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**On the elliptic equation** $Lu - k + K\exp[2u] = 0$

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 12, nº 2 (1985), p. 191-224
<http://www.numdam.org/item?id=ASNSP_1985_4_12_2_191_0>

# On the Elliptic Equation $Lu - k + K \exp[2u] = 0$.

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## 1. – Introduction.

Let $(M, g)$ be a Riemannian manifold of dimension 2 and $K$ be a given function on $M$. One may ask the following question: can we find a new metric $g_1$ on $M$ such that $K$ is the Gaussian curvature of $g_1$ and $g_1$ is *conformal* to $g$ (i.e. there exists a positive function $\varphi$ on $M$ such that $g_1 = \varphi g$)? If one writes $\varphi = \exp[2u]$, then this is equivalent to the problem of solving the elliptic equation

$$\Delta_g u - k + K \exp[2u] = 0 \tag{1.1}$$

on $M$, where $\Delta_g$, $k$ are the Laplace-Beltrami operator and the Gaussian curvature of $M$ in the metric $g$. This problem has been considered by many authors. When $M$ is compact, we refer to Kazdan an Warner [KW] for details and references (see also a recent survey by Kazdan [K]). When $M$ is $\mathbf{R}^2$, and $g$ is the usual metric, equation (1.1) reduces to

$$\Delta u + K \exp[2u] = 0 \tag{1.2}$$

on $\mathbf{R}^2$. For equation (1.2), there have been some non-existence results due to various authors, e.g. Ahlfors [A], Wittich [W], Osserman [Os], Sattinger [S], Oleinik [O], Ni [$N_1$], etc. The first existence result for equation (1.2) seems due to Ni [$N_1$] in the case $K \leqslant 0$. It complements previous non-existence results. The approach used in [$N_1$] is via the barrier method, i.e. super- and sub-solutions. Using an entirely different approach, namely, weighted Sobolev spaces, McOwen [$M_1$] refined some of Ni's results by obtaining more precise asymptotic behaviour of the solutions given by his method.

(*) Both authors are partially supported by NSF. The first author is also partially supported by the Alfred P. Sloan Foundation.

Pervenuto alla Redazione il 10 Febbraio 1984 e in forma definitiva il 14 dicembre 1984.

More recently, Aviles [Av] and McOwen [$M_2$] used this weighted Sobolev space approach to obtain some existence results when $K$ changes sign and decays at $\infty$.

Returning to the more general equation (1.1), we observe that, under some appropriate hypothesis on the metric $g$, equation (1.1) may be reduced to equation (1.1) via the Uniformization Theorem. However, in order to apply previous results to this reduced equation (1.2), it is essential to have asymptotic estimates for this conformal diffeomorphism $\varphi$ guaranteed by the Uniformization Theorem. For example, the following estimates

$$\frac{|x|}{C} \leqslant \varphi(x) \leqslant C|x|$$

for some positive constant $C$ and for $|x|$ large, would be sufficient. This kind of estimates have been established by Ahlfors and Bers [AB] under the condition that the metric $g_{ij} = \lambda^2(x)\delta_{ij}$ outside a compact set in $\mathbb{R}^2$ for some $\lambda > 0$. We would also like to point out that, in the above reduction, it is *important* that $k$ is the Gaussian curvature of $(\mathbb{R}^2, g)$. Therefore, from a differential equations point of viaw, a natural question arises: can we solve equation (1.1) if $k$ or $\Delta_g$ is perturbed? i.e. what happens to the solvability of (1.1) if we replace the curvature $k$ by something similar or if $\Delta$- is replaced by an elliptic operator of divergence form? The Uniformization Theorem approach does not seem to give an answer.

The purpose of this present paper is to study the solvability of equation (1.1) as well as to investigate the above question. We shall free ourselves from geometry and discuss the equation

$$Lu - k + K\exp[2u] = 0 \tag{1.3}$$

on $\mathbb{R}^2$, where $L = \partial_i(a_{ij}\partial_j)$, $(a_{ij})$ is uniformly elliptic, $k$, $K$ are given locally bounded functions with appropriate conditions at $\infty$. It turns out we can extend the results in [$N_1$] to the equation (1.3). We again adapt the barrier method used in [$N_2$]. However, in the present case, it is much more involved to construct barriers. The new ingredient that we need is the construction of entire solutions to the linear equation $Lu = f$, with sharp asymptotic behavior at $\infty$. We accomplish this by adapting a technique of Friedman [F]. Incidentally, a refinement of the argument gives the existence of a global fundamental solution for $L$, with estimates depending only on the uniform ellipticity constants of $L$. While this fact is not needed in the paper, we sketch its proof in an appendix, since we have been unable to find a proof of it in the literature, and it may prove useful in the future.

Once sharp results for the linear equation $Lu = f$ are obtained, we use a rearrangement inequality of Talenti [T], to be able to construct solutions to (1.3) under optimal assumptions on $K$.

In the case when $L$ is degenerate, i.e. $(a_{ij})$ is uniformly elliptic on compact sets and its eigenvalues behave like $|x|^b$ at $\infty$ with $b \neq 0$, this barrier method also works. In these cases, we employ the theory of degenerate elliptic equations developed by Fabes, Jerison, Kenig and Serapioni [FKS], [FJK].

Weaker versions of our present results and methods in the case when $L$ is uniformly elliptic were announced by us in the AMS Summer Meeting of 1982 at Toronto (in the special session « Nonlinear partial differential equations in physics and geometry ») August, 1982, and at the 803rd AMS meeting at New York City (in the special session « Variational problems in Riemannian geometry »), April, 1983 (see [$KN_2$], [$KN_3$]). The methods and the flavor of this paper are quite close to our previous work on the corresponding problem for prescribing scalar curvatures [$KN_1$].

The organization of this paper is as follows. In section 2, we treat a model case of (1.3) when $L \equiv \Delta$, the standard Laplacian. In section 3, we extend these results to the case $L$ is uniformly elliptic. In sections 4 and 5, we treat equation (1.3) in the case when $L$ is degenerate. Finally, in the appendix we prove the existence of a global fundamental solution for uniformly elliptic operators, with precise asymptotic behavior.

In closing, we should mention that L. Karp [$K_p$] has some results related to ours.

ACKNOWLEDGEMENT. Part of this research was carried out while the first author was visiting the Centre for Mathematical Analysis at the Australian National University, Canberra, Australia. The first author wishes to thank the Centre for its financial support, and its staff for their warm hospitality during his stay in Canberra.

## 2. – The special case $L \equiv \Delta$.

To make our approach more transparent, we first study the case $L \equiv \Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$, i.e. the equation

$$\Delta u - k + K \exp[2u] = 0 \tag{2.1}$$

where $k$, $K$ are given, and we are only interested in entire solutions on $\mathbb{R}^2$. In this section, we shall present some existence results using the barrier method. We also include a non-existence result and an example which (together) show that our existence result (Theorem 2.2) is sharp.

THEOREM 2.2. *Suppose*

$$|k| \leqslant \frac{C}{|x|^{2+\varepsilon}}$$

*at* $\infty$ *for some* $\varepsilon > 0$ *and* $K \leqslant 0$ *on* $\mathbb{R}^2$. *If* $K \not\equiv 0$ *and*

$$|K| \leqslant \frac{C}{|x|^{2+\alpha}}$$

*at* $\infty$ *for some* $\alpha > (1/\pi)\int_{\mathbb{R}^2} k(x)\,dx$, *then for every* $0 < a < \frac{1}{3}\left(\alpha - (1/\pi)\int_{\mathbb{R}^2} k\right)$, *equation* (2.1) *possesses an entire solution* $u$ *with*

$$u(x) = \left(\frac{1}{2\pi}\int_{\mathbb{R}^2} k(y)\,dy + a\right)\log|x| + O(1)$$

*at* $\infty$.

PROOF. We use the super-and sub-solution method (See, e.g. [KN$_1$], or [N$_2$]) which is by now well-known. First, we construct a sub-solution. Let

$$f_2(x) = \frac{\delta}{(1 + |x|^2)^{1+a/2}}$$

where $\delta = a^2$. An easy computation shows that

$$\int_{\mathbb{R}^2} f_2(x)\,dx = \frac{2\pi\delta}{a} = 2\pi a\,.$$

Let $v_2$ solve $\Delta v_2 = k + f_2$; more precisely, let

$$v_2(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x - y|\cdot[k(y) + f_2(y)]\,dy\,.$$

By dividing $\mathbb{R}^2$ into the three regions $\{y \in \mathbb{R}^2\colon |y| > 2|x|\}$, $\{y \in \mathbb{R}^2\colon 2|x| > |y| > |x|/2\}$ and $\{y \in \mathbb{R}^2\colon |y| < |x|/2\}$, we may estimate $v_2$. It is not hard to obtain that, at $\infty$

$$v_2(x) = \frac{1}{2\pi}\left(\int_{\mathbb{R}^2} (k + f_2)\right)\log|x| + O(1)\,.$$

For simplicity, we set

$$\tilde{k} = \frac{1}{2\pi}\int_{\mathbf{R}^2} k\,,$$

and then

$$v_2(x) = (\tilde{k} + a)\log|x| + O(1)$$

at $\infty$.

Now, let $u_2 = v_2 + C_2$ where $C_2$ is a constant to be chosen later. We compute

$$\Delta u_2 - k + K\exp[2u_2] = f_2 + K\exp[2v_2]\exp[2C_2]\,.$$

From the estimate of $v_2$ and the hypothesis on the behavior of $K$ at $\infty$, we have

$$|K|\exp[2v_2]\exp[2C_2] \leqslant \frac{C\exp[2C_2]}{|x|^{2+\alpha-2(\tilde{k}+a)}}$$

at $\infty$. Since

$$2 + a < 2 + \alpha - 2(\tilde{k} + a)$$

we always have $f_2 + K\exp[2v_2] > 0$ at $\infty$. For the finite part, since $f_2 > 0$ in $\mathbf{R}^2$, $f_2$ is bounded below by a positive constant. It is thus possible to choose $C_2$ to be a large negative constant so that

$$f_2 + K\exp[2v_2]\exp[2C_2] > 0$$

on $\mathbf{R}^2$. Therefore, $u_2 = v_2 + C_2$ where $C_2$ is a large negative constant is a subsolution.

Next, we construct a super-solution $u_1$ with $u_1 \geqslant u_2$ on $\mathbf{R}^2$. Without loss of generality, we may assume $K(0) \not\equiv 0$. Let $B_\varrho$ be a ball of radius $\varrho$ centered at 0 and contained in the interior of the support of $K$, and let $f_1(x) = f_1(|x|)$ be a radial function, positive in $B_{\varrho/2}$ with

$$f_1(|x|) \leqslant -|k(x)|,\quad |f_1(r)| \leqslant C/r^{2+\varepsilon}$$

outside $B_\varrho$ and

$$\int_{\mathbf{R}^2} f_1(x)\,dx = \int_{\mathbf{R}^2}(k + f_2)(x)\,dx$$

(i.e. $f_1$ is large and positive in $B_{\varrho/2}$).

