n u_i u_{i1} - 2\lambda u_1(\mu - u) \\ &= 2u_1 u_{11} + 2 \sum_{i=2}^n u_i u_{i1} - 2\lambda u_1(\mu - u) \geq 0. \end{aligned}$$

Consider the equation $\Delta u = -\lambda u - 2(\nabla u \cdot \nabla \log f_1)$, where both Δu and u are smooth up to the boundary and thus attain finite values on $\partial\Omega$. Hence, $(\nabla u \cdot \nabla \log f_1) = (1/f_1) \left[u_1(f_1)_1 + \sum_{2 \leq i \leq n} u_i(f_1)_i \right]$ achieves finite values on $\partial\Omega$ as well. Nevertheless, since $f_1 \equiv 0$ on $\partial\Omega$, we have $(f_1)_i = 0 \ \forall 2 \leq i \leq n$ (l_i , $2 \leq i \leq n$, is in the tangential direction). This implies that $\{(1/f_1)u_i(f_1)_i\}$ must be finite. By the Hopf lemma, $(f_1)_1 = \partial f_1 / \partial x_1 \neq 0$ on $\partial\Omega$, we get the important observation that

$$(2.11) \quad u_1 \equiv 0 \quad \text{on } \partial\Omega.$$

Using (2.11) one can rewrite (2.10) as follows

$$(2.12) \quad \frac{\partial G}{\partial x_1}(p) = 2 \sum_{i=2}^n u_i u_{i1} \geq 0.$$

From the definition of second fundamental form of a hypersurface in $\mathbb{R}^n$, one can derive

$$(2.13) \quad u_{i1} = - \sum h_{ij} u_j + \sum b_{ij} u_i, \quad 2 \leq i, j \leq n$$

where (b_{ij}) is a skew symmetric matrix i.e. $b_{ij} = -b_{ji}$. Putting (2.13) into (2.12), we have

$$(2.14) \quad \frac{\partial G}{\partial x_1}(p) = -2 \sum_{i,j=2}^n u_i h_{ij} u_j \geq 0.$$

This contradicts the convexity of $\partial\Omega$. Thus $u_i(p) = 0$ for all $2 \leq i \leq n$, and yields our inequality $G \leq \sup_{\Omega} \lambda(\mu - u)^2$.

Theorem 2.1 follows from the above two propositions.

3. - Lower bound.

In this section, we shall derive our lower bound $\pi^2/4d^2 \leq \lambda_2 - \lambda_1$.

Recall our basic estimate (Theorem 2.1) which says that for $\mu \geq \sup u$:

$$(3.1) \quad |\nabla u|^2 \leq \lambda \left\{ \sup_{\Omega} (\mu - u)^2 - (\mu - u)^2 \right\}.$$

In particular, we have

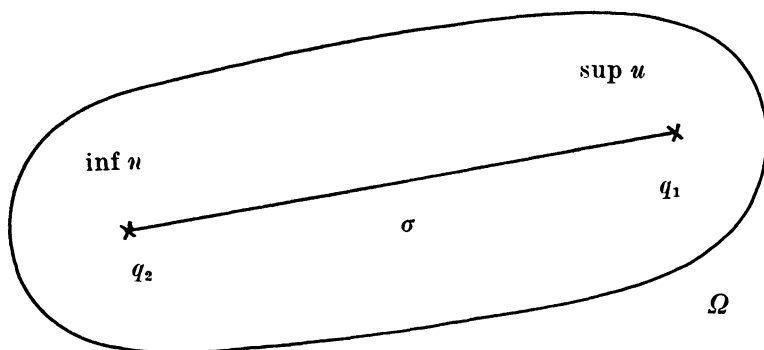
$$(3.2) \quad |\nabla u|^2 \leq \lambda \{ (\sup u - \inf u)^2 - (\sup u - u)^2 \}.$$

Furthermore,

$$(3.3) \quad \sqrt{\lambda} \geq \frac{|\nabla u|}{\sqrt{(\sup u - \inf u)^2 - (\sup u - u)^2}}.$$

