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On Dyson's lemma

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Dedicated to the memory
of Aldo Andreotti

I. – Introduction.

I.1. A classical problem in Diophantine approximation is the search for rational approximations to algebraic numbers. Let α be an algebraic number with $\deg\alpha \geqq 2$. Roth's well known theorem states that for every $\varepsilon > 0$ there exists a positive constant $q_0 = q_0(\alpha, \varepsilon)$ such that

$$\left|\alpha - \frac{p}{q}\right| > q^{-2-\varepsilon}$$

for all integers p and q with $q \geqq q_0$. However, Roth's theorem is not effective, in the sense that it does not furnish a method to calculate $q_0(\alpha, \varepsilon)$. Weaker results of the following kind:

$$\left|\alpha - \frac{p}{q}\right| > q^{-\lambda(r)-\varepsilon} \qquad \text{for } q \geqq q_0(\alpha, \varepsilon)\,,$$

where $r = \deg\alpha$, were previously obtained by Liouville $(\lambda(r) = r)$, by Thue $(\lambda(r) = r/2 + 1)$, by Siegel $\left(\lambda(r) = \min\limits_{s\in\mathbf{N}} (r/(s+1) + s)\right)$, and by Dyson and Gelfond $(\lambda(r) = \sqrt{2r})$ independently. All the above results, except Liouville's elementary exponent $\lambda(r) = r$, are equally ineffective.

The problem of finding effective lower bounds for $|\alpha - p/q|$, improving on Liouville's theorem, has been the object of deep work by Baker, Feldman, Bombieri, G. V. Chudnovsky and others, but so far only partial results have been obtained. We refer to the recent papers [1] and [2] for a thorough discussion of these results, and of the different techniques employed in this context.

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I.2. In the approaches of Thue, Siegel, Dyson and Gelfond one considers two approximations p_1/q_1 and p_2/q_2 to α, and an auxiliary polynomial $P(x_1, x_2)$ with integral coefficients vanishing to a high order at (α, α) and to a low order at $(p_1/q_1, p_2/q_2)$. Later Roth succeeded in extending a similar construction to the case of several variables, which enabled him to prove the theorem mentioned above.

Unlike Siegel's and Roth's approaches, where one needs rational approximations p/q to α with q large compared with the height of α, Dyson's method [4] is essentially free from considerations of heights, as Bombieri has remarked in [1]. Dyson's construction of the auxiliary polynomial is based on a lemma, which relates the vanishing of some derivatives of a polynomial in two variables at certain points with the degrees of the polynomial itself.

By improving the Thue-Siegel method, Bombieri [1] gets effective exponents of Thue-Siegel's type for all generators of some number fields. A crucial point in Bombieri's method is the use of a refined form of Dyson's lemma, which we now describe.

I.3. We follow here Bombieri's notation. Let $P(x_1, x_2)$ be a polynomial with complex coefficients, P not identically zero. Let

$$\deg_{x_j} P \leq d_j \qquad (j = 1, 2)\,, \tag{1}$$

and let $\xi_1, \ldots, \xi_m \in \mathbb{C}^2$, $\xi_h = (\xi_{h1}, \xi_{h2})$, be m *admissible* points, i.e. such that

$$\xi_{h1} \neq \xi_{h'1} \qquad \text{and} \qquad \xi_{h2} \neq \xi_{h'2} \qquad \text{for } h \neq h'\,. \tag{2}$$

For real $\theta_1, \theta_2 > 0$; $t_1, \ldots, t_m \geq 0$, let

$$\frac{\partial^{i_1+i_2} P}{\partial x_1^{i_1}\, \partial x_2^{i_2}}(\xi_h) = 0 \tag{3}$$

for all (i_1, i_2) satisfying

$$\theta_1 \frac{i_1}{d_1} + \theta_2 \frac{i_2}{d_2} < t_h \qquad (h = 1, \ldots, m)\,. \tag{4}$$

Further, define

$$\varphi(t) = \varphi(t; \theta_1, \theta_2) = \underset{\theta_1 z_1 + \theta_2 z_2 \leq t}{\int_0^1 \int_0^1} dz_1\, dz_2\,. \tag{5}$$

Then we have the following

DYSON'S LEMMA. *If $P(x_1, x_2)$ is a non-identically vanishing polynomial with complex coefficients satisfying* (1)-(4), *then*

$$\sum_{h=1}^{m} \varphi(t_h) \leqq 1 + \max\left(\frac{m}{2} - 1, 0\right) \min\left(\frac{d_1}{d_2}, \frac{d_2}{d_1}\right), \tag{6}$$

where φ is defined by (5).

This is Bombieri's version of Dyson's lemma ([1], Theorem 1). The meaning of the lemma is the following. Since the number of (i_1, i_2) satisfying

$$0 \leqq i_j \leqq d_j, \qquad \theta_1 \frac{i_1}{d_1} + \theta_2 \frac{i_2}{d_2} < t$$

is asymptotic to $\varphi(t) d_1 d_2$ when $d_1, d_2 \to \infty$, (3) can be viewed as a system of approximately $\varphi(t_h) d_1 d_2$ homogeneous linear equations in the coefficients of P. Thus we have approximately $d_1 d_2 \sum_{h=1}^{m} \varphi(t_h)$ equations in $\leqq (d_1 + 1) \cdot (d_2 + 1) \sim d_1 d_2$ unknowns. Since P is not identically zero, it is natural to expect, under suitable conditions, that

$$d_1 d_2 \left(\sum_{h=1}^{m} \varphi(t_h) - 1 \right)$$

will not be too large.

I.4. Dyson's and Bombieri's proofs of the above lemma are based upon the theory of generalized Wronskians. The main purpose of the present paper is to give a new and more intrinsic approach to the lemma, by applying the theory of singular points for plane algebraic curves instead of the theory of Wronskian determinants. We shall obtain an improved form of Dyson's lemma (see our Main Theorem below). Our method will show that the structure of the inequalities obtained depends on the factorization of $P(x_1, x_2)$, so that any information about the latter would imply sharper forms of the remainder term. Moreover, the admissibility condition (2) for the points $\xi_1, \ldots, \xi_m$ is shown to be unnecessary if the curve $P(x_1, x_2) = 0$ has no components $x_1 =$ constant nor $x_2 =$ constant. Condition (2) can also be replaced by a natural upper bound for $t_1, \ldots, t_m$.

As an application, we shall prove in Section VI some results on effective measures of irrationality for algebraic numbers, following the approach of [1] and [2].

We also mention that in a recent paper [6], Esnault and Viehweg have obtained an important generalization of Dyson's inequality (6) to the case of a polynomial $P(x_1, \dots, x_n)$ in several variables. Their method, again stemming from algebraic geometry, is based on the classification theory of higher dimensional varieties.

I.5. We now give a precise statement of our main results. For convenience we slightly change our notation. Again let $P \in \mathbf{C}[x_1, x_2]$ be a polynomial, not identically vanishing, with

$$\deg_{x_j} P \leqq d_j \qquad (j = 1, 2)\,. \tag{7}$$

Let $\xi_1, \dots, \xi_m \in \mathbf{C}^2$, $\xi_h = (\xi_{h1}, \xi_{h2})$, be m distinct points such that

$$\frac{\partial^{i_1+i_2} P}{\partial x_1^{i_1}\, \partial x_2^{i_2}}(\xi_h) = 0 \tag{8}$$

for all (i_1, i_2) satisfying

$$\frac{i_1}{\lambda_1} + \frac{i_2}{\lambda_2} < t_h \qquad (h = 1, \dots, m)\,, \tag{9}$$

where $\lambda_1, \lambda_2 > 0$; $t_1, \dots, t_m \geqq 0$ are real numbers. Since, up to factorials,

$$\frac{\partial^{i_1+i_2} P}{\partial x_1^{i_1}\, \partial x_2^{i_2}}(\xi_h)$$

is the coefficient of $(x_1 - \xi_{h1})^{i_1}(x_2 - \xi_{h2})^{i_2}$ in the Taylor expansion of P, and since P is not identically zero, there exists some (i_1, i_2) for which $i_1 \leqq d_1$, $i_2 \leqq d_2$ and

$$\frac{\partial^{i_1+i_2} P}{\partial x_1^{i_1}\, \partial x_2^{i_2}}(\xi_h) \neq 0\,.$$

This shows that (8)-(9) implies

$$t_h \leqq \frac{d_1}{\lambda_1} + \frac{d_2}{\lambda_2}\,.$$

We define, for $0 \leqq t \leqq d_1/\lambda_1 + d_2/\lambda_2$,

$$f(t) = \underset{z_1+z_2 \leqq t}{\int_0^{d_1/\lambda_1} \int_0^{d_2/\lambda_2}} dz_1\, dz_2\,, \tag{10}$$

and note that if

$$\begin{cases} u = u(t) = \max\left(t - \dfrac{d_1}{\lambda_1},\, 0\right), \\ v = v(t) = \max\left(t - \dfrac{d_2}{\lambda_2},\, 0\right), \end{cases} \tag{11}$$

then

$$f(t) = \tfrac{1}{2}(t^2 - u^2 - v^2) \leqq \tfrac{1}{2}t^2 . \tag{12}$$

We also note that $f(t) = (d_1/\lambda_1)(d_2/\lambda_2)\varphi(t)$, where φ is defined by (5) with $\theta_j = d_j/\lambda_j$. Thus, with the present notation, Dyson's inequality (6) states that if $\xi_1, \ldots, \xi_m$ are admissible, then

$$\sum_{h=1}^{m} f(t_h) \leqq \frac{d_1}{\lambda_1}\frac{d_2}{\lambda_2} + \max\left(\frac{m}{2} - 1, 0\right)\frac{(\min(d_1, d_2))^2}{\lambda_1\lambda_2} . \tag{13}$$

We need not assume that $\xi_1, \ldots, \xi_m$ are admissible: we denote by m_2 the number of distinct values in the set $\{\xi_{h1}\}$ $(h = 1, \ldots, m)$, by m_1 the number of distinct values in the set $\{\xi_{h2}\}$ $(h = 1, \ldots, m)$, and we let

$$m' = \max(m_1, m_2) .$$

For each $h = 1, \ldots, m$, we denote by ϱ_h the number of h', $1 \leqq h' \leqq m$, such that $\xi_{h'2} = \xi_{h2}$ (including $h' = h$); similarly, we denote by σ_h the number of h' such that $\xi_{h'1} = \xi_{h1}$. Thus we have $\varrho_h = \sigma_h = 1$ $(h = 1, \ldots, m)$ if and only if $\xi_1, \ldots, \xi_m$ are admissible.

