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Abrupt and Smooth Separation of Free Boundaries in Flow Problems (*).

HANS WILHELM ALT - LUIS A. CAFFARELLI - AVNER FRIEDMAN

Introduction.

For some classes of free boundary problems it has been established that the solution u in the given domain Ω is smooth up to the free boundary Γ and that Γ is smooth in Ω (see, for instance [13], and the references given there). Restricting ourselves to the case of two dimensions, consider the situation where Γ initiates from a point X on $\partial\Omega$. Then it is natural to ask how smooth Γ is at the detachment point X . For example, in the case of the rectangular dam problem, with $\Omega = \{(x, y); 0 < x < a, 0 < y < b\}$, Γ has a tangent at X ; Γ and $\partial\Omega$ are orthogonal at X if X is the highest point of the higher reservoir, but have the same tangent if X is the point above the lower reservoir where Γ meets $\partial\Omega$ (see [15]; for general dams see [7]). Denoting the latter point by $X = (a, y_*)$ and the free boundary by $x = \psi(y)$, we then have that $\psi(y)$ is defined for $y_* \leq y \leq H$ ($H \leq b$) and $\psi'(y_*) = 0$. On the other hand (see [15])

$$(0.1) \quad (y - y_*)^\varepsilon \psi''(y) \rightarrow \infty \quad \text{if } y \downarrow y_*$$

for any $\varepsilon > 0$. Thus we can say that the detachment or separation at X is, roughly, C^1 but not $C^{1+\varepsilon}$ for any $\varepsilon > 0$.

In flow problems of jets and cavities one is given a fixed curve N called a nozzle (for jets) or a nose (for cavities) with end-point A , and the problem is to find a stream function ψ and its free boundary Γ such that $\psi \geq 0$ and $\mathfrak{L}\psi = 0$ in $\{\psi > 0\}$ (= the flow region) where $\mathfrak{L}$ is an appropriate elliptic oper-

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ator and $|\nabla\psi| = \lambda_\mu$ on Γ , where λ_μ is a given function depending on a parameter μ . The parameter μ must be chosen in such a way that the free boundary curve Γ initiates precisely at the separation point A ; this property is called *continuous fit*. If further Γ and N have the same tangent at A and $\nabla\psi$ is continuous in $\{\psi \geq 0\}$ -neighborhood of A , then we speak of *C^1 smooth fit*.

For two dimensional problems (with ideal incompressible, irrotational flow) the existence of such solutions was established by many authors, mainly over the last fifty years (see [10], [16] and the references given there). More recently Alt, Caffarelli and Friedman [2-6] have established the existence of solutions with C^1 smooth fit for problems which include 3-dimensional axially symmetric flows; their work includes also flows under the force of gravity as well as compressible flows; irrotational flows were studied in [14].

In this paper we consider the question of C^2 smooth fit of N and Γ at the separation point A . Our method is sufficiently general to include all the flow problems with C^1 smooth fit considered in [2-6] and [14]. The main result is the following alternative: either

- (i) the curvature κ along Γ tends to the curvature of N at A (which means a C^2 smooth fit), or
- (ii) the curvature κ of Γ tends to $+\infty$ in absolute value as one approaches A along Γ .

Case (ii) is called *abrupt separation*.

In the special case of two dimensional flows, this result is due to Villat (see [10; p. 81] or [16; p. 372]). Carter [11] extended this result to the case of flow in gravity field, assuming that N is an analytic curve. The axially symmetric case is discussed in [8] where formal expansions of the solution are assumed.

In § 1 we state the general variational problem for which our results will be established. In § 2 we express the curvature κ of Γ in terms of derivatives of a function ω (the «logarithmic hodograph variable»). In § 3 we derive a nonlinear elliptic system for ω of the form

$$(0.2) \quad \omega_{\bar{t}} = a\omega_t + b\bar{\omega}_{\bar{t}} + c$$

where $t = \sqrt{f}$ and $f = \varphi + i\psi$ is the «complex potential»; the coefficients a, b, c depend on ω . In §§ 4, 5 we obtain some regularity results for ω near $t = 0$. Then, in § 6, we obtain some expansion for ω near $t = 0$, working with the independent variable $s = \log t$ and with a second order elliptic equation for $\operatorname{Re} \omega$ (or $\operatorname{Im} \omega$) in the strip $\{s = \sigma + i\tau, -\infty < \sigma < 0, 0 < \tau < \pi/2\}$. Combining this extension with the formula for κ (in § 2), the proof of (i), (ii) then easily follows.

In § 7 we prove (i), (ii) for incompressible flows under weaker differentiability assumptions on N .

1. – The variational problem; the main result.

We denote a point (x, y) in R^2 by $z = x + iy$. Consider a functional

$$J(u) = \int_{\Omega} (f(z, \nabla u) + Q(z) \chi_{\{u > 0\}})$$

and an admissible class

$$K = \{u \in H^{1,2}(\Omega), u = u^0 \text{ on } S\}$$

where Ω is a bounded domain in R^2 whose boundary $\partial\Omega$ is locally a Lipschitz graph, S is a subset of $\partial\Omega$ with nonempty interior, u^0 is a nonnegative function in $H^{1,2}(\Omega)$, $Q \in C^{1,\alpha}(\bar{\Omega})$, $Q > 0$ in $\bar{\Omega}$, and

$$\begin{aligned} f(z, p) &= F(z, |p|^2), \quad f \in C^{2,\alpha}, \\ (1.1) \quad F(z, 0) &= 0, \\ c_0 |\xi|^2 &\leq \sum f_{z_i z_i}(z, p) \xi_i \xi_i \leq C_0 |\xi|^2 \quad (0 < c_0 < C_0 < \infty) \end{aligned}$$

for all $z \in \bar{\Omega}$, $p \in R^2$, $\xi \in R^2$.

Consider the variational problem:

Find ψ such that

$$\psi \in K, \quad J(\psi) = \min_{u \in K} J(u).$$

Then, any minimizer ψ is a nonnegative function, $\psi \in C^{0,1}(\Omega)$ and its free boundary

$$\Gamma = \partial\{\psi > 0\} \cap \Omega$$

is in $C^{2,\alpha}$; for $F(z, t) = t$ this was proved in [1] and, for general F , in [6]. We recall that ψ satisfies

$$\begin{aligned} (1.3) \quad \nabla f_z(z, \nabla \psi) &= 0 \quad \text{in } \{\psi > 0\}, \\ \psi &= 0, \quad f_z(z, \nabla \psi) \nabla \psi - f(z, \nabla \psi) = Q(z) \text{ on } \Gamma. \end{aligned}$$

Setting

$$(1.4) \quad \varrho(z, t) = [F_s(z, s)]_{s=t^2}$$

we see that

$$(1.5) \quad \operatorname{div} (\varrho(z, |\nabla \psi|) \nabla \psi) = 0 \quad \text{in } \{\psi > 0\},$$

$$(1.6) \quad 2\varrho(z, |\nabla \psi|)(|\nabla \psi|^2 - F(z, |\nabla \psi|^2)) = Q(z) \quad \text{on } \Gamma.$$

Using the convexity of f one deduces that (1.6) determines uniquely $|\nabla \psi|$ on the free boundary, say

$$(1.7) \quad |\nabla \psi| = \lambda(z) \quad \text{on } \Gamma.$$

We now specialize to flow problems and take for definiteness a jet problem of axially symmetric compressible flow. In this case the nozzle N is a given $C^{1,\alpha}$ y -graph, say

$$(1.8) \quad N: x = h_0(y) \quad (y_* \leq y < y_1),$$

$Q(z) = \lambda_0$ is a positive parameter, and $F = F_0(|p|^2/y^2)$ with an appropriate function F_0 . The free boundary is a y -graph

$$(1.9) \quad \Gamma: x = k_0(y) \quad (y_0 < y < y_*)$$

with

$$(1.10) \quad x_* \equiv h_0(y_*) = k_0(y_* - 0).$$

Actually, the parameter λ_0 is not a priori given but is chosen in such a way that (1.10) will be satisfied. The condition (1.10) is called the *continuous fit*. We also recall that

$$(1.11) \quad \begin{aligned} &N \cup \Gamma \text{ is } C^1 \text{ at } (x_*, y_*), \text{ and} \\ &\nabla \psi \text{ is continuous in a } \{\psi \geq 0\}\text{-neighborhood of } (x_*, y_*); \end{aligned}$$

we refer to (1.11) as the *C^1 smooth fit*.

In establishing the existence of a solution for the above compressible flow problem one must actually assume that the second order differential equation satisfied by the potential function φ is elliptic (rather than hyperbolic); this means that the flow is subsonic. Technically this condition states

(see [6; (11.6)] that the function $\tilde{\varrho}$ defined by

$$\tilde{\varrho}(z, t^2) = 1/\varrho(z, t)$$

satisfies

$$\tilde{\varrho}(z, s) + 2s\tilde{\varrho}'_s(z, s) > 0 ,$$

or, equivalently,

$$(1.12) \quad \frac{t\varrho_t(z, t)}{\varrho(z, t)} < 1 .$$

For flows under gravity one replaces $Q(z) = \lambda_0$ by $Q(z) = \sqrt{\lambda_0 + x}$.

In the sequel we assume that F and Q are general functions such that (1.1) hold, and that a C^1 smooth fit (with the notation (1.8)-(1.11)) is already known. We further assume the subsonic condition (1.12) at the point of detachment point $A = (x_*, y_*)$.

Finally, for technical reasons we assume that

$$(1.13) \quad \begin{aligned} &N \text{ is in } C^{3,\alpha} \text{ near } A , \\ &\varrho(z, t) \text{ is in } C^{3,\alpha} \text{ near } z = A , \quad t = |\nabla\psi(A)| . \\ &\beta(z) \text{ is in } C^{3,\alpha} \text{ near } z = A . \end{aligned}$$

We denote by $\kappa(z)$ the curvature of Γ at a point z .

Our main result is:

THEOREM 1.1. *Either (i) $\kappa(z)$ converges to the curvature of N at A , as $z \rightarrow A$, or (ii) $\kappa(z)$ converges to $+\infty$ or to $-\infty$ as $z \rightarrow A$; more precisely,*

$$|z - A|^{\frac{1}{2}}\kappa(z) \rightarrow \gamma \quad \text{if } z \rightarrow A , \quad \gamma \neq 0 .$$

In § 7 we shall prove Theorem 1.1 for the incompressible case assuming only that N is in $C^{2+\alpha}$ and Q is in $C^{1+\alpha}$.

