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Harmonic Maps into a Hemisphere.

M. GIAQUINTA - J. SOUČEK

As it is well known, the regularity of a weakly harmonic map from an n -dimensional Riemannian manifold M^n into a N -dimensional Riemannian manifold M^N depends on the geometry of M^N (cfr. [3], [4], [2]).

If the sectional curvature of M^N is non-positive, then any harmonic mapping is smooth (see [5], where existence of a regular harmonic map, but essentially also a priori estimates, are proved). In general only partial regularity holds. More precisely, bounded energy minimizing maps are regular (in the interior) except for a closed set Σ of Hausdorff dimension at most $n - 3$ (and if $n = 3$ Σ is discrete), see [17] [8]; while they are regular at the boundary provided boundary values are smooth, see [18] [15].

In 1977 Hildebrandt, Kaul and Widman [11] proved existence and regularity of harmonic maps under the assumption that the image of the map is contained in a ball $\mathcal{B}_R(q) = \{p \in M^N: \text{dist}(p, q) < R\}$ which lies within normal range of all its points (or more generally is disjoint to the cut locus of the center q), and for which

$$(0.1) \quad R < \frac{\pi}{2\sqrt{\kappa}}$$

where $\kappa \geq 0$ is an upper bound for the sectional curvature of M^N . They also showed that the map $U^* = x/|x|$ from the unit ball B of $\mathbb{R}^n$ into the equator of the standard sphere $S^n = \{y \in \mathbb{R}^{n+1}: |y| = 1\} \subset \mathbb{R}^{n+1}$ is a critical point for the energy. We shall refer to U^* as the equator map.

More recently Jäger and Kaul [13] have shown that U^* is an absolute minimum (with respect to variations with the same boundary value) if $n \geq 7$, but it is even unstable if $3 \leq n \leq 6$; and Baldes [1] has considered the equator

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map from B into ellipsoids

$$A = \left\{ (y, y_{n+1}) \in \mathbb{R}^{n+1} : |y|^2 + \frac{y_{n+1}^2}{a^2} = 1 \right\} \subset \mathbb{R}^{n+1},$$

showing that U^* is strictly stable for the energy functional if $a^2 > 4(n-1)/(n-2)^2$ and unstable if $a^2 < 4(n-1)/(n-2)^2$.

In this paper we shall be concerned with the special case of energy minimizing harmonic maps from a domain Ω in some Riemannian manifold into the N -dimensional hemisphere $S_+^N = \{y \in \mathbb{R}^{N+1} : |y| = 1, y_N \geq 0\} \subset \mathbb{R}^{N+1}$ and more precisely with the regularity of such maps. We note that our case, though quite special, corresponds to having equality sign in (0.1).

The main results of this paper are the following

THEOREM 1. *Every energy minimizing map u from a domain Ω in some Riemannian manifold into the hemisphere S_+^N is regular provided $3 \leq n \leq 6$.*

A simple re-reading of the proof of theorems 1 and 2 of [8], see also [17], then allows us to state

THEOREM 1'. *Every energy minimizing map u from a domain Ω in some Riemannian manifold into the hemisphere S_+^N is regular except for a closed set Σ of Hausdorff dimension at most $n-7$. Σ is discrete for $n=7$.*

Finally we prove a Bernstein type theorem (compare e.g. with [10])

THEOREM 2. *Every energy minimizing map u from $\mathbb{R}^n$ into S_+^N is constant, if $n \leq 6$.*

For the sake of simplicity, in the following we shall suppose that Ω is a domain in $\mathbb{R}^n$ (compare with the remark in section 1).

The reader will recognize that the methods used in the proofs follow closely those developed in the theory of minimal surfaces (compare for example [9]).

We would like to thank E. Giusti and W. Jäger for discussions in connection with this work.

After this paper was ready, J. Jost informed us that R. Shoen and K. Uhlenbeck have proved a result similar to the one of theorem 1 and 1'.