Further, let

$$P_1(x_1, x_2), \ldots, P_n(x_1, x_2)$$

be all the irreducible factors of P satisfying

$$\deg_{x_j} P_k \geqq 1 \qquad (j = 1, 2; \quad k = 1, \ldots, n) ,$$

with $P_k/P_{k'}$ not constant for $k \neq k'$, and let e_k denote the exponent of P_k in the canonical factorization of P. We define

$$Q(x_1, x_2) = \prod_{k=1}^{n} P_k(x_1, x_2)^{e_k} ,$$

so that the curve $Q = 0$ is $P = 0$ deprived of the lines parallel to the axes. Let

$$\delta_j = \deg_{x_j} Q \qquad (j = 1, 2) ,$$

and

$$e = \max_{1 \leqq k \leqq n} e_k .$$

We shall prove the following

MAIN THEOREM. *With the above notation, if* $P \in \mathbb{C}[x_1, x_2]$ *does not vanish* *identically and satisfies* (7), (8) *and* (9), *then*

$$(14) \qquad \sum_{h=1}^{m} f(t_h) \leqq \frac{d_1}{\lambda_1}\frac{d_2}{\lambda_2} + \max\left(\frac{m'}{2}-1,\, 0\right)\frac{e}{\max(\lambda_1, \lambda_2)}\min\left(\frac{\delta_1}{\lambda_1}, \frac{\delta_2}{\lambda_2}\right),$$

provided at least one of the following additional assumptions holds:

(A) $P = Q$;

(B) $\varrho_h = \sigma_h = 1$ $(h = 1, \ldots, m)$, *i.e.* $\xi_1, \ldots, \xi_m$ *are admissible*;

(C) $t_h \leqq \min\left(\dfrac{d_1}{\varrho_h \lambda_1}, \dfrac{d_2}{\sigma_h \lambda_2}\right) \qquad (h = 1, \ldots, m).$

Since Q divides P and $\deg_{x_j} P_k \geqq 1$, we have

$$e \leqq \sum_{1}^{n} e_k \leqq \delta_j \leqq d_j \qquad (j = 1, 2),$$

so that (14) is sharper than (13).

I.6. We shall assume the above conditions (A), (B) or (C) in Theorems 1, 2 and 3 respectively. Algebraic geometry is employed in the proof of Theorem 1. Theorems 2 and 3 will be proved by combining Theorem 1 with some estimates involving the lines parallel to the axes occurring as components of the curve $P(x_1, x_2) = 0$. Such lines affect only the main term in (14), the remainder term arising from $Q(x_1, x_2)$ only.

We point out that the remainder term in (14) is essentially the best possible if m_1 and m_2 are approximately equal. We shall obtain other forms of the remainder term (see (28) and (29)), which may be more convenient if m_1 and m_2 are not of the same size.

We also remark that both (B) and (C) are redundant, in the sense that if for a given h^* the line $x_2 = \xi_{h^*2}$ (resp. $x_1 = \xi_{h^*1}$) is not a component of the curve $P = 0$, no assumption involving ϱ_{h^*} (resp. σ_{h^*}) is required. However, in the applications to Diophantine approximation no information on the factorization of P is *a priori* available; thus the above formulations of (B) and (C) are most appropriate in practice.

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I am also indebted to E. Bombieri for very helpful conversations about the significance of Dyson's lemma in Diophantine approximation, and for suggesting the applications given in Section VI.

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II. – Lemmas on singular points.

II.1. As we have already remarked, (8)-(9) is a vanishing condition for certain Taylor coefficients of P at ξ_h, and therefore means that the Newton polygon of the algebraic curve $P(x_1, x_2) = 0$ at the singular point ξ_h lies on or above the line $i_1/\lambda_1 + i_2/\lambda_2 = t_h$ in the (i_1, i_2)-plane. We require two lemmas relating the Newton polygons to the local behaviour of curves at singular points.

LEMMA 1. *Let $P(x, y) = \sum a_{ij}x^iy^j$ and $Q(x, y) = \sum b_{ij}x^iy^j$ be polynomials with complex coefficients such that the curves $P = 0$ and $Q = 0$ have no common components through the origin. Let*

$$a_{ij} = 0 \qquad \text{for all } (i, j) \text{ satisfying } \frac{i}{\alpha} + \frac{j}{\beta} < 1 ,$$

$$b_{ij} = 0 \qquad \text{for all } (i, j) \text{ satisfying } \frac{i}{\alpha'} + \frac{j}{\beta'} < 1 ,$$

for positive $\alpha, \beta, \alpha', \beta'$ (not necessarily integers) such that $\alpha\beta' = \alpha'\beta$. Then the intersection multiplicity of $P = 0$ and $Q = 0$ at the origin is at least $\alpha\beta' = \alpha'\beta$.

PROOF. We denote by Γ, resp. Γ', the curves given by $P = 0$ and $Q = 0$, by $\nu_0(\Gamma, \Gamma')$ their intersection multiplicity at the origin, and by ν_0 and ν_0' the multiplicities of the origin as a singular point of Γ and Γ' respectively. If for some (i, j) such that $a_{ij} = 0$ we introduce a small non-zero coefficient in the equation of Γ, we change Γ into a curve $\tilde{\Gamma}$ by a small perturbation. If we change Γ' into $\tilde{\Gamma}'$ similarly, it is well known that

$$\nu_0(\Gamma, \Gamma') \geqq \nu_0(\tilde{\Gamma}, \tilde{\Gamma}') ,$$

provided the perturbations are small enough. Hence we may assume

$a_{ij} \neq 0$ for all (i, j) on or just above the line $i/\alpha + j/\beta = 1$, and similarly for b_{ij}.

We assume e.g. $\alpha \geqq \beta$, whence $\alpha' \geqq \beta'$, and denote by B and B' the least integers $\geqq \beta$ and β' respectively. Clearly $\nu_0 = B$, $\nu_0' = B'$. Since $\nu_0(\Gamma, \Gamma') \geqq \nu_0 \nu_0' = BB'$, the lemma is true if $\alpha \leqq B$ or $\alpha' \leqq B'$; we therefore assume $\alpha > B$ and $\alpha' > B'$. We now blow up the origin in $\mathbf{C}^2$. Since, by our assumptions, the line $x = 0$ is not tangent to Γ at the origin, the equation of the proper transform Γ_1 of Γ can be obtained by substituting xy for y in the equation of Γ and then removing the highest common power of x. Thus, if we write the equation of Γ_1 as $\sum c_{ij} x^i y^j = 0$, we have $c_{ij} = a_{i+B-j,j}$; hence $c_{ij} = 0$ for all (i, j) satisfying $i/\alpha_1 + j/\beta_1 < 1$, where

$$\alpha_1 = \alpha - B\,, \qquad \beta_1 = \frac{\alpha - B}{\alpha - \beta}\,\beta\,.$$

Similarly, the coefficients in the equation of the proper transform Γ_1' of Γ' vanish for $i/\alpha_1' + j/\beta_1' < 1$, where

$$\alpha_1' = \alpha' - B'\,, \qquad \beta_1' = \frac{\alpha' - B'}{\alpha' - \beta'}\,\beta'\,.$$

From $\alpha\beta' = \alpha'\beta$ we obtain

$$\alpha_1 \beta_1' = (\alpha - B)\,\frac{\alpha' - B'}{\alpha' - \beta'}\,\beta' = (\alpha' - B')\,\frac{\alpha - B}{\alpha - \beta}\,\beta = \alpha_1' \beta_1\,.$$

As a simple consequence of a theorem of Max Noether (see [3], p. 690, Satz 13) we have

$$\nu_0(\Gamma, \Gamma') \geqq \nu_0 \nu_0' + \nu_0(\Gamma_1, \Gamma_1')\,.$$

Note that

$$\begin{aligned} \nu_0 \nu_0' = BB' &= B\beta' + \frac{B\alpha' - B\beta'}{\alpha' - \beta'}\,(B' - \beta') \\ &\geqq B\beta' + \frac{\alpha'\beta - B\beta'}{\alpha' - \beta'}\,(B' - \beta') \\ &= B\beta' + \frac{(\alpha - B)\,\beta'}{\alpha' - \beta'}\,(B' - \beta')\,. \end{aligned}$$

If we assume the lemma to be true for Γ_1 and Γ_1', we have

$$\nu_0(\Gamma_1, \Gamma_1') \geqq \alpha_1 \beta_1' = (\alpha - B)\,\frac{\alpha' - B'}{\alpha' - \beta'}\,\beta'\,,$$

whence

$$\begin{aligned}\nu_0(\Gamma, \Gamma') &\geqq \nu_0\nu_0' + \nu_0(\Gamma_1, \Gamma_1') \\ &\geqq B\beta' + \frac{(\alpha - B)\beta'}{\alpha' - \beta'}(B' - \beta') + (\alpha - B)\frac{\alpha' - B'}{\alpha' - \beta'}\beta' \\ &= B\beta' + \frac{\alpha - B}{\alpha' - \beta'}\beta'(\alpha' - \beta') = \alpha\beta' .\end{aligned}$$

Thus if the lemma is true for Γ_1 and Γ_1', it is also true for Γ and Γ'. We iterate this process. Since

$$\frac{\alpha_1'}{\beta_1'} = \frac{\alpha_1}{\beta_1} = \frac{\alpha}{\beta} - 1 ,$$

after $[\alpha/\beta]$ steps we interchange x and y. Also

$$\alpha_1 = \alpha - B \leqq \alpha - 1 , \qquad \beta_1 = \frac{\alpha - B}{\alpha - \beta}\beta \leqq \beta ;$$

hence the lemma follows easily by induction. Q.E.D.