Let $A = \sup u - \inf u$ and $W = \sup u - u$. One can rewrite (3.3) as

$$(3.4) \quad \sqrt{\lambda} \geq \frac{|\nabla W|}{\sqrt{A^2 - W^2}}.$$



Let q_1, q_2 be two points of $\bar{\Omega}$ such that $u(q_1) = \sup u$, $u(q_2) = \inf u$, and σ is the line segment joining them. σ lies in Ω since it is convex by assumption. We integrate both sides of (3.3) along σ from q_1 to q_2 and obtain

$$\int_{\sup u}^{\inf u} \frac{|\nabla u| ds}{\sqrt{(\sup u - \inf u)^2 - (\sup u - u)^2}} \leq \int_{q_1}^{q_2} \sqrt{\lambda} ds.$$

Changing variables, we have

$$\int_0^A \frac{|dW|}{\sqrt{A^2 - W^2}} \leq \int_{q_1}^{q_2} \sqrt{\lambda} ds.$$

By elementary calculus, one has

$$\frac{2}{\pi} \leq \sqrt{\lambda} \ell(\sigma) \leq \sqrt{\lambda} d,$$

where $\ell(\sigma)$ = length of σ , d = diameter of Ω . This proves $\lambda_2 - \lambda_1 \geq \pi^2/4d^2$ as has been claimed.

4. - Upper bound.

The major step to establish our upper bound $\lambda_2 - \lambda_1 \leq 4\pi^2/D^2 + \frac{4(M-m)}{n}$ is the following.

LEMMA 4.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and V a bounded nonnegative potential defined on $\bar{\Omega}$. Suppose λ_1, λ_2 are the first and second nonzero eigenvalues of the Dirichlet boundary problem*

$$(4.1) \quad \begin{cases} \Delta f - Vf = -\lambda f \\ f \equiv 0 \quad \text{on } \partial\Omega, \end{cases}$$

then

$$\lambda_2 - \lambda_1 \leq \frac{4}{n}(\lambda_1 - m).$$

where $m = \inf_{\bar{\Omega}} V$.

Some results of this sort in the case of $V \equiv 0$ were given by Payne, Pólya and Weinberger [6].

PROOF. Let f_1 be the first eigenfunction of (4.1). Take a trial function $f = x_i f_1 - a f_1$, where x_i is any fixed coordinate function for some $1 \leq i \leq n$ and a is a constant chosen to satisfy $\int_{\Omega} f \cdot f_1 = 0$. The following computation shows that

$$(4.1) \quad \begin{aligned} -\Delta f + Vf &= -2 \frac{\partial f_1}{\partial x_i} + (x_i - a)(-\Delta f_1 + Vf_1) \\ &= -2 \frac{\partial f_1}{\partial x_i} + \lambda_1(x_i - a)f_1 \\ &= -2 \frac{\partial f_1}{\partial x_i} + \lambda_1 f. \end{aligned}$$

Multiplying both sides of (4.1) by f , integrating over Ω and then dividing by $\int_{\Omega} f^2$, we have

$$(4.2) \quad \frac{\int_{\Omega} f(-\Delta f + Vf)}{\int_{\Omega} f^2} = \frac{-2 \int_{\Omega} (\partial f_1 / \partial x_i) \cdot f}{\int_{\Omega} f^2} + \lambda_1.$$

The following formula is well-known,

$$(4.3) \quad \lambda_2 = \inf_{\substack{g \perp f_1 \\ g=0 \text{ on } \partial\Omega}} \frac{\int_{\Omega} (-g \Delta g + Vg^2)}{\int_{\Omega} g^2} = \inf_{\substack{g \perp f_1 \\ g=0 \text{ on } \partial\Omega}} \frac{\int_{\Omega} |\nabla g|^2 + \int_{\Omega} Vg^2}{\int_{\Omega} g^2}.$$

