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Semilinear Elliptic Eigenvalue Problems on an Infinite Strip with an Application to Stratified Fluids.

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1. – Introduction.

In this paper, we consider the semilinear elliptic eigenvalue problem

$$-\nabla\cdot(a(y)\nabla u) = \lambda b(y)u + F(y, u, \lambda) \quad \text{in } S = R\times(0,1)\,, \tag{1.1}$$

$$u(x, 0) = u(x, 1) = 0\,, \qquad x\in\mathbb{R}\,, \tag{1.2}$$

$$u(x, y) \to 0 \quad \text{as } (x, y)\to\infty \quad \text{in } S. \tag{1.3}$$

We assume that $a, b\colon [0,1]\to\mathbb{R}$ and $F\colon [0,1]\times\mathbb{R}^2\to\mathbb{R}$ are as smooth as we need, and that $a>0$, $b\geqslant 0$ on $[0,1]$ with b not identically zero. We consider two cases for F:

$$\frac{F(y, u, \lambda)}{|u|^{1+\sigma}} \to A(y, \lambda) \qquad \text{as } u\to 0\,, \tag{1.4}$$

$$\frac{F(y, u, \lambda)}{|u|^{\sigma}u} \to B(y, \lambda) \qquad \text{as } u\to 0 \tag{1.5}$$

uniformly for $y\in[0,1]$ and λ in compact sets. Here $\sigma>0$ is an arbitrary fixed number. We shall give quite precise conditions on A (or B) to ensure the existence of an unbounded, connected set $\mathscr{C}\subset\mathbb{R}\times\big(H_0^1(S)\cap C_0(\bar{S})\big)$ of non-trivial classical solutions (λ, u) to (1.1)-(1.3). The particular case $\sigma=1$ in (1.4) arises in the problem for solitary waves in stratified fluids, and will be considered in section 7.

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Consider the problem of x-independent solutions of (1.1)-(1.2):

$$-\frac{d}{dy}\left(a(y)\frac{d}{dy}m(y)\right) = \lambda b(y)m(y) + F(y, m, \lambda)\,, \qquad y \in (0,1)\,, \tag{1.6}$$

$$m(0) = m(1) = 0\,. \tag{1.7}$$

Nontrivial solutions to this equation do not satisfy (1.3). Set

$$\frac{1}{\mu} = \max_{m \in H_0^1(0,1)} \frac{\int_0^1 bm^2}{\int_0^1 a(m')^2} > 0\,, \tag{1.8}$$

and let $w(y)$ denote the corresponding positive eigenfunction normalized by $\int_0^1 aw^2 = 1$. The eigenvalue μ is simple, and the global bifurcation theory of Rabinowitz [1] gives the existence of unbounded, connected sets $\mathscr{D}^+$ and $\mathscr{D}^-$ in $\mathbb{R} \times H_0^1(0, 1)$, containing $(\mu, 0)$ in their closure, such that $(\lambda, m) \in \mathscr{D}^+$ satisfies (1.6)-(1.7) and $m > 0$ on $(0, 1)$. Similarly, $m < 0$ for elements of $\mathscr{D}^-$. If we restrict attention for the moment to F satisfying (1.4), then one can ask in which direction the bifurcation occurs. A simple calculation shows that

$$\tilde{A} \equiv \int_0^1 A(y, \mu)w(y)^{2+\sigma}dy \tag{1.9}$$

greater than zero ensures that $\lambda < \mu$ for all small solutions $(\lambda, u) \in \mathscr{D}^+$. If $\tilde{A} < 0$, then $\lambda < \mu$ for small solutions in $\mathscr{D}^-$ and $\lambda > \mu$ for small solutions in $\mathscr{D}^+$. An important point to note is that $\tilde{A}$ may be readily computed as soon as a, b, and F are given. Indeed, the first eigenvalue μ and corresponding positive eigenfunction $w(y)$ may be computed numerically to great accuracy, while $A(y, \mu)$ is known from (1.4) as soon as μ is known. Our results will depend only on the *sign* of $\tilde{A}$.

In this paper, we are interested in solutions to (1.1)-(1.3) which are even functions of x, are negative or positive on S, and have certain monotonicity properties. More precisely, we shall say that (λ, u) satisfies

$(P+)$ if u is an even function of x, $u > 0$ on S, $u \in C_0^1(\bar{S})$ and $u_x(x, y) < 0$ on $(0, \infty) \times (0, 1)$.