We now introduce the Milnor number of a curve at a singular point. If π is a point on the curve $P(x, y) = 0$, and if $P = 0$ has no multiple components through π, the Milnor number μ_π can be defined as the intersection multiplicity of $\partial P/\partial x = 0$ and $\partial P/\partial y = 0$ at π (see [8], § 7).

LEMMA 2. *Let $P(x, y) = \sum a_{ij}x^iy^j = 0$ be an algebraic curve with no multiple components through the origin; let*

$$a_{ij} = 0 \qquad \textit{for all } (i, j) \textit{ satisfying } \frac{i}{\alpha} + \frac{j}{\beta} < 1 ,$$

for real numbers $\alpha, \beta \geqq 1$. Then the Milnor number μ_0 of $P = 0$ at the origin satisfies

$$\mu_0 \geqq (\alpha - 1)(\beta - 1) .$$

PROOF. If we write $\partial P/\partial x$ as $\sum b_{ij}x^iy^j$, it is obvious that $b_{ij} = 0$ for $i/(\alpha - 1) + j/(\beta(1 - 1/\alpha)) < 1$; similarly the coefficients of $\partial P/\partial y$ vanish for $i/(\alpha(1 - 1/\beta)) + j/(\beta - 1) < 1$. By Lemma 1, the intersection multiplicity of $\partial P/\partial x = 0$ and $\partial P/\partial y = 0$ at the origin is at least $(\alpha - 1)(\beta - 1)$.

Q.E.D.

It is interesting to remark that Lemma 2 is a special case of a difficult theorem of Kouchnirenko [7]. We also note that, as is clear by simple examples or by referring to Kouchnirenko's theorem, the inequalities expressed by Lemmas 1 and 2 are the best possible.

II.2. We shall use here some elementary Puiseux expansion arguments. We refer to [5] for details. Let π be a point on the algebraic curve Γ, and let r be a branch of Γ at π. If we take π as the origin and the tangent to r as the line $y = 0$, then r has a parametrization:

$$\begin{cases} x = t^{\nu} , \\ y = at^{\nu+c} + \text{(higher terms)} \qquad (a \neq 0) , \end{cases}$$

where ν and c are positive integers. ν is called the order and c the class of the branch r. It is easily seen that the branch r' corresponding to r in the dual curve has order c and class ν. Also, $\nu + c$ is clearly the intersection multiplicity of r with its tangent. Since $\nu + c$ is self-dual, it also represents the intersection multiplicity of the dual branch r' with its tangent.

Given a line l through π, we introduce a number $c_\pi(l)$, which we define to be the sum of the classes of those branches of Γ that have tangent l at π.

LEMMA 3. *Let Γ have no multiple components through π. Let ν_π be the multiplicity of π on Γ, let $\nu_\pi(\Gamma, l)$ denote the intersection multiplicity of Γ with a line l at π, and let $c_\pi(l)$ be defined as above. Then*

$$\nu_\pi + c_\pi(l) = \nu_\pi(\Gamma, l) .$$

PROOF. Let r be a branch of Γ at π, with order ν and class c. If r has tangent l, the intersection multiplicity $\nu_\pi(r, l)$ is $\nu + c$, as we have already remarked. Otherwise we have $\nu_\pi(r, l) = \nu$. Summing over all the branches r, the lemma follows. Q.E.D.

We now define the local class c_π of Γ at π. Let the curve Γ' and the line π' correspond to Γ and π by duality. If $r_1, \ldots, r_\omega$ are the branches of Γ at π, denote by $r'_1, \ldots, r'_\omega$ the corresponding branches of Γ'. Then c_π is the sum of the intersection multiplicities of π' with $r'_1, \ldots, r'_\omega$. Since π' is tangent to $r'_1, \ldots, r'_\omega$, by a remark above we have

$$c_\pi = \sum_1^\omega (\nu_i + c_i) ,$$

where ν_i is the order and c_i is the class of the branch r_i.

LEMMA 4. *With the above notation we have*

$$\mu_\pi + c_\pi \geqq \nu_\pi^2 + 1 ,$$

where μ_π is the Milnor number.

PROOF. As usual, to each singular point π of Γ we associate a positive integer δ_π which represents the number of double points of Γ concentrated at π in the genus formula (see [8], p. 85). By a theorem of Milnor ([8], Theorem 10.5) we have

$$\mu_\pi = 2\delta_\pi - \omega + 1 ,$$

where ω is the number of branches of Γ at π. Since each branch r_i has class $c_i \geqq 1$, we have

$$c_\pi = \sum_1^\omega \nu_i + \sum_1^\omega c_i = \nu_\pi + \sum_1^\omega c_i \geqq \nu_\pi + \omega .$$

Moreover

$$\delta_\pi \geqq \tfrac{1}{2}\nu_\pi(\nu_\pi - 1) ,$$

whence

$$\mu_\pi + c_\pi \geqq 2\delta_\pi + \nu_\pi + 1 \geqq \nu_\pi^2 + 1 . \qquad \text{Q.E.D.}$$

II.3. In the proof of Theorem 1, the local analysis of the singularities will be carried out using the above Lemmas 1-4. For the global control of the singularities we shall use a suitable version of Plücker's first formula, which for convenience we state as a lemma. As usual, if $P(x, y) = 0$ is an algebraic curve, we call $\deg P$ the order of the curve. The class of the curve is the order of the dual curve.

LEMMA 5 (*Plücker's first formula*). *Let Γ be an algebraic curve with no multiple components, with order D and class C. For each singular point π, let μ_π be the Milnor number and ν_π the multiplicity of π on Γ. Then*

$$\sum_\pi (\mu_\pi + \nu_\pi - 1) + C = D(D-1) ,$$

where the sum runs over all the singular points of Γ in the projective plane.

PROOF. First assume Γ irreducible. Let δ_π be as in the proof of Lemma 4, and let ω_π denote the number of branches of Γ at π. If g is the genus of Γ, we have

$$g = \tfrac{1}{2}(D-1)(D-2) - \sum_\pi \delta_\pi$$

and

$$2D + 2g - 2 = C + \sum_{\pi} (\nu_\pi - \omega_\pi)$$

([5], p. 421), the latter equation being Plücker's first formula for arbitrary singularities. Eliminating g we obtain

$$\sum_{\pi} (2\delta_\pi + \nu_\pi - \omega_\pi) + C = D(D-1)\,.$$

It is easy to see that this formula holds even if Γ is reducible but has no multiple components (by induction on the irreducible components of Γ). Since, by Milnor's theorem, $\mu_\pi = 2\delta_\pi - \omega_\pi + 1$, we have

$$2\delta_\pi + \nu_\pi - \omega_\pi = \mu_\pi + \nu_\pi - 1\,,$$

and the lemma follows. Q.E.D.

III. - The case $P = Q$.

III.1. We return to the notation and assumptions of Section I.5. $P(x_1, x_2)$ is a polynomial with complex coefficients, not identically vanishing, such that

$$\deg_{x_j} P \leqq d_j \qquad (j = 1, 2)\,,$$

and satisfying condition (8)-(9) at distinct points

$$\xi_h = (\xi_{h1}, \xi_{h2}) \qquad (h = 1, \ldots, m)\,.$$

The coordinates ξ_{h1} are not necessarily distinct: we denote by

$$x_1^{(1)}, \ldots, x_1^{(m_2)}$$

the distinct values in the set $\{\xi_{h1}\}$. Similarly, we denote by

$$x_2^{(1)}, \ldots, x_2^{(m_1)}$$

the distinct values in the set $\{\xi_{h2}\}$.

We assume here $P = Q$, i.e. that among the components of the curve $P = 0$ there are no lines parallel to the axes. Thus, if

$$P(x_1, x_2) = \prod_{k=1}^{n} P_k(x_1, x_2)^{e_k}$$

denotes the factorization of P into powers of distinct irreducible factors, and

$$\delta_j^{(k)} = \deg_{x_j} P_k ,$$

we have

$$\delta_j^{(k)} \geqq 1 \qquad (j = 1, 2;\ k = 1, \ldots, n) .$$

Also

$$\delta_j = \deg_{x_j} P = \sum_{k=1}^{n} e_k \delta_j^{(k)} \leqq d_j \qquad (j = 1, 2) . \tag{15}$$

For any h and k, we denote by $t_h^{(k)}$ the greatest number such that

$$\frac{\partial^{i_1+i_2} P_k}{\partial x_1^{i_1} \partial x_2^{i_2}} (\xi_h) = 0 \tag{16}$$

for all (i_1, i_2) satisfying

$$\frac{i_1}{\lambda_1} + \frac{i_2}{\lambda_2} < t_h^{(k)} . \tag{17}$$

It follows that

$$t_h \leqq \sum_{k=1}^{n} e_k t_h^{(k)} \qquad (h = 1, \ldots, m) . \tag{18}$$

For if $X(x, y) = \sum a_{pq} x^p y^q$ is a polynomial such that $a_{pq} = 0$ for all (p, q) satisfying $p/\lambda_1 + q/\lambda_2 < t$ and $a_{pq} \neq 0$ for some (p, q) satisfying $p/\lambda_1 + q/\lambda_2 = t$, and similarly $Y(x, y) = \sum b_{rs} x^r y^s$ is such that $b_{rs} = 0$ for all $r/\lambda_1 + s/\lambda_2 < \tau$ and $b_{rs} \neq 0$ for some $r/\lambda_1 + s/\lambda_2 = \tau$, then the product $X(x, y) Y(x, y) = \sum c_{ij} x^i y^j$ is such that

$$c_{ij} = \sum_{(p,q)+(r,s)=(i,j)} a_{pq} b_{rs} = 0$$

for all $i/\lambda_1 + j/\lambda_2 < t + \tau$, and $c_{ij} \neq 0$ for some $i/\lambda_1 + j/\lambda_2 = t + \tau$, as is clear e.g. by considering the vector sum $(i, j) = (p, q) + (r, s)$ with the largest p and r for which $p/\lambda_1 + q/\lambda_2 = t$, $a_{pq} \neq 0$, and $r/\lambda_1 + s/\lambda_2 = \tau$, $b_{rs} \neq 0$. This proves (18).