REMARK 1.1. Let $z = g(\zeta)$ be a conformal mapping which maps the unit disc into a neighborhood of A such that the inverse image of N is the real line segment

$$\hat{N} = \{-1 \leq \xi \leq 0, \eta = 0\}$$

where $\zeta = \xi + i\eta$. We can choose g such that

$$(1.14) \quad |\nabla g(0)| = 1 .$$

We can express S locally in terms of ζ :

$$\begin{aligned} J(u) &= \int \left(F(z, |\nabla u(z)|^2) + Q(z) \chi_{\{u>0\}} \right) dx dy \\ &= \int \left(F \left(g(\zeta), \frac{|\nabla \hat{u}(\zeta)|^2}{tg'(\zeta)^2} \right) |g'(\zeta)|^2 + \hat{Q}(\zeta) |g'(\zeta)|^2 \chi_{\{\hat{u}>0\}} \right) d\xi d\eta = \mathcal{J}(\hat{u}) \end{aligned}$$

where $\hat{v}(\zeta) = v(t)$. For the minimizer $\hat{\psi}$, the free boundary $\hat{\Gamma}$ is a C^1 curve

$$\hat{\Gamma} : \eta = k(\xi)$$

with

$$k(0) = 0, \quad k'(0) = 0$$

and $\{\hat{\psi} > 0\}$ is a region neighboring $\hat{N} \cup \hat{\Gamma}$ from above (say). This mapping g is in $C^{3,\alpha}$ (by (1.13)) and therefore the regularity assumptions for the original problem remain intact for the transformed problem. Clearly (1.1) also remains valid. Further, since (1.14) holds, the condition (1.12) also remains valid at $\zeta = 0$. We conclude that in proving Theorem 1.1 we may assume, without loss of generality, that

$$(1.15) \quad N = \{-\infty < x \leq 0, y = 0\}.$$

REMARK 1.2. In proving Theorem 1.1 we may assume that (A is the point $z = 0$ and)

$$(1.16) \quad |\nabla \psi(0)| = 1.$$

Indeed, otherwise we replace ψ by ψ/λ_0 and $\varrho(x, t)$ by $\varrho(x, \lambda_0 t)$ where $\lambda_0 = |\nabla \psi(0)|$. We may further assume that

$$(1.17) \quad \varrho(0, 1) = 1,$$

since otherwise we divide the differential equation (1.5) by $\varrho(0, 1)$ and work with $\varrho(z, t)/\varrho(0, 1)$. Notice that after the normalizations (1.15)-(1.17), the condition (1.12) at the point of detachment becomes

$$(1.18) \quad \varrho_t(0, 1) < 1.$$

REMARK 1.3. The proof of Theorem 1.1 extends to functionals which depend on u , such as

$$(1.18) \quad \int (f(z, \nabla u) - c(z)u + Q(z)\chi_{\{u>0\}})$$

where $c \geq 0$; this corresponds to irrotational flows.

Sections 2-6 are devoted to proving Theorem 1.1.

2. - The curvature of the free boundary.

As usual we define

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Denote by ν the outward normal to the flow region $\Omega_0 = \{\psi > 0\}$ at Γ and denote by τ the unit tangent vector obtained by rotating ν clockwise by $\pi/2$, that is, $\tau = -i\nu$. Then

$$\nu = -\frac{\psi_{\bar{z}}}{|\psi_z|}, \quad \tau = \frac{i\psi_{\bar{z}}}{|\psi_z|}.$$

The curvature κ of Γ in the direction $-\nu$ is defined by

$$-\kappa\nu = \partial_{\bar{\tau}}\tau$$

(κ is a real number, ≥ 0 if Ω_0 is convex). Thus

$$-\kappa\nu = -i\partial_{\bar{\tau}}\nu = -i(\nu_z\tau + \nu_{\bar{z}}\bar{\tau}) = -\nu_z\nu + \nu_{\bar{z}}\bar{\nu},$$

or

$$\begin{aligned} \kappa &= \nu_z - \nu_{\bar{z}}\bar{\nu}^2 = -\left(\frac{\psi_{\bar{z}}}{|\psi_z|}\right)_z + \left(\frac{\psi_{\bar{z}}}{|\psi_z|}\right)_{\bar{z}}\bar{\nu}^2 \\ &= -\left(\frac{1}{|\psi_z|}\right)_z \psi_{\bar{z}} + \left(\frac{1}{|\psi_z|}\right)_{\bar{z}} \psi_z \frac{\psi_z^2}{|\psi_z|^2} - \frac{\psi_{z\bar{z}}}{|\psi_z|} + \frac{\psi_{\bar{z}\bar{z}}}{|\psi_z|} \bar{\nu}^2 \\ &= -2i \operatorname{Im} \left\{ \left(\frac{1}{|\psi_z|}\right)_z \psi_{\bar{z}} \right\} - \frac{\psi_{z\bar{z}}}{|\psi_z|} + \frac{\bar{\psi}_{z\bar{z}}}{|\psi_z|} \bar{\nu}^2. \end{aligned}$$

Hence:

LEMMA 2.1. *There holds:*

$$(2.1) \quad \kappa = -\frac{\psi_{z\bar{z}}}{|\psi_z|} + \frac{\psi_{zz}}{|\psi_z|} \nu^2 + \text{imag}.$$

where «imag» stands for an imaginary number.

We shall compute the right hand side of (2.1) in terms of the function

$$(2.2) \quad \omega = i \log \frac{2i\psi_z}{\lambda(z)}$$

where $\lambda(z)$ is the function occurring in (1.7). We sometimes write ω in terms of its real and imaginary parts

$$\omega = \omega_1 + i\omega_2.$$

Since $2|\psi_z| = |\nabla\psi| = \lambda(z)$ on Γ and $2\psi_z = -i\psi_\nu$ on N (recall (1.15)), we see that

$$(2.3) \quad \begin{aligned} \omega_2 &= 0 & \text{on } \Gamma, \\ \omega_1 &= 0 & \text{on } N. \end{aligned}$$

In view of (1.5), we can define a function φ by

$$(2.4) \quad \begin{aligned} \varphi_x &= \varrho(z, |\nabla\psi|) \psi_\nu \\ \varphi_\nu &= -\varrho(z, |\nabla\psi|) \psi_x \end{aligned}$$

and further normalize it by $\varphi(0) = 0$. The physical interpretation of φ is the real velocity potential. We also introduce the «complex potential»

$$(2.5) \quad f = \varphi + i\psi$$

and an independent variable t :

$$(2.6) \quad t = \sqrt{f(z)}.$$

We easily compute that

$$(2.7) \quad \varphi_z = i\varrho\psi_z, \quad \varphi_{\bar{z}} = -i\varrho\psi_{\bar{z}},$$

$$(2.8) \quad f_z = (1 + \varrho)i\psi_z, \quad f_{\bar{z}} = (1 - \varrho)i\psi_{\bar{z}}.$$

By the normalization in Remarks 1.1, 1.2 we then have

$$(2.9) \quad f_z(0) = \psi_z(0) = \lambda(0) = 1, \quad f_{\bar{z}}(0) = 0.$$

Therefore the mapping (2.6) is one-to-one with

$$(2.10) \quad f(z) \approx z, \quad t \approx \sqrt{z} \text{ near } z = 0.$$

Furthermore, in a small disc about the origin in the t -plane I' is an interval

$$I' = \{0 < \operatorname{Re} t < \delta_0, \operatorname{Im} t = 0\}$$

and N is an interval

$$N' = \{\operatorname{Re} t = 0, 0 < \operatorname{Im} t < \delta_0\};$$

(2.3) reads:

$$(2.11) \quad \begin{aligned} \omega_2 &= 0 \text{ on the positive } t_1\text{-axis, near } 0, \\ \omega_1 &= 0 \text{ on the positive } it_2\text{-axis, near } 0. \end{aligned}$$

From (2.6),

$$(2.12) \quad t_z = \frac{f_z}{2t}, \quad t_{\bar{z}} = \frac{\bar{f}_{\bar{z}}}{2\bar{t}}$$

so that, by (2.8),

$$t_z = i \frac{1 + \varrho}{2t} \psi_z, \quad t_{\bar{z}} = i \frac{1 - \varrho}{2\bar{t}} \psi_{\bar{z}},$$

and then

$$(2.14) \quad \begin{aligned} \psi_{z\bar{z}} &= (\psi_z)_{\bar{z}} = \psi_{z\bar{t}} t_{\bar{z}} + \psi_{z\bar{t}} \bar{t}_{\bar{z}} = i \left[\psi_{z\bar{t}} \frac{1 - \varrho}{2\bar{t}} \psi_{\bar{z}} - \psi_{z\bar{t}} \frac{1 + \varrho}{2\bar{t}} \psi_{\bar{z}} \right] \\ &= \frac{i}{2} \psi_{\bar{z}} \left[\frac{\psi_{z\bar{t}}}{\bar{t}} (1 - \varrho) - \frac{\psi_{z\bar{t}}}{\bar{t}} (1 + \varrho) \right]. \end{aligned}$$

Similarly,

$$(2.15) \quad \begin{aligned} \psi_{zz} &= \psi_{zt} t_z + \psi_{zt} \bar{t}_z = i \left[\psi_{zt} \frac{1 + \varrho}{2t} \psi_z - \psi_{zt} \frac{1 - \varrho}{2t} \psi_z \right] \\ &= \frac{i}{2} \psi_z \left[\frac{\psi_{zt}}{t} (1 + \varrho) - \frac{\psi_{zt}}{t} (1 - \varrho) \right]. \end{aligned}$$

Setting

$$(2.16) \quad v = i \log 2i\psi_z$$

we can write

$$(2.17) \quad 2i\psi_z = \exp[-iv], \quad -2i\psi_{\bar{z}} = \exp[i\bar{v}].$$

Hence

$$\psi_{zt} = -i\psi_z v_t, \quad \psi_{z\bar{t}} = -i\psi_z v_{\bar{t}}.$$

Substituting this in (2.14) and (2.15), we get

$$\begin{aligned} \psi_{z\bar{z}} &= \frac{i\psi_z}{2} \left[-\frac{i\psi_z v_t}{t} (1 - \varrho) + \frac{i\psi_z v_{\bar{t}}}{\bar{t}} (1 + \varrho) \right] = \frac{|\psi_z|^2}{2} \left[\frac{v_t}{t} (1 - \varrho) - \frac{v_{\bar{t}}}{\bar{t}} (1 + \varrho) \right], \\ \psi_{zz} &= \frac{i\psi_z}{2} \left[-\frac{i\psi_z v_t}{t} (1 + \varrho) + \frac{i\psi_z v_{\bar{t}}}{\bar{t}} (1 - \varrho) \right] = \frac{(\psi_z)^2}{2} \left[\frac{v_t}{t} (1 + \varrho) - \frac{v_{\bar{t}}}{\bar{t}} (1 - \varrho) \right]. \end{aligned}$$