(4.3) together with (4.2) and the fact that $f \perp f_1$ imply

$$(4.4) \quad \lambda_2 \leq -2 \left(\frac{\int_{\Omega} (\partial f_1 / \partial x_i) \cdot f}{\int_{\Omega} f^2} \right) + \lambda_1,$$

$$(4.5) \quad \lambda_2 - \lambda_1 \leq -2 \left(\frac{\int_{\Omega} (\partial f_1 / \partial x_i) \cdot f}{\int_{\Omega} f^2} \right).$$

Substituting $f = x_i f_1 - a f_1$ and integrating by parts, gives

$$\begin{aligned} (4.6) \quad \int_{\Omega} f \cdot \frac{\partial f_1}{\partial x_i} &= \int_{\Omega} (x_i f_1 - a f_1) \frac{\partial f_1}{\partial x_i} = \frac{1}{2} \int_{\Omega} (x_i - a) \frac{\partial (f_1^2)}{\partial x_i} \\ &= \frac{1}{2} \int_{\Omega} x_i \frac{\partial (f_1^2)}{\partial x_i} \\ &= -\frac{1}{2} \int_{\Omega} f_1^2. \end{aligned}$$

We can always normalize f_1 such that $\int_{\Omega} f_1^2 = 1$. Combining (4.5) and (4.6), we have

$$(4.7) \quad \lambda_2 - \lambda_1 \leq \frac{1}{\int_{\Omega} f^2}.$$

Again from (4.6) $\int_{\Omega} f (\partial f_1 / \partial x_i) = -\frac{1}{2}$; moreover, the Schwarz lemma says that

$$(4.8) \quad \left(\int_{\Omega} \left(\frac{\partial f_1}{\partial x_i} \right)^2 \right) \left(\int_{\Omega} f^2 \right) \geq \frac{1}{4}.$$

This implies that

$$(4.9) \quad \left(\int_{\Omega} |\nabla f_1|^2 \right) \cdot \left(\int_{\Omega} f^2 \right) \geq \frac{n}{4}$$

since $|\nabla f_1|^2 = \sum_{j=2}^n (\partial f_1 / \partial x_j)^2$. Bringing (4.7) and (4.9) together, we have

$$(4.10) \quad \lambda_2 - \lambda_1 \leq \frac{4}{n} \int_{\Omega} |\nabla f_1|^2.$$

Since $-\Delta f_1 + V f_1 = \lambda_1 f_1$ and $V \geq m$, it is easy to see that $\int_{\Omega} |\nabla f_1|^2 \leq \lambda_1 - m$.

Using this fact, one can conclude from (4.10) that

$$\lambda_2 - \lambda_1 \leq \frac{4}{n} (\lambda_1 - m).$$

This completes the proof.

REMARK. It is in general true that $\lambda_{i+1} - \lambda_i \leq \left(2 \sum_{j=1}^i \lambda_j\right) / i$, where $1 \leq i \leq n-1$.

PROOF OF UPPER BOUND OF $\lambda_2 - \lambda_1$. Recall the identity (4.3)

$$\lambda_1 = \inf_{f=0 \text{ on } \partial\Omega} \frac{\int_{\Omega} |\nabla f|^2 + \int_{\Omega} V f^2}{\int_{\Omega} f^2}.$$

Let us choose g vanishing on $\partial\Omega$ s.t. $\int_{\Omega} |\nabla g|^2 / \int_{\Omega} g^2 = \mu_1$, where μ_1 is the first-nonnzero eigenvalue of the Dirichlet problem (1.1) on Ω with $V \equiv 0$. Clearly we have

$$\lambda_1 \leq \frac{\int_{\Omega} |\nabla g|^2 + \int_{\Omega} V g^2}{\int_{\Omega} g^2} \leq \frac{\int_{\Omega} |\nabla g|^2}{\int_{\Omega} g^2} + M = \mu_1 + M.$$