Here $C_0^1(\bar{S})$ denotes the continuously differentiable functions on $\bar{S}$ which vanish on ∂S and at infinity. We say that (λ, u) satisfies $(P-)$ if $(\lambda, -u)$

satisfies $(P+)$. In [2], Kirchgässner showed that $\tilde{A} > 0$ leads to a local branch of solutions (λ, u) satisfying (1.1)-(1.3) and $(P+)$ with $\lambda < \mu$. If the sign of $\tilde{A}$ is reversed, then there is a local branch of solutions satisfying $(P-)$. In Theorems 1.1 and 1.2 below, we state a global version of these results. We prove in Theorem 2.3(*c*) that there are no solutions of (1.1)-(1.3) satisfying $(P+)$ or $(P-)$ if $\lambda > \mu$. Therefore, it is natural to enquire about the behavier of $\mathscr{C}$ near to $\lambda = \mu$. For $p \in [1, \infty]$, the point $\lambda = \mu$ will be called a *bifurcation point* of $\mathscr{C}$ in $L^p(S) \cap C_0(\bar{S})$ if $(\mu, 0)$ is an element of the closure of $\mathscr{C}$ in $\mathbb{R} \times (L^p(S) \cap C_0(\bar{S}))$, and an *asymptotic bifurcation point* of $\mathscr{C}$ if $|u_k|_{L^p(S)} + |u_k|_{C_0(S)} \to \infty$ as $\lambda_k \to \mu$.

THEOREM 1.1. *Assume that* (1.4) *holds and* $\tilde{A} = \int_0^1 A(y, \mu) w(y)^{2+\sigma} dy > 0$.

(*a*) *There exists an unbounded connected set* $\mathscr{C}$ *in* $\mathbb{R} \times (H_0^1(S) \cap C_0(\bar{S}))$ *and in* $\mathbb{R} \times (L^p(S) \cap C_0(\bar{S}))$, $p \in [1, \infty)$, *of solutions* (λ, u) *of* (1.1)-(1.3) *satisfying* $(P+)$.

(*b*) $\{\lambda\colon (\lambda, u) \in \mathscr{C}\} \subset (-\infty, \mu)$.

(*c*) *The point* $\lambda = \mu$ *is a bifurcation point of* $\mathscr{C}$ *in* $L^p(S) \cap C_0(\bar{S})$ *for all* $p \in [1, \infty] \cap (\sigma/2, \infty]$, *while asymptotic bifurcation for* $\mathscr{C}$ *occurs for all* $p \in [1, \infty) \cap (0, \sigma/2)$. *If* $p = \sigma/2 \geqslant 1$, *then neither occurs.*

(*d*) *If* $|\lambda - \mu| + |u|_{C_0(\bar{S})}$ *is sufficiently small, then there are no solutions* (λ, u) *of* (1.1)-(1.3) *satisfying* $(P-)$.

(*e*) *If* (1.4) *holds and* $\tilde{A} < 0$, *then* (*a*)-(*d*) *hold with the rôles of* $(P+)$ *and* $(P-)$ *reversed.*

COROLLARY 1.2. *Assume that* (1.5) *holds.*

(*a*) *If* $\tilde{B} \equiv \int_0^1 B(y, \mu) w(y)^{2+\sigma} dy > 0$, *then there is a branch* $\mathscr{C}$ *of solutions* (λ, u) *of* (1.1)-(1.3) *satisfying* $(P+)$, *and a branch* $\tilde{\mathscr{C}}$ *of solutions satisfying* $(P-)$. *Both branches* $\mathscr{C}$ *and* $\tilde{\mathscr{C}}$ *satisfy* (*a*)-(*c*) *of Theorem* 1.1.

(*b*) *If* $\tilde{B} < 0$ *and* $|\lambda - \mu| + |u|_{C_0(\bar{S})}$ *is sufficiently small, then there are no solutions of* (1.1)-(1.3) *satisfying* $(P+)$ *or* $(P-)$.

If one ignores the exceptional cases $\tilde{A} = 0$ or $\tilde{B} = 0$, then Theorem 1.1 and Corollary 1.2 give necessary and sufficient conditions for a global branch of $(P+)$ or $(P-)$ solutions to bifurcate from $\lambda = \mu$ (in $L^p(S) \cap C_0(\bar{S})$ for sufficiently large p); for example, if $\tilde{A} > 0$, there is a global branch of $(P+)$ solutions bifurcating from $(\mu, 0)$ while $\tilde{A} < 0$ ensures there are no $(P+)$ solutions near to $(\mu, 0)$. *We shall restrict our attention throughout this paper* *to proving parts* (*a*)-(*d*) *of Theorem* 1.1 since (*e*) follows immediately upon replacing u by $-u$ in (1.1)-(1.3). Corollary 1.2 follows immediately from Theorem 1.1. The motivation for this paper is twofold: a global version of the results in [2], and a theory applicable to the problem of solitary waves in stratified fluids.