Substituting these expressions into (2.1) we obtain

$$\begin{aligned} \kappa &= -\frac{|\psi_z|}{2} \left[\frac{v_t}{t} (1 - \varrho) - \frac{v_{\bar{t}}}{\bar{t}} (1 + \varrho) \right] + \frac{|\psi_z|}{2} \left[\frac{v_t}{t} (1 + \varrho) - \frac{v_{\bar{t}}}{\bar{t}} (1 - \varrho) \right] + \text{imag.} \\ &= 2\varrho \frac{|\psi_z|}{2} \left(\frac{v_t}{t} + \frac{v_{\bar{t}}}{\bar{t}} \right) + \text{imag.} \end{aligned}$$

Noting that

$$\left(\frac{v_t}{t} + \frac{v_{\bar{t}}}{\bar{t}} \right) - \left(\frac{\omega_t}{t} + \frac{\omega_{\bar{t}}}{\bar{t}} \right) = \frac{i}{\lambda} \left(\frac{\lambda_t}{t} + \frac{\lambda_{\bar{t}}}{\bar{t}} \right) = \frac{i}{\lambda} 2 \operatorname{Re} \left\{ \frac{\lambda_t}{t} \right\} = \text{imag.},$$

we conclude:

THEOREM 2.2. *There holds:*

$$(2.18) \quad \kappa = \varrho(z, 2|\psi_z|)|\psi_z| \left(\frac{\omega_t}{t} + \frac{\omega_{\bar{t}}}{\bar{t}} \right) + \text{imag.}$$

3. - Differential equation for ω .

From (2.2) we have

$$(3.1) \quad \lambda(z) \exp[-i\omega] = 2i\psi_z = \psi_y + i\psi_x,$$

$$(3.2) \quad \omega_2 = \log \frac{2|\psi_z|}{\lambda};$$

also

$$(3.3) \quad \begin{aligned} \psi_x &= \frac{\lambda}{2i} (\exp[-i\omega] - \exp[i\bar{\omega}]) \\ \psi_y &= \frac{\lambda}{2} (\exp[-i\omega] + \exp[i\bar{\omega}]) . \end{aligned}$$

The first equation for ω is derived from $\psi_{yx} = \psi_{xy}$:

$$(\lambda \exp[-i\omega] + \lambda \exp[i\bar{\omega}])_x - \frac{1}{i} (\lambda \exp[-i\omega] - \lambda \exp[i\bar{\omega}])_y = 0 ,$$

or

$$(3.4) \quad (\lambda \exp[-i\omega])_{\bar{z}} + (\lambda \exp[i\bar{\omega}])_z = 0 \quad \left(\text{or } \operatorname{Re} \{ (\lambda \exp[-i\omega])_{\bar{z}} \} = 0 \right) .$$

The second equation for ω is derived from the differential equation (1.5) for ψ :

$$(\varrho \lambda (\exp[-i\omega] - \exp[i\bar{\omega}]))_x + i (\varrho \lambda (\exp[-i\omega] + \exp[i\bar{\omega}]))_y = 0$$

or

$$(3.5) \quad (\varrho \lambda \exp[-i\omega])_{\bar{z}} - (\varrho \lambda \exp[i\bar{\omega}])_z = 0 \quad \left(\text{or } \operatorname{Im} \{ (\varrho \lambda \exp[-i\omega])_{\bar{z}} \} = 0 \right) .$$

Notice that by (3.2)

$$\varrho(z, 2|\psi_z|) = \varrho(z, \lambda(z) \exp[\omega_2]) = \exp[\sigma(z, \omega_2)]$$

if

$$(3.6) \quad \sigma(z, t) = \log \varrho(z, \lambda(z) \exp[t]) .$$

We also write

$$(3.7) \quad \lambda(z) = \exp[\mu(z)] .$$

Recall that by the normalization made in Remarks 1.1, 1.2,

$$(3.8) \quad \mu(0) = 0 , \quad \sigma(0, 0) = 0 ,$$

and that (1.18) becomes

$$(3.9) \quad \sigma_t(0, 0) < 1 .$$

We can now rewrite (3.4) in the form

$$\exp\mu - i\omega_{\bar{z}} + \exp\mu + i\bar{\omega}_z = 0,$$

or

$$(3.10) \quad \exp[-i\omega](\omega + i\mu)_{\bar{z}} - \exp[i\bar{\omega}](\bar{\omega} - i\mu)_z = 0$$

Similarly (3.5) becomes

$$(3.11) \quad \exp[-i\omega](\omega + i\mu + i\sigma(z, \omega_2))_{\bar{z}} + \exp[i\bar{\omega}](\bar{\omega} - i\mu - i\sigma(z, \omega_2))_z = 0.$$

Since the left hand side of (3.10) is real and that of (3.11) is imaginary, we obtain the complex differential equation for ω by adding (3.10) and (3.11). This gives

$$(3.12) \quad \left(\omega + i\mu + \frac{i}{2} \sigma(z, \omega_2) \right)_{\bar{z}} - \exp[2i\omega_1] \frac{i}{2} \sigma(z, \omega_2)_z = 0.$$

Denoting by σ' the derivative of $\sigma(z, \omega_2)$ with respect to the variable ω_2 , we have

$$\sigma(z, \omega_2)_z = \sigma_z + \sigma' \omega_{2,z},$$

$$\sigma(z, \omega_2)_{\bar{z}} = \sigma_{\bar{z}} + \sigma' \omega_{2,\bar{z}}.$$

Therefore (3.12) becomes

$$(3.13) \quad \omega_{\bar{z}} + \frac{i\sigma'}{2} \omega_{2,\bar{z}} - \exp[2i\omega_1] \frac{i\sigma'}{2} \omega_{2,z} = i \left(\exp[2i\omega_1] \frac{\sigma_z}{2} - \frac{\sigma_{\bar{z}}}{2} - \mu_{\bar{z}} \right).$$

Since, by (2.12),

$$\omega_{\bar{z}} = \omega_t \frac{f_{\bar{z}}}{2t} + \omega_{\bar{t}} \frac{\bar{f}_{\bar{z}}}{2\bar{t}},$$

$$\omega_z = \omega_t \frac{f_z}{2t} + \omega_{\bar{t}} \frac{\bar{f}_z}{2\bar{t}},$$

we obtain

$$\begin{aligned} \omega_{\bar{t}} \frac{\bar{f}_{\bar{z}}}{\bar{t}} + \omega_t \frac{f_{\bar{z}}}{t} + \frac{i\sigma'}{2} \left(\omega_{2,\bar{t}} \frac{\bar{f}_{\bar{z}}}{\bar{t}} + \omega_{2,t} \frac{f_{\bar{z}}}{t} \right) - \exp[2i\omega_1] \frac{i\sigma'}{2} \left(\omega_{2,\bar{t}} \frac{\bar{f}_z}{\bar{t}} + \omega_{2,t} \frac{f_z}{t} \right) \\ = i \left(\exp[2i\omega_1] \sigma_z - \sigma_{\bar{z}} - 2\mu_{\bar{z}} \right), \end{aligned}$$

or, noticing that $f_z \neq 0$ in a neighborhood of 0,

$$(3.14) \quad \omega_{\bar{t}} + \frac{\bar{t}}{t} \frac{\bar{f}_z}{f_z} \omega_t + \frac{i\sigma'}{2} \left(\omega_{2,\bar{t}} + \frac{\bar{t}\bar{f}_z}{t f_z} \omega_{2,t} \right) - \exp[2i\omega_1] \frac{i\sigma'}{2} \left(\frac{\bar{f}_z}{f_z} \omega_{2,\bar{t}} + \frac{\bar{t}\bar{f}_z}{t f_z} \omega_{2,t} \right) \\ = \frac{i\bar{t}}{\bar{f}_z} (\exp[2i\omega_1] \sigma_z - \sigma_{\bar{z}} - 2\mu_{\bar{z}}).$$

Now, by (2.8),

$$\frac{f_z}{\bar{f}_z} = -\frac{1-\varrho}{1+\varrho}.$$

Also, from the definition of ω ,

$$\omega_1 = \arg(-i\psi_z^-)$$

so that

$$\exp[2i\omega_1] = \left(\frac{-i\psi_z^-}{|\psi_z^-|} \right)^2 = -\frac{\psi_z^-}{\psi_z}.$$

Hence by (2.8),

$$\exp[2i\omega_1] \frac{\bar{f}_z}{f_z} = -\frac{\psi_z^-}{\psi_z} \cdot \frac{i(1-\varrho)\psi_z}{\psi_z - i(1+\varrho)\psi_z^-} = -\frac{1-\varrho}{1+\varrho}, \\ \exp[2i\omega_1] \frac{f_z}{\bar{f}_z} = -\frac{\psi_z^-}{\psi_z} \cdot \frac{i(1+\varrho)\psi_z}{\psi_z - i(1+\varrho)\psi_z^-} = 1.$$

Therefore (3.14) reduces to

$$(3.15) \quad \omega_{\bar{t}} + i \frac{\sigma'}{1+\varrho} \omega_{2,\bar{t}} - \frac{\bar{t}}{t} \left(\frac{1-\varrho}{1+\varrho} \omega_t + \frac{i\sigma'}{1+\varrho} \omega_{2,t} \right) \\ = \frac{i\bar{t}}{\bar{f}_z} (\exp[2i\omega_1] \sigma_z - \sigma_{\bar{z}} - 2\mu_{\bar{z}}) \text{ in } Q_1,$$

where Q_i denotes the i -th quadrant in R^2 . Since $\omega_2 = 0$ on the positive real axis (near the origin),

$$\omega(t) = \bar{\omega}(\bar{t})$$

defines a continuous extension of ω into Q_4 . We also let

$$z(t) = z(\bar{t}) \quad \text{for } t \in Q_4.$$

Since $\omega_{\bar{t}} = \overline{\omega_{\bar{t}}(t)}$ and $\omega_{2,\bar{t}}(t) = -\omega_{2,t}(\bar{t})$ (and similarly for t derivatives), ω satisfies in Q_4 an equation like (3.15) with the same left hand side and with the right hand side

$$(3.16) \quad -\frac{i\bar{t}}{f_z}(\exp[-2i\omega_1]\sigma_{\bar{z}} - \sigma_z - 2\mu_z) \quad (\text{in } Q_4).$$

Since $\omega_1 = 0$ on the imaginary axis, we extend ω into $Q_2 \cup Q_3$ by

$$\omega(t) = -\bar{\omega}(-\bar{t}) \quad \text{for } \operatorname{Re} t < 0.$$

Since $\omega_{\bar{t}}(t) = \overline{\omega_{\bar{t}}(-\bar{t})}$ and $\omega_{2,\bar{t}} = -\omega_{2,t}(-\bar{t})$, we see that ω satisfies in a neighborhood of 0 and equation like (3.15) with the same left hand side and with the right hand side

$$(3.17) \quad \frac{i\bar{t}}{f_z}(\exp[-2i\omega_1]\sigma_{\bar{z}} - \sigma_z - 2\mu_z) \quad \text{in } Q_2,$$

$$(3.18) \quad -\frac{i\bar{t}}{f_z}(\exp[2i\omega_1]\sigma_z - \sigma_{\bar{z}} - 2\mu_{\bar{z}}) \quad \text{in } Q_3.$$

4. - Initial estimates for $\nabla^i \omega$.