By (16)-(17), the intersection multiplicity of the line $x_2 = \xi_{h2}$ with $P_k = 0$ at ξ_h is at least $\lambda_1 t_h^{(k)}$. Also, the total number of intersections at finite distance of any line $x_2 =$ constant with $P_k = 0$ does not exceed $\delta_1^{(k)}$. Hence, for any $k = 1, \ldots, n$ and $q = 1, \ldots, m_1$, we have

$$\lambda_1 \sum_{\xi_{h2} = x_2^{(q)}} t_h^{(k)} \leqq \delta_1^{(k)} . \tag{19}$$

In particular, for any h and k we get

$$t_h^{(k)} \leqq \frac{\delta_1^{(k)}}{\lambda_1},$$

and similarly

$$t_h^{(k)} \leqq \frac{\delta_2^{(k)}}{\lambda_2}.$$

Thus, defining

$$\theta^{(k)} = \min\left(\frac{\delta_1^{(k)}}{\lambda_1}, \frac{\delta_2^{(k)}}{\lambda_2}\right), \tag{20}$$

we have

$$t_h^{(k)} \leqq \theta^{(k)} \qquad (h = 1, \ldots, m;\ k = 1, \ldots, n). \tag{21}$$

We note that

$$\sum_{k=1}^{n} e_k \theta^{(k)} \leqq \sum_{k=1}^{n} e_k \frac{\delta_j^{(k)}}{\lambda_j} = \frac{\delta_j}{\lambda_j} \qquad (j = 1, 2),$$

whence, by (18) and (21),

$$t_h \leqq \sum_{k=1}^{n} e_k \theta^{(k)} \leqq \min\left(\frac{\delta_1}{\lambda_1}, \frac{\delta_2}{\lambda_2}\right) \leqq \min\left(\frac{d_1}{\lambda_1}, \frac{d_2}{\lambda_2}\right). \tag{22}$$

Therefore, for the function f defined by (10) we have here

$$f(t_h) = \tfrac{1}{2} t_h^2 \qquad (h = 1, \ldots, m).$$

From (19) we obtain

$$\sum_{h=1}^{m} t_h^{(k)} = \sum_{a=1}^{m_1} \sum_{\xi_{h2} = x_2^{(a)}} t_h^{(k)} \leqq m_1 \frac{\delta_1^{(k)}}{\lambda_1}. \tag{23}$$

Similarly

$$\sum_{h=1}^{m} t_h^{(k)} \leqq m_2 \frac{\delta_2^{(k)}}{\lambda_2}. \tag{24}$$

Denoting

$$m' = \max(m_1, m_2),$$

we have, by (20),

$$\sum_{h=1}^{m} t_h^{(k)} \leqq m' \theta^{(k)}. \tag{25}$$

Applying the above intersection argument to the curve $P = 0$, or using

(15), (18), (23) and (24) together, we see that

$$\sum_{h=1}^{m} t_h \leqq \min\left(m_1\frac{\delta_1}{\lambda_1},\ m_2\frac{\delta_2}{\lambda_2}\right). \tag{26}$$

III.2. We can now prove our first theorem.

THEOREM 1. *Let* $P(x_1, x_2) = \prod_{k=1}^{n} P_k(x_1, x_2)^{e_k}$ *satisfy* (8)-(9) *at distinct points* $\xi_h = (\xi_{h1}, \xi_{h2})$ $(h = 1, \dots, m)$. *Let* $P_k(x_1, x_2)$ *be irreducible,* $P_k/P_{k'}$ *not constant for* $k \neq k'$, *and*

$$\delta_j^{(k)} = \deg_{x_j} P_k \geqq 1 \qquad (j = 1, 2;\ k = 1, \dots, n).$$

Let

$$\delta_j = \deg_{x_j} P$$

and

$$m' = \max(m_1, m_2),$$

where m_1 *is the number of distinct* ξ_{h2} *and* m_2 *is the number of distinct* ξ_{h1}. *Further, let*

$$e = \max_{1\leqq k\leqq n} e_k.$$

Then

$$\sum_{h=1}^{m} \tfrac{1}{2} t_h^2 \leqq \frac{\delta_1}{\lambda_1}\frac{\delta_2}{\lambda_2} + \max\left(\frac{m'}{2}-1,\ 0\right)\frac{e}{\max(\lambda_1, \lambda_2)}\min\left(\frac{\delta_1}{\lambda_1},\ \frac{\delta_2}{\lambda_2}\right). \tag{27}$$

REMARK 1. The remainder term in (27) arises from the upper bound (25). Using either (23) or (24) in place of (25), it will be clear that the remainder term in (27) can also be replaced by any one of the following four quantities:

$$\left\{\begin{array}{l} \max\left(\dfrac{m_1}{2}-1,\ 0\right)\dfrac{e\delta_1}{\lambda_1\lambda_2}, \\[2ex] \max\left(\dfrac{m_2}{2}-1,\ 0\right)\dfrac{e\delta_2}{\lambda_1\lambda_2}, \\[2ex] \dfrac{m_1}{2}\dfrac{e\delta_1}{\lambda_1^2}, \\[2ex] \dfrac{m_2}{2}\dfrac{e\delta_2}{\lambda_2^2}, \end{array}\right. \tag{28}$$

and therefore by

$$\frac{e}{2\max(\lambda_1,\lambda_2)}\min\left(m_1\frac{\delta_1}{\lambda_1},\, m_2\frac{\delta_2}{\lambda_2}\right). \tag{29}$$

REMARK 2. In the special case when the curve $P = 0$ has no multiple components ($e = 1$), we shall prove an inequality somewhat stronger than all the above, namely

$$\begin{aligned}\sum_{h=1}^{m}\tfrac{1}{2}t_h^2 \leqq \frac{\delta_1}{\lambda_1}\frac{\delta_2}{\lambda_2} \\ +\min\left\{\frac{1}{2\lambda_1}\min\left(m_1\frac{\delta_1}{\lambda_1},\, m_2\frac{\delta_2}{\lambda_2}\right)-\frac{\delta_2}{\lambda_1\lambda_2},\, \frac{1}{2\lambda_2}\min\left(m_1\frac{\delta_1}{\lambda_1},\, m_2\frac{\delta_2}{\lambda_2}\right)-\frac{\delta_1}{\lambda_1\lambda_2}\right\}.\end{aligned} \tag{30}$$

PROOF OF THEOREM 1. We first assume $e = 1$, i.e. that $P = 0$ has no multiple components. By condition (8)-(9), we may clearly assume

$$\max(\lambda_1 t_h,\, \lambda_2 t_h) \geqq 1\,.$$

Hence, for the Milnor number μ_{ξ_h} of $P = 0$ at ξ_h we have by Lemma 2

$$\mu_{\xi_h} \geqq (\lambda_1 t_h - 1)(\lambda_2 t_h - 1) \qquad (h = 1, \ldots, m)\,. \tag{31}$$

We denote by l_h the line $x_2 = \xi_{h2}$. By (8)-(9), the intersection multiplicity of l_h with $P = 0$ at ξ_h is at least $\lambda_1 t_h$. Hence Lemma 3 yields

$$\nu_{\xi_h} + c_{\xi_h}(l_h) \geqq \lambda_1 t_h \qquad (h = 1, \ldots, m)\,. \tag{32}$$

On introducing homogeneous coordinates (x_0, x_1, x_2), it is easy to see that the points $(0, 1, 0)$ and $(0, 0, 1)$ at infinity are on $P = 0$ with multiplicities

$$\nu_{(0,1,0)} = D - \delta_1\,, \qquad \nu_{(0,0,1)} = D - \delta_2\,, \tag{33}$$

where $D = \deg P$. Thus, if $\delta_1 < D$, for the Milnor number and the local class at $(0, 1, 0)$ we have, by Lemma 4,

$$\mu_{(0,1,0)} + c_{(0,1,0)} \geqq (D - \delta_1)^2 + 1\,,$$

whence

$$\mu_{(0,1,0)} + \nu_{(0,1,0)} - 1 + c_{(0,1,0)} \geqq (D - \delta_1)^2 + D - \delta_1\,. \tag{34}$$

If $\delta_2 < D$, again by Lemma 2 we have

$$\mu_{(0,0,1)} \geqq (D-\delta_2-1)^2 ,$$

whence

$$\mu_{(0,0,1)} + \nu_{(0,0,1)} - 1 \geqq (D-\delta_2)^2 - D + \delta_2 . \tag{35}$$

For any point p in the projective plane such that no line containing p is a component of the curve $P=0$, the class C of the curve is given by the number of intersections of the dual curve with the line corresponding to p by duality. Choosing $p=(0,1,0)$ we obtain

$$C \geqq \sum_{h=1}^{m} c_{\xi_h}(l_h) + c_{(0,1,0)} .$$

Then Lemma 5 yields

$$\begin{aligned} D(D-1) = \sum_{\pi} (\mu_\pi + \nu_\pi - 1) + C \geqq \sum_{h=1}^{m} \left(\mu_{\xi_h} + \nu_{\xi_h} - 1 + c_{\xi_h}(l_h)\right) \\ + \begin{cases} \mu_{(0,1,0)} + \nu_{(0,1,0)} - 1 + c_{(0,1,0)} , & \text{if } \delta_1 < D , \\ 0 , & \text{if } \delta_1 = D , \end{cases} \\ + \begin{cases} \mu_{(0,0,1)} + \nu_{(0,0,1)} - 1 , & \text{if } \delta_2 < D , \\ 0 , & \text{if } \delta_2 = D . \end{cases} \end{aligned}$$

By (31), (32), (34) and (35) we have

$$\begin{aligned} D(D-1) \geqq \sum_{h=1}^{m} \{(\lambda_1 t_h - 1)(\lambda_2 t_h - 1) + \lambda_1 t_h - 1\} \\ + (D-\delta_1)^2 + D - \delta_1 + (D-\delta_2)^2 - D + \delta_2 \\ = \sum_{h=1}^{m} (\lambda_1 t_h - 1)\lambda_2 t_h + (D-\delta_1)^2 + (D-\delta_2)^2 - \delta_1 + \delta_2 . \end{aligned}$$

It follows that

$$\sum_{h=1}^{m} (\lambda_1 t_h - 1)\lambda_2 t_h \leqq D(D-1) - (D-\delta_1)^2 - (D-\delta_2)^2 + \delta_1 - \delta_2 .$$

Since the right side is maximal for $D = \delta_1 + \delta_2$, we get

$$\sum_{h=1}^{m} (\lambda_1 t_h - 1)\lambda_2 t_h \leqq 2(\delta_1 - 1)\delta_2 .$$

Hence

$$\sum_{h=1}^{m} \tfrac{1}{2} t_h^2 \leqq \frac{(\delta_1-1)\delta_2}{\lambda_1\lambda_2} + \frac{1}{2\lambda_1} \sum_{h=1}^{m} t_h . \tag{36}$$

Combining (26) and (36) we have

$$\sum_{h=1}^{m} \tfrac{1}{2} t_h^2 \leqq \frac{\delta_1}{\lambda_1}\frac{\delta_2}{\lambda_2} + \frac{1}{2\lambda_1}\min\left(m_1\frac{\delta_1}{\lambda_1},\ m_2\frac{\delta_2}{\lambda_2}\right) - \frac{\delta_2}{\lambda_1\lambda_2}\,.$$

On interchanging the variables x_1, x_2 throughout, we obtain a similar inequality with δ_1, δ_2; λ_1, λ_2 and m_1, m_2 interchanged. This proves (30).