Let

$$v_1(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \log|x-y|\, f_1(y)\, dy\,.$$

Similarly, $\Delta v_1 = f_1$ and at $\infty$

$$v_1(x) = \left(\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} f_1\right)\cdot \log|x| + O(1)\,.$$

Let $u_1 = v_1 + C_1$ where $C_1$ is a large positive constant to be chosen later. We compute,

$$\Delta u_1 - k + K\exp[2u_1] = f_1 - k + K\exp[2v_1]\exp[2C_1]\,.$$

Outside $B_\varrho$, $f_1 \leqslant -|k| \leqslant k$. Thus $f_1 - k + K\exp[2v_1]\exp[2C_1] \leqslant 0$ there since $K \leqslant 0$ on $\mathbb{R}^2$. In $B_\varrho$, $f_1 - k$ is (fixed and therefore) bounded while $k \leqslant -C < 0$ on $B_\varrho$, and so

$$K\exp[2v_1] \leqslant -C' < 0$$

on $\overline{B_\varrho}$, thus, we may choose $C_1 > 0$ so large that

$$K\exp[2v_1]\exp[2C_1] \leqslant -C'\exp[2C_1] \leqslant -(f_1 - k)$$

on $\overline{B_\varrho}$. With this choice of $C_1$, we have

$$\Delta u_1 - k + K\exp[2u_1] \leqslant 0$$

on $\mathbb{R}^2$ and $u_1$ is a super-solution. Now, comparing $u_1$ and $u_2$, we see that at infinity,

$$v_1(x) = \left(\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} f_1\right)\cdot\log|x| + O(1) = \left[\frac{1}{2\pi}\int\limits_{\mathbb{R}^2}(k + f_2)\right]\log|x| + O(1) = v_2(x) + O(1)\,.$$

It is therefore possible to choose $C_1 > 0$ even larger to assure that $u_1 > u_2$ on $\mathbb{R}^2$ (the finite part does not create any difficulties at all).

Now, our assertion follows from, e.g. Theorem 5 in [KN$_1$] or Theorem 2.10 in [N$_2$].

REMARK 2.3:

(i) Note that there is no sign condition imposed on $k$, and the constant $\alpha$

could be large negative $\left(\text{it depends only on } \int_{\mathbf{R}^2} k\right)$. In this case $|K|$ is actually allowed to have polynomial growth at $\infty$.

(ii) The bound on $\alpha$ is actually best possible by the nonexistence result below and the example at the end of this section.

(iii) Let

$$v(x) = \int \log|x-y|\, k(y)\, dy = \left(\int_{\mathbf{R}^2} k\right) \log|x| + O(1)\,,$$

It is easy to see that by setting $w = u - v$, equation (2.1) reduces to equation (1.2) and therefore Theorem 2.2 follows from the previous work in [$\mathrm{N}_1$]. We prefer our first proof because it generalizes to the more general equation (1.3) in its present form and thereby our approach is made transparent. Furthermore, the idea in our first proof may be applied to handle the situation where $k$ is *not* integrable (see the remark below and Theorem 2.4).

(iv) Our method is simple, elementary and yet very flexible. For $k$, $K$ with other asymptotic behaviors, it is still possible to use this barrier method to obtain some results; we shall only include one of those results here, just to illustrate this point.

THEOREM 2.4. *Suppose $k$, $K$ are both negative on $\mathbf{R}^2$. If there exist constants $C_1, C_2, C_3, C_4, l_1, l_2, l_3$ all positive, such that*

$$-\frac{C_1}{|x|^2} \geq k(x) \geq -C_2|x|^{l_1}$$

*and*

$$-\frac{C_3}{|x|^{l_2}} \geq K(x) \geq -C_4|x|^{l_3}$$

*at $\infty$, then, equation* (2.1) *possesses at least one entire solution on $\mathbf{R}^2$.*

PROOF. The proof goes along the same line as that of Theorem 2.2. First, let $f_1$ be a function positive in the unit ball $B_1$, identically zero outside $B_1$ and with

$$\frac{1}{2\pi}\int_{\mathbf{R}^2} f_1(x)\, dx = a\,,$$

where $a$ is an arbitrary number which is bigger than $\frac{1}{2}\max\{l_1 + l_2, 2 + l_2\}$.

Let

$$v_1(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| \cdot f_1(y)\, dy ,$$

then $\Delta v_1 = f_1$ and at $\infty$,

$$v_1(x) = a \log|x| + O(1) .$$

Set $u_1 = v_1 + C_5$, where $C_5 > 0$ will be chosen later. Now, compute

$$\Delta u_1 - k + K \exp[2u_1] = f_1 - k + K \exp[2v_1] \exp[2C_5] .$$

At infinity, we have

$$K \exp[2v_1] \exp[2C_5] \leqslant C \exp[2C_5]\, |x|^{2a-l_2}(-C_3) = -C \exp[2C_5]\, |x|^{2a-l_2} .$$

Thus, at $\infty$,

$$f_1 - k + K \exp[2u_1] \leqslant C_2|x|^{l_1} - C \exp[2C_5]\, |x|^{2a-l_2} < 0$$

since $2a - l_2 > l_1$. For the finite region, since $f_1 - k$ is fixed, and $K < 0$, we can choose $C_5 > 0$ so large that the above inequality

$$f_1 - k + K \exp[2v_1] \exp[2C_5] < 0$$

also holds on, say, $|x| \leqslant R$. With this choice of $C_5$, we see that $u_1 = v_1 + C_5$ is a super-solution.

Now, we are ready to construct a sub-solution $u_2$ with $u_2 \leqslant u_1$ on $\mathbb{R}^2$. Let $f_2(x) \leqslant 0$ on $\mathbb{R}^n$ and

$$|f_2(x)| \leqslant \tfrac{1}{2}|k(x)|$$

for every $x \in \mathbb{R}^2$, with

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} f_2(x)\, dx = -a$$

(Note that this is possible since $k(x) \leqslant -C/|x|^2$ at $\infty$). Set $u_2 = v_2 - C_6$, where $C_6 < 0$ will be chosen later and

$$v_2(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f_2(y) \log|x-y|\, dy .$$

Then, similarly, $\Delta v_2 = f_2$ and at $\infty$,

$$v_2(x) = -a \log |x| + O(1) .$$

Therefore, at $\infty$

$$\Delta u_2 - k + K \exp[2u_2] \geq -\tfrac{1}{2}k + K \exp[2C_6]\,|x|^{-2a} \geq \frac{C}{|x|^2} - C\frac{1}{|x|^{2a-l_3}} > 0$$

since $C_6 < 0$ and $2a - l_3 > 2$. For the finite region, say $|x| \leq R$, we can choose $C_6$ *large negative so that*

$$-\tfrac{1}{2}k + K \exp[2v_2] \exp[2C_6] > 0$$

since $k < 0$. Thus, $u_2$ is a subsolution if $C_6$ is large negative. Since $a > 0$, $u_1 > u_2$ at $\infty$. Thus $u_1 \geq u_2$ on $\mathbb{R}^2$ (by varying $C_6$ again if necessary) and our result follows.

The argument we use in proving the following non-existence result goes back to Wittich [W] and has been used by various authors (see, e.g. [S], [$N_1$]). We present here a variant of this technique.

THEOREM 2.5. *If* $k \geq 0$ *on* $\mathbb{R}^2$ *with*

$$k(x) \geq \frac{C_0}{|x|^{2+\varepsilon}}$$

*for* $|x| \geq R_0$ *where* $C_0, \varepsilon$ *are positive constants, then equation* (2.1) *possesses no entire solution on* $\mathbb{R}^2$ *for* $K \leq 0$ *on* $\mathbb{R}^2$ *with*

$$|K(x)| \geq \frac{C}{|x|^{2+\alpha}}$$

*at* $\infty$, *for some* $\alpha < 2C_0/\varepsilon R_0^{\varepsilon}$.

REMARK 2.6. Following $Ni$[$\mathrm{N}_1$], the pointwise conditions on $k$ and $K$ may be replaced by some appropriate integral conditions. To this end, we define, as in [$\mathrm{N}_1$],

$$\bar{v}(r) = \frac{1}{2\pi r}\int_{S_r} v(x)\,dS\,, \qquad \tilde{K}(r) = \left(\frac{1}{2\pi r}\int_{S_r}\frac{1}{K(x)}\,dS\right)^{-1}$$

where $S_r$ is the sphere of radius $r$ centered at the origin, and $dS$ denotes the volume element of $S_r$. Note that $\tilde{K} \equiv K$ if $K$ is radially symmetric and use the convention that $\tilde{K}(r) = 0$ if

$$\int_{S_r} \frac{1}{K(x)}\, dS = \infty .$$

Now, the conclusion of Theorem 2.5 still holds if we replace $k$ by $\bar{k}$ and $K$ by $\tilde{K}$.

PROOF OF THEOREM 2.5. Suppose $u$ is a solution of (2.1) on $\mathbb{R}^2$. Set $v = -u$, equation (2.1) becomes

$$\Delta v + k - K \exp[-2v] = 0 .$$

Using Green's identity, Jensen's and Schwarz's inequalities we arrive at (see p. 350 [$N_1$] for details)

$$\frac{1}{r}(r\bar{v}')' + \bar{k} \leqslant \tilde{K} \exp[-2\bar{v}] .$$

*Relabel* this, writing $v$, $k$, $K$ instead of $\bar{v}$, $\bar{k}$, $\tilde{K}$. We have

$$\frac{1}{r}(rv')' + k \leqslant K \exp[-2v] .$$

Making the change of variables $r = \exp[t]$, $m(t) = v(\exp[t])$, we get

$$\ddot{m} + \exp[2t]\, k(\exp[t]) \leqslant \exp[2t]\, K(\exp[t]) \exp[-2m] , \tag{2.7}$$

where the dot $\cdot$ means differentiation with respect to $t$. Observe that $v' \leqslant 0$, $\dot{m} \leqslant 0$ (since $k \geqslant 0$, $K \leqslant 0$).