A number of results have appeared recently for the stratified fluid problem [2]-[4], but they are limited by assumptions on the density $\varrho_\infty(y)$, or to small-amplitude solutions, or to variational solutions. We consider this problem in section 7 and show that Theorem 1.1 (with $\sigma = 1$) is applicable for *any* suitably smooth density ϱ_∞. The case of a discontinuous density is considered in [8]. Although a variational approach has been used in [3] to get large-amplitude solutions for a special class of smooth densities, it was not shown that the solutions form a connected branch. The connectedness of $\mathscr{C}$ is useful for proving physical properties of the solutions, and an example of this is given in section 7.2. Although elements of $\mathscr{C}$ satisfy (7.11)-(7.13), only a strict subset $\mathscr{D}$ will in general satisfy condition (7.19) which is necessary for a solution to be physically relevant. The set $\mathscr{D}$ will be defined as the maximal connected subset of $\mathscr{C}$ satisfying (7.19) and containing $(\mu, 0)$ in its closure. If $\mathscr{D}$ is properly contained in $\mathscr{C}$, then nontrivial elements of $\partial\mathscr{D}$ exist, and they will be solutions for which equality holds at a point $\tilde{y} \in (0, 1)$ in (7.19). Such waves are physically relevant, but the wave profile is not smooth at the point $(0, \tilde{y})$ [36], and so may be referred to as «extreme waves» as in [5]-[7]. Further properties of extreme waves appear in [36].

Equations such as (1.1) posed on a strip or in all of $\mathbb{R}^n$ have been studied recently by a number of authors [2]-[20]. The methods vary from variational ones for large-amplitude solution to implicit functions theorems for small solutions. The particular case of a strip has been considered in [2]-[3], [15]-[16], [18]-[20]. Some of the results and notation in this paper (particularly the decomposition (1.10)) are motivated by recent work of Kirchgässner [2]. The approach here is similar to that in [20], but the techniques are very different and more difficult. The case $F(y, u, \lambda) \equiv A|u|^{1+\sigma}$, A constant, is covered by the results in [20]. In section 2, we begin by studying (1.1) with Dirichlet boundary conditions on a rectangle S_n, and use the theory of Rabinowitz [1] to find an unbounded, connected branch $\mathscr{C}_n$ of nontrivial solutions which bifurcate from a point $(\mu_n, 0)$. The solutions u are positive on S_n, are even functions of x, and satisfy $u_x(x, y) < 0$ on $(0, n] \times (0, 1)$. The eigenvalues μ_n are shown to converge to μ as $n \to \infty$. If one takes a bounded open set $U \subset \mathbb{R} \times \big(L^{1+\sigma}(S) \times C_0(\bar{S})\big)$ with $(\mu, 0) \in U$, then there exist $(\lambda_n, u_n) \in C_n \cap \partial U$ for all large n. We may assume that $\lambda_n \to \lambda$ and $u_n \rightharpoonup u$ in $L^{1+\sigma}(S)$ as $n \to \infty$. In Theorem 2.3 we show that

$\lambda \leqslant \mu$, and that $\lambda < \mu$ implies the convergence of u_n in $H_0^1(S) \cap L^p(S) \cap C_0(\bar{S})$, $p \in [1, \infty)$, to a solution (λ, u) of (1.1)-(1.3) satisfying $(P+)$. Section 3 is devoted to the remaining case $\lambda = \mu$. We prove in Theorem 3.2 that if $(\mu, u) \in \mathbb{R} \times C_0(\bar{S})$ satisfies (1.1)-(1.3) and $u \geqslant 0$ on S with $u_x(x, y) \leqslant 0$, $(x, y) \in (0, \infty) \times (0, 1)$, then $u \equiv 0$ in S. Hence, if $\lambda_n \to \mu$ and $u_n \rightharpoonup u$ in $L^{1+\sigma}(S)$ with $|u_n|_{C_0(\bar{S})}$ bounded, then $u_n \rightharpoonup 0$ in $L^{1+\sigma}(S)$. In section 4, we show that this convergence is strong in $L^p(S)$ for $p \in [1, \infty) \cap (\sigma/2, \infty)$; in particular, $(\lambda_n, u_n) \to (\mu, 0)$ in $\mathbb{R} \times (L^{1+\sigma}(S) \cap C_0(\bar{S}))$. (The main tool is a decomposition of u_n in the form