We now turn to the general case. Let

$$D^{(k)} = \deg P_k \qquad (k = 1, \ldots, n)\,.$$

For any $k \neq k'$, the curves $P_k = 0$ and $P_{k'} = 0$ have $D^{(k)}D^{(k')}$ intersections, of which at least

$$(D^{(k)} - \delta_1^{(k)})(D^{(k')} - \delta_1^{(k')}) + (D^{(k)} - \delta_2^{(k)})(D^{(k')} - \delta_2^{(k')})$$

are at infinity, by (33). Hence the number of intersections at finite distance does not exceed

$$D^{(k)}D^{(k')} - (D^{(k)} - \delta_1^{(k)})(D^{(k')} - \delta_1^{(k')}) - (D^{(k)} - \delta_2^{(k)})(D^{(k')} - \delta_2^{(k')}) \leqq \delta_1^{(k)}\delta_2^{(k')} + \delta_1^{(k')}\delta_2^{(k)}$$

(note that the left side is maximal for $D^{(k)} = \delta_1^{(k)} + \delta_2^{(k)}$, $D^{(k')} = \delta_1^{(k')} + \delta_2^{(k')}$). Moreover, by (16)-(17) and Lemma 1, the intersections at ξ_h are at least $\lambda_1\lambda_2 t_h^{(k)} t_h^{(k')}$. It follows that

$$\lambda_1\lambda_2\sum_{h=1}^{m} t_h^{(k)} t_h^{(k')} \leqq \delta_1^{(k)}\delta_2^{(k')} + \delta_1^{(k')}\delta_2^{(k)} \qquad (k \neq k')\,.$$

Also, for any k we have, by (36),

$$\sum_{h=1}^{m} \tfrac{1}{2} t_h^{(k)2} \leqq \frac{(\delta_1^{(k)} - 1)\delta_2^{(k)}}{\lambda_1\lambda_2} + \frac{1}{2\lambda_1}\sum_{h=1}^{m} t_h^{(k)}\,.$$

Hence, by (15) and (18),

$$\begin{aligned}
\sum_{h=1}^{m} \tfrac{1}{2} t_h^2 \leqq \sum_{h=1}^{m} \tfrac{1}{2}\left(\sum_{k=1}^{n} e_k t_h^{(k)}\right)^2 &= \sum_{k=1}^{n} e_k^2 \sum_{h=1}^{m} \tfrac{1}{2} t_h^{(k)2} + \sum_{k<k'} e_k e_{k'} \sum_{h=1}^{m} t_h^{(k)} t_h^{(k')} \\
&\leqq \sum_{k=1}^{n} e_k^2\left(\frac{\delta_1^{(k)}\delta_2^{(k)}}{\lambda_1\lambda_2} + \frac{1}{2\lambda_1}\sum_{h=1}^{m} t_h^{(k)} - \frac{\delta_2^{(k)}}{\lambda_1\lambda_2}\right) + \sum_{k<k'} e_k e_{k'} \frac{\delta_1^{(k)}\delta_2^{(k')} + \delta_1^{(k')}\delta_2^{(k)}}{\lambda_1\lambda_2} \\
&= \frac{\delta_1\delta_2}{\lambda_1\lambda_2} + \frac{1}{\lambda_1}\sum_{k=1}^{n} e_k^2\left(\tfrac{1}{2}\sum_{h=1}^{m} t_h^{(k)} - \frac{\delta_2^{(k)}}{\lambda_2}\right).
\end{aligned}$$

From (20) and (25) we obtain

$$\tfrac{1}{2}\sum_{h=1}^{m} t_h^{(k)} - \frac{\delta_2^{(k)}}{\lambda_2} \leqq \frac{m'}{2}\theta^{(k)} - \theta^{(k)} = \left(\frac{m'}{2}-1\right)\theta^{(k)}.$$

Therefore, by (22),

$$\sum_{k=1}^{n} e_k^2\left(\tfrac{1}{2}\sum_{h=1}^{m} t_h^{(k)} - \frac{\delta_2^{(k)}}{\lambda_2}\right) \leqq \left(\frac{m'}{2}-1\right)\sum_{k=1}^{n} e_k^2\theta^{(k)} \leqq \max\left(\frac{m'}{2}-1,\ 0\right) e\sum_{k=1}^{n} e_k\theta^{(k)}$$
$$\leqq \max\left(\frac{m'}{2}-1,\ 0\right) e\min\left(\frac{\delta_1}{\lambda_1}, \frac{\delta_2}{\lambda_2}\right).$$

This gives

$$\sum_{h=1}^{m} \tfrac{1}{2} t_h^2 \leqq \frac{\delta_1\delta_2}{\lambda_1\lambda_2} + \max\left(\frac{m'}{2}-1,\ 0\right)\frac{e}{\lambda_1}\min\left(\frac{\delta_1}{\lambda_1}, \frac{\delta_2}{\lambda_2}\right).$$

A similar inequality holds with δ_1, δ_2 and λ_1, λ_2 interchanged. This proves (27). Q.E.D.

IV. – The admissible case.

IV.1. Lemma 6. *Let $a_1, \ldots, a_m$; $b_1, \ldots, b_m$ be non-negative real numbers, and let*

$$\begin{cases} U_i = \max\left(a_i + b_i - \sum_1^m a_h,\ \ 0\right) \\ V_i = \max\left(a_i + b_i - \sum_1^m b_h,\ \ 0\right) \end{cases} \qquad (i = 1, \ldots, m).$$

Then

$$\sum_1^m \{(a_h + b_h)^2 - U_h^2 - V_h^2\} \leqq 2\left(\sum_1^m a_h\right)\left(\sum_1^m b_h\right).$$

Proof. If $U_i = V_i = 0$ $(i = 1, \ldots, m)$, then

$$a_i + b_i \leqq \min\left(\sum_1^m a_h,\ \sum_1^m b_h\right),$$

whence

$$\sum (a_h + b_h)^2 \leqq \left\{\min\left(\sum a_h,\ \sum b_h\right)\right\}\sum (a_h + b_h)$$
$$\leqq \left(\sum b_h\right)\left(\sum a_h\right) + \left(\sum a_h\right)\left(\sum b_h\right)$$
$$= 2\left(\sum a_h\right)\left(\sum b_h\right).$$

Otherwise at least one of the numbers U_i, V_i, say U_1, is positive. Then $\sum_2^m a_h < b_1$; hence for $i = 2, \ldots, m$ we have

$$a_i \leqq \sum_2^m a_h < b_1 \leqq \sum_{h \neq i} b_h .$$

Therefore $V_2 = \ldots = V_m = 0$. Similarly, if we assume $V_1 > 0$ we get $U_2 = \ldots = U_m = 0$. In this case we have

$$b_1 - U_1 = \sum_2^m a_h , \qquad a_1 - V_1 = \sum_2^m b_h ,$$

whence

$$\sum_2^m (a_h + b_h)^2 \leqq \left(\sum_2^m (a_h + b_h)\right)^2 = (a_1 + b_1 - U_1 - V_1)^2 .$$

It follows that

$$\begin{aligned}(a_1 + b_1)^2 - U_1^2 - V_1^2 + \sum_2^m (a_h + b_h)^2 & \\ &\leqq 2\{(a_1 + b_1)^2 - (a_1 + b_1) U_1 - (a_1 + b_1) V_1 + U_1 V_1\} \\ &= 2(a_1 + b_1 - U_1)(a_1 + b_1 - V_1) \\ &= 2\left(\sum_1^m a_h\right)\left(\sum_1^m b_h\right) .\end{aligned}$$

It remains to consider the case $V_1 = V_2 = \ldots = V_m = 0$. We now define, for $i = 1, \ldots, m$,

$$b_i' = b_i - U_i = \min\left(b_i, \sum_{h \neq i} a_h\right) .$$

We have, for each i,

$$a_i + b_i' \leqq a_i + \sum_{h \neq i} a_h = \sum_1^m a_h .$$

If there exists $j \neq i$ such that $b_j' = \sum_{h \neq j} a_h$, then

$$a_i \leqq \sum_{h \neq j} a_h = b_j' \leqq \sum_{h \neq i} b_h' .$$

Otherwise for each $j \neq i$ we have $b_j' = b_j$; this, together with $V_i = 0$, yields again

$$a_i \leqq \sum_{h \neq i} b_h = \sum_{h \neq i} b_h' .$$

Hence

$$a_i + b'_i \leqq \min\left(\sum_1^m a_h, \sum_1^m b'_h\right) \qquad (i = 1, \dots, m).$$

This implies

$$\sum_1^m (a_h + b'_h)^2 \leqq 2\left(\sum_1^m a_h\right)\left(\sum_1^m b'_h\right).$$

Also

$$\begin{aligned}(a_i + b_i)^2 - U_i^2 &= (a_i + b'_i)^2 + 2U_i(a_i + b'_i)\\ &\leqq (a_i + b'_i)^2 + 2U_i \sum_1^m a_h,\end{aligned}$$

whence

$$\begin{aligned}\sum_1^m \{(a_h + b_h)^2 - U_h^2\} &\leqq \sum_1^m (a_h + b'_h)^2 + 2\left(\sum_1^m a_h\right)\left(\sum_1^m U_h\right)\\ &\leqq 2\left(\sum_1^m a_h\right)\left(\sum_1^m b'_h\right) + 2\left(\sum_1^m a_h\right)\left(\sum_1^m U_h\right)\\ &= 2\left(\sum_1^m a_h\right)\left(\sum_1^m b_h\right).\end{aligned}$$

Q.E.D.