Let $t_0 = \log R_0$. Then

$$k(\exp[t]) \geqslant \frac{C_0}{\exp[(2+\varepsilon)t]}$$

ofr $t \geqslant t_0$. From (2.7), we conclude, for $t \geqslant t_0$

$$\ddot{m} \leqslant -C_0 \exp[-\varepsilon t] .$$

Integrating,

$$\dot{m}(t) \leqslant \frac{C_0}{\varepsilon}(\exp[-\varepsilon t] - \exp[-\varepsilon t_0])$$

for $t \geqslant t_0$. Integrating again, we obtain

$$-m(t) \geqslant \frac{C_0}{\varepsilon R_0^\varepsilon}(t - t_0) - \left[m(t_0) + \frac{C_0}{\varepsilon^2 R_0^\varepsilon}\right] \tag{2.8}$$

for all $t \geqslant t_0$. Denote the constants by

$$A = \frac{C_0}{\varepsilon R_0^\varepsilon}, \qquad B = m(t_0) + \frac{C_0}{\varepsilon_0^2 R_0^\varepsilon},$$

(2.8) reduces to

$$-m(t) \geqslant A(t - t_0) - B \tag{2.9}$$

for all $t \geqslant t_0$: By (2.7) and our assumptions on $k$,

$$\ddot{m} \leqslant [\exp[2t]\, K(\exp[t]) \exp[-(2-\delta)m]] \exp[-\delta m] \tag{2.10}$$

where $\delta > 0$ is a small constant to be chosen later. By (2.9) and our hypothesis on $K$, we derive the following estimate for $t \geqslant t_0$

$$\begin{aligned} &\exp[2t]\, K(\exp[t]) \exp[-(2-\delta)m] \\ &\qquad \leqslant -C\exp[-(2-\delta)[At_0 + B]] \exp[[(2-\delta)A - \alpha]t]\,. \\ &\qquad \leqslant -C \end{aligned}$$

if we choose $\delta = 2 - \alpha/A$. (Note that $\delta > 0$ by the assumption on $\alpha$). Thus (2.10) becomes, for $t \geqslant t_0$,

$$\ddot{m} \leqslant -C\exp[-\delta m]\,. \tag{2.11}$$

We now proceed to get a contradiction. Multiplying (2.11) by $\dot{m}$ (which reverses the inequality) and integrating,

$$\frac{-\dot{m}(t)}{\sqrt{2C/\delta \exp[-\delta m(t)] + D}} \geqslant 1 \tag{2.12}$$

with $D = \dot{m}^2(t_0) - (2C/\delta)\exp[-\delta m(t_0)]$. Notice that $(2C/\delta)\exp[-\delta m(t)] + D$ is nondecreasing and is $\dot{m}^2(t_0)$ (which may be taken to be positive) at $t = t_0$.

Integrating (2.12) from $t_0$ to $t$, we obtain

$$t - t_0 \leqslant \int_{-\infty}^{m_0} \frac{dm}{[(2C/\delta)\exp[-\delta m] + D)]^{\frac{1}{2}}} < \infty$$

for all $t \geqslant t_0$. Letting $t \to \infty$ we obtain a contradiction.

EXAMPLE 2.13. Define

$$k(x) = \begin{cases} C_0/|x|^{2+\varepsilon}, & \text{if } |x| \geqslant R_0 \\ 0 & , \text{ if } |x| < R_0 \end{cases} \tag{2.14}$$

where $\varepsilon > 0$ is an arbitrary constant. Then the constant in Theorem 2.2 is

$$\frac{1}{\pi}\int_{\mathbf{R}^2} k(x)\,dx = \frac{2C_0}{\varepsilon R_0^\varepsilon}$$

which is precisely the constant that appeared in Theorem 2.5. Thus we learn that the exponent

$$2 + \frac{1}{\pi}\int_{\mathbf{R}^2} k(x)\,dx$$

is the dividing point for existence and non-existence. We should also remark that although the function $k$ in (2.14) is not continuous, it is clear that we can smooth out the discontinuity of $k$ and yet maintain its integral as close as we want to the original one.

## 3. – The uniformly elliptic case.

The main goal of this section is to extend the result in § 2 to the equation

$$Lu - k + K\exp[2u] = 0 \tag{3.1}$$

on $\mathbf{R}^2$, where $L \equiv \partial_i(a_{ij}(x)\partial_j)$, (we use the summation convention), and the matrix $(a_{ij}(x))$ is symmetric, bounded, measurable, and uniformly elliptic on $\mathbf{R}^2$. The first difficulty one encounters in trying to generalize the proofs in section 2 is the existence of global solutions to $Lu = f$ in $\mathbf{R}^2$, with precise asymptotic behavior at $\infty$. In order to do so we adapt a technique due to Friedman [F]. A refinement of the argument also yields the existence of a global fundamental solution for $L$, with estimates which depend only on the ellipticity constants, for $L$. This is not actually needed in our paper, but, since we have not been able to find a proof of this statement in the literature, we will include it in an appendix, for future reference.

Using our results on the linear equation $Lu = f$, we then proceed to solve (3.1), obtaining a generalization of Theorem 2.2. When $L = \Delta$, our theorem reduces to the sharp result in Theorem 2.2.

We assume throughout that

$$\lambda|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j, \text{ for all } \xi \in \mathbb{R}^2,$$

and that, if $A(x) = \left(a_{ij}(x)\right)_{i,j=1}^2$, then $\|A(x)\|_\infty \leqslant \Lambda$. Sometimes we will refer to $\lambda$ and $\Lambda$ as the ellipticity constants of $L$.

DEFINITION 3.2. Consider the reflection (Kelvin transformation) given by $y = x/|x|^2$. Solutions $u$ to $Lu(x) = f(x)$ in $|x| > 1$ correspond, under this reflection to solutions to $\tilde{L}v(y) = f(x(y))/J(x(y))$ in $0 < |y| < 1$, where $x(y) = y/|y|^2$, $\tilde{L} = \partial_k \tilde{a}_{kl}(y)\partial_l$, and

$$\tilde{a}_{kl}(y) = \frac{1}{J(x(y))} a_{ij}(x(y)) \frac{\partial y_k}{\partial x_i}(x(y)) \cdot \frac{\partial y_l}{\partial x_j}(x(y)); \qquad \frac{\partial y_k}{\partial x_i} = \frac{1}{|x|^2}\left(\delta_{ij} - \frac{2x_i x_k}{|x|^2}\right),$$

$$J(x) = \frac{1}{|x|^4}$$

(see [KN$_1$], Lemma 2). Since the matrix $\left(\delta_{ik} - 2x_i x_k/|x|^2\right)$ is unitary, $(\tilde{a}_{kl})$ satisfies the same ellipticity estimates as $(a_{ij})$. Let $\tilde{g}$ be the Green's function for $\tilde{L}$, with pole at 0, in the ball $|y| < 1$. Let $g(x) = -\tilde{g}(x/|x|^2)$. $g(x)$ will be called the Green's function for $L$ in $|x| > 1$, with pole at $\infty$. It satisfies the following properties:

$$\begin{cases} Lg = 0 \text{ in } |x| > 1 \\ g|_{|x|=1} = 0 \end{cases}$$

Moreover, there exist constants $C_1$ and $C_2$, which depend only on $\lambda$ and $\Lambda$ (and which can be estimated explicitly in terms of $\lambda$ and $\Lambda$), such that $C_1 \log|x| \leqslant g(x) \leqslant C_2 \log|x|$, for $|x| > 2$. For this last property, see [LSW].

THEOREM 3.3. *Suppose that $f(x)$ is locally bounded on $\mathbb{R}^2$, $f$ is integrable on $\mathbb{R}^2$, and $\int_{|x|>1} |f(x)|^p\, |x|^{4(p-1)} dx < \infty$, for some $p > 1$. Then, there exists a solution $u$ to $Lu = f$ on $\mathbb{R}^2$. Moreover, the solution $u$ has the asymptotic behavior $u(x) = \left(\int f\right) \cdot g(x) = O(1)$, as $x \to \infty$, where $g(x)$ is the Green's function for $L$ in $|x| > 1$, with pole at $\infty$.*

PROOF. As mentioned before, we adapt a technique due to Friedman ([F]). We proceed by steps.

*Step* 1. Solve

$$\begin{cases} Lw_1 = f \\ w_1|_{|x|=1} = 0 \end{cases} \quad \text{in } |x| > 1 \text{ with } |w_1(x)| = O(1)$$

In order to do so, we solve

$$\begin{cases} \tilde{L}\tilde{w}_1 = f(x(y))/J(x(y)) & \text{in } |y| < 1 \\ \tilde{w}_1|_{|y|=1} = 0 \end{cases}$$

Since $|f(x(y))|/J(x(y)) = |f(x(y))|/|y|^4 \in L^p(dy)$ for some $p > 1$, by our assumption on $f$ the existence and boundedness of $\tilde{w}_1$ follows from [LSW]. We then set $w_1(x) = \tilde{w}_1(x/|x|^2)$.

*Step* 2. Solve

$$\begin{cases} Lz_1 = f & \text{in } |x| < 2 \\ z_1 = w_1 & \text{on } |x| = 2 \end{cases} \quad \text{and} \begin{cases} Lz_2 = 0 & \text{in } |x| < 2 \\ z_2 = g & \text{on } |x| = 2 \end{cases}$$

This step is clear by [LSW].

*Step* 3. Let $S^1 = \{x \in \mathbb{R}^2 \colon |x| = 1\}$. Define an operator $W$ in $C(S^1; \mathbb{R})$ by the following procedure.

Fix an $h \in C(S^1; \mathbb{R})$,

$$\text{(i) Solve} \begin{cases} Lw = 0 & \text{in } |x| > 1, \\ w = h & \text{on } |x| = 1, \end{cases} \qquad \|w\|_{L^\infty(|x|<1)} \leqslant \|h\|_\infty .$$

(To do so, it is enough to solve

$$\begin{cases} \tilde{L}\tilde{w} = 0 & \text{in } |y| < 1, \\ \tilde{w} = h & \text{on } |y| = 1, \end{cases} \qquad \text{and then set } w(x) = \tilde{w}\left(\frac{x}{|x|^2}\right). \Big)$$

$$\text{(ii) Solve} \begin{cases} Lz = 0 & \text{in } |x| < 2 \\ z = w & \text{on } |x| = 2 \end{cases}$$

(This step is clear by [LSW]).

(iii) Define $W(h) = z|_{|x|=1}$. Since $W(h)$ is a Hölder continuous function, it is easy to see that $W$ is a compact linear operator. Also $\|W\| \leqslant 1$, and $W(1) = 1$. Thus, $\|W\| = 1$. Clearly $\lambda = 1$ is an eigenvalue. Suppose that $h$ is a corresponding eigenfunction, i.e. $W(h) = h$, and suppose that $h$ is non-constant. Then $w$ and $z$ are both non-constant, and $z|_{|x|=1} = h$, while $z|_{|x|=1} = w|_{|x|=2}$, $\max_{|x|=2} |w(x)| \leqslant \max_{|x|=1} |h(x)|$. This clearly violates the strong maximum principle. Thus, $h$ must be a constant, and the eigenspace is 1-dimensional. By the Fredholm theory, the corange of I-W has dimension 1, the kernel of I-W* also has dimension 1, and if $0 \neq \hat{h} \in \ker(\text{I-W}^*)$, $\zeta$ is in the range of I-W if and only if $\hat{h}(\zeta) = 0$.