Using a theorem of Cheng [2], we have

$$\mu_1 \leq \frac{n^2 \pi^2}{D^2}, \quad \text{when } n = \dim \Omega$$

and D = the diameter of the largest inscribed ball in Ω . With Lemma 4.1, we can now establish our upper bound for $\lambda_2 - \lambda_1$ asserted in Theorem 1.1.

$$\lambda_2 - \lambda_1 \leq \frac{4}{n} (\lambda_1 - m) \leq \frac{4}{n} [\mu_1 + M - m] \leq \frac{4}{n} \left[\frac{n^2 \pi^2}{D^2} + M - m \right] \leq \frac{4n\pi^2}{D^2} + \frac{4}{n} (M - m).$$

5. - Gap of eigenvalues over R^n .

In this section, we extend the estimate for eigenvalues of bounded domain to eigenvalues of R^n . We need the following well-known fact.

PROPOSITION 5.1. *Let $\lambda_2(R)$ be the second eigenvalue of $\Delta - V$ defined on the ball $B(R)$ with Dirichlet boundary condition. Then $\lambda_2(R)$ is a con-*

tinuous piecewise smooth function of R when $R > 0$. When it is smooth,

$$(5.1) \quad \frac{d}{dR} \lambda_2(R) = - \int_{\partial B(R)} \left(\frac{\partial \varphi}{\partial r} \right)^2$$

where φ is a normalized second eigenfunction of $\Delta - V$ defined on $B(R)$.

PROOF. Let $\varphi(x; r_2)$ be the normalized second eigenfunction of $\Delta - V$ defined on the ball $B(r_2)$ with Dirichlet boundary condition. In polar coordinates, φ is a function of the form $\varphi(\theta, r_1; r_2)$ where $\theta \in S^{n-1}$, the unit sphere, and $0 < r_1 \leq r_2 < \infty$.

It is well-known that we can assume φ to be piecewise smooth as a function of r_2 . At the points where u is smooth, we can differentiate the equation for φ and obtain

$$(5.2) \quad \int_{B(r_2)} \varphi \Delta \left(\frac{\partial \varphi}{\partial r_2} \right) = \int_{B(r_2)} \varphi (V - \lambda_2) \left(\frac{\partial \varphi}{\partial r_2} \right) - \int_{B(r_2)} \frac{d\lambda_2}{dr_2} \varphi^2.$$

Integrating by parts, we derive

$$(5.3) \quad \frac{d\lambda_2}{dr_2} = \int_{\partial B(r_2)} \frac{\partial \varphi}{\partial r_1} \frac{\partial \varphi}{\partial r_2}.$$

Notice that $\varphi(\theta, r, r) = 0$ for all r . Hence

$$(5.4) \quad 0 = \frac{d}{dr} \varphi(\theta; r, r) = \frac{\partial \varphi}{\partial r_1}(\theta; r, r) + \frac{\partial \varphi}{\partial r_2}(\theta, r, r).$$

Putting this into (5.3) we have

$$(5.5) \quad \frac{d\lambda_2}{dr_2}(R) = - \int_{\partial B(r_2)} \left(\frac{\partial \varphi_2}{\partial r_1} \right)^2.$$

PROPOSITION 5.2. Let φ be an eigenfunction of $\Delta - V$ defined on the ball $B(R) \subset \mathbb{R}^n$ with Dirichlet boundary condition and eigenvalue λ . Then

(i) If $2n - 2 \geq k > 2$,

$$(5.6) \quad \int_{\partial B(R)} \left(\frac{\partial \varphi}{\partial r} \right)^2 \leq R^{-k+n-1} \left[-k \int_{B(R)} r^{k-n} (V - \lambda) \varphi^2 - \int_{B(R)} r^{k-n+1} \frac{\partial V}{\partial r} \varphi^2 \right].$$