We recall that ω is a continuous function in a neighborhood of $t = 0$, satisfying

$$(4.1) \quad L\omega \equiv \omega_{\bar{t}} + \frac{i\sigma'}{1+\varrho} \omega_{2,\bar{t}} - \frac{\bar{t}}{t} \left(\frac{1-\varrho}{1+\varrho} \omega_t + \frac{i\sigma'}{1+\varrho} \omega_{2,t} \right) = \Phi(t, \omega)$$

where

$$(4.2) \quad \Phi(t, \omega) = \bar{t}\Phi_0(t, \omega)$$

and

$$(4.3) \quad \begin{aligned} \Phi_0 &= \frac{i}{f_z}(\exp[2i\omega_1]\sigma_z - \sigma_{\bar{z}} - 2\mu_z) && \text{in } Q_1, \\ &= -\frac{i}{f_z}(\exp[-2i\omega_1]\sigma_{\bar{z}} - \sigma_z - 2\mu_z) && \text{in } Q_4, \\ &= \frac{i}{f_z}(\exp[-2i\omega_1]\sigma_{\bar{z}} - \sigma_z - 2\mu_z) && \text{in } Q_2, \\ &= -\frac{i}{f_z}(\exp[2i\omega_1]\sigma_z - \sigma_{\bar{z}} - 2\mu_z) && \text{in } Q_3. \end{aligned}$$

L has continuous coefficients except for the factor $\bar{t}/t$ in front of the t -derivative.

All the estimates below are for z near $z = 0$ and t near $t = 0$.

LEMMA 4.1. *There holds:*

$$(4.4) \quad |\nabla_z^j \psi(z)| \leq \frac{C}{|z|^{j-1}} \quad (j = 2, 3)$$

and, consequently,

$$(4.5) \quad |\nabla_t^j \omega(t)| \leq \frac{C}{|t|^j} \quad (j = 1, 2).$$

PROOF. Let $\psi(z_0) > 0$. Consider the scale function

$$\psi_r(z) = \frac{1}{r} \psi(z_0 + rz) \quad \left(r = \frac{|z_0|}{4}\right) > 0$$

in the unit disc B_1 : Since $|\nabla \psi| \leq C$ and $\psi(0) = 0$, we have

$$|\psi_r| \leq C.$$

If $\psi_r > 0$ in B_1 then by interior elliptic estimates for the uniformly elliptic equation (1.5) we get

$$(4.6) \quad |\nabla_z^j \psi_r| \leq C \text{ in } B_{\frac{1}{2}} \quad (j = 2, 3).$$

If B_1 intersects the scaled N , then it cannot intersect the free boundary. We then apply interior-boundary elliptic estimates in order to deduce (4.6) noting that ϱ is in $C^{2,\alpha}$ (by (1.13)). Finally it remains to consider the case where B_1 intersects the scaled Γ . By a regularity theorem for free boundaries [5] the free boundary is a uniformly $C^{3,\alpha}$ graph and we again deduce (4.6). Notice finally that (4.4) follows from (4.6) with $z = 0$.

To prove (4.5) observe that by (2.2), (2.12),

$$(4.7) \quad |\nabla_t \omega| \leq C(|\nabla_z \psi_z| + 1)|\nabla_t z| \leq C|t|(|\nabla_z \psi(z)| + 1).$$

Substituting t from (2.10) into (4.7), (4.5) follows for $j = 1$. The proof for $j = 2$ is similar.

Consider a system

$$(4.8) \quad M\omega = \omega_{\bar{t}} + \mu\omega_t + \nu\bar{\omega}_{\bar{t}} + \alpha\omega + \beta\bar{\omega} = \gamma \quad \text{in } B_1,$$

with measurable coefficients satisfying

$$(4.9) \quad |\mu| + |\nu| \leq \theta < 1 \quad (\text{ellipticity}),$$

$$(4.10) \quad |\alpha| + |\beta| \leq C_1, \quad \gamma \in L^q(B_1).$$

LEMMA 4.2. *There exists a number $p \in (2, \infty)$ depending only on θ such that if $q \geq p$ then for any solution ω in $C^0(B_1) \cap H^{1,2}(B_1)$ of (4.8), and for any disc B_r with $r < 1$ there holds: $\omega \in W^{1,p}(B_r)$ and*

$$(4.11) \quad |\omega|_{W^{1,p}(B_r)} \leq C(|\gamma|_{L^p(B_1)} + |\omega|_{L^p(B_1)}),$$

where C depends only on r, θ and C_1 ; further, $p \rightarrow \infty$ if $\theta \rightarrow 0$ and $p \rightarrow 2$ if $\theta \rightarrow 1$.

PROOF. Let $\eta \in C_0^1(B_1)$, $\eta = 1$ in B_r and set $v = \eta\omega$. Then

$$Mv = \lambda\eta + \omega\eta_{\bar{t}} + \omega\mu\eta_t + \bar{\omega}\nu\eta_{\bar{t}} \equiv \tilde{\gamma}.$$

The solution v can be expressed in terms of solutions of simpler equations

$$(4.12) \quad u_{\bar{t}} = \tilde{\mu}u_t + \tilde{\nu}\tilde{\gamma}$$

with $|\tilde{\mu}| \leq \theta$ and $\tilde{\mu}, \tilde{\nu}$ depending on μ, ν, α, β ; $|\tilde{\nu}| \leq C$ (see [9; pp. 260-266]). We now apply Theorem 1 of [9; p. 269] to u .

REMARK 4.1. The system (4.1) can easily be written as an elliptic system

$$(4.13) \quad \begin{aligned} u_{t_1} &= a_{11}v_{t_1} + a_{12}v_{t_2} + \Psi_1(t, u, v), \\ -u_{t_2} &= a_{21}v_{t_1} + a_{22}v_{t_2} + \Psi_2(t, u, v), \end{aligned}$$

i.e., $a_{12} > 0$ and $4a_{12}a_{21} - (a_{11} + a_{22})^2 > 0$ (cf. (5.7), (5.8) below). Hence, by [9; p. 257] this system can also be written in the form (4.8) (with (4.9) (4.10)). Thus Lemma 4.2 applies, in principle, to the system (4.1). However the estimate (4.5) is not strong enough to deduce that $\omega \in H^{1,2}$ in a neighborhood of 0. Thus we cannot deduce at this point that $\omega \in W^{1,p}$ (and, consequently, that $\omega \in C^\beta$ for some $\beta > 0$). We shall therefore proceed in a different way to establish these regularity properties of ω .

We begin by some improvement of (4.5).

LEMMA 4.3. *There holds:*

$$(4.14) \quad |\nabla_t \omega(t)| \leq \frac{C}{|t|} \max_{|t|/2 \leq |t'| \leq 2|t|} |\omega(t')| + C.$$

PROOF. Consider the scaled function

$$\tilde{\omega}(t) = \omega(rt) \quad \text{in } R \equiv \left\{ \frac{1}{3} < |t| < 3 \right\}.$$

It satisfies

$$(4.15) \quad \tilde{L}\tilde{\omega} = r\tilde{\Phi}$$

where $\tilde{\Phi}$ and the coefficients of $\tilde{L}$ are the same as Φ and the coefficients of L (in (4.1)) evaluated at rt . By Lemma 4.2,

$$(4.16) \quad |\tilde{\omega}|_{W^{1,p}(R')} \leq C|\tilde{\omega}|_{L^\infty(R)} + Cr$$

where R' is any subring of R ; C depends on R' .

Let us now consider $\tilde{\omega}(t)$ near the positive real axis $t_1 > 0$ (where $t = t_1 + it_2$). Differentiating (4.15) with respect to t_1 we obtain

$$(4.17) \quad \begin{aligned} \tilde{L}(D_{t_1}\tilde{\omega}) = & -i \left(\frac{\sigma'}{1+\varrho} \right)' \tilde{\omega}_{2,\bar{i}}(t) \tilde{\omega}_{2,t_1}(t) \\ & - i \nabla_z \left(\frac{\sigma'}{1+\varrho} \right) \nabla_{rt} z(rt) D_{t_1}(rt) \tilde{\omega}_{2,\bar{i}}(t) \\ & + D_{t_1} \left(\frac{\bar{t}}{t} \right) \left[\frac{1-\varrho(z, \omega_2(rt))}{1+\varrho(z, \omega_2(rt))} \tilde{\omega}_t(t) + \frac{i\sigma'(z, \omega_2(rt))}{1+\varrho(z, \omega_2(rt))} \tilde{\omega}_{2,t}(t) \right] \\ & + \frac{\bar{t}}{t} \left(\frac{1-\varrho}{1+\varrho} \right)' \tilde{\omega}_t(t) \tilde{\omega}_{2,t_1}(t) + \frac{\bar{t}}{t} \nabla_z \left(\frac{1-\varrho}{1+\varrho} \right) \nabla_{rt} z(rt) D_{t_1}(rt) \tilde{\omega}_t(t) \\ & + A_1 + A_2 + rD_{t_1}\tilde{\Phi} \equiv \sum_{k=1}^5 J_k + A_1 + A_2 + rD_{t_1}\tilde{\Phi}. \end{aligned}$$

Here the terms A_1, A_2 are similar to J_4, J_5 , and

$$\left(\frac{\sigma'}{1+\varrho} \right)', \quad \left(\frac{1-\varrho}{1+\varrho} \right)'$$

indicate derivatives with respect to the last variable; one evaluates these derivatives at the point corresponding to rt .