*Step* 4. $\hat{h}(z_2) \neq 0$. If not, $z_2$ is in the range of I-W. Thus we can find $h$ so that $z_2 + W(h) = h$. Define now $u(x)$ on $\mathbb{R}^2$ by

$$u(x) = \begin{cases} z_2 + z & \text{for } |x| < 2 \\ w + g & \text{for } |x| > 1 . \end{cases}$$

Then, $u$ is well defined, since $z_2 + z = w + g$ for $1 < |x| < 2$. Moreover, $Lu = 0$ in $\mathbb{R}^2$. But, $w$ is bounded, and $g(x)$ behaves like $\log|x|$ at $\infty$-which contradicts the minimum principle.

*Step* 5. Let $\lambda = -\hat{h}(z_1)/\hat{h}(z_2)$. Then, there exists $h$ so that $z_1 + \lambda z_2 + + W(h) = h$. Let $w, z$ be the corresponding solutions in (i), (ii) of step 3. Define

$$u = \begin{cases} z_1 + \lambda z_2 + z & \text{in } |x| < 2 \\ w_1 + \lambda g + w & \text{in } |x| > 1 . \end{cases}$$

Then, it is easy to check that $u$ is an entire solution of $Lu = f$ in $\mathbb{R}^2$.

*Step* 6. $\lambda = \int f$. If we show this equality, since $w_1$ and $w$ are bounded in $|x| > 1$, the theorem will follow. Let

$$\varphi \in C_0^\infty(\{|y| < 1\}),\ \varphi \geqslant 0,\ \varphi \equiv 1 \text{ near } 0, \quad \text{and } v(y) = u(y/|y|^2) .$$

Then,

$$\int \tilde{a}_{kl}(y)\,\partial_k v(y)\cdot\partial_l\varphi(y)\,dy = \int \tilde{a}_{kl}(y)\,\partial_k \tilde{w}_1(y)\cdot\partial_l\varphi(y)\,dy$$
$$-\lambda\int \tilde{a}_{kl}(y)\,\partial_k\tilde{g}(y)\cdot\partial_l\varphi(y)\,dy + \int \tilde{a}_{kl}\partial_k\tilde{w}(y)\cdot\partial_l\varphi(y)\,dy = -\int \frac{f(x(y))}{J(x(y))}\cdot\varphi(y)\,dy + \lambda .$$

Let $\tilde{\varphi}(x) = \varphi(x/|x|^2)$, and change variables. Then,

$$\int a_{ij}(x)\,\partial_i u(x)\cdot\partial_j\tilde{\varphi}(x)\,dx = -\int f(x)\tilde{\varphi}(x)\,dx + \lambda .$$

Set now $\psi(x) = \tilde{\varphi}(x) - 1$, so that $\psi(x) \in C_0^\infty(\mathbb{R}^2)$. Then

$$\int a_{ij}(x)\,\partial_i u(x)\cdot\partial_j\tilde{\varphi}(x)\,dx = \int a_{ij}(x)\,\partial_i u(x)\cdot\partial_j\psi(x)\,dx = -\int f(x)\,\psi(x)\,dx .$$

Thus,

$$-\int f(x)\tilde{\varphi}(x)\,dx + \int f(x)\,dx = -\int f(x)\tilde{\varphi}(x)\,dx + \lambda ,$$

and our assertion follows.

Before we proceed to apply Theorem 3.3 to equation (3.1), we need a shapr estimate on the function $g(x)$ defined in 3.2.

Assume that $|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j$. Then, we have

LEMMA 3.4. *For every $\beta$, with $0 < \beta < 4\pi$, we have that*

$$\int_{|x|>1} \exp[\beta g(x)] \frac{dx}{|x|^4} < +\infty .$$

PROOF. We first note that the result is sharp, since when $(a_{ij}(x)) = I$, $g(x) = 1/2\pi \log|x|$. By reflection, we have to show that

$$\int_{|y|<1} \exp[\beta(-\tilde{g}(y))]\,dy < +\infty, \text{ where} \begin{cases} (-\tilde{L})(-\tilde{g}) = \delta_0 & \text{in } |y| < 1 , \\ (-\tilde{g}) \qquad\quad = 0 & \text{on } |y| = 1 , \end{cases}$$

and $\tilde{L}$ is given by a matrix $\tilde{a}_{kl}$, with $\tilde{a}_{kl}(y)\xi_k\xi_l \geqslant |\xi|^2$. Pick $\varphi \in C_0^\infty(|y| < \frac{1}{2})$, $0 \leqslant \varphi$, $\varphi$ radial, and $\int\varphi = 1$. Let $\varphi_\varepsilon(y) = (1/\varepsilon^2)\,\varphi(y/\varepsilon)$. Let $v_\varepsilon$ solve

$$\begin{cases} -\tilde{L}v_\varepsilon = \varphi_\varepsilon \text{in} & |y| < 1 \\ v_\varepsilon \qquad = 0 & \text{on } |y| = 1 . \end{cases}$$

The results of [LSW] show that $v_\varepsilon$ converges to $(-\tilde{g})$ as $\varepsilon \to 0$ uniformly on compact subsets of $\{|y| \leqslant 1\} - \{0\}$. Therefore, we have, by Fatou's lemma

$$\int_{|y|<1} \exp[\beta(-\tilde{g}(y))]\,dy \leqslant \liminf_{\varepsilon \leftarrow 0} \int_{|y|<1} \exp[\beta v_\varepsilon(y)\,dy .$$

We are therefore reduced to studying $\int_{|y|<1} \exp[\beta v_\varepsilon(y)]\,dy$, as $\varepsilon \to 0$. We will appeal to the results and notations of [T]. Note that, by the minimum principle, $v_\varepsilon \geqslant 0$. Let now $v_\varepsilon^*(y)$ be the spherically symmetric decreasing rearrangement of $v_\varepsilon$, i.e. the function from $\{|y| \leqslant 1\}$ into $[0, +\infty)$, whose level sets $\{y\colon |y| < 1, v_\varepsilon^*(y) > t\}$ are concentric discs with the same measure as the level sets $\{y\colon |y| < 1,\ v_\varepsilon(y) > t\}$, and such that $v_\varepsilon^*(y) = 0$ for $|y| = 1$. Clearly, we have

$$\int_{|y|<1} \exp[\beta v_\varepsilon(y)]\,dy = \int_{|y|<1} \exp[\beta v_\varepsilon^*(y)]\,dy .$$

The main result in [T] shows that $v_\varepsilon^*(y) \leqslant w_\varepsilon(y)$ pointwise, where $w_\varepsilon(y)$ solves

$$\begin{cases} -\Delta w_\varepsilon = \varphi_\varepsilon & \text{in } |y| < 1 \\ w_\varepsilon \qquad = 0 & \text{on } |y| = 1 . \end{cases}$$

Therefore,

$$\int_{|y|<1} \exp[\beta v_\varepsilon(y)]\, dy \leqslant \int_{|y|<1} \exp[\beta w_\varepsilon(y)]\, dy\ ,$$

since $\beta > 0$. We claim that

$$w_\varepsilon(y) \leqslant \frac{1}{2\pi} \log \frac{1}{|y|}\,.$$

In fact,

$$w_\varepsilon(y) = \frac{1}{2\pi} \int_{|z|<1} \log \frac{1}{|y-z|}\, \varphi_\varepsilon(z)\, dz$$

and $\varphi_\varepsilon$ is radial. Therefore, by the superharmonicity of $(1/2\pi)\log(1/|y-z|)$ as a function of $z$, this is smaller than $(1/2\pi)\log(1/|y|)$. The lemma now since follows

$$\int_{|y|<1} \exp[\beta w_\varepsilon(y)]\, dy \leqslant \int_{|y|<1} \exp\left[(\beta/2\pi)\log\frac{1}{|y|}\right] dy = \int_{|y|<1} \frac{1}{|y|^{\beta/(2\pi)}}\, dy < +\infty$$

if $\beta < 4\pi$.

COROLLARY 3.5. *If $\lambda|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j$, then for every $\beta$, with $0<\beta<\lambda 4\pi$,*

$$\int_{|x|>1} \exp[\beta g(x)]\, \frac{dx}{|x|^4} < \infty\,.$$

PROOF. Let $L_\lambda = \partial_i(a_{ij}(x)/\lambda)\ \partial_j$, and $g_\lambda$ the Green's function for $L_\lambda$ with pole at $\infty$ then $g(x) = g_\lambda(x)/\lambda$. The corollary now follows from 3.4.

We now proceed to apply the results in Theorem 3.3 and Corollary 3.5 to study equation (3.1).

THEOREM 3.6. *If*

$$|k(x)| \leqslant \frac{C}{|x|^{2+\varepsilon}}$$

*at $\infty$, for some positive constants $\varepsilon$, $C$, and $K \leqslant 0$ on $\mathbb{R}^2$, with*

$$|K(x)| \leqslant A\, \frac{\exp[\alpha g(x)]}{|x|^4} \tag{3.7}$$

*at $\infty$ for some $A > 0$ and*

$$\alpha < 4\pi\lambda - 2\int k \tag{3.8}$$

*then for every $a$, $0 < a < \lambda 4\pi$, $a < \frac{1}{2}[4\pi\lambda - 2\int k - \alpha]$, equation* (3.1) *possesses an entire solution $u$, with*

$$u(x) = \Big(\int_{\mathbf{R}^2} k(y)\,dy + \frac{a}{2}\Big) g(x) + O(1) \ \ at\ \infty .$$

PROOF. We follow the outline of the proof of Theorem 2.2. First we construct a subsolution. Set

$$h_2(x) = \begin{cases} 1 & \text{for } |x| < 1 . \\ \dfrac{\exp\left[[4\pi\lambda - a]\, g(x)\right]}{|x|^4} & \text{for } |x| \geqslant 1 . \end{cases}$$