1. From now on we shall use stereographic coordinates on the sphere S^N . Therefore the energy functional for a map u from a domain Ω in $\mathbb{R}^n$ into S_+^N is given (apart from a factor 2) by

$$E(u; \Omega) = \int_{\Omega} \frac{|Du|^2}{(1 + |u|^2)^2} dx$$

where $Du = \partial_1 u, \dots, \partial_n u$ and $\partial_\alpha = \partial/\partial x_\alpha$ are the standard partial derivatives in $\mathbb{R}^n$. The map u is a weakly energy minimizing map with image on the hemisphere if $u \in H_{\text{loc}}^{1,2}(\Omega, \mathbb{R}^N)$, $|u| \leq 1$, and

$$E(u; \text{supp}(u - v)) \leq E(v; \text{supp}(u - v))$$

for any v with $|v| \leq 1$ and $\text{supp}(u - v) \subset \subset \Omega$.

As it is geometrically clear, and a simple calculation shows

$$E(v; \Omega) = E(\tilde{v}; \Omega)$$

for any bounded $v \in H^{1,2}(\Omega, \mathbb{R}^N)$, where $\tilde{v}$ is the reflected map through the equator, i.e.

$$\tilde{v} = \begin{cases} v & \text{if } |v| \leq 1 \\ \frac{v}{|v|^2} & \text{if } |v| \geq 1. \end{cases}$$

The we have: if $u \in H^{1,2}(\Omega, \mathbb{R}^N)$, $|u| \leq 1$, Ω bounded, minimizes the energy among maps v with $v = u$ on $\partial\Omega$ and image into the hemisphere, i.e. $|v| \leq 1$, then u minimizes the energy in the class $\{v \in H^{1,2}(\Omega, \mathbb{R}^N) : v \text{ bounded, } v = u \text{ on } \partial\Omega\}$.

Therefore u is a solution of the Euler equation for E

$$(1.1) \quad \int_{\Omega} \left[\frac{Du \cdot D\varphi}{(1 + |u|^2)^2} - 2 \frac{|Du|^2}{(1 + |u|^2)^2} \frac{u \cdot \varphi}{1 + |u|^2} \right] dx = 0 \quad \forall \varphi \in H_0^1 \cap L^\infty(\Omega, \mathbb{R}^N).$$

Now let us assume that u has a singular point $x_0 \in \Omega$. We may suppose without loss in generality that x_0 is isolated; in fact from the results in [8] or [17] we know that if in dimension $n = \bar{n}$ harmonic maps are regular then they have at most isolated singularities in dimension $n = \bar{n} + 1$; moreover in dimension $n = 3$ the singular set, if non empty, is discrete.

By translating the point x_0 to the origin and blowing-up, we then produce (see [8] [17]) a new energy minimizing (with respect to its own boundary values) map, defined on the unit ball B of $\mathbb{R}^n$ (still called u) with $|u| \leq 1$, u singular at zero and *homogeneous of degree zero*.

If we choose in (1.1) $\varphi(x) = u(x)\eta(x)$ where $\eta = \eta(|x|)$ is a smooth function with compact support in B , because of the 0-homogeneity of u , we immediately get

$$(1.2) \quad \int_{\partial\Omega} \frac{|Du|^2}{(1 + |u|^2)^2} \frac{1 - |u|^2}{1 + |u|^2} d\mathcal{H}^{n-1} = 0$$

which obviously implies that either u is constant or $|u|$ is *identically* 1. Note that u is regular in $B - \{0\}$. The first possibility being excluded, roughly because the limit of singular points is a singular point (see [8]), we may state that *if our original energy minimizing map had a singularity at x_0 , then we can produce a 0-homogeneous energy minimizing map u , with $|u| = 1$, singular at zero.*

Now we shall esclude that possibility in dimension less or equal than 6 by means of the stability property of u itself, deducing the regularity of u for $n \leq 6$. This procedure is very similar to the one in the theory of minimal surfaces (see e.g. [9] [19]).