(ii) If $k \geq 2n-2$ and $k > 2$,

$$(5'7) \quad \int_{\partial B(R)} \left(\frac{\partial \varphi}{\partial r} \right)^2 \leq \frac{k-2n+2}{2} R^{-k+n-1} \int_{B(R)} \varphi^2 \Delta r^{k-n} \\ + (2n-2-2k) R^{-k+n-1} \int_{B(R)} r^{k-n} (V-\lambda) \varphi^2 - R^{-k+n-1} \int_{B(R)} r^{k+n-1} \frac{\partial V}{\partial r} \varphi^2.$$

PROOF. Let $d\theta$ be the volume element of the unit sphere S^{n-1} in $\mathbb{R}^n$ and Δ_θ be the spherical Laplacian. Then

$$(5.8) \quad \frac{\partial^2 \varphi}{\partial r^2} + \frac{n-1}{r} \frac{\partial \varphi}{\partial r} + \frac{\Delta_\theta \varphi}{r^2} = (V-\lambda) \varphi$$

and

$$(5.9) \quad \frac{d}{dr} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r} \right)^2 (r, \theta) d\theta = 2 \int_{S^{n-1}} \frac{\partial \varphi}{\partial r} \frac{\partial^2 \varphi}{\partial r^2} d\theta \\ = -2 \int_{S^{n-1}} \frac{n-1}{r} \left(\frac{\partial \varphi}{\partial r} \right)^2 d\theta - 2 \int_{S^{n-1}} r^{-2} \frac{\partial \varphi}{\partial r} \Delta_\theta \varphi d\theta + 2 \int_{S^{n-1}} (V-\lambda) \varphi \frac{\partial \varphi}{\partial r} d\theta \\ = -2(n-1)r^{-1} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r} \right)^2 d\theta + r^{-2} \frac{\partial}{\partial r} \int_{S^{n-1}} |\nabla_\theta \varphi|^2 d\theta + \int_{S^{n-1}} (V-\lambda) \frac{\partial \varphi^2}{\partial r} d\theta \\ = -2(n-1)r^{-1} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r} \right)^2 d\theta + r^{-2} \frac{\partial}{\partial r} \int_{S^{n-1}} |\nabla_\theta \varphi|^2 d\theta + \frac{d}{dr} \int_{S^{n-1}} (V-\lambda) \varphi^2 d\theta - \int_{S^{n-1}} \frac{\partial V}{\partial r} \varphi^2 d\theta.$$

Multiplying this equation by r^k (with $k > 2$) and integrating from 0 to R , we have

$$(5.10) \quad \int_0^R r^k \frac{d}{dr} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r} \right)^2 (r, \theta) d\theta dr = -2(n-1) \int_0^R r^{k-1} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r} \right)^2 d\theta dr \\ + \int_0^R r^{k-2} \frac{d}{dr} \int_{S^{n-1}} |\nabla_\theta \varphi|^2 d\theta dr + \int_0^R r^k \frac{d}{dr} \int_{S^{n-1}} (V-\lambda) \varphi^2 d\theta dr - \int_0^R r^k \int_{S^{n-1}} \left(\frac{\partial V}{\partial r} \right) \varphi^2 d\theta dr.$$

Integrating by parts, we have the following

$$(5.11) \quad R^k \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r} \right)^2 (R, \theta) d\theta = k \int_0^R r^{k-1} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r} \right)^2 (r, \theta) d\theta dr \\ + \int_0^R r^k \left[\frac{d}{dr} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r} \right)^2 (r, \theta) d\theta \right] dr,$$

$$(5.12) \quad 0 = \int_0^R (k-2) r^{k-3} \int_{S^{n-1}} |\nabla_{\theta} \varphi|^2 d\theta dr + \int_0^R r^{k-2} \left[\frac{d}{dr} \int_{S^{n-1}} |\nabla_{\theta} \varphi|^2 d\theta \right] dr$$

$$(5.13) \quad 0 = k \int_0^R r^{k-1} \int_{S^{n-1}} (V-\lambda) \varphi^2 d\theta dr + \int_0^R r^k \left[\frac{d}{dr} \int_{S^{n-1}} (V-\lambda) \varphi^2 d\theta \right] dr.$$