By Corollary 3.5, $\int h_2(x)\,dx < \infty$. Moreover, there exists $p > 1$ such that

$$\int_{|x|>1} |h_2(x)|^p \, |x|^{4(p-1)}\, dx < \infty .$$

Choose now $\delta = \delta(a) > 0$ so that $\delta \int h_2(x)\,dx = a/2$. Let now $f_2(x) = \delta h_2(x)$. Let $v_2(x)$ be the solution given in Theorem 3.3 to $Lv_2 = k + f_2$. By Theorem 3.3, and our choice of $f_2$, we have $v_2(x) = \left(\int k + a/2\right) g(x) + O(1)$ at $\infty$. For simplicity, set $\tilde{k} = \int k$. Let $u_2(x) = v_2(x) + A_2$, where the constant $A_2 < 0$ is to be chosen later. We compute, at $\infty$,

$$\begin{aligned} Lu_2 - k + K\exp[2u_2] &\geqslant f_2 + K\exp[2v_2]\exp[2A_2] \\ &\geqslant \delta\, \frac{\exp\left[[4\pi\lambda - a]\, g(x)\right]}{|x|^4} - A'\, \frac{\exp[\alpha g(x)]}{|x|^4} \cdot \exp[(2\tilde{k} + a)\, g(x)] \\ &= \frac{1}{|x|^4} \exp\left[[\alpha + 2\tilde{k} + a]\, g(x)\right] \cdot \left\{\delta \exp\left[[4\pi\lambda - \alpha - 2\tilde{k} - 2a]\, g(x)\right] - A'\right\} . \end{aligned}$$

Since $g(x) \to \infty$ as $|x| \to \infty$, this last expression will be $\geqslant 0$ at $\infty$, provided $4\pi\lambda - \alpha - 2\tilde{k} - 2a > 0$, or $a < \frac{1}{2}[4\pi\lambda - 2\tilde{k} - \alpha]$. This holds by our assumptions on $\alpha$ and $a$. For the finite part, $f_2$ is bounded from below by a positive constant. It is thus possible to choose $A_2$ to be a large negative constant so that

$$f_2 + K\exp[2v_2]\exp[2A_2] > 0$$

on $\mathbf{R}^2$. Therefore, $u_2$ is a subsolution.

Next, we construct a super-solution $u_1$ with $u_1 \geqslant u_2$ on $\mathbf{R}^2$. Without loss of generality, we may assume $K \not\equiv 0$ (otherwise (3.1) reduces to a linear equation, which can be solved using Theorem 3.3), and that $K(0) \neq 0$. Suppose that $B_\varrho = \{x \in \mathbb{R}^2 : |x| < \varrho\} \subset \operatorname{supp} K$. Let $f_1(x) = f_1(|x|)$ be a radial function, positive in $B_{\varrho/2}$, with

$$f_1(|x|) \leqslant -|k(x)|, \text{ and } |f_1(|x|)| \leqslant C/|x|^{2+\varepsilon}$$

outside $B_\varrho$, and

$$\int_{\mathbb{R}^2} f_1(|x|)\,dx = \int_{\mathbb{R}^2} (k + f_2)(x)\,dx = \tilde{k} + a/2 .$$

Let $v_1(x)$ be the solution to $Lv_1 = f_1$ on $\mathbb{R}^2$, given by Theorem 3.3. At $\infty$ $v_1(x) = (\tilde{k} + a/2)g(x) + O(1)$. Let $u_1 = v_1 + A_1$, where $A_1$ is a large positive constant to be chosen later. We compute,

$$Lu_1 - k + K\exp[2u_1] = f_1 - k + K\exp[2v_1]\exp[2A_1] .$$

Outside $B_\varrho$, $f_1 \leqslant -|k| \leqslant k$. Thus, $f_1 - k + K\exp[2v_1]\exp[2A_1] \leqslant 0$ there, since $K \leqslant 0$ on $\mathbb{R}^2$. In $B_\varrho$, $f_1 - k$ is bounded, while $K \leqslant -C < 0$ on $\bar{B}_\varrho$, and so

$$K\exp[2v_1] \leqslant C' \text{ on } \bar{B}_\varrho .$$

Thus, we may choose $A_1 > 0$ so large that

$$K\exp[2v_1]\exp[2A_1] \leqslant -C'\exp[2A_1] \leqslant -(f_1 - k) \text{ on } \bar{B}_\varrho .$$

With this choice of $A_1$, we have

$$Lu_1 - k + K\exp[2u_1] \leqslant 0 \text{ on } \mathbb{R}^2 ,$$

and $u_1$ is a supersolution. Now, comparing $u_1$ and $u_2$, we see that at infinity,

$$v_1(x) = v_2(x) + O(1) .$$

It is therefore possible to choose $A_1 > 0$ even larger to assure that $u_1 > u_2$ on $\mathbb{R}^2$ (the finite part does not create any difficulties at all). The Theorem now follows from Theorem 5 in [KN$_1$].

REMARK 3.9. When

$$L = \Delta,\ \lambda = 1,\ g(x) = \frac{1}{2\pi}\log|x|,\quad \frac{\exp[\alpha g(x)]}{|x|^4} = \frac{1}{|x|^{4-\alpha/(2\pi)}} .$$

Let $\beta = 2 - (\alpha/2\pi)$, then $\beta > (1/\pi)\int k$ if and only if $\alpha < 4\pi - 2\int k$. Thus, in this case, (3.8) reduces to the condition in Theorem 2.2. Hence, Theorem 3.6 is sharp.

REMARK 3.10. Recall that $C_1 \log |x| \leqslant g(x) \leqslant C_2 \log |x|$ for $|x| > 2$, where $C_1$ and $C_2$ depend only on $\lambda$, $\Lambda$ and can be explicitly estimated in terms of $\lambda$, $\Lambda$. Then, if $4\pi\lambda - 2\int k \geqslant 0$, a sufficient condition for (3.7) to hold is that $|K(x)| \leqslant A|x|^{(\alpha C_1 - 4)}$ at $\infty$, for some $0 \leqslant \alpha < 4\pi\lambda - 2\int k$. On the other hand, if $4\pi\lambda - 2\int k < 0$, a sufficient condition for (3.7) to hold is that $|K(x)| \leqslant A|x|^{(\alpha C_2 - 4)}$ at $\infty$, for some $\alpha < 4\pi\lambda - 2\int k$.

REMARK 3.11. Remark 2.3 and Theorem 2.4 can be carried out in this more general setting in an almost word by word fashion. The details are omitted.

REMARK 3.12. It is easy to apply Theorem 3.6 to the geometric equation (1.1) to obtain existence results. The details are left to the interested readers.

## 4. – The degenerate case $b < 0$.

We now consider operators $L = \partial_i(a_{ij}(x)\,\partial_j)$ where $(a_{ij}(x))$ is a uniformly elliptic matrix on compact sets, and whose eigenvalues behave like $|x|^b$ at $\infty$ for some $b < 0$.

We shall follow the approach used in § 3. The only real modification is the appeal to the results in [FKS], [FJK] on degenerate elliptic equations. Because of the similarity of the arguments with those used in § 3, we shall be sketchy, concentrating on the main differences.

Following the approach in § 3, we first consider the problem of finding solutions to

$$Lu = f \text{ on } \mathbb{R}^2 ,$$

where

$$|f(x)| \leqslant c/|x|^a \text{ at } \infty, \qquad a = 2 + \varepsilon,\ \varepsilon > 0 .$$

We start out constructing a Green's function $g(x)$, with pole at $\infty$.

LEMMA 4.1. *There exists a Green's function $g(x)$ for $L$ on $|x| > 1$, with pole at $\infty$. $g$ satisfies*

$$\begin{cases} Lg = 0 & \text{in } |x| > 1 , \\ g = 0 & \text{on } |x| = 1 , \end{cases}$$

*and $C_1|x|^{-b} \leqslant g(x) \leqslant C_2|x|^{-b}$, for $|x| > 2$, where $C_1$, $C_2$ depend only on the eigenvalues of the uniformly elliptic matrix $(a_{ij}(x)/|x|^b)$ for $|x| > 2$.*

PROOF. $g$ will be constructed by reflection. Let $\tilde{L}$ be the operator with matrix

$$\tilde{a}_{kl}(y) = \frac{1}{J(x(y))}\cdot a_{ij}(x(y))\cdot\frac{\partial y_k}{\partial x_i}(x(y))\cdot\frac{\partial y_l}{\partial x_j}(x(y)),$$

where

$$y = \frac{x}{|x|^2}, \quad J(x) = \frac{1}{|x|^4}, \quad \frac{\partial y_k}{\partial x_i} = \frac{1}{|x|^2}\left(\delta_{ik} - \frac{2x_i x_k}{|x|^2}\right).$$

The size of the eigenvalues of $(\tilde{a}_{kl}(y))$ is $|y|^{-b}$, with $-b > 0$, as $y \to 0$. Therefore, the results in [FKS], [FJK] apply to $\tilde{L}$. Let $\tilde{g}(y)$ be the Green's function for $\tilde{L}$ in $|y| < 1$, with pole at the origin. The existence of $\tilde{g}$ is guaranteed by the results in [FJK]. Also, by [FJK], for $|y| < \frac{1}{2}$

$$-\tilde{g}(y) \approx \int_{|y|}^{1} \frac{s}{\tilde{w}(B(0,s))}\,ds,$$

where $\tilde{w}(B(0,s)) = \int_{|z|<s} |z|^{-b}dz \approx s^{2-b}$. Hence, $-\tilde{g}(y) \approx |y|^b$ for $|y| < \frac{1}{2}$. Set now $g(x) = -\tilde{g}(x/|x|^2)$. Then, $g(x)$ satisfies all the required properties

THEOREM 4.2. *Suppose that $f$ is locally bounded on $\mathbb{R}^2$, and that $|f(x)| \leqslant C/|x|^a$, $a = 2 + \varepsilon$, $\varepsilon > 0$ at $\infty$. Then, there exists a solution $u$ to $Lu = f$ on $\mathbb{R}^2$. Moreover, the solution $u$ has the asymptotic behavior $u(x) = (\int f)\, g(x) + o(g(x))$ as $x \to \infty$, where $g(x)$ is the function constructed in Lemma* 4.1.

PROOF. We proceed by steps, paralleling Theorem 3.3.

*Step* 1. Solve

$$\begin{cases} Lw_1 = f & \text{in } |x| > 1 \\ w_1\big|_{|x|=1} = 0 \end{cases}$$

with $|w_1(x)| \leqslant C/|x|^{\varepsilon+b}$ at $\infty$.