We calculate the first and second variation of the energy at u . For any smooth function $\varphi(x)$ with compact support in B we consider the function

$$E(t) = E(u + t\varphi; B)$$

and we have

$$\begin{aligned} \delta E(\varphi) &:= \frac{d}{dt} E|_{t=0} = \frac{1}{2} \int_B [Du \cdot D\varphi - u \cdot \varphi |Du|^2] dx \\ \delta^2 E(\varphi, \varphi) &:= \frac{d^2}{dt^2} E|_{t=0} = 0 = \frac{1}{2} \int_B [|D\varphi|^2 - 4u \cdot \varphi Du \cdot D\varphi - |Du|^2 |\varphi|^2 \\ &\quad + 3(u \cdot \varphi)^2 |Du|^2] dx. \end{aligned}$$

The meaning of the notation is obvious.

Since u is stationary

$$\delta E(\varphi) = 0$$

for any $\varphi \in H_0^1 \cap L^\infty(B, \mathbb{R}^N)$ (actually, because of the 0-homogeneity of u , for any $\varphi \in H^1 \cap L^\infty(B, \mathbb{R}^N)$), and, because u is a minimum point,

$$\delta^2 E(\varphi, \varphi) \geq 0 \quad \forall \varphi \in H_0^1 \cap L^\infty(B, \mathbb{R}^N).$$

Now we choose in the second variation $\varphi(x) = u(x)|Du(x)|\eta(|x|)$ where $\eta(t)$ is any smooth function with compact support in $(0, 1)$ and after a few standard calculations we conclude that

$$(1.3) \quad \delta^2 E = \frac{1}{2} \int_B \{c^2 |D\eta|^2 - \eta^2 [c^4 + \frac{1}{2} \Delta c^2 - |Dc|^2]\} dx \geq 0$$

where $c(x) = |Du(x)|$.

LEMMA 1. *We have:*

$$(1.4) \quad c^4 + \frac{1}{2} \Delta c^2 - |Dc|^2 \geq \frac{c^2}{|x|^2} + \frac{c^4}{n-1}.$$

PROOF. First we note that

$$\begin{aligned} \frac{1}{2} \Delta c^2 + c^4 &= \sum_{i=1}^N \sum_{\alpha, \beta=1}^n (\partial_\alpha \partial_\beta u^i)^2 \\ c^2 |Dc|^2 &= \sum_{\beta=1}^n \left(\sum_{j=1}^N \sum_{\alpha=1}^n \partial_\alpha u^j \partial_\beta \partial_\alpha u^j \right)^2 \end{aligned}$$

hence

$$(1.5) \quad \frac{1}{2} \Delta c^2 + c^4 - |Dc|^2 = \frac{1}{2c^2} \sum_{\alpha, \beta, \gamma=1}^n \sum_{i, j=1}^n [\partial_\alpha u^i \partial_\beta \partial_\gamma u^j - \partial_\beta u^j \partial_\alpha \partial_\gamma u^i]^2.$$

If $x_0 \in B - \{0\}$ by orthogonal changes of coordinates we can assume that

$$x_0 = (r, 0, 0, \dots, 0) \quad r = |x_0|, \quad u(x_0) = (1, 0, \dots, 0).$$

Therefore, if the letters $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ and $\bar{i}, \bar{j}$ run from 2 to n and from 2 to N respectively, we have, taking into account the 0-homogeneity of u :

$$(1.6) \quad \partial_1 u^i(x_0) = 0$$

$$(1.7) \quad \partial_1 \partial_{\bar{\alpha}} u^i(x_0) = -\frac{1}{r} \partial_{\bar{\alpha}} u^i(x_0)$$

Since $u^i \partial_\alpha u^i = 0$ and $\partial_\beta (u^i \partial_\alpha u^i) = 0$, we get moreover

$$(1.8) \quad \partial_\alpha u^1(x_0) = 0$$

and $\partial_\alpha u^i \partial_\beta u^i + u^i \partial_\alpha \partial_\beta u^i = 0$, i.e.

$$(1.9) \quad \partial_\alpha \partial_\beta u^1(x_0) = -\frac{1}{r} \sum_{k=1}^N \partial_\alpha u^k(x_0) \partial_\beta u^k(x_0).$$