It will be useful to express $z_t, z_{\bar{t}}$ in terms of t . From the relations

$$z_t t_z + z_{\bar{t}} \bar{t}_z = 1, \quad z_t t_{\bar{z}} + z_{\bar{t}} \bar{t}_{\bar{z}} = 0$$

we obtain

$$z_t = \frac{\bar{t}_{\bar{z}}}{D}, \quad z_{\bar{t}} = -\frac{t_{\bar{z}}}{D}$$

where $D = |t_z|^2 - |\bar{t}_z|^2$, and using (2.12) we get

$$(4.18) \quad \begin{aligned} z_t &= 2t \frac{\bar{f}_z}{|f_z|^2 - |\bar{f}_z|^2}, \\ z_{\bar{t}} &= -2\bar{t} \frac{f_z}{|f_z|^2 - |\bar{f}_z|^2}. \end{aligned}$$

The first term J_1 is $\tilde{\omega}_{2,t_1}$ times a bounded coefficient, since

$$|\tilde{\omega}_{2,\bar{t}}(t)| = r|\omega_{2,\bar{t}}(rt)| \leq C$$

by Lemma 4.1. Next, by (4.18),

$$|J_2| \leq Cr|t| \cdot r|\tilde{\omega}_{2,t}(t)| \leq Cr^2,$$

where Lemma 4.1 was again used. Next,

$$|J_3| \leq C(|\tilde{\omega}_t| + |\tilde{\omega}_{2,t}|).$$

J_4 , like J_1 , is equal to $\tilde{\omega}_{2,t_1}$ times a bounded coefficient, and J_5 , like J_2 is bounded by Cr^2 .

The terms A_1, A_2 are evaluated as J_4, J_5 . Thus it remains to consider $rD_{t_1}\tilde{\Phi}$. This consists of terms $r^2D_{t_1}\tilde{\omega}_j$ times bounded coefficients coming from differentiating $\exp[2i\omega_1]$, σ and f_z , and terms involving derivatives of $\sigma_z, \sigma_{\bar{z}}, \mu_z, \mu_{\bar{z}}$ with respect to $z, \bar{z}$. Since t_1 is a tangential direction, these derivatives are bounded functions.

Putting together the above estimates for the right hand side of (4.17), we deduce that Lemma 4.2 can be applied to yield

$$(4.19) \quad |D_{t_1}\tilde{\omega}|_{W^{1,p}(R')} \leq C|\tilde{\omega}|_{L^\infty(R)} + Cr.$$

Actually the same proof is valid also in any subset of R which stays away from the real and imaginary axis. Hence, in particular, (4.19) holds if

$$R'' = R_0 \cap \{|t_2| \leq C|t_1|, t_1 > 0\}$$

where R_0 is any subring of R ; this gives

$$(4.20) \quad |D_{t_1}\omega(t)| \leq \frac{C}{|t|} \max_{|t|/2 \leq |t'| \leq 2|t|} |\omega(t')| + C, \quad t \in R^*$$

where

$$R^* = \{t = t_1 + it_2, |t_2| < C|t_1|, t_1 > 0\}.$$

From (4.1) we can express ω_{t_2} in terms of ω_{t_1} :

$$\begin{aligned} (4.21) \quad \omega_{t_1} \left(i + i \frac{\bar{t}}{t} \frac{1-\varrho}{1+\varrho} \right) + i \omega_{2,t_2} \left(\frac{i\sigma'}{1+\varrho} + \frac{\bar{t}}{t} \frac{i\sigma'}{1+\varrho} \right) \\ = -\omega_{t_1} \left(1 - \frac{\bar{t}}{t} \frac{1-\varrho}{1+\varrho} \right) - \left(\frac{i\sigma'}{1+\varrho} - \frac{\bar{t}}{t} \frac{i\sigma'}{1+\varrho} \right) \omega_{2,t_1} + 2\Phi; \end{aligned}$$

since $\sigma'(0, 0) \geq 0$ we can indeed solve for ω_{2,t_2} and ω_{1,t_2} in terms of the right hand side. Using (4.20) we then conclude that also $D_{t_1}\omega$ can be estimated by the right hand side of (4.20). Since the same estimate can be derived with respect to the negative t_1 -axis and the imaginary axis, the proof of Lemma 4.3 is complete.

LEMMA 4.4. *There holds:*

$$(4.22) \quad |\nabla_{t_1}^2 \omega(t)| \leq \frac{C}{|t|^2} \max_{|t|/2 < |t'| < 2|t|} |\omega(t')| + \frac{C}{|t|}.$$

PROOF. Apply D_{t_1} to equation (4.17). Examining each of the terms, as we did for the right hand side of (4.17), and making use of Lemma 4.3 and of (4.19), we find that Lemma 4.2 can be applied to yield

$$|D_{t_1}^2 \omega|_{W^{1,p}(R^*)} \leq C \frac{1}{r} |\tilde{\omega}|_{L^\infty(R)} + C$$

in appropriate regions R'' , R , thus giving (cf. (4.20))

$$|D_{t_1}^2 \omega(t)| \leq \frac{C}{|t|^2} \max_{|t|/2 \leq |t'| \leq 2|t|} |\omega(t')| + \frac{C}{|t|}$$

for any $t \in R^*$. Applying D_{t_1} to (4.21) and solving for $D_{t_1}\omega_1$, $D_{t_1}\omega_2$ we obtain also the estimate

$$(4.24) \quad |D_{t_1} D_{t_2} \omega(t)| \leq \frac{C}{|t|^2} \max_{|t|/2 \leq |t'| \leq 2|t|} |\omega(t')| + \frac{C}{|t|}.$$

Finally we apply D_{t_1} to (4.21) and use (4.24) in order to estimate $D_{t_1}^2 \omega$ in the same way.

5. - Initial regularity for ω ; reduction to a second order elliptic equation.

In equation (4.1) the leading coefficients of ω_t and $\omega_{2,t}$ are discontinuous at the origin $t = 0$, with discontinuity of the type $\bar{t}/t$. We are interested in «transforming away» this discontinuity so that we can work with elliptic equations with smooth coefficients. An appropriate transformation is given by the conformal mapping

$$(5.1) \quad t = e^s, \quad s = \xi + i\eta,$$

which maps $B_1 \cap Q$, in the t -plane into the half strip $\{-\infty < \xi < 0, 0 < \eta < \pi/2\}$. Clearly

$$(5.2) \quad t\omega_t = \omega_s, \quad \bar{t}\omega_{\bar{t}} = \omega_{\bar{s}},$$

and (4.1) becomes

$$(5.3) \quad \hat{L}\omega \equiv \omega_{\bar{s}} + \frac{i\sigma'}{1+\varrho} \omega_{2,\bar{s}} - \left(\frac{\bar{t}}{t}\right)^2 \left(\frac{1-\varrho}{1+\varrho} \omega_s + \frac{i\sigma'}{1+\varrho} \omega_{2,s}\right) = \bar{t}^2 \Phi_0.$$

Let

$$(5.4) \quad L_0\omega = \omega_{\bar{s}} + i\theta\omega_{2,\bar{s}} - \left(\frac{\bar{t}}{t}\right)^2 i\theta\omega_{2,s}$$

where

$$\theta = \frac{\sigma'(0,0)}{2} \quad \left(0 \leq \theta < \frac{1}{2}\right).$$

Then we can rewrite (5.3) in the form

$$(5.5) \quad L_0\omega = F_0$$

where

$$(5.6) \quad F_0 = \bar{t}^2 \Phi_0 + F_0^*(t, \omega, \nabla_s \omega)$$

and the coefficients of $\nabla_s \omega$ in F_0^* converge to 0 if $t \rightarrow 0$. Since

$$\left(\frac{\bar{t}}{t}\right)^2 = \exp[-4i\eta] = C - iS$$

where $C = \cos 4\eta$, $S = \sin 4\eta$, (5.5) can be written in the form

$$\omega_{1,\xi} + i\omega_{1,\eta} + i\omega_{2,\xi} - \omega_{2,\eta} + i\theta\omega_{2,\xi} - \theta\omega_{2,\eta} + (-iC\theta - S\theta)(\omega_{2,\xi} - i\omega_{2,\eta}) = 2F_0,$$

or

$$(5.7) \quad \omega_{1,\xi} - (1 + \theta + \theta C)\omega_{2,\eta} - \theta S\omega_{2,\xi} = F_1,$$

$$(5.8) \quad \omega_{1,\eta} + (1 + \theta - \theta C)\omega_{2,\xi} + \theta S\omega_{2,\eta} = F_2$$

where

$$2F_0 = F_1 + iF_2.$$

Applying $\partial/\partial\eta$ to (5.7), $\partial/\partial\xi$ to (5.8) and subtracting, we obtain

$$(5.9) \quad \begin{aligned} \mathfrak{L}\omega_2 &\equiv ((1 + \theta - \theta C)\omega_{2,\xi} + \theta S\omega_{2,\eta})_\xi \\ &\quad + ((1 + \theta + \theta C)\omega_{2,\eta} + \theta S\omega_{2,\xi})_\eta = F_{2,\xi} - F_{1,\eta}. \end{aligned}$$

Next we solve from (5.7), (5.8) for $\omega_{2,\xi}, \omega_{2,\eta}$:

$$\omega_{2,\xi} = \frac{1}{1 + 2\theta} \{-(1 + \theta + \theta C)\omega_{1,\eta} - \theta S\omega_{1,\xi} + (1 + \theta + \theta C)F_2 + \theta SF_1\},$$

$$\omega_{2,\eta} = \frac{1}{1 + 2\theta} \{(1 + \theta - \theta C)\omega_{1,\xi} + \theta S\omega_{1,\eta} - (1 + \theta - \theta C)F_1 - \theta SF_2\}.$$

Hence the relation $\partial\omega_{2,\xi}/\partial\eta = \partial\omega_{2,\eta}/\partial\xi$ gives

$$(5.10) \quad \mathfrak{L}\omega_1 = F_3$$

where F_3 is a linear combination of F_1, F_2 and their first derivatives, with coefficients involving C, S .

Note that the coefficients of $\mathfrak{L}$ are smooth; in fact, this will be of crucial importance later on.