We do this once more by reflection. Let $\tilde{w}_1$ solve

$$\begin{cases} \tilde{L}\tilde{w}_1 = f(x(y))/J(x(y)) & \text{in } |y| < 1\,. \\ \tilde{w}_1 = 0 & \text{on } |y| = 1\,. \end{cases}$$

Since

$$\frac{f(x(y))}{J(x(y))} \leqslant C|y|^{a-4} \in L^1_{\mathrm{loc}}(dy),$$

Lemma 2.7 in [FJK] implies that $\tilde{w}_1(y)$ exists and is given by

$$\tilde{w}_1(z) = \int_{|y|<1} \tilde{g}(z, y)\, h(y)\, dy$$

where $h(y) = f(x(y))/J(x(y))$. As in the proof of Theorem 1 in [KN$_1$], we estimate $\tilde{w}_1$ as follows. Write

$$\tilde{w}_1(z) = \int_{|y|<\frac{1}{2}} \tilde{g}(z, y) h(y)\, dy + \int_{\frac{1}{2}<|y|<1} \tilde{g}(z, y) h(y)\, dy\,, \tag{4.3}$$

the second term is bounded, and we only have to bound the first one for $|z| < \frac{2}{3}$.

We first give pointwise estimates for $\tilde{g}(z, y)$ for $|y| < \frac{1}{2}$, $|z| < \frac{2}{3}$. Let $\tilde{w}(y) = |y|^{-b}$, $\tilde{w}(B(z, s)) = \int_{|z-y|<s} |y|^{-b} dy$. It is easy to check that

$$\tilde{w}(B(z, s)) = \begin{cases} s^{2-b} & |z| \leqslant s\,. \\ s^2|z|^{-b} & |z| > s\,. \end{cases}$$

Moreover, by Theorem 3.3 of [FJK],

$$|\tilde{g}(z, y)| \approx \int_{|z-y|}^{\alpha} \frac{s}{\tilde{w}(B(z, s))}\, ds$$

whenever $|z - y| < d/2$, $d = \big|1 - |z|\big|$. We then see that

$$|\tilde{g}(z, y)| \approx \begin{cases} |z-y|^b & \text{for } |z-y| > |z|/2\,. \\ |z|^b \log \dfrac{|z|}{|z-y|} + |z|^b & \text{for } |z-y| < |z|/2\,. \end{cases}$$

We can now estimate the first term in (4.3). Write

$$\int_{|y|<\frac{1}{2}} |\tilde{g}(z, y)| \frac{dy}{|y|^{2-\varepsilon}} = \left( \int_{|y|<|z|/2} + \int_{|z|/2<|y|<2|z|} + \int_{2|z|<|y<\frac{1}{2}|} \right)\left(\frac{|\tilde{g}(z, y)|}{|y|^{2-\varepsilon}}\, dy\right) = \mathrm{I} + \mathrm{II} + \mathrm{III}\,.$$

In the region where $|y| < |z|/2$, $|z - y| \approx |z|$, so we get

$$|\mathrm{I}| \leqslant C|z|^b\, |z|^\varepsilon.$$

In the region where $|z|/2 < |y| < 2|z|$, $|z| \approx |y|$ and $|z - y| \leqslant C|z|$, thus.

$$|\mathrm{II}| \leqslant C|z|^b |z|^\varepsilon + \frac{|z|^b}{|z|^{2-\varepsilon}} \int\limits_{|z|/2<|y|<2|z|} \left| |\log \frac{|z|}{|z-y|} \right| dy \leqslant C|z|^b |z|^\varepsilon .$$

In the region where $|y| > 2|z|$, $|z - y| \approx |y|$, and hence

$$|\mathrm{III}| \leqslant C \int_{2|z|}^{1} |y|^{b-2+\varepsilon} dy \approx C|z|^b |z|^\varepsilon$$

provided $\varepsilon < - b$. We now set $w_1(x) = \tilde{w}_1(x/|x|^2)$.

*Step* 2. Solve

$$\begin{cases} Lz_1 = f & \text{in } |x| < 2 \\ z_1 = w_1 & \text{on } |x| = 2 \end{cases}$$

$$\begin{cases} Lz_2 = 0 & \text{in } |x| < 2 \\ z_2 = g & \text{on } |x| = 2 \end{cases}$$

*Step* 3. Define an operator $W$ on $C(S^1; \mathbb{R})$ by the following procedure: Fix an $h \in C(S^1; \mathbb{R})$

(i) Solve

$$\begin{cases} Lw = 0 & \text{in } |x| > 1 , \\ w = & \text{on } |x| = 1 \end{cases} \qquad \|w\|_{L^\infty(|x|<1)} \leqslant \|h\|_\infty .$$

(This can be done by reflection, using the results in [FJK]).

(ii) Solve

$$\begin{cases} Lz = 0 & \text{in } |x| < 2 \\ z = w & \text{on } |x| = 2 \end{cases}$$

(iii) Define $W(h) = z|_{|x|=1}$ .

$W$ is compact, $\|W\| = 1$, $W(1) = 1$, $1$ is the only eigenvalue of $W$, the corresponding eigenspace is one dimensional and consists only of constants. If $0 \neq \hat{h} \in \ker(I - W^*)$, $\zeta$ is in the range of $I - W$ if and only if $\hat{h}(\zeta) = 0$.

*Step* 4. $\hat{h}(z_2) \neq 0$. The proof is the same as step 4 in Theorem 3.3, since $g(x) \to \infty$ at $\infty$.

*Step* 5. Let $\lambda - = \hat{h}(z_1)/\hat{h}(z_2)$. Pick $h$ so that $z_1 + \lambda z_2 + W(h) = h$. Let $w$, $z$ be the corresponding solutions in (i), (ii) of step 3. Define

$$u = \begin{cases} z_1 + \lambda z_2 + z & \text{in } |x| < 2 \\ w_1 + \lambda g + w & \text{in } |x| > 1 \end{cases}$$

Then, $Lu = f$.

*Step* 6. $\lambda = \int f$. If we show this equality, since $w_1 = O(g(x))$ by Lemma 4.1, and step 1, and $w$ is bounded, the theorem will follow. The equality $\lambda = \int f$ is proved in the same way as in Step 6 of Theorem 3.3.

We can now solve the nonlinear equation

$$Lu - k + K \exp[2u] = 0 \text{ on } \mathbb{R}^2 , \tag{4.4}$$

where $L = \partial_i(a_{ij}\partial_j)$ and $(a_{ij})$ is uniformly elliptic on compact sets, and its eigenvalues behave like $|x|^b$ at $\infty$, $b < 0$.

THEOREM 4.5. *If*

$$|k(x)| \leqslant C/|x|^{2+\varepsilon} \text{ at } \infty ,$$

$K \leqslant 0$ *on* $\mathbb{R}^2$ *and*

$$|K(x)| \leqslant C \exp[-\alpha g(x)] \textit{ at } \infty, \qquad \textit{where } \alpha > 2\int k \tag{4.6}$$

*then, for every* $a > 0$, *with* $a < \min\{-b, \varepsilon\}$, $a < (1/4\pi)(\alpha - 2\int k)$, *equation* (4.4) *possesses an entire solution* $u$ *on* $\mathbb{R}^2$, *with*

$$o(g(x)) + \left(\int k + 2\pi a\right) g(x) \leqslant u(x) \leqslant \left(\int k + 4\pi a\right) g(x) + o(g(x)) \text{ at } \infty .$$

PROOF. We follow the proof of Theorem 3.6. First we construct a subsolution. Let

$$f_2(x) = \frac{a^2}{(1 + |x|^2)^{1+a/2}} ,$$

where $a < \min\{-b, \varepsilon\}$. Since $\int f_2 = 2\pi a$, if $v_2$ is the solution of $Lv_2 = k + f_2$ constructed in Theorem 4.2, then at $\infty$, $v_2(x) = (\int k + 2\pi a) g(x) + o(g(x))$. Set $u_2 = v_2 + A_2$, where the constant $A_2 < 0$ is to be chosen later. We see that

$$Lu_2 - k + K \exp[2u_2] = f_2 + K \exp[2v_2] \exp[2A_2] .$$

At $\infty$, we have

$$f_2 + K \exp[2v_2] \exp[2A_2] \geqslant \frac{a}{(1+|x|^2)^{1+a/2}} - C \exp\left[\left[-\alpha + 2\int k + 4\pi a + o(1)\right] g(x)\right].$$

Since $a < (1/4\pi)(\alpha - 2\int k)$, the number $-\alpha + 2\int k + 4\pi a + o(1)$ remains smaller than a fixed negative constant for $x$ large. Thus, as $g(x) \approx |x|^{-b}$ for $x$ large, $f_2(x) + K \exp[2v_2] \exp[2A_2] \geqslant 0$ for $x$ large. On the finite part, since $f_2 \geqslant C > 0$, we just have to choose $A_2 < 0$, with $|A_2|$ large, to guarantee this inequality. We have thus constructed a subsolution $u_2$, with

$$u_2(x) = \left(\int k + 2\pi a\right) g(x) + o(g(x)) \text{ at } \infty .$$

Next we shall construct a super-solution $u_1 \geqslant u_2$. Without loss of generality, we can assume $K \not\equiv 0$ (Otherwise we are done by Theorem 4.2). As before, let $B_\varrho \subset \operatorname{int}(\operatorname{supp} K)$, and let $f_1(x)$ satisfy

(i) $f_1(x) \leqslant -|k(x)|$ in $\mathbb{R}^2 \setminus B_\varrho$, $\quad |f_1(x)| \leqslant \dfrac{C}{|x|^{2+\varepsilon}}$ at $\infty$.

(ii) $\displaystyle\int f_1 = \int k + 4\pi a$

Let now $v_1$ be the solution to $Lv_1 = f_1$ given by Theorem 4.2. At $\infty$, $v_1(x) = (\int k + 4\pi a) g(x) + o(g(x))$. Set $u_1 = v_1 + A_1$, $A_1 > 0$, a constant to be chosen later.

$$Lu_1 - k + K \exp[2u_1] = f_1 - k + K \exp[2v_1] \exp[2A_1]$$

In $\mathbb{R}^2 \setminus B_\varrho$, $f_1 - k \leqslant 0$, $K \leqslant 0$, so $f_1 - k + K \exp[2v_1] \exp[2A_1] \leqslant 0$. In $B_\varrho$, $f_1 - k$ is bounded, $K \leqslant -C < 0$. It is therefore possible to choose $A_1 > 0$ so large that $f_1 - k + K \exp[2v_1] \exp[2A_1] \leqslant 0$. Thus, $u_1$ is a supersolution. Furthermore, at $\infty$, $v_1(x) = (\int k + 4\pi a) g(x) + o(g(x))$, while $u_1(x) = (\int k + 2\pi a) g(x) + o(g(x))$, thus $v_1(x) \geqslant u_1(x)$ at $\infty$. It is thus possible to choose $A_1 > 0$, even larger so that $u_1(x) \geqslant u_2(x)$ on $\mathbb{R}^2$. Now, standard results (see Theorem 5 in [KN$_1$]) imply the existence of a solution $u$ of equation (4.4), with $u_1 \geqslant u \geqslant u_2$.