We illustrate the geometrical meaning of Lemma 6. If the curve $P = 0$ consists only of lines parallel to the axes, each of them containing just one of the points $\xi_1, \dots, \xi_m$, we may write

$$P(x_1, x_2) = \prod_{h=1}^m (x_1 - \xi_{h1})^{\varepsilon_h}(x_2 - \xi_{h2})^{\eta_h},$$

where the multiplicities ε_h, η_h are non-negative integers. Denoting here $d_j = \deg_{x_j} P$, we have

$$d_1 = \sum_1^m \varepsilon_h, \qquad d_2 = \sum_1^m \eta_h,$$

and clearly

$$t_h \leqq \frac{\varepsilon_h}{\lambda_1} + \frac{\eta_h}{\lambda_2} \qquad (h = 1, \dots, m).$$

Therefore, by (10) and (12),

$$f(t_h) \leqq f\left(\frac{\varepsilon_h}{\lambda_1} + \frac{\eta_h}{\lambda_2}\right) = \frac{1}{2}\left\{\left(\frac{\varepsilon_h}{\lambda_1} + \frac{\eta_h}{\lambda_2}\right)^2 - U_h^2 - V_h^2\right\},$$

where

$$\begin{cases} U_i = \max\left(\dfrac{\varepsilon_i}{\lambda_1} + \dfrac{\eta_i}{\lambda_2} - \sum\limits_1^m \dfrac{\varepsilon_h}{\lambda_1},\ 0\right) \\ V_i\ = \max\left(\dfrac{\varepsilon_i}{\lambda_1} + \dfrac{\eta_i}{\lambda_2} - \sum\limits_1^m \dfrac{\eta_h}{\lambda_2},\ 0\right) \end{cases} \qquad (i = 1, \dots, m)\,.$$

From Lemma 6 with $a_h = \varepsilon_h/\lambda_1$, $b_h = \eta_h/\lambda_2$, we obtain

$$\sum_1^m f\left(\frac{\varepsilon_h}{\lambda_1} + \frac{\eta_h}{\lambda_2}\right) \leqq \left(\sum_1^m \frac{\varepsilon_h}{\lambda_1}\right)\left(\sum_1^m \frac{\eta_h}{\lambda_2}\right).$$

Hence

$$\sum_1^m f(t_h) \leqq \frac{d_1}{\lambda_1}\frac{d_2}{\lambda_2}\,,$$

i.e. Dyson's inequality with no remainder term.

IV.2. We can now deal with case (B) of the Main Theorem. More precisely, under the assumptions of Section I.5, we now let

$$P(x_1, x_2) = Q(x_1, x_2) \prod_{p=1}^{M_2} (x_1 - x_1^{(p)})^{\varepsilon_p} \prod_{q=1}^{M_1} (x_2 - x_2^{(q)})^{\eta_q}\,. \tag{37}$$

Here $x_1^{(1)}, \dots, x_1^{(m_2)}$ are the distinct ξ_{h1}; $x_2^{(1)}, \dots, x_2^{(m_1)}$ are the distinct ξ_{h2}; $m_1 \leqq M_1$, $m_2 \leqq M_2$; ε_p, η_q are non-negative integers; each of the lines $x_1 = x_1^{(p)}$ with $\varepsilon_p > 0$ and $x_2 = x_2^{(q)}$ with $\eta_q > 0$ contains at most one of the points ξ_h, and the curve $Q = 0$ is free from lines parallel to the axes.

We recall that

$$\deg_{x_j} P \leqq d_j\,,$$

$$\deg_{x_j} Q = \delta_j\,,$$

and that P satisfies condition (8)-(9).

For each $h = 1, \dots, m$, let $p(h)$ and $q(h)$ be defined by

$$\begin{cases} \xi_{h1} = x_1^{(p(h))} \\ \xi_{h2} = x_2^{(q(h))}\,, \end{cases}$$

and let

$$\begin{cases} \tilde{\varepsilon}_h = \varepsilon_{p(h)} \\ \tilde{\eta}_h = \eta_{q(h)}\,, \end{cases}$$

so that $\tilde{\varepsilon}_h$, $\tilde{\eta}_h$ are the multiplicities in $P = 0$ of the lines parallel to the axes containing ξ_h. Hence we may clearly assume

$$t_h \geqq \frac{\tilde{\varepsilon}_h}{\lambda_1} + \frac{\tilde{\eta}_h}{\lambda_2} \qquad (h = 1, \ldots, m) .$$

Let

$$\tau_h = t_h - \left(\frac{\tilde{\varepsilon}_h}{\lambda_1} + \frac{\tilde{\eta}_h}{\lambda_2}\right),$$

whence, by the argument given in the proof of (18),

$$\frac{\partial^{i_1+i_2} Q}{\partial x_1^{i_1} \partial x_2^{i_2}} (\xi_h) = 0$$

for all (i_1, i_2) satisfying

$$\frac{i_1}{\lambda_1} + \frac{i_2}{\lambda_2} < \tau_h .$$

We define the remainder term $R_Q(\boldsymbol{\xi}; \boldsymbol{\tau}; \boldsymbol{\lambda})$ by

$$\sum_{h=1}^{m} \tfrac{1}{2} \tau_h^2 = \frac{\delta_1 \delta_2}{\lambda_1 \lambda_2} + R_Q(\boldsymbol{\xi}; \boldsymbol{\tau}; \boldsymbol{\lambda}) . \tag{38}$$

Thus, by Theorem 1,

$$R_Q(\boldsymbol{\xi}; \boldsymbol{\tau}; \boldsymbol{\lambda}) \leqq \max\left(\frac{m'}{2} - 1, 0\right) \frac{e}{\max(\lambda_1, \lambda_2)} \min\left(\frac{\delta_1}{\lambda_1}, \frac{\delta_2}{\lambda_2}\right), \tag{39}$$

where e is the largest exponent in the factorization of Q.

$R_Q(\boldsymbol{\xi}; \boldsymbol{\tau}; \boldsymbol{\lambda})$ can also be bounded from above by any one of the quantities (28) or (29).

We have

$$\begin{cases} \displaystyle\sum_{h=1}^{m} \tilde{\varepsilon}_h \leqq \sum_{p=1}^{M_2} \varepsilon_p \leqq d_1 - \delta_1 , \\ \displaystyle\sum_{h=1}^{m} \tilde{\eta}_h \leqq \sum_{q=1}^{M_1} \eta_q \leqq d_2 - \delta_2 , \end{cases} \tag{40}$$

and, by (22),

$$\tau_h \leqq \min\left(\frac{\delta_1}{\lambda_1}, \frac{\delta_2}{\lambda_2}\right). \tag{41}$$

We define here

$$(42)\qquad \begin{cases} U_i = \max\left(\dfrac{\tilde{\varepsilon}_i}{\lambda_1} + \dfrac{\tilde{\eta}_i}{\lambda_2} - \sum\limits_{h=1}^{m} \dfrac{\tilde{\varepsilon}_h}{\lambda_1},\ 0\right) \\ V_i = \max\left(\dfrac{\tilde{\varepsilon}_i}{\lambda_1} + \dfrac{\tilde{\eta}_i}{\lambda_2} - \sum\limits_{h=1}^{m} \dfrac{\tilde{\eta}_h}{\lambda_2},\ 0\right) \end{cases} \qquad (i = 1, \dots, m)\,.$$

Then, by Lemma 6,

$$(43)\qquad \sum_{h=1}^{m} \tfrac{1}{2}\left\{\left(\frac{\tilde{\varepsilon}_h}{\lambda_1} + \frac{\tilde{\eta}_h}{\lambda_2}\right)^2 - U_h^2 - V_h^2\right\} \leqq \left(\sum_{h=1}^{m} \frac{\tilde{\varepsilon}_h}{\lambda_1}\right)\left(\sum_{h=1}^{m} \frac{\tilde{\eta}_h}{\lambda_2}\right).$$

IV.3. Theorem 2. *Let $P(x_1, x_2)$ be as in Section* IV.2. *In particular, P is given by* (37), *where each of the lines $x_1 = x_1^{(p)}$ with $\varepsilon_p > 0$ and $x_2 = x_2^{(q)}$ with $\eta_q > 0$ contains at most one of the points ξ_h. Then*

$$\sum_{h=1}^{m} f(t_h) \leqq \frac{d_1}{\lambda_1}\frac{d_2}{\lambda_2} + \max\left(\frac{m'}{2} - 1,\ 0\right)\frac{e}{\max(\lambda_1, \lambda_2)} \min\left(\frac{\delta_1}{\lambda_1}, \frac{\delta_2}{\lambda_2}\right).$$

Proof. Let

$$\begin{cases} u_h = \max\left(t_h - \dfrac{d_1}{\lambda_1},\ 0\right) \\ v_h = \max\left(t_h - \dfrac{d_2}{\lambda_2},\ 0\right) \end{cases} \qquad (h = 1, \dots, m)\,,$$

whence, by (11) and (12),

$$f(t_h) = \tfrac{1}{2}(t_h^2 - u_h^2 - v_h^2)\,.$$

Since

$$t_h = \tau_h + \frac{\tilde{\varepsilon}_h}{\lambda_1} + \frac{\tilde{\eta}_h}{\lambda_2}\,,$$

we have

$$(44)\qquad f(t_h) = \tfrac{1}{2}\tau_h^2 + \tfrac{1}{2}\left\{\left(\frac{\tilde{\varepsilon}_h}{\lambda_1} + \frac{\tilde{\eta}_h}{\lambda_2}\right)^2 - U_h^2 - V_h^2\right\} + \frac{\tau_h \tilde{\varepsilon}_h}{\lambda_1} + \tfrac{1}{2}V_h^2 - \tfrac{1}{2}v_h^2 + \frac{\tau_h \tilde{\eta}_h}{\lambda_2} + \tfrac{1}{2}U_h^2 - \tfrac{1}{2}u_h^2\,,$$

where U_h, V_h are defined by (42).