Using (1.6) ... (1.9) we see immediately that at x_0

$$\begin{aligned} \sum_{\alpha, \beta, \gamma=1}^n \sum_{i, j=1}^N [\partial_\alpha u^i \partial_\beta \partial_\gamma u^j - \partial_\beta u^j \partial_\alpha \partial_\gamma u^i]^2 &= \\ &= \sum_{\bar{\alpha}, \bar{\beta}, \bar{\gamma}} \sum_{\bar{i}, \bar{j}} [...]^2 + \frac{2}{r^2} c^4 = \frac{2}{n-1} c^6 + \frac{2}{r^2} c^4 + \sum_{\bar{\alpha}, \bar{\beta}, \bar{\gamma}} \sum_{\bar{i}, \bar{j}} [...]^2 \end{aligned}$$

and the result follows from (1.5).

PROOF OF THEOREM 1: From (1.3) (1.4) we deduce for any $\eta(t)$ with compact support in $(0, 1)$

$$(1.10) \quad \int_B \left\{ c^2 |D\eta(|x|)|^2 - \eta^2(|x|) \left(\frac{c^2}{|x|^2} + \frac{c^4}{n+1} \right) \right\} dx \geq 0.$$

Now either $c^2 \equiv 0$, i.e. u is constant and therefore regular, or, since c^2 is homogeneous of degree -2

$$(1.11) \quad \int_0^1 r^{n-3} \eta^2(r) dr - \left(1 + \frac{1}{n-1} \frac{\int_B c^4 d\mathcal{H}^{n-1}}{\int_B c^2 d\mathcal{H}^{n-1}} \right) \int_0^1 r^{n-4} \eta^2 dr \geq 0$$

for any smooth η with compact support in $[0, 1)$.

First we observe that by an approximation and rescaling argument, setting

$$1 + \frac{1}{n-1} \frac{\int_B c^4 d\mathcal{H}^{n-1}}{\int_B c^2 d\mathcal{H}^{n-1}} = 1 + \sigma, \quad \sigma > 0$$

(1.11) implies that

$$(1.12) \quad \int_0^\infty r^{n-3} \eta^3(r) dr - (1 + \sigma) \int_0^\infty r^{n-5} \eta^2(r) dr \geq 0$$

for any function $\eta: (0, +\infty) \rightarrow \mathbb{R}$ for which the integrals in (1.12) converge. Now we choose

$$\eta(r) = \begin{cases} r^\alpha & \text{for } r < 1 \\ r^\beta & \text{for } r \geq 1 \end{cases}$$

where

$$\alpha = \frac{1}{2}(4 - n + \varepsilon), \quad \beta = \frac{1}{2}(4 - n - \varepsilon) \quad \varepsilon > 0$$

and from (1.12) we deduce that

$$\frac{\alpha^2}{n-4+2\alpha} - \frac{\beta^2}{n-4+2\beta} \geq (1+\sigma) \left[\frac{1}{n-4+2\alpha} - \frac{1}{n-4+2\beta} \right] \quad \forall \varepsilon > 0$$

i.e.

$$\frac{(4-n)^2}{2} \geq 2(1+\sigma)$$

that is $n \geq 4 + 2\sqrt{1+\sigma}$ q.e.d.

REMARK. More generally we can consider bounded minimum points of the energy functional given by

$$(1.13) \quad E(u; \Omega) = \int_B \gamma^{\alpha\beta}(x) g_{ij}(u) D_\alpha u^i D_\beta u^j \sqrt{\gamma} dx$$

where summation over repeated indices is understood and $\gamma_{\alpha\beta}, g_{ij}$ are symmetric positive definite smooth and bounded matrices; $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$ and $\gamma = \det \gamma_{\alpha\beta}$. If $x_0 \in \Omega$ is a singular point, we can proceed as in [8] and we end up with a minimum point u_0 of the functional

$$\int_B \gamma^{\alpha\beta}(x_0) g_{ij}(u_0) D_\alpha u_0^i D_\beta u_0^j \sqrt{\gamma} dx$$

moreover u_0 is singular at zero and 0-homogeneous.