Putting (5.11), (5.12) and (5.13) into (5.10), we have

$$(5.14) \quad R^k \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r} \right)^2 (R, \theta) d\theta = (k-2n+2) \int_0^R r^{k-1} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r} \right)^2 d\theta dr \\ - (k-2) \int_0^R r^{k-3} \int_{S^{n-1}} |\nabla_{\theta} \varphi|^2 d\theta dr - k \int_0^R r^{k-1} \int_{S^{n-1}} (V-\lambda) \varphi^2 d\theta dr - \int_0^R \int_{S^{n-1}} r^k \frac{\partial V}{\partial r} \varphi^2 d\theta dr.$$

Hence,

$$(5.15) \quad R^{k-n+1} \int_{\partial B(R)} \left(\frac{\partial \varphi}{\partial r} \right)^2 \leq (k-2n+2) \int_{B(R)} r^{k-n} \left(\frac{\partial \varphi}{\partial r} \right)^2 - k \int_{B(R)} r^{k-n} (V-\lambda) \varphi^2 \\ - \int_{B(R)} r^{k-n+1} \frac{\partial V}{\partial r} \varphi^2.$$

By the divergence theorem,

$$(5.16) \quad 0 = \int_{\partial B(R)} r^{k-n} \varphi \frac{\partial \varphi}{\partial r} = \int_{B(R)} r^{k-n} |\nabla \varphi|^2 + \int_{B(R)} \varphi \nabla r^{k-n} \cdot \nabla \varphi + \int_{B(R)} r^{k-n} \varphi \Delta \varphi.$$

Hence,

$$(5.17) \quad \int_{B(R)} r^{k-n} \left(\frac{\partial \varphi}{\partial r} \right)^2 \leq \int_{B(R)} r^{k-n} |\nabla \varphi|^2 = - \int_{B(R)} \varphi \nabla r^{k-n} \cdot \nabla \varphi - \int_{B(R)} r^{k-n} \varphi \Delta \varphi \\ = \frac{1}{2} \int_{B(R)} \varphi^2 \Delta r^{k-n} - \int_{B(R)} r^{k-n} (V-\lambda) \varphi^2.$$

The proposition follows from (5.15) and (5.17).

It is straightforward to derive from Theorem 1.1 and the last two propositions the following theorem.

THEOREM 5.1. *Let V be a C^1 -function defined on R^n with $n > 4$. Let $\lambda_2(\varrho)$ be the second eigenvalue of the operator $-\Delta + V$ defined on the ball*

$B(\rho)$ with Dirichlet boundary condition. Suppose that V is convex in the ball $B(R)$, then

$$(i) \quad \lambda_2 - \lambda_1 \geq \frac{\pi^2}{4R^2} - \frac{1}{k-n} R^{-k+n} \cdot \sup \left\{ -k|x|^{k-n}(\lambda_2(|x|) - V(x)) - |x|^{k-n+1} \frac{\partial V}{\partial r} \right\}_+.$$

where $2n-2 \geq k > n$ and $\{f\}_+$ stands for the positive part of f .

(ii) When $k \geq 2n-2$, $k > n$ and $k > 2$,

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{4R^2} - \frac{1}{k-n} R^{-k+n} \cdot \sup_{|x| < R} \left\{ \frac{k-2n+2}{2} \Delta r^{k-n} + (2n-2-2k)|x|^{k-n}(V(x) - \lambda_2(x)) - r^{k-n+1} \frac{\partial V}{\partial r} \right\}_+.$$

REMARK. If $\lim_{|x| \rightarrow \infty} V(x) = \infty$ and $\partial V / \partial r \geq 0$, $k-n > 2$ and R large, we can obtain a positive lower estimate for $\lambda_2 - \lambda_1$. Note also that

$$\lambda_2(R) \leq \lambda_2(1) \leq \frac{n+4}{n} \lambda_1(1) - \frac{4}{n} \inf_{B(1)} V \leq n(n+4)\pi^2 + \frac{n+4}{n} \sup_{B(1)} V - \frac{4}{n} \inf_{B(1)} V.$$

Hence $(\lambda_2(R) - V(x))_+$ can be estimated easily if $\lim_{|x| \rightarrow \infty} V(x) = \infty$.