REMARK 4.7. We have actually shown that (4.4) has infinitely many solutions.

REMARK 4.8. Since $C_1|x|^{-b} \leqslant g(x) \leqslant C_2|x|^{-b}$ for $|x| > 2$, if $\int k \geqslant 0$, a sufficient condition for (4.6) to hold is that

$$|K(x)| \leqslant C \exp[-\alpha C_2 |x|^{-b}]$$

for some $\alpha > 2\int k$, while if $\int k < 0$, a sufficient condition for (4.6) to hold is that

$$|K(x)| \leqslant C \exp[-\alpha C_1 |x|^{-b}],$$

for some $0 \geqslant \alpha > 2\int k$.

## 5. – The degenerate case $b > 0$.

In this section, we consider operators $L = \partial_i(a_{ij}\partial_j)$ with the opposite asymptotic behavior as that in the previous section, namely, $(a_{ij})$ is uniformly elliptic on compact sets and its eigenvalues behave like $|x|^b$ at $\infty$ for some $b > 0$. This case turns out to be similar to our previous paper [KN$_1$] although it actually has a mixed flavour of two dimensional and higher dimensional results.

We shall follow closely the treatment in [KN$_1$].

LEMMA 5.1. *There exists a function* $G(x) \in H^1_{\text{loc}}(|x| > \frac{1}{2})$ *with* $LG = 0$ *in* $|x| > \frac{1}{2}$, $G(x) \approx 1/|x|^b$ *at* $\infty$.

PROOF. Extend $a_{ij}(x)$ to all of $\mathbb{R}^2$ in such a way that the eigenvalues of $(a_{ij})$ are $\approx |x|^b$ in all of $\mathbb{R}^2$. Let $w(x) = |x|^b$, the results of [FJK] and [FKS] apply. Let $G_k(x)$ be the Green function for $L$ on $|x| < k$ with pole at the origin, an easy rescaling argument shows thet $G_k(ky) = \widetilde{G}_k(y)$ where $\widetilde{G}_k(y)$ is the Green function, with pole at 0 in $|y| < 1$, of

$$L_k = \partial_{y_i}(a_{ij}(ky)\partial_{y_j}).$$

Let $\bar{a}_{ij}(y) = a_{ij}(ky)$. The eigenvalues of $L_k$ are $\approx k^b|y|^b$. By Theorem 3.3 in [FJK],

$$\widetilde{G}_k(y) \approx \int_{|y|}^{1} \frac{s}{\overline{w}(B(0,s))}\, ds$$

for $|y| < \frac{1}{2}$, where

$$\overline{w}(B(0,s)) = \int_{|y|<s} k^b |y|^b\, dy \approx k^b s^{2+b},$$

and hence

$$\tilde{G}_k(y) \approx \frac{1}{k^b|y|^b}$$

for $|y| < \frac{1}{2}$. But then,

$$G_k(x) \approx \frac{1}{|x|^b}$$

for $0 < |x| < k/2$, independently of $k$. The lemma follows as in Lemma 1 in [KN$_1$].

LEMMA 5.2. *If $w$ is a positive solution of $Lw = 0$ on $|x| > \frac{1}{2}$, then $u$ solves*

$$\begin{cases} Lu = f & \text{in } |x| > 1 \\ u = 0 & \text{on } |x| = 1 \end{cases} \tag{5.3}$$

*iff $v(y) = u(x(y))/w(x(y))$ solves*

$$\begin{cases} (\bar{a}_{kl} v_{y_k})_{y_l} = \dfrac{w(x(y))\, f(x(y))}{J(x(y))} & \text{in } 0 < |y| < 1 \\ v = 0 & \text{on } |y| = 1 . \end{cases} \tag{5.4}$$

*where $y = y(x) = x/|x|^2$, $x = x(y) = y/|y|^2$, $J(x) = |x|^{-4}$ and*

$$\bar{a}_{kl}(y) = \frac{w^2(x(y))}{J(x(y))}\, a_{ij}(x(y)) \frac{\partial y_k}{\partial x_i}(x(y)) \cdot \frac{\partial y_l}{\partial x_j}(x(y)) , \tag{5.5}$$

$$\frac{\partial y_k}{\partial x_i} = \frac{1}{|x|^2}\left(\delta_{ik} - 2\frac{x_i x_k}{|x|^2}\right).$$

The proof is a simple change of variables.

THEOREM 5.6. *Assume that*

$$|f(x)| \leqslant C/|x|^{2-b+\varepsilon} \tag{5.7}$$

*at $\infty$, then there exists a unique bounded solution of* (5.3) *with*

$$|u(x)| \leqslant C/|x|^{\varepsilon} \text{ as } |x| \to \infty . \tag{5.8}$$

PROOF. Let $w(x) = G(x)$, where $G(x)$ is given by Lemma 5.1. Set

$$h(y) = G(x)\, f(x)/J(x) ,$$

then $|h(y)| \leqslant C/|y|^{2-\varepsilon}$. Let $\tilde{L} = \partial_{y_k}(\bar{a}_{kl}(y)\, \partial_{y_l})$, the eigenvalues of $(\bar{a}_{kl}(y))$ are of the order

$$\frac{|x|^4\,|x|^{-2b}}{|x|^4}\cdot|x|^b=|y|^b.$$

Let $g(z, y)$ be the Green function for $\tilde{L}$ in $|y|<1$ and

$$v(z)=\int_{|y|<1} g(z,y)h(y)\,dy=\Big(\int_{|y|<\frac12}+\int_{\frac12<|y|<1}\Big)(g(z,y)h(y)\,dy)=v_1(z)+v_2(z)\,,$$

then $v(z)$ solves (5.4) by [FJK], and the last equality defines $v_1(z)$ and $v_2(z)$. As in step 1 of the previous section, we can estimate $v_1$, $v_2$ since the behavior of $g(z, y)$ is similar to $\tilde{g}$ there,

$$g(z,y)\approx\begin{cases}|z-y|^{-b}, & \text{for } |z-y|>|z|/2\\ |z|^{-b}\log\dfrac{|z|}{|z-y|}+|z|^{-b}, & \text{for } |z-y|<|z|/2\,.\end{cases}$$

Imitating the proof in Step 1 in § 4, we obtain that $v_2$ is bounded and

$$|v_1(z)|\leqslant|z|^{-b+\varepsilon}\qquad\text{for } |z|<\tfrac{2}{3}\,,$$

thus

$$|u(x)|\leqslant\frac{C}{|x|^\varepsilon}\qquad\text{at } \infty.$$

THEOREM 5.9. *Given a continuous function $h$ on $|x|=1$, there exists a unique bounded solution $u$ of*

$$\begin{cases}Lu=0 & \textit{on } |x|>1\\ \;u=h & \textit{on } |x|=1\end{cases}\tag{5.10}$$

*with*

$$|u(x)|\leqslant C\|h\|_\infty/|x|^b\tag{5.11}$$

*at* $\infty$.

PROOF. The proof is almost identical with that of Theorem 5.6, thus we omit it here.

THEOREM 5.12. *Assume that $f$ satisfies* (5.7) *at $\infty$ with $b>\varepsilon>0$. Then there exists a bounded solution of $Lu=f$ with the asymptotic behavior* (5.8) *at* $\infty$.

PROOF. The proof is almost identical with that of Theorem 3 in [$KN_1$], the condition $b>\varepsilon>0$ implies that (5.11) decays faster than (5.8).

We now can use Theorem 5.12 to show that the nonlinear equation

$$Lu - k + K \exp[2u] = 0 \tag{5.13}$$

has infinitely many solutions on $\mathbb{R}^2$. Our main result is the following

THEOREM 5.14. *If there exists an $\varepsilon$ such that $b > \varepsilon > 0$ and such that, for $|x|$ large*

$$|k|,\ |K| \leqslant \frac{C}{|x|^{2-b+\varepsilon}}. \tag{5.15}$$

*for some constant $C > 0$, and $K \leqslant 0$ on $\mathbb{R}^2$, equation* (5.13) *possesses infinitely many bounded solutions on* $\mathbb{R}^2$.

PROOF. Let

$$f(x) = \frac{1}{(1 + |x|^2)^{(2-b+\varepsilon)/2}}.$$

By Theorem 5.12, there exist $v_1$, $v_2$ solving $Lv_1 = k - f$, $Lv_2 = k + f$ respectively and with

$$|v_i(x)| \leqslant \frac{C}{|x|^{\varepsilon}}, \qquad i = 1, 2,$$

at $\infty$. Set $u_1 = v_1 + C_1$; then

$$Lu_1 - k + K \exp[2u_1] = -f + K \exp[2v_1] \exp[2C_1] < 0$$

on $\mathbb{R}^2$, for any constant $C_1$. Set $u_2 = v_2 + C_2$, then

$$\begin{aligned} Lu_2 - k + K \exp[2u_2] &= f + K \exp[2v_2] \exp[2C_2] \\ &= \frac{1}{(1 + |x|^2)^{(2-b+\varepsilon)/2}} - |K| \cdot \exp[2v_2] \exp[2C_2]. \end{aligned}$$

Since $k$ satisfies (5.15) and $v_2$ is bounded (and fixed), we can choose $C_2 < 0$ with $|C_2|$ large to ensure

$$\frac{1}{(1 + |x|^2)^{(2-b+\varepsilon)/2}} - |K| \exp[2v_2] \exp[2C_2] > 0.$$

Thus, with this choice of $C_2$, $u_2$ is a subsolutions. Set now $C_1 = 0$ and choose $C_2 < 0$, $|C_2|$ large so that $u_2 = v_2 + C_2$ is a subsolution and $u_2 \leqslant u_1$ (since both $v_1$, $v_2$ are bounded). We then obtain a solution of (5.13) with $u_2 \leqslant u \leqslant u_1$.

For our second solution, we choose $C_1' < 0$, with $|C_1'|$ so large that $u_1' = v_1 + C_1' < v_2 + C_2$, and $C_2' < 0$ with $|C_2'|$ so large that $u_2' = v_2 + C_2' < u_1'$. Then there exists $u'$ such that $u_1' \geqslant u' \geqslant u_2'$ and $u'$ is a solution of (5.13) on $\mathbb{R}^2$. It is clear that $u > u'$, they are thus distinct. In this way, we obtain infinitely many bounded solutions of (5.13).

## Appendix. – On the existence of global fundamental solutions for uniformly elliptic operators.

In this appendix we indicate how to modify the arguments in Section 3, to obtain the existence of a global fundamental solution in $\mathbb{R}^2$ for uniformly elliptic operators $L = \partial_i(a_{ij}(x)\partial_j)$, where

$$\lambda|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j, \qquad \text{for all } \xi \in \mathbb{R}^2,$$

and such that $A(x) = (a_{ij}(x))$ satisfies $\|A(x)\|_\infty \leqslant \Lambda$.