By a change of coordinates we may always assume that $\gamma_{\alpha\beta}(x_0) = \delta_{\alpha\beta}$. So inserting in the Euler equation

$$\int_B \{g_{ij}(u_0) D_\alpha u_0^i D_\alpha \varphi^j + g_{ij, u^i} D_\alpha u_0^i D_\alpha u_0^j \varphi^2\} dx = 0 \quad \forall \varphi \in H_0^1 \cap L^\infty(B, \mathbb{R}^N)$$

$\varphi = u_0 \gamma$ where $\gamma = \gamma(|x|)$ is a smooth function, we deduce that

$$\int_B [2g_{ij}(u_0) + u_0^i g_{ij, u^i}(u_0)] D_\alpha u_0^i D_\alpha u_0^j \gamma dx = 0.$$

Therefore if we assume that

$$(1.14) \quad [g_{ij}(\sigma) + \tfrac{1}{2} \sigma_0^i g_{ij, \sigma^i}(\sigma)] \xi_i \xi_j \geq \nu |\xi|^2 \quad \forall \xi; \nu > 0$$

We can conclude that u_0 is constant, i.e. the original minimizing map u was not singular. So we can summarize: *a bounded minimizing map of the energy functional (1.13) is regular in any dimension provided (1.14) holds.*

This corresponds, roughly speaking, to the case of the target manifold $(\mathbb{R}^N, g_{ij})$ with non positive sectional curvature; and it provides a different proof of theorem 5.2 of [7], compare also with [17] [14].

The same argument gives a different proof of the (interior) regularity result of [11] for energy minimizing maps and, because of the uniqueness theorem in [12] for any external, too. We note, anyway, that the method does not provide a priori estimates.

2. In this section we shall prove theorem 2. Let $u: \mathbb{R}^n \rightarrow \mathbb{R}^N$, $|u| \leq 1$, be a (locally) energy minimizing map, i.e.

$$E(u; \text{supp } \varphi) \leq E(u + \varphi; \text{supp } \varphi)$$

for any φ with compact support.

We begin by recalling some results from [8]. A simple inspection of the proof of lemma 2 of [8] permits to state the following monotonicity

LEMMA 2. For every ϱ, R , $0 < \varrho < R$, we have

$$(2.1) \quad \int_{\partial B_1} |u(Rx) - u(\varrho x)|^2 d\mathcal{H}^{n-1} \leq \gamma_0 \left(\log \frac{R}{\varrho} \right) [\varphi(R; u) - \varphi(\varrho; u)]$$

where

$$\varphi(t; u) = t^{2-n} \int_{B_t} \frac{|Du|^2}{(1 + |u|^2)^2} dx.$$

and $B_t = \{x \in \mathbb{R}^n: |x| < t\}$.

For $j \in \mathbb{N}$ set now

$$u_j(x) = u(jx).$$

Since $|u_j| \leq 1$, as in lemma 1 of [8], we get that for any R the L^2 and L^p norms of the gradient of u (p being a suitable number larger than two) are equibounded in B_R . Hence, by means of a diagonal process, we show (see [8]) that there exists a subsequence u_{r_j} locally weakly converging in $H^{1,2}$ to some u_0 and moreover u_0 locally minimizes the energy functional E .

From [16] (step 2 of the proof of theorem 3) we deduce that actually u_{r_j} converge strongly to u_0 in $H^{1,2}(B_R, \mathbb{R}^N)$ for any R , therefore for any R

$$(2.2) \quad E(u_{r_j}; B_R) \rightarrow E(u_0; B_R).$$

Now we shall show that u_0 is homogeneous of degree zero.