$$u_n(x, y) = v_n(x) w(y; \lambda_n) + z_n(x, y) \tag{1.10}$$

where w satisfies (2.11)-(2.12) and

$$v_n(x) = \int_0^1 a(y) u_n(x, y) w(y; \lambda_n) dy .)$$

We show that if λ_n is near to μ, then the dominant term in (1.10) is $v_n(x) w(y; \lambda_n)$, and we then study the *ordinary* differential equation (4.18) for $v_n(x)$. Since $(\lambda_n, u_n) \in \partial U$ and $(\lambda_n, u_n) \to (\mu, 0)$, we must have $(\mu, 0) \in \partial U$. However, $(\mu, 0)$ lies in the interior of U. This contradiction means that $\lambda_n \to \lambda < \mu$, and so there is a solution (λ, u) of (1.1)-(1.3) satisfying $(P+)$ on the boundary of every bounded, open set U which contains $(\mu, 0)$ in its interior. In section 5, we combine this with a standard result [5, Theorem A6] to prove the existence of an unbounded, connected set $\mathscr{C} \subset \mathbb{R} \times (L^{1+\sigma}(S) \cap C_0(\bar{S}))$ of solutions (λ, u) of (1.1)-(1.3) satisfying $(P+)$. The remaining parts of Theorem 1.1 are proven in sections 5 and 6. Section 7 is devoted to applying our results to the problem of solitary waves in stratified fluids.

2. – Preliminary estimates.

2.1 *Notation.*

For a domain $\Omega \subset \mathbb{R}^2$, a non-negative integer m and $p \in [1, \infty]$, we let $W^{m,p}(\Omega)$ denote the Sobolev space of functions which along with their weak derivatives of orders up to and including m lie in $L^p(\Omega)$. The case $m = 0$ is written as $L^p(\Omega)$. The completion of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$ is denoted by $H_0^1(\Omega)$. The spaces $C^k(\bar{\Omega})$ and $C^{k,\beta}(\bar{\Omega})$, k a non-negative integer and $\beta \in (0, 1]$, have their usual meaning. We let $C_0^k(\bar{\Omega})$ and $C_0^{k,\beta}(\bar{\Omega})$ denote those elements of $C^k(\bar{\Omega})$ and $C^{k,\beta}(\bar{\Omega})$, respectively, which vanish on $\partial\Omega$; if Ω is unbounded, we add the condition that $u(x, y) \to 0$ as $(x, y) \to \infty$ in Ω. The rectangle $(-1, 1) \times (0, 1)$ will be denoted by Q.

For any $c > 0$, let S_c denote the open rectangle $(-c, c) \times (0, 1)$. Finally, a function $u \in L^1_{\text{loc}}(\Omega)$ is said to be a weak solution of $-\nabla \cdot (a\nabla u) = f$ in Ω for $f \in L^1_{\text{loc}}(\Omega)$ if

$$-\iint_\Omega u\nabla \cdot (a\nabla\varphi) = \iint_\Omega f\varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

2.2 *The case of bounded rectangles.*

In this section, we shall study the following equation

$$-\nabla \cdot \big(a(y)\nabla u\big) = \lambda b(y) u + F(y, u, \lambda) \quad \text{in } S_n, \tag{2.1}$$

$$u = 0 \quad \text{on } \partial S_n, \tag{2.2}$$

where a, b, and F are as before, and n is a large positive integer, say $n \geqslant 2$. The remark after Theorems 1.1 and 1.2 allows us to restrict attention to the case

$$\frac{F(y, u, \lambda)}{|u|^{1+\sigma}} \to A(y, \lambda) \quad \text{as } u \to 0 \tag{2.3}$$

uniformly for $y \in [0, 1]$ and bounded λ. Here $\sigma > 0$ is arbitrary, but fixed.