DEFINITION A-1. *A function $F(x)$ is called a fundamental solution for $L$, with pole at the origin if*

(i) $F \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus 0)$, $F \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for all $p < 2$, and, for every $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\int a_{ij}(x)\,\partial_i F(x)\,\partial_j\varphi(x)\,dx = -\varphi(0)$$

(ii) $|F(x)| \leqslant C \log|x|$ as $x \to \infty$, for some $C > 0$.

It is easy to see that if $F_1(x)$ and $F_2(x)$ are two fundamental solutions for $L$ then $F_1(x) - F_2(x) \equiv C$. Let $g(x)$ be the function defined in 3.2. We then have:

THEOREM A-2. *There exists a unique fundamental solution $F$ for $L$, with pole at the origin, with the property that $\lim_{|x|\to\infty} F(x) - g(x) = 0$. Moreover, there exist constants $C_1$, $C_2$, $C_3$, $C_4$, $R_1 < 1$, $R_2 > 1$, which depend only on $\lambda$, $\Lambda$ (and which can be explicitly estimated in terms of $\lambda$ and $\Lambda$), such that*

$$C_1 \log(1/|x|) \leqslant -F(x) \leqslant C_2 \log(1/|x|) \qquad \text{for } |x| < R_1$$

$$C_3 \log|x| \leqslant F(x) \leqslant C_4 \log|x| \qquad \text{for } |x| > R_2$$

PROOF. The uniqueness of $F$ follows from the remark after A.1. For the existence, we follow the proof of Theorem 3.3.

*Step* 1. With $R_0$ large, to be chosen later depending only on $\lambda$, $\Lambda$, let $z_1$ solve

$$\begin{cases} Lz_1 = \delta \text{ in } |x| < R_0 , \\ z_1|_{|x|=R_0} = 0 , \end{cases}$$

and let $z_2$ solve

$$\begin{cases} Lz_2 = 0 \text{ in } |x| < R_0 . \\ z_2|_{|x|=R_0} = g|_{|x|=R_0} . \end{cases}$$

*Step* 2. Define a linear operator $W$ on $C(S^1, \mathbb{R})$ as in Step 3 of Theorem 3.3, but replacing $|x| = 2$ by $|x| = R_0$. $W$ satisfies all the properties mentioned in Step 3 of 3.3.

*Step* 3. $\hat{h}(z_2) \neq 0$. This follows as in Step 4 of 3.3.

*Step* 4. Let $\lambda = -\hat{h}(z_1)/\hat{h}(z_2)$. Then, there exists $h$ so that $z_1 + \lambda z_2 + W(h) = h$. Define now

$$F_1(x) = \begin{cases} z_1 + \lambda z_2 + z & \text{for } |x| < R_0 \\ \lambda g + w & \text{for } |x| > 1 \end{cases}$$

It is easy to see that $F_1$ is a fundamental solution. Note that the function $h$ that we found above is only unique modulo constants.

*Step* 5. $\lambda = 1$. This is the same as step 6 in 3.3.

*Step* 6. We now show that if $h$ in Step 4 is chosen so that $w(x) \to 0$ at $\infty$, and $R_0$ is large, then the unique $F(x)$ thus obtained satisfies al the required properties.

In order to do so, we define a new linear operator $V$ on $C(S^1, \mathbb{R})$, in the following way: let $h \in C(S^1, \mathbf{R})$, and let $\tilde{w}$ be as in i) of step 2. Let $\tilde{v}(y) = \tilde{w}(y) - \int h\, d\mu$, where $d\mu$ is the unique probability measure on $S^1$ with the property that $\tilde{w}(0) = \int h\, d\mu$. Thus, $\tilde{L}\tilde{v} = 0$, and $\tilde{v}(0) = 0$. By the DeGiorgi-Nash-Moser estimates, $|\tilde{v}(y)| < C\|h\|_\infty |y|^\delta$, where $C$, $\delta$ depend only on $\lambda$, $\Lambda$. Now, let $R_0$ be chosen so that $\sup\limits_{|y|=1/R_0} |\tilde{v}(y)| \leqslant (\frac{1}{2})\|h\|_\infty$. Let now $v(x) = \tilde{v}(x/|x|^2)$, so that

$$\begin{cases} Lv = 0 \quad \text{for } |x| > 1 . \\ v|_{|x|=1} = h - \int h d\mu . \end{cases}$$

Note that $\sup\limits_{|y|=R_0} |v(x)| \leqslant (\frac{1}{2})\|h\|_\infty$. Let now $\zeta$ solve

$$\begin{cases} L\zeta = 0 \quad |x| < R_0 , \\ \zeta|_{|x|=R_0} = v|_{|x|=R_0} , \end{cases}$$

and define $Vh = \zeta|_{|x|=1}$. Clearly, $\|V\| \leqslant \frac{1}{2}$, and $V(h) = W(h) - \int h\, d\mu$. The equation in $h$, $\theta + Vh = h$ has a unique solution for each $\theta$, and $\|h\|_\infty \leqslant 2\|\theta\|_\infty$. We are now in a position to show that, if $\hat{h}$ is as in Step 2, $\hat{h}(1) \neq 0$. Suppose not. Then, we can find $h_1$ so that $1 + W(h_1) = h_1$. Let now $\theta$ be an arbitrary element in $C(S^1)$. Pick $h$ such that $\theta + V(h) = h$, or $\theta + W(h) - \int h\, d\mu = h$. Now, $\int h\, d\mu = (\int h\, d\mu)\,(h_1 - W(h_1)) = (\int h\, d\mu)\, h_1 - W([\int h\, d\mu]\, h_1)$. Thus, $\theta + W(h) - (\int h\, d\mu)\, h_1 + W([\int h\, d\mu]\, h_1) = h$ or $\theta + W(h + [\int h\, d\mu]\, h_1) = h + [\int h\, d\mu]\, h_1$, which is a contradiction for $\theta \notin \operatorname{range}(I - W)$. With $\lambda$ as in step 4, i.e. $\lambda = 1$ by step 5, let $\theta$ be the unique solution of $z_1 + \lambda z_2 + V(\theta) = 0$. Note that $\|\theta\|_\infty \leqslant 2\|z_1 + \lambda z_2\|_\infty$. We claim that $\int \theta\, d\mu = 0$. In fact, $z_1 + \lambda z_2 + W(\theta) - \int \theta\, d\mu = \theta$, or $z_1 + \lambda z_2 - \int \theta\, d\mu = \theta - W(\theta)$. Operator on both sides of this equality with $\hat{h}$. By our choice of $\lambda$, $\hat{h}(z_1 + \lambda z_2) = 0$. By the definition of $\hat{h}$, $\hat{h}(\theta - W(\theta)) = 0$. Thus, $\hat{h}(\int \theta\, d\mu) = \hat{h}(1) \cdot \int \theta\, d\mu = 0$. Since $\hat{h}(1) \neq 0$, our claim follows. Thus, $\theta$ in fact is a solution to $z_1 + \lambda z_2 + W(\theta) = \theta$. This solution has the following additional properties: if $w$ is as in ii) of step 2, $w(x) \to 0$ as $x \to \infty$. Let now

$$F(x) = \begin{cases} z_1 + \lambda z_2 + z & |x| < R_0 . \\ \lambda g + w & |x| > 1 . \end{cases}$$

Then clearly $F(x)$ is the fundamental solution with $\lim_{x \to \infty} F(x) - g(x) = 0$. We now prove that it satisfies all the required bounds. We recall that $\lambda = 1$. For $|x| < R_0$, $F(x) = z_1(x) + z_2(x) + z(x)$;

$$\sup_{|x| < R_0} |z(x)| \leqslant \sup_{|x| = R_0} |w(x)| \leqslant \|\theta\|_\infty \leqslant 2\|(z_1(x) + z_2(x))|_{|x|=1}\|_\infty .$$

It is known (see [LSW]) that $z_1$ is negative, and $B \log 1/|x| < -z_1(x) \leqslant A \log(1/|x|)$, for $|x| < \eta_1$, where $A$, $B$, $\eta_1$ depend only on ellipticity. Similar bounds hold for $\tilde{g}$, and so $|z_2(x)| \leqslant C$, for $|x| < R_0$, where $C$ depends only on ellipticity. From these estimates, the required bounds near the origin for $F$, follow. At $\infty$, note that $\max_{|x|>1} |w(x)| \leqslant \max_{|x|=1} |\theta(x)| \leqslant C$, where $C$ depends only on ellipticity, and that $D \log |x| \leqslant g(x) \leqslant E \log |x|$ for $|x| > 2$, where $D$, $E$ depend only on ellipticity. The Theorem follows.

We are now ready to define the fundamental solution for $L$, with an arbitrary pole.

DEFINITION A.3. For $z \in \mathbb{R}^2$, the function $F(x, z) = F_z(x - z)$ is called the fundamental solution for $L$, with pole at $z$. Here, $F_z(y)$ is the fundamental solution with pole at the origin, given by Theorem A.2 for the operator $L_z$, with coefficient matrix $A_z(x) = (a_{ij}(x + z))$.

THEOREM A.4. *$F(x, z)$ is continuous in $z$, for $z \neq x$. Moreover,*

$$C_1 \log \frac{1}{|x-z|} \leqslant -F(x,z) \leqslant C_2 \log \frac{1}{|x-z|}, \qquad \text{for } |x-z| < R_1$$

$$C_3 \log |x-z| \leqslant F(x,z) \leqslant C_4 \log |x-z|, \qquad \text{for } |x-z| > R_2 .$$

PROOF. The estimates are an immediate consequence of the definition and Theorem A.2. The continuity is a tedious exercise in elliptic equations.

REMARK A.5. If $|f(x)| \leqslant C/|x|^{2+\varepsilon}$ at $\infty$, and $u(x) = \int F(x,z)(z)\,dz$, it is easy to see that $Lu = f$ in $\mathbb{R}^2$.

*Note added in proof.*

The sharp asymptotic behavior of the conformal diffeomorphism in the classical Uniformization Theorem is recently obtained in the following paper (for uniformly elliptic metrics):

LIN, FANG-HUA and NI, WEI-MING, *On the least growth of harmonic functions and the boundary behavior of Riemann mappings*, Comm. PDE, **10** (1985), pp. 767-786.

Using results there, we observe that the Uniformization Theorem approach to solving equation (1.1) (mentioned in the Introduction of our present paper) does not seem to give the optimal results for the nonlinear equation (1.1) while our present paper does.

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