Note also that

$$(5.11) \quad \omega_2 = 0 \quad \text{on } \{\eta = 0\},$$

$$(5.12) \quad \omega_1 = 0 \quad \text{on } \left\{\eta = \frac{\pi}{2}\right\}.$$

From (5.7), (5.8) we also infer the Neumann conditions

$$(5.13) \quad -\omega_{2,\eta}(1 + \theta + \theta C) = F_1 \quad \text{on } \left\{\eta = \frac{\pi}{2}\right\},$$

$$(5.14) \quad \omega_{1,\eta} = F_2 \quad \text{on } \{\eta = 0\}.$$

We proceed to estimate F_1, F_2 . Notice that F_0^* contains perturbation terms such as

$$(1 - \varrho)\omega_s, \quad \left(\frac{\sigma'}{1 + \varrho} - \frac{\sigma'(0, 0)}{1 + \varrho(0, 1)} \right) \omega_{2, \bar{s}}.$$

The first term is bounded by

$$|\varrho'| |\omega_2 \omega_s| + |\nabla_z \varrho| |z \omega_s|$$

and the second term is estimated in the same way. Since $|z| \leq C|t|^2$, and since also $|\bar{t}^2 \Phi_0| \leq C|t|^2$, we conclude from (5.6) that

$$|F_0| \leq C|t|^2 + C|\omega| |\omega_s| + C|t|^2 |\omega_s|.$$

By Lemma 4.3

$$|\omega|_s = |t| |\omega_t(t)| \leq C \max_{|t|/2 < |t'| < 2|t|} |\omega(t')| + C|t|$$

and $\omega(r') \rightarrow 0$ if $|t'| \rightarrow 0$. Since

$$|t| = \exp[\xi] = \exp[-|\xi|],$$

we deduce that

$$(5.15) \quad \begin{aligned} F_0 &= a_0 \omega + O(\exp[2\xi]), \\ a_0 &\rightarrow 0 \quad \text{if } t \rightarrow 0 \text{ (i.e., if } \xi \rightarrow -\infty). \end{aligned}$$

Hence the Neumann condition (5.13) becomes

$$(5.16) \quad \begin{aligned} \omega_{2,\eta} + a_1 \omega_2 &= O(\exp[2\xi]) \quad \text{on } \left\{ \eta = \frac{\pi}{2} \right\}, \\ a_1(\xi) &\rightarrow 0 \quad \text{if } \xi \rightarrow -\infty. \end{aligned}$$

Similarly, the Neumann condition (5.14) becomes

$$(5.17) \quad \begin{aligned} \omega_{1,\eta} + a_2 \omega_1 &= O(\exp[2\xi]) \quad \text{on } \{\eta = 0\}, \\ a_2(\xi) &\rightarrow 0 \quad \text{if } \xi \rightarrow -\infty. \end{aligned}$$

We next turn to the differential equation (5.9) and proceed to estimate the right hand side. Consider a typical term that appears when we diffe-

rentiate the perturbation part F_0^* of F_0 with respect to s :

$$\frac{\partial}{\partial s} ((1 - \varrho) \omega_s) = -\varrho' \omega_{2,s} \omega_s - \nabla_z \varrho \nabla_t z t_s \omega_s + (1 - \varrho) \omega_{ss}.$$

By Lemma 4.3

$$|\omega_s| = |t \omega_t| \leq C \left(\max_{|t|/2 \leq |t'| \leq |t|} |\omega(t')| + |t| \right)$$

and $\omega(t') \rightarrow 0$ if $t' \rightarrow 0$. Also

$$|\nabla_z \varrho \nabla_t z t_s \omega_s| \leq C |t|^2 |\omega_s| \leq C |t|^2$$

since $|\nabla_t z| \leq C |t|$, $t_s = t$ and $|\omega_s| = |t \omega_t| \leq C$ (by Lemma 4.1).

Finally,

$$\begin{aligned} (1 - \varrho) \omega_{ss} &= |(1 - \varrho) t (t \omega_t)_t| \leq C (|z| + |\omega_2|) (|t^2 \omega_{tt} + t \omega_t|) \\ &\leq C |t|^2 + C |\omega_2| \left(\max_{|t|/2 \leq |t'| \leq |t|} |\omega(t')| + C |t| \right) \end{aligned}$$

by Lemmas 4.1-4.3 and $\max |\omega(t')| \rightarrow 0$ if $t' \rightarrow 0$. We conclude that

$$\partial_s ((1 - \varrho) \omega_s) = O(\exp[2\xi]) + A_1 \omega_2 + A_2 \omega_{2,s}$$

where $A_i \rightarrow 0$ if $\xi \rightarrow -\infty$.

The other terms in the derivative of F_0^* are estimated in the same way. Finally, we can estimate

$$\frac{\partial}{\partial s} \bar{t}^2 \Phi_0 \text{ by } C \exp[2\xi].$$

We conclude that

$$\begin{aligned} (5.18) \quad \mathfrak{L} \omega_2 + A_1 \omega_2 + A_2 \nabla_s \omega_2 &= O(\exp[2\xi]), \\ A_1 \rightarrow 0, \quad A_2 \rightarrow 0 &\text{ if } \xi \rightarrow -\infty. \end{aligned}$$

Similarly (since the coefficients depend only on ω_2)

$$\begin{aligned} (5.19) \quad \mathfrak{L} \omega_1 &= A_3 \omega_2 + A_4 \nabla_s \omega_2 = O(\exp[2\xi]), \\ A_3 \rightarrow 0, \quad A_4 \rightarrow 0 &\text{ if } \xi \rightarrow -\infty. \end{aligned}$$

Setting

$$T = \left\{ \xi + i\eta; -\infty < \xi < 0, 0 < \eta < \frac{\pi}{2} \right\}$$

we can summarize:

LEMMA 5.1. (i) ω_2 satisfies (5.18) in T with the boundary conditions (5.11), (5.16); (ii) ω_1 satisfies (5.19) in T with the boundary conditions (5.12), (5.17).

We wish to use Lemma 5.1 together with a comparison argument in order to estimate the decay of ω at $\xi = -\infty$. In order to construct comparison functions we try to find solutions

$$u = \exp[\lambda\xi]g(\eta), \quad \lambda \geq 0$$

of $\mathfrak{L}u = 0$. This leads to the equation

$$(5.20) \quad \begin{aligned} M_\lambda g &= [(1 + \theta - \theta C)\lambda^2 + 4\theta C\lambda]g + (2\theta\lambda S - 4\theta S)g' \\ &\quad + (1 + \theta + \theta C)g'' = 0 \end{aligned}$$

for g .

LEMMA 5.2. If $\lambda_1 \geq 0$, $M_{\lambda_1}g = 0$ for $\eta_1 \leq \eta \leq \eta_2$ and $g(\eta_1) = g(\eta_2) = 0$, $g(\eta) > 0$ in (η_1, η_2) , then any solution $\tilde{g}$ of $M_{\lambda_2}\tilde{g} = 0$ in (η_1, η_2) , with $\lambda_2 > \lambda_1$, must have at least one zero in the interval (η_1, η_2) .

PROOF. Consider the functions

$$u = \exp[\lambda_1\xi]g, \quad v = \exp[\lambda_2\xi]\tilde{g}.$$

If the assertion is not true then we may assume that $\tilde{g} > 0$ in (η_1, η_2) , so that for some constant $C > 0$, $Cv \geq u$ on the boundary of T . For any $\varepsilon > 0$ the function $h = Cv - u + \varepsilon$ is nonnegative on the boundary of $T \cap \{\xi > -M_\varepsilon\}$ (here $M_\varepsilon \rightarrow \infty$ if $\varepsilon \rightarrow 0$) and $\mathfrak{L}h = 0$. By the maximum principle we get $h > 0$, or (as $\varepsilon \rightarrow 0$)

$$C \exp[\lambda_2\xi]\tilde{g} \geq \exp[\lambda_1\xi]g$$

which is impossible if $\xi \rightarrow -\infty$ (since $\lambda_2 > \lambda_1$).

Consider the eigenvalue problem

$$(5.21) \quad \begin{aligned} M_\lambda g &= 0 & \text{if } 0 < \eta < \pi, \\ g(0) &= 0, & g(\pi) = 0. \end{aligned}$$

If $\lambda = 0$ then the equation becomes

$$-4\theta Sg' + (1 + \theta + \theta C)g'' = 0.$$

If a solution g of this equation with $g(0) = 0$, $g'(0) > 0$ has a positive maximum, say at $\eta = \eta_0$, then $(g - g(\eta_0))^{(j)}(\eta_0) = 0$ for $j = 0, 1$, so that, by uniqueness,

$$g - g(\eta_0) \equiv 0;$$

a contradiction.

Thus for λ small (5.21) has no eigenvalues. Lemma 5.2 can now be used to establish (as in the classical Sturm-Liouville theory) the existence of a sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 < \dots$$

and eigenfunctions $g_1, g_2, \dots$, such that the n -th eigenfunction has precisely $n - 1$ zeros in the interval $(0, \pi)$. One can easily check that

$$(5.22) \quad \lambda_2 = 2, \quad g_2(\eta) = \sin 2\eta;$$

LEMMA 5.3. $\lambda_1 = 1$.

PROOF ⁽¹⁾. Let g be a solution of

$$M_1 g = 0 \quad \text{if } 0 < \eta < \pi, \quad g(0) = 0, \quad g'(0) \neq 0.$$

It suffices to show that $g(\pi) = 0$. Using the identities

$$\sin 4x \sin kx = \frac{1}{2} [\cos (k - 4)x - \cos (k + 4)x]$$

$$\cos 4x \sin kx = \frac{1}{2} [\sin (k + 4)x + \sin (k - 4)x]$$

in the relation

$$\int_0^\pi \sin kx M_1 g(x) dx = 0 \quad (k \text{ positive integer})$$

and integrating by parts, we obtain the following recursive relations for the

⁽¹⁾ The proof is due to Joel Friedman.

Fourier coefficients $a_m = \int_0^\pi g(x) \sin mx \, dx$:

$$(5.23) \quad \begin{aligned} \theta a_{k+4} \left[\frac{(k+4)^2}{2} - (k+4) - \frac{3}{2} \right] + (1+\theta) a_k [k^2 - 1] \\ + \theta a_{k-4} \left[\frac{(k-4)^2}{2} + (k-4) - \frac{3}{2} \right] = k(1+2\theta)g(\pi)(-1)^{k+1}. \end{aligned}$$

For $k=1$ and $k=5$ this gives

$$a_5 = \frac{1+2\theta}{6\theta} g(\pi),$$

$$30\theta a_9 + 24(1+\theta)a_5 = 5(1+2\theta)g(\pi).$$

Suppose now that $g(\pi) \neq 0$. Then we may normalize g so that $g(\pi) = 1$ and then deduce that $|a_5| > 2$, $|a_9/a_5| > 2$, provided θ is small enough. We can now proceed to prove inductively that

$$(5.24) \quad \left| \frac{a_k}{a_{k-4}} \right| \geq 2 \quad \text{for } k = 1 + 4m, \quad m \geq 3.$$

Indeed, from (5.23) we obtain

$$\left| \frac{a_{k+4}}{a_k} \right| \geq \frac{1}{\theta} - \left| \frac{a_{k-4}}{a_k} \right| - \frac{3}{k|a_k|\theta} \quad (k \geq 9),$$

so that the assertion follows if θ is small enough (independently of k). We conclude from (5.24) that $a_k \rightarrow \infty$ if $k = 1 + 4m \rightarrow \infty$, a contradiction.