6. - Appendix.

A) Here we shall give a quick argument to verify the « standard » fact that $u = f_2/f_1$ is smooth up to the boundary $\partial\Omega$. In the whole discussion, we assume Ω to be smooth convex. Our conditions in Theorem 2.1 allow us to apply the classical Hopf lemma to f_1 .

Let us choose local coordinates $\{x_1, x_2, \dots, x_n\}$ on a sufficiently small open set U such that $U \cap \partial\Omega = U \cap \{x_1 = 0\}$. Since f_1 is identically equal to zero on $\partial\Omega$ and $f > 0$ in Ω , by the Hopf lemma we have $\partial f_1 / \partial x_1 < 0$ on $\partial\Omega$. Furthermore, f_1 is smooth up to the boundary, thus one can consider f_1 as a smooth function which is defined on U restricted to $U \cap \bar{\Omega}$. Using the Malgrange preparation theorem [5], together with the fact that $\partial f_1 / \partial x_1 \neq 0$ on $\partial\Omega$, we have locally

$$(6.1) \quad f_1 = g_1 \cdot x_1,$$

where g_1 is a unit which is smooth on $\bar{\Omega} \cap U$.

Moreover, f_2 is identically zero on $\partial\Omega$; applying the Malgrange's theorem again, one can write locally

$$(6.2) \quad f_2 = g_2 \cdot x_1 \cdot h_2,$$

where g_2 is a unit which is smooth in $\bar{\Omega} \cap U$, and h_2 is also a smooth function in $\bar{\Omega} \cap U$. Now it is clear

$$u = \frac{f_2}{f_1} = \frac{g_2 \cdot h_2}{g_1}$$

must be smooth on $U \cap \bar{\Omega}$.

B) Here we give a proof of a theorem of Brascamp and Lieb.

Let f_1 be the first positive eigenfunction of the operator $\Delta - V$ on a convex domain Ω with Dirichlet condition. Then $u = \log f$ satisfies the equation

$$\Delta u = (V - \lambda) - |\nabla u|^2.$$

By convexity of Ω , it is easy to see that u is concave in a neighborhood of $\partial\Omega$. If we consider the Hessian of u as a function of the frame bundle of Ω , it achieves a maximum in the interior of Ω . At such a point,

$$(6.3) \quad 0 \geq \Delta u_{ii} = (V_{ii}) - 2 \sum_j u_{ij}^2$$

and

$$(6.4) \quad u_{ij} = 0 \quad \text{for } i \neq j,$$

Hence

$$(6.5) \quad u_{ii}^2 \geq \frac{1}{2} \min_{\Omega} V_{ii}.$$

By using (6.5), we can prove the concavity of u by the method of continuity. In fact, we can find family Ω_t and V_t so that $\Omega_1 = \Omega$ and $V_1 = V$. Furthermore, we may assume Ω_0 is a ball in Ω and V_0 is a quadratic function so that by computation, the theorem is valid in this case. In fact, we can let $V_t = tV + (1-t)V_0$ and $\Omega_t = \{(1-t)x_0 + tx_1: x_1 \in \Omega_1 \text{ and } x_0 \in \Omega_0\}$. Then $\min_{\Omega} (V_t)_{ii} > 0$.

If for $t < 1$, u_t is not concave, $(u_t)_{ii}$ at the maximum point will be positive by (6.5). This is not possible if we have a sequence $t_\alpha \rightarrow t$ with $\max (u_{t_\alpha})_{ii} \leq 0$. Hence we have proven the log concavity of f_1 .

The proof actually shows that $(\log f_1)_{ii} \leq -\sqrt{\frac{1}{2} \min_{\Omega} V_{ii}}$:

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