If $U_h = 0$ then, by (40) and (41),

$$t_h = \tau_h + \frac{\tilde{\varepsilon}_h}{\lambda_1} + \frac{\tilde{\eta}_h}{\lambda_2} \leqq \frac{\delta_1}{\lambda_1} + \sum_1^m \frac{\tilde{\varepsilon}_i}{\lambda_1} \leqq \frac{d_1}{\lambda_1},$$

whence $u_h = 0$. If $U_h > 0$, $u_h > 0$, then

$$U_h - u_h = \frac{d_1}{\lambda_1} - \sum_1^m \frac{\tilde{\varepsilon}_i}{\lambda_1} - \tau_h = \frac{\Delta_1}{\lambda_1} - \tau_h,$$

where, again by (40),

$$\Delta_1 = d_1 - \sum_1^m \tilde{\varepsilon}_i \geqq \delta_1 .$$

If $U_h > 0$, $u_h = 0$, then

$$U_h = t_h - \tau_h - \sum_1^m \frac{\tilde{\varepsilon}_i}{\lambda_1} = \frac{\Delta_1}{\lambda_1} - \tau_h - \left(\frac{d_1}{\lambda_1} - t_h\right).$$

It follows that

$$0 \leqq U_h - u_h \leqq \frac{\Delta_1}{\lambda_1} - \tau_h .$$

Therefore

$$\frac{\tau_h \tilde{\eta}_h}{\lambda_2} \leqq \frac{\Delta_1 \tilde{\eta}_h}{\lambda_1 \lambda_2} - \frac{U_h \tilde{\eta}_h}{\lambda_2} + \frac{u_h \tilde{\eta}_h}{\lambda_2},$$

whence

$$\frac{\tau_h \tilde{\eta}_h}{\lambda_2} + \tfrac{1}{2} U_h^2 - \tfrac{1}{2} u_h^2 \leqq \frac{\Delta_1 \tilde{\eta}_h}{\lambda_1 \lambda_2} + \tfrac{1}{2}\left\{\left(\frac{\tilde{\eta}_h}{\lambda_2} - U_h\right)^2 - \left(\frac{\tilde{\eta}_h}{\lambda_2} - u_h\right)^2\right\} \leqq \frac{\Delta_1 \tilde{\eta}_h}{\lambda_1 \lambda_2},$$

because $u_h \leqq U_h \leqq \tilde{\eta}_h/\lambda_2$. Similarly

$$\frac{\tau_h \tilde{\varepsilon}_h}{\lambda_1} + \tfrac{1}{2} V_h^2 - \tfrac{1}{2} v_h^2 \leqq \frac{\Delta_2 \tilde{\varepsilon}_h}{\lambda_1 \lambda_2},$$

where

$$\Delta_2 = d_2 - \sum_1^m \tilde{\eta}_i \geqq \delta_2 .$$

Then (38), (43) and (44) yield

$$\begin{aligned}\sum_{h=1}^m f(t_h) &\leqq \frac{\Delta_1 \Delta_2}{\lambda_1 \lambda_2} + R_Q(\boldsymbol{\xi}; \boldsymbol{\tau}; \boldsymbol{\lambda}) + \frac{1}{\lambda_1 \lambda_2}\left(\sum_1^m \tilde{\varepsilon}_h\right)\left(\sum_1^m \tilde{\eta}_h\right) + \frac{1}{\lambda_1 \lambda_2}\left(\Delta_1 \sum_1^m \tilde{\eta}_h + \Delta_2 \sum_1^m \tilde{\varepsilon}_h\right) \\ &= \frac{1}{\lambda_1 \lambda_2}\left(\Delta_1 + \sum_1^m \tilde{\varepsilon}_h\right)\left(\Delta_2 + \sum_1^m \tilde{\eta}_h\right) + R_Q(\boldsymbol{\xi}; \boldsymbol{\tau}; \boldsymbol{\lambda}) = \frac{d_1 d_2}{\lambda_1 \lambda_2} + R_Q(\boldsymbol{\xi}; \boldsymbol{\tau}; \boldsymbol{\lambda}) .\end{aligned}$$

This, together with (39), proves Theorem 2. Q.E.D.

V. – An upper bound for $t_1, \ldots, t_m$.

V.1. We complete the proof of the Main Theorem. Here we follow closely the notation of Section IV.2. However, the assumption of Section IV.2 that each component of the curve $P = 0$ parallel to an axis contains at most one of $\xi_1, \ldots, \xi_m$ will be replaced here by an upper bound for $t_1, \ldots, t_m$.

Under the assumptions of Section I.5 we let again

$$P(x_1, x_2) = Q(x_1, x_2) \prod_{p=1}^{M_2} (x_1 - x_1^{(p)})^{\varepsilon_p} \prod_{q=1}^{M_1} (x_2 - x_2^{(q)})^{\eta_q},$$

where $x_1^{(1)}, \ldots, x_1^{(m_2)}$ are the distinct ξ_{h1}; $x_2^{(1)}, \ldots, x_2^{(m_1)}$ are the distinct ξ_{h2}; $m_1 \leqq M_1$, $m_2 \leqq M_2$; ε_p, η_q are non-negative integers, and the curve $Q = 0$ is free from lines parallel to the axes.

Also, P satisfies (8)-(9), and d_j, δ_j, $\tilde{\varepsilon}_h$, $\tilde{\eta}_h$ are as in Section IV.2. Of course, since each line may now contain several points ξ_h, the inequalities (40) no longer hold.

We recall that for each $h = 1, \ldots, m$, ϱ_h (resp. σ_h) denotes the number of h' such that $\xi_{h'2} = \xi_{h2}$ (resp. $\xi_{h'1} = \xi_{h1}$). Thus, in place of (40), we now have

$$\left\{\begin{aligned} &\sum_{h=1}^{m} \frac{\tilde{\varepsilon}_h}{\sigma_h} \leqq \sum_{p=1}^{M_2} \varepsilon_p \leqq d_1 - \delta_1, \\ &\sum_{h=1}^{m} \frac{\tilde{\eta}_h}{\varrho_h} \leqq \sum_{q=1}^{M_1} \eta_q \leqq d_2 - \delta_2. \end{aligned}\right. \tag{45}$$

Here we define τ_h to be the greatest number such that

$$\frac{\partial^{i_1+i_2} Q}{\partial x_1^{i_1} \partial x_2^{i_2}} (\xi_h) = 0$$

for all $(i_1 . i_2)$ satisfying

$$\frac{i_1}{\lambda_1} + \frac{i_2}{\lambda_2} < \tau_h \qquad (h = 1, \ldots, m).$$

Hence, by (18),

$$t_h \leqq \tau_h + \frac{\tilde{\varepsilon}_h}{\lambda_1} + \frac{\tilde{\eta}_h}{\lambda_2}. \tag{46}$$

Again we define $R_Q(\boldsymbol{\xi}; \boldsymbol{\tau}; \boldsymbol{\lambda})$ by (38), so that (39) holds.

V.2. THEOREM 3. *Let $P(x_1, x_2)$ be as in Section* V.1, *and let*

$$t_h \leqq \frac{d_1}{\varrho_h \lambda_1} \qquad \textit{if } \tilde{\eta}_h > 0, \tag{47}$$

and

$$(48) \qquad t_h \leqq \frac{d_2}{\sigma_h \lambda_2} \qquad \text{if } \tilde{\varepsilon}_h > 0\,.$$

Then

$$\sum_{h=1}^{m} \tfrac{1}{2} t_h^2 \leqq \frac{d_1}{\lambda_1}\frac{d_2}{\lambda_2} + \max\left(\frac{m'}{2}-1,\,0\right)\frac{e}{\max(\lambda_1,\,\lambda_2)}\min\left(\frac{\delta_1}{\lambda_1},\,\frac{\delta_2}{\lambda_2}\right).$$

PROOF. By (46), (47) and (48) we have

$$t_h^2 - \tau_h^2 \leqq (t_h + \tau_h)\left(\frac{\tilde{\varepsilon}_h}{\lambda_1} + \frac{\tilde{\eta}_h}{\lambda_2}\right) \leqq \frac{d_2 \tilde{\varepsilon}_h}{\lambda_1 \lambda_2 \sigma_h} + \frac{d_1 \tilde{\eta}_h}{\lambda_1 \lambda_2 \varrho_h} + \frac{\tau_h \tilde{\varepsilon}_h}{\lambda_1} + \frac{\tau_h \tilde{\eta}_h}{\lambda_2}\,.$$

Therefore

$$(49) \qquad \sum_{h=1}^{m} \tfrac{1}{2} t_h^2 \leqq \sum_{h=1}^{m} \tfrac{1}{2} \tau_h^2 + \frac{1}{2\lambda_1\lambda_2}\left(d_1 \sum_{h=1}^{m} \frac{\tilde{\eta}_h}{\varrho_h} + d_2 \sum_{h=1}^{m} \frac{\tilde{\varepsilon}_h}{\sigma_h}\right) + \frac{1}{2\lambda_1}\sum_{h=1}^{m}\tau_h\tilde{\varepsilon}_h + \frac{1}{2\lambda_2}\sum_{h=1}^{m}\tau_h\tilde{\eta}_h\,.$$