Denote $g(\eta)$ also by $g_\theta(\eta)$.

Having proved that $g_\theta(\pi) = 0$ if θ is small enough, we now appeal to the fact that $g_\theta(\pi)$ is analytic in θ in order to conclude that $g_\theta(\pi) = 0$ if $0 \leq \theta < \frac{1}{2}$.

LEMMA 5.4. *For some positive and sufficiently small μ there holds:*

$$(5.25) \quad |\omega(s)| \leq C \exp[\mu\xi] \quad (C \text{ constant}).$$

PROOF. For $\mu > 0$ and $\delta > 0$ denote by $g_{\mu,\delta}$ the solution of

$$\begin{aligned} M_\mu g &= -\delta \quad \text{in } 0 \leq \eta \leq \pi/2, \\ g(0) &= 0, \quad g'(0) = 1. \end{aligned}$$

By an argument given above

$$(5.26) \quad g'_{\mu,\delta}(\eta) \geq C > 0 \quad \text{if } 0 \leq \eta \leq \pi/2$$

for some constant $C > 0$, provided μ and δ are small enough. It follows that the function

$$u = \exp[\mu\xi]g_{\mu,\delta}(\eta)$$

satisfies

$$\mathfrak{L}u = -\delta \exp[\mu\xi] \quad \text{in } T.$$

Consequently by (5.18)

$$\mathfrak{L}u + A_1u + A_2\nabla_s u \leq -\frac{\delta}{2} \exp[\mu\xi] \quad \text{in } T \cap \{\xi \leq -\xi_0\}$$

where ξ_0 is a sufficiently large constant. Also by (5.26), we can choose ξ such that

$$u_\eta + a_1u \geq c_1 \exp[\mu\xi] \quad \text{if } \eta = \frac{\pi}{2}, \quad \xi \leq -\xi_0$$

where a_1 is as in (5.16). Since finally $u(\xi, 0) \geq 0$, we can compare $Cu + \varepsilon$ with ω_2 ; C is chosen such that $Cu > |\omega_2|$ on $\xi = -\xi_0$ and ε is any positive number. The maximum principle yields $\omega_2 \leq Cu + \varepsilon$ and, taking $\varepsilon \rightarrow 0$, we obtain

$$(5.27) \quad |\omega_2(s)| \leq C \exp[\mu\xi].$$

To estimate $\omega_1(s)$ we proceed a little differently since the equation (5.19) for ω_1 involves ω_2 and $\nabla_s \omega_2$.

Consider the function $U = \omega_1^2 + \omega_2^2$. It satisfies

$$\begin{aligned} \mathfrak{L}U + 2\omega_1(A_1\omega_2 + A_2\nabla_s \omega_2) + 2\omega_2(A_3\omega_2 + A_4\nabla_s \omega_2) \\ = 2 \sum a_{ij}\omega_{k,i}\omega_{k,j} + O(\exp[2\xi]) \end{aligned}$$

where a_{ij} are the leading coefficients in $\mathfrak{L}$. Since (a_{ii}) is uniformly positive whereas the A_i converge to zero as $\xi \rightarrow -\infty$, we easily obtain

$$\mathfrak{L}U + \tilde{a}U \geq O(\exp[2\xi]) \quad (\tilde{a} \rightarrow 0 \text{ if } \xi \rightarrow -\infty)$$

if $\xi \leq -\xi_0$. Next, on $\{\eta = 0\}$,

$$U_\eta = \omega_1\omega_{1,\eta} = -a_2\omega_1^2 + O(\exp[2\xi]) = -a_2U + O(\exp[2\xi])$$

and on $\{\eta = \pi/2\}$

$$U = \omega_2^2 = O(\exp[2\mu\xi]), \quad \text{by (5.27).}$$

Hence we can apply the proof of (5.27) with a smaller μ ; this completes the proof of Lemma 5.4.

COROLLARY 5.5. $\omega(t)$ belongs to C^β (for some $0 < \beta < 1$) in a neighborhood of $t = 0$.

Indeed, Lemma 5.4 gives

$$(5.28) \quad |\omega(t)| \leq C|t|^\mu,$$

and together with (4.14) the assertion follows.

Lemma 5.4 allows us to estimate the coefficients A_i and ω_s in (5.18), (5.19). Indeed, recall that

$$\begin{aligned} |A_s(\xi + i\eta)| &\leq C \max_{\substack{|\xi - \xi'| \leq 2 \\ 0 \leq \eta' \leq \pi/2}} |\omega(\xi' + i\eta')| + C|t| \\ &\leq C \exp[\mu\xi] \quad (\text{by Lemma 5.4}), \end{aligned}$$

and, by Lemma 4.3,

$$\begin{aligned} |\omega_s(\xi + i\eta)| &= |t\omega_t| \leq C \max_{\substack{|\xi - \xi'| \leq 2 \\ 0 \leq \eta' \leq \pi/2}} |\omega(\xi' + i\eta')| + C|t| \\ &\leq C \exp[\mu\xi]. \end{aligned}$$

It follows that

$$(5.29) \quad \mathfrak{L}\omega_i = O(\exp[2\mu\xi]) + O(\exp[2\xi]).$$

Similarly, we have

$$(5.30) \quad \omega_{2,n} = G_2(\xi) \quad \text{on } \left\{ \eta = \frac{\pi}{2} \right\},$$

$$(5.31) \quad \omega_{1,\eta} = G_1(\xi) \quad \text{on } \{\eta = 0\}$$

where

$$(5.32) \quad D_\xi^j G_i = O(\exp[2\mu\xi]) + O(\exp[2\xi]) \quad (j = 0, 1);$$

the proof of (5.32) for $j = 1$ requires Lemma 4.4.

The above argument is valid for any $\mu > 0$, provided one knows that

$$(5.33) \quad |\omega(s)| \leq C \exp [\mu \xi].$$

We can thus assert:

LEMMA 5.6. *If (5.33) holds for some $\mu > 0$ then the ω_i satisfy (5.29) and (5.30), (5.31) where the G_i satisfy the estimates in (5.32).*

6. - Asymptotic expansion.

We shall write the equation $\mathcal{L}u = f$ as a system

$$(6.1) \quad U_\xi - AU = F$$

where $A = (a_{ij})$ is a 2×2 matrix with

$$\begin{aligned} a_{11} &= 0, & a_{12} &= 1, \\ a_{21} &= -\frac{1}{1 + \theta - \theta C} [(1 + \theta + \theta C) D_\eta^2 - 4\theta S D_\eta], \\ a_{22} &= -\frac{1}{1 + \theta - \theta C} [2\theta S D_\eta + 4\theta C] \end{aligned}$$

and

$$(6.2) \quad U = \begin{pmatrix} u \\ u_\xi \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \\ 1 + \theta - \theta C \end{pmatrix}.$$

The domain $D(A)$ and A consists of all column vectors (u, v) with

$$(6.3) \quad \begin{aligned} u &\in C^{2,\varrho} \left[0, \frac{\pi}{2} \right], & u(0) &= 0, & u' \left(\frac{\pi}{2} \right) &= 0, \\ v &\in C^{1,\varrho} \left[0, \frac{\pi}{2} \right], \end{aligned}$$

where ϱ is chosen to be any number in $(0, \alpha)$, α as in (1.13). The norm we shall use is

$$\|U\| = |u|_{C^{2,\varrho}[0, \pi/2]} + |v|_{C^{1,\varrho}[0, \pi/2]}.$$

By standard results (see [12; p. 224] the spectrum of A , when restricted to functions F as in (6.2) with $f \in C^{0,q}[0, \pi/2]$, consists of discrete set of eigenvalues λ . We also observe that

$$AU - \lambda U = 0$$

if and only if U is the column vector $(u, \lambda u)$ where

$$u = \exp[\lambda \xi]g(\eta)$$

and

$$(6.4) \quad M_\lambda g = 0 \quad \left(0 \leq \eta \leq \frac{\pi}{2}\right),$$

$$(6.5) \quad g(0) = 0, \quad g'\left(\frac{\pi}{2}\right) = 0.$$

Note that if $g(\eta)$ is a solution of (6.4), (6.5) then its extension to $(\pi/2, \pi)$ by reflection yields a solution of (5.21). This means that the eigenvalues of (6.4), (6.5) form a subset of the eigenvalues of (5.21). The eigenfunction for the first eigenvalue $\lambda_1 = 1$ for (5.21) is clearly symmetric about $\eta = \pi/2$ (since $g_{\lambda_1}(\pi - \eta)$ is also an eigenfunction) and thus

$$g'_{\lambda_1}\left(\frac{\pi}{2}\right) = 0.$$

On the other hand $g_{\lambda_2}(\eta)$ is not an eigenfunction of (6.4), (6.5). Recalling (5.22) we conclude:

LEMMA 6.1. *The first two eigenvalues of (6.4), (6.5) are λ_1, λ_3 , and $\lambda_1 = 1, \lambda_3 > 2$.*

THEOREM 6.2. *For any $0 < \mu < 1$ there holds:*

$$(6.6) \quad \left\| \begin{pmatrix} \omega \\ \omega_\xi \end{pmatrix} - \gamma g_{\lambda_1}(\eta) \exp[\lambda_1 \xi] \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \right\| = O(\exp[(2\mu - \varepsilon)\xi]) \quad (\lambda_1 = 1)$$

where γ is a constant and ε is any positive number.

PROOF. Let $\tilde{\omega}_2$ be a solution of

$$\begin{aligned} \mathfrak{L}\tilde{\omega}_2 &= 0 && \text{in } T, \\ \tilde{\omega}_2 &= 0 && \text{on } \{\eta = 0\}, \\ \frac{\partial \tilde{\omega}_2}{\partial \eta} &= G_2 && \text{on } \left\{\eta = \frac{\pi}{2}\right\}. \end{aligned}$$

By elliptic estimates

$$(6.7) \quad |\tilde{\omega}|_{C^{1,\alpha}(T_2)} \leq C |G_2|_{C^{1,\alpha}(T_1)}$$

where $T_j = T \cap \{\xi < -j\}$.