Note that by (2.1) $\varphi(r_j R, u) = R^{2-n} E(R; u_j)$ is increasing in the first argument and by (2.2)

$$\lim_{j \rightarrow \infty} \varphi(r_j R; u) = R^{2-n} E(u_0; B_R)$$

If $\varrho < R$, for every j there exists a $m_j < 0$ such that

$$r_j \varrho > r_{j+m_j} R.$$

Then

$$\varphi(r_{j+m}; R; u) \leq \varphi(r_j; u) \leq \varphi(r_j R; u)$$

so that

$$\lim_{j \rightarrow \infty} \varphi(r_j \varrho; u) = R^{2-n} E(u_0; B_R).$$

Hence we have proved that

$$\varrho^{2-n} E(\varrho; u_0)$$

is independent of ϱ and so, from (2.1), we conclude that u_0 is homogeneous of degree zero.

Now we have

$$(2.3) \quad (Rr_j)^{2-n} E(Rr_j; u) = R^{2-n} E(r; u_r) \uparrow R^{2-n} E(R; u_0).$$

Suppose that $n \leq 6$; in this case, since u_0 must be a smooth function by theorem 1, and it is 0-homogeneous and bounded, it follows that u_0 is constant. Using the monotonicity of $t \rightarrow \varphi(t; u)$ we get from (2.3)

$$R^{2-n} \int_{B_R} |Du|^2 dx = 0$$

for every $R > 0$, i.e. u is constant.

REFERENCES

- [1] A. BALDES, *Stability properties of the equator map from a ball into ann ellipsoid*, preprint.
- [2] J. EELLS, *Regularity of certain harmonic maps*, Proc. Durham Symp. on Global Riem. Geo., July 1982, to appear.
- [3] J. EELLS - L. LEMAIRE, *A report on harmonic maps*, Bull. London Math. Soc., **40** (1978), pp. 1-68.
- [4] J. EELLS - J. LEMAIRE, *Selected topics in harmonic maps*, N.S.F. Conf. Board Math. Sci., to appear.
- [5] J. EELLS - J. H. SAMPSON, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., **86** (1964), pp. 109-160.
- [6] M. GIAQUINTA, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Math. Studies, no. 105, Princeton Univ. Press.
- [7] M. GIAQUINTA - E. GIUSTI, *On the regularity of the minima of variational integrals*, Acta Math., **448** (1982), pp. 31-46.
- [8] M. GIAQUINTA - E. GIUSTI, *The singular set of the minima of certain quadratic functionals*, Ann. Sc. Norm. Sup. Pisa, to appear.

- [9] E. GIUSTI, *Minimal surfaces and functions of bounded variation*, Birkhäuser, New York, to appear.
- [10] S. HILDEBRANDT, *Liouville theorems for harmonic mappings and an approach to Bernstein theorem*, Annals of Math. Studies, no. 102, Princeton (1982), pp. 107-132.
- [11] S. HILDEBRANDT - H. KAUL - K.-O. WIDMAN, *An existence theorem for harmonic mappings of Riemannian manifolds*, Acta Math., **438** (1977), pp. 1-16.
- [12] W. JÄGER - H. KAUL, *Uniqueness and stability of harmonic maps and their Sacobi fields*, Manuscripta Math., **28** (1979), pp. 269-291.
- [13] W. JÄGER - H. KAUL, *Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems*, reprint.
- [14] J. JOST, *Existence proofs for harmonic mappings with the help of a maximum principle*, Math. Z., to appear.
- [15] J. JOST - M. MEIER, *Boundary regularity for minima of certain quadratic functionals*, Math. Ann., **262** (1983), pp. 549-561.
- [16] M. MEIER, *Liouville theorems, partial regularity, and Hölder continuity of weak solutions to quasilinear elliptic systems*, preprint.
- [17] R. SCHOEN - K. UHLENBECK, *A regularity theory for harmonic maps*, J. Diff. Geo., **47** (1982), pp. 307-335.
- [18] R. SCHOEN - K. UHLENBECK, *Boundary regularity and miscellaneous results on harmonic maps*, J. Diff. Geom., to appear.
- [19] J. SIMONS, *Minimal varieties in riemannian manifolds*, Annals of Math., **88** (1968), pp. 62-105.

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