Applying the intersection argument given in the proof of (19) to the curve $Q = 0$, we obtain

$$\lambda_1 \sum_{\xi_{h2} = x_2^{(q)}} \tau_h \leqq \delta_1 \qquad (q = 1, \ldots, m_1)\,,$$

whence

$$\frac{1}{2\lambda_2}\sum_{h=1}^{m}\tau_h\tilde{\eta}_h = \frac{1}{2\lambda_2}\sum_{q=1}^{m_1}\eta_q \sum_{\xi_{h2}=x_2^{(q)}}\tau_h \leqq \frac{\delta_1}{2\lambda_1\lambda_2}\sum_{q=1}^{M_1}\eta_q\,.$$

Similarly

$$\frac{1}{2\lambda_1}\sum_{h=1}^{m}\tau_h\tilde{\varepsilon}_h \leqq \frac{\delta_2}{2\lambda_1\lambda_2}\sum_{p=1}^{M_2}\varepsilon_p\,.$$

Hence, by (38), (45) and (49),

$$\begin{aligned}\sum_{h=1}^{m} \tfrac{1}{2} t_h^2 &\leqq \frac{\delta_1\delta_2}{\lambda_1\lambda_2} + R_Q(\boldsymbol{\xi};\,\boldsymbol{\tau};\,\boldsymbol{\lambda}) + \frac{1}{2\lambda_1\lambda_2}\left(d_1\sum_1^{M_1}\eta_q + d_2\sum_1^{M_2}\varepsilon_p + \delta_1\sum_1^{M_1}\eta_q + \delta_2\sum_1^{M_2}\varepsilon_p\right)\\ &\leqq \frac{\delta_1\delta_2}{\lambda_1\lambda_2} + R_Q(\boldsymbol{\xi};\,\boldsymbol{\tau};\,\boldsymbol{\lambda}) + \frac{1}{2\lambda_1\lambda_2}\left\{(d_1+\delta_1)(d_2-\delta_2) + (d_2+\delta_2)(d_1-\delta_1)\right\}\\ &= \frac{d_1 d_2}{\lambda_1\lambda_2} + R_Q(\boldsymbol{\xi};\,\boldsymbol{\tau};\,\boldsymbol{\lambda})\,.\end{aligned}$$

By (39), this proves Theorem 3, and therefore completes the proof of the Main Theorem. Q.E.D.

VI. – Effective measures of irrationality.

VI.1. Let $k \subset K$ be number fields with $[K:k] \geqq 2$. Bombieri's papers [1] and [2] deal with the problem of approximating effectively some $\alpha \in K \setminus k$ by numbers $\beta \in k$. However, owing to the admissibility condition for $\xi_1, \ldots, \xi_m$ in Dyson's lemma, several results in [1] and [2] are subject to the restriction that α be a generator of K over k. Here we show that this restriction can be dispensed with by means of condition (C) of our Main Theorem (see also [2], p. 196). We are now interested in applying (C) rather than using the improved form of the remainder term in (14).

Let $k \subset K$ satisfy $[K:k] = r \geqq 2$; let $\beta_1, \beta_2 \in k$ and $\alpha_1, \alpha_2 \in K$, with α_1 of degree r over k.

We first improve upon Lemma 1 of [2], by showing that for the conclusion of that lemma, the assumption that α_2 be of degree r over k is unnecessary. To this purpose, however, the condition $t \leqq \min(\theta, \theta^{-1})$ in Lemma 1 of [2] must be suitably altered.

Let $\alpha_2 \notin k$, and let s be the degree of α_2 over k. Moreover, let t, θ, δ and τ be given, with

$$\delta > 0\,, \qquad 0 < t \leqq \sqrt{\frac{2}{r}}\,, \qquad t \leqq \min\left(\theta, \frac{s}{r}\theta^{-1}\right),$$

and

$$\sqrt{2 - rt^2 + (r-1)\delta} < \tau \leqq \min(\theta, \theta^{-1})\,. \tag{50}$$

LEMMA 7. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2, t, \theta, \delta$ and τ be as above. Let $P \in k[x_1, x_2]$ be a polynomial not identically vanishing, such that*

$$\deg_{x_j} P \leqq d_j \qquad (j = 1, 2)\,,$$

with $d_2 \leqq \delta d_1$. If

$$\frac{\partial^{i_1+i_2} P}{\partial x_1^{i_1}\, \partial x_2^{i_2}}(\alpha_1, \alpha_2) = 0$$

for all (i_1, i_2) satisfying

$$\theta^{-1}\frac{i_1}{d_1} + \theta\frac{i_2}{d_2} < t\,,$$

then there exists (i_1^*, i_2^*) such that

$$\frac{\partial^{i_1^*+i_2^*} P}{\partial x_1^{i_1^*}\, \partial x_2^{i_2^*}}(\beta_1, \beta_2) \neq 0$$

and

$$\theta^{-1}\frac{i_1^*}{d_1}+\theta\frac{i_2^*}{d_2}<\tau\,.$$

PROOF. Let $(\alpha_{h1}, \alpha_{h2})$, $h=1, \ldots, r$, be the conjugates of (α_1, α_2) over k. Since α_2 has degree s over k, we may assume

$$\alpha_{h2}=\alpha_{h'2} \qquad \text{if and only if } h\equiv h' \bmod s\,,$$

whereas $\alpha_{11}, \ldots, \alpha_{r1}$ are all distinct. Clearly

$$\frac{\partial^{i_1+i_2}P}{\partial x_1^{i_1}\,\partial x_2^{i_2}}(\alpha_{h1}, \alpha_{h2})=0$$

for $\theta^{-1}(i_1/d_1)+\theta(i_2/d_2)<t$ and $h=1, \ldots, r$. If we had

$$\frac{\partial^{i_1+i_2}P}{\partial x_1^{i_1}\,\partial x_2^{i_2}}(\beta_1, \beta_2)=0$$

for all $\theta^{-1}(i_1/d_1)+\theta(i_2/d_2)<\tau$, we should obtain

$$\frac{1}{2}\tau^2+\frac{r}{2}t^2\leqq 1+\frac{r-1}{2}\frac{d_2}{d_1}\leqq 1+\frac{r-1}{2}\delta\,,$$

by (C) of the Main Theorem with $m=r+1$,

$$\xi_h=(\alpha_{h1}, \alpha_{h2}) \quad (h=1, \ldots, r)\,, \qquad \xi_{r+1}=(\beta_1, \beta_2)\,,$$

$$\lambda_1=\theta d_1\,, \qquad \lambda_2=\theta^{-1}d_2\,, \qquad t_1=\ldots=t_r=t\,, \qquad t_{r+1}=\tau\,,$$

$$\varrho_1=\ldots=\varrho_r=r/s\,, \qquad \varrho_{r+1}=1\,, \qquad \sigma_1=\ldots=\sigma_{r+1}=1\,.$$

Therefore

$$\tau\leqq\sqrt{2-rt^2+(r-1)\delta}\,,$$

which contradicts (50). Q.E.D.

Lemma 1 of [2] is the special case $s=r$ of our Lemma 7.

VI.2. We follow here the notation and terminology of [2]. We refer to [2], p. 177, for the definition of the absolute value $|\cdot|_v$, where v is a place of k, and for the height $h(\beta)$ of a number $\beta\in k$. Also, μ is an effective measure of irrationality for a number $\alpha \in K \setminus k$ over k with respect to v, if for every $\varepsilon > 0$ we can effectively find $c(\alpha, \varepsilon) > 0$ such that

$$|\alpha - \beta|_v > c(\alpha, \varepsilon)\, h(\beta)^{-\mu-\varepsilon}$$

for all $\beta \in k$.

$\mu_{\text{eff}}(\alpha, K/k; v)$ is the infimum of all such effective μ, and

$$\mu_{\text{eff}}(K/k; v) = \sup_{\alpha} \mu_{\text{eff}}(\alpha, K/k; v)\,.$$

THEOREM 4. *Let $k \subset K$ be number fields with $[K:k] = r \geqq 2$, and let $\alpha_1 \in K$ be of degree r over k. Let t satisfy*

$$\sqrt{\frac{2}{r+1}} < t < \sqrt{\frac{2}{r}}\,.$$

If there exist $\beta \in k$ and $\theta > 0$ such that

$$|\alpha_1 - \beta|_v < C_1 h(\beta)^{-(2/\theta)/(t-\sqrt{2-rt^2})} \tag{51}$$

where $C_1 = C_1(t, \theta; \alpha_1)$ is suitably defined, then

$$\mu_{\text{eff}}(\alpha_2, K/k; v) \leqq \frac{2\theta}{t - \sqrt{2 - rt^2}} \tag{52}$$

for all $\alpha_2 \in K \setminus k$, and therefore

$$\mu_{\text{eff}}(K/k; v) \leqq \frac{2\theta}{t - \sqrt{2 - rt^2}}\,.$$

We only sketch the proof of Theorem 4, since the arguments are similar to the proof of Theorem 2 of [2]. Also, the exact definition of $C_1(t, \theta; \alpha_1)$ can be found in [2].

We can find an interval

$$\sqrt{2 - rt^2 + (r-1)\delta} < \tau < \sqrt{2 - rt^2} + \varepsilon_0 < t$$

such that (51) holds with $\sqrt{2 - rt^2}$ replaced by τ. We cannot have $\theta \leqq t$, since this would contradict the Liouville bound

$$|\alpha_1 - \beta|_v > C_2 h(\beta)^{-r}\,.$$

If $\theta \geqq \frac{s}{rt}$, where s is the degree of α_2 over k, then

$$\frac{2\theta}{t-\sqrt{2-rt^2}} > \frac{2\theta}{t} \geqq \frac{2s}{rt^2} > s\,,$$

and (52) is a trivial consequence of the Liouville bound

$$\mu_{\text{eff}}(\alpha_2, K/k;\, v) \leqq s\,.$$

Hence we may assume

$$t < \min\left(\theta, \frac{s}{r}\theta^{-1}\right).$$

Then Theorem 4 is proved by combining the Thue-Siegel principle of [2] with our Lemma 7, following the arguments given in the proof of Theorem 2 of [2].

The upper bound for $\mu_{\text{eff}}(K/k;\, v)$ given by Theorem 4 can be used to extend to all $\alpha \in K\setminus k$ the results proved in [1] and [2] for the generators of K over k.

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