The function $u = \omega_2 - \tilde{\omega}_2$ satisfies:

$$(6.8) \quad u_\eta \left(\xi, \frac{\pi}{2} \right) = 0, \quad u(\xi, 0) = 0.$$

Further, if

$$(6.9) \quad |\omega(s)| = O(\exp[\mu\xi]), \quad \mu < 1$$

then

$$\mathbb{L}u = f$$

where

$$(6.10) \quad |f(\xi + i\eta)|_{C^{0,\alpha}[0,\pi/2]} \leq C(\exp[2\mu\xi] + \exp[2\xi]).$$

Indeed, Lemmas 4.3, 4.4 can be extended to derive Hölder estimates (with exponent ϱ) for the derivatives $\nabla_j^i \omega$ ($j = 1, 2$). Then we can proceed as in Lemma 5.6 in order to derive Hölder estimates in addition to the L^∞ estimates; note that the inequality (6.7) is used in the derivation of (6.10).

We are now in a position to apply a well known theorem on asymptotic expansion of solutions of evolution equations (see [18] or [12; p. 228]; cf. also [17]) asserting that (recall that $\lambda_2 = 2$)

$$\int_0^\infty \left\| \exp[2\mu - \varepsilon]|\xi| \left[U - \gamma g_{\lambda_1}(\eta) \exp[\lambda_1 \xi] \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \right] \right\|^2 d\xi < \infty$$

where γ is a constant and ε is any positive number; here $U = (u, u_\xi)$. It follows that

$$\int_{|\xi - \xi'| < 1} \left\| U - \gamma g_{\lambda_1}(\eta) \exp[\lambda_1 \xi] \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \right\|^2 d\xi \leq C \exp[2(2\mu - \varepsilon)\xi];$$

hence

$$\int_{|\xi - \xi'| < 1} [|u(\xi, \eta)|_{C^{2,\alpha}[0,\pi/2]} + |u_\xi(\xi, \eta)|_{C^{1,\alpha}[0,\pi/2]}]^2 d\xi \leq C \exp[2(2\mu - \varepsilon)\xi].$$

By elliptic estimates we now obtain the pointwise estimates

$$(6.11) \quad |u(\xi, \eta)|_{C^{2, \alpha}[0, \pi/2]} + |u(\xi, \eta)|_{C^{1, \alpha}[0, \pi/2]} \leq C \exp[(2\mu - \varepsilon)\xi].$$

Thus for μ as in (6.9) it is proved that (6.6) holds for ω_2 with γ real. The proof for ω_1 (with different real γ) is similar, and (6.6) (with complex γ) follows by adding the two estimates. (6.6) now implies that (6.9) is satisfied for a new $\tilde{\mu} = \min(1, 2\mu)$. Therefore starting with Lemma 5.4, Theorem 6.2 follows by iteration.

We now consider two cases:

Case 1. $\gamma \neq 0$.

Then on the free boundary

$$(6.12) \quad \frac{\omega_t}{t} + \frac{\omega_{\bar{t}}}{\bar{t}} = \frac{\omega_s + \omega_{\bar{s}}}{t^2} = \frac{2\tilde{\gamma}}{t^2} \exp[\xi] + O(\exp[\mu\xi])$$

where $\mu = 2 - \varepsilon$ and

$$2\tilde{\gamma} = \gamma g'_{\lambda_1}(0) \neq 0;$$

γ is real since $\omega_{1,\xi}/t^2$ is real. Substituting (6.12) into (2.18) we find that

$$(6.13) \quad \kappa(z) = \frac{\tilde{\gamma}}{t} (1 + o(1)) = \frac{\tilde{\gamma}}{x^{\frac{1}{2}}} (1 + o(1)) \quad (x \downarrow 0);$$

this yields the assertion (ii) of Theorem 1.1.

Case 2. $\gamma = 0$.

In this case (6.6) shows that

$$\left\| \begin{pmatrix} \omega \\ \omega_{\xi/\varepsilon} \end{pmatrix} \right\| = O(\exp[2 - \varepsilon)\xi])$$

for any $\varepsilon > 0$.

Let

$$q = \lim_{\substack{t \in Q_1 \\ t \rightarrow 0}} \Phi_0(t, \omega(t))$$

where Φ_0 is defined in (4.3), and consider the equation

$$(6.14) \quad \omega_{\bar{t}} + i\theta\omega_{2,\bar{t}} - \frac{\bar{t}}{t} i\theta\omega_{2,t} = q\bar{t}.$$

This equation has a solution $\tilde{\omega}$ of the form

$$\tilde{\omega} = a(t^2 - \bar{t}^2)$$

where a is a complex number. Setting $W = \omega - \tilde{\omega}$ and recalling (5.4)-(5.6) we find that

$$L_0 W = \tilde{F}$$

where $\tilde{F} = O(\exp[(2 + \delta)\xi])$ together with its $C^{0,q}$ norm in the variable η ; here δ is some positive number. A similar estimate holds for the Neumann data G . Applying the same arguments as before we deduce (since $\lambda_3 > 2$; see Lemma 6.1) that

$$\|W\| = O(\exp[(2 + \delta - \varepsilon)\xi])$$

for any $0 < \varepsilon < \delta$. It follows that

$$\frac{\omega_t}{t} + \frac{\omega_{\bar{t}}}{\bar{t}} = \frac{\tilde{\omega}_t}{t} + \frac{\tilde{\omega}_{\bar{t}}}{\bar{t}} + O(\exp[(\delta - \varepsilon)\xi]) .$$

Since finally

$$\frac{\tilde{\omega}_t}{t} + \frac{\tilde{\omega}_{\bar{t}}}{\bar{t}} = 0 ,$$

it follows (by (2.18)) that $\varkappa(z) \rightarrow 0$ along Γ if $z \rightarrow 0$. This proves the assertion (i) of Theorem 1.1.

COROLLARY 6.3. *$N \cup \Gamma$ is in $C^{1, \frac{1}{2}}$ and ψ is in $C^{1, \frac{1}{2}}$ in some $\{\psi \geq 0\}$ -neighborhood of the separation point A .*

Indeed, the first assertion follows easily from (6.13) and the second assertion follows from the regularity theory for elliptic equations.

7. - The incompressible case.

We consider the special case of incompressible flow. The functional $J(\psi)$ has the form

$$(7.1) \quad \int |\nabla \psi|^2 + Q^2(z) \chi_{\{\psi > 0\}} dx dy$$

for two dimensional flows, and

$$(7.2) \quad \int \left(\left| \frac{\nabla \psi}{y} \right|^2 + Q^2(z) \chi_{\{\psi > 0\}} \right) y dx dy$$

for 3-dimensional axially symmetric flow. We shall prove Theorem 1.1 under weaker differentiability assumptions on N and λ :

THEOREM 7.1. *Assume that $N \in C^{2,\alpha}$ and $Q \in C^{1,\alpha}$. Then the assertions of Theorem 1.1 remain valid.*

PROOF. For definiteness we consider only the case (7.2) (which is more difficult). In this case $\varrho = 1/y$. We shall not perform a conformal mapping as in remark 1.1, since this would complicate the form of ϱ .

We define the function f in a different way than before, namely,

$$f = \varphi + i\psi/y$$

but keep the same definitions of ω and t as in (2.2), (2.6) (with $\lambda = yQ$) [For the functional (7.1) we take $f = \varphi + i\psi$, $\lambda = Q$.] Then

$$\begin{aligned}\varphi_z &= i\psi_z/y, \\ \omega_{\bar{z}} &= \frac{1}{4y} - \frac{3}{8} \frac{\psi}{y^2 f_z} \equiv \delta(z).\end{aligned}$$

From the relation

$$\omega_{\bar{z}} = \omega_t t_z + \omega_{\bar{t}} \bar{t}_z$$

we get

$$\omega_{\bar{t}} + a\omega_t = b$$

where

$$\begin{aligned}a &= \frac{t_z}{\bar{t}_z} = \frac{\bar{t}}{t} \frac{\psi/4y^2}{\varphi_z - \psi/4y^2}, \\ b &= \frac{\delta}{\bar{t}_z} = \bar{t} \frac{\delta}{\varphi_z - \psi/4y^2}.\end{aligned}$$

Observe that

$$|a| \leq C|\psi(z)| \leq C|\operatorname{Im} f| \leq C|t_1 t_2|,$$

so that

$a = 0$ on the positive real axis
and on the positive imaginary axis.

We now extend ω into Q_4 by

$$\omega(t) = \overline{\omega(\bar{t})}$$

and then introduce the function

$$W = (1 + ih') \exp[-i\omega]$$

where $N: y = h(x)$. Since W is real on the t_2 -axis, we can extend it into $Q_2 \cup Q_3$ by

$$W(t) = -\overline{W(-\bar{t})}$$

so that it remains continuous; further

$$(7.3) \quad W_{\bar{t}} - AW_t = \bar{t}\Phi_0(t, \omega)$$

where Φ_0 is similar to the function in (4.3) and

$$(7.4) \quad |A(t)| \leq C|t_1 t_2|.$$

Equation (7.3) thus replaces (4.1); it has more regular coefficients.

By standard L^p elliptic estimates we immediately deduce that W belongs to $W^{1,p}(B_{\delta_0})$ for any $p < \infty$ (for some δ_0 -neighborhood B_{δ_0} of $t = 0$). Notice also that (since $A = 0$ on the t_j axes) the function G_j in (5.30), (5.31) are now given directly by Φ_0 , so that the $C^{1,\alpha}$ norm of G_j can be estimated by the C^α norm of $\nabla_s \omega$.

We can rewrite (5.9) in the form

$$\mathfrak{L}'\omega_2 = f'$$

where $\mathfrak{L}'$ is a perturbation of the Laplacian (the perturbation term is of the form $A\nabla_s D_s \omega$) and f' contains terms from $\nabla_s(\bar{t}^2 \Phi_0)$, as well as terms with $\nabla_s A D_s \omega$.

The above remarks show that we can repeat the proof of Theorem 1.1 without using Lemma 4.4. Thus the assumptions that $N \in C^{2,\alpha}$ and $Q \in C^{1,\alpha}$ are sufficient for establishing the theorem.

REMARK 7.1. Theorem 7.1 can be established also by studying (7.3) directly. In case $A \equiv 0$ we represent the solution by means of a fundamental solution, after subtracting a special quadratic solution of (7.3) with $\Phi_0(t, \omega) = q_i$ in Q_i , where $q_i = \lim_{\substack{t \rightarrow 0 \\ t \in Q_i}} \Phi_0(t, \omega(t))$. The case $A \equiv 0$ can be reduced to the case $A \not\equiv 0$ by a suitable diffeomorphism.

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