Unless stated otherwise, we shall assume through the remainder of this paper that

$$\tilde{A} \equiv \int_0^1 A(y, \mu) w(y)^{2+\sigma} dy > 0 \tag{2.4}$$

where $w(y)$ denotes the positive eigenfunction of the linear equation

$$-\frac{d}{dy}\left(a(y)\,\frac{dw}{dy}\right) = \mu b(y) w(y) \quad \text{on } (0, 1), \tag{2.5}$$

$$w(0) = w(1) = 0. \tag{2.6}$$

Equivalently, $w(y)$ is the positive maximizer of the variational problem

$$\frac{1}{\mu} = \max_{w \in H_0^1(0,1)} \frac{\int_0^1 bw^2}{\int_0^1 a(w')^2}. \tag{2.7}$$

We are interested in solutions (λ, u) to (2.1)-(2.3) which satisfy

(P_n+) $u>0$ on S_n, u is even in x, $u \in C_0^1(\bar{S}_n)$, and $u_x(x,y)<0$ on $(0,n]\times(0,1)$.

LEMMA 2.1. *Assume that $u \in C_0(\bar{S})$ is a weak solution of* (2.1) *which is non-negative and not identically zero on S_n. Then*

$$(2.8)\quad (a)\ u \in C_0^{2,\beta}(\bar{S}_n),\ \beta \in (0,1) \quad \textit{and} \quad |u|_{C_0^{2,\beta}} \leqslant g(\lambda, |u|_{C(\bar{Q})}, \beta),$$

where g is a continuous function on $\mathbb{R}\times[0,\infty)\times(0,1)$ with $g(\cdot,0,\cdot)=0$.

(*b*) *The solution (λ, u) satisfies (P_n+) and*

$$(2.9)\quad u_{xy}(-n,-1),\ u_{xy}(n,1)>0 \quad \textit{and} \quad u_{xy}(-n,1),\ u_{xy}(n,-1)<0.$$

(*c*) *If $u \in C_0(\bar{S})$ is a weak solution of* (1.1)-(1.3) *which is non-negative and not identically zero, is an even function x, and $u_x \leqslant 0$ on $(0,\infty)\times(0,1)$, then (λ, u) satisfies $(P+)$ and* (2.8) *holds, with S_n replaced by S.*

PROOF. (*a*) We sketch the proof and refer the reader to Lemma 2.1 and Theorem 2.2 of [20] for the details. If we denote the right-hand side of (2.1) by G, then $G \in L^q(S_n)$ for all $q \in (1,\infty)$, and so the estimates of Agmon [37] give

$$|u|_{W^{2,q}((m-\frac{1}{2},m+\frac{1}{2})\times(0,1))} \leqslant \text{const}\left(|u|_{L^q((m-1,m+1)\times(0,1))} + |G|_{L^q((m-1,m+1)\times(0,1))}\right)$$

for any integers m with $(m-1,m+1)\subset(-n,n)$. The particular case $m=0$ and a Sobolev embedding theorem gives

$$|u|_{C^{1,\beta}([m-\frac{1}{4},m+\frac{1}{4}]\times[0,1])} \leqslant h(\lambda, |u|_{C(\bar{Q})}, \beta),$$

where h has the same properties as g. Since u takes its maximum on $x=0$ by (*b*) below, we have $|u|_{C_0(\bar{S}_n)} \leqslant h(\lambda, |u|_{C(\bar{Q})}, \beta)$. It follows that

$$|G|_{C_0(\bar{S})} \leqslant j(\lambda, |u|_{C(\bar{Q})}, \beta)$$

and the use of this with the interior and boundary estimates in [20] yields (2.8).

(*b*) Since $u \geqslant 0$ in S_n by hypothesis, a variant of the maximum principle due to Serrin [38] ensures that $u>0$ in S_n. The remaining conditions for membership in (P_n+) and (2.9) follow the proof of Theorem 7.2 of [20].

(*c*) The proof that $u > 0$ is as before. If one differentiates (1.1) with respect to x, there results a linear elliptic equation for u_x. Since $u_x \leqslant 0$ on $(0, \infty) \times (0, 1)$, the maximum principle [38] ensures that $u_x \equiv 0$ or $u_x < 0$ on this set. The result $u > 0$ shows that the latter holds. q.e.d.

We now proceed to the existence of a global branch of solutions to (2.1)-(2.2).

THEOREM 2.2. (*a*) *If*

$$\frac{1}{\mu_n} = \max_{u \in H_0^1(S_n)} \frac{\iint_{S_n} bu^2}{\iint_{S_n} a|\nabla u|^2}, \tag{2.10}$$

then $\mu_n > 0$ is the smallest eigenvalue of the equation

$$\begin{aligned} -\nabla(a(y)\nabla u) &= \lambda u \quad \text{on } S_n, \\ u &= 0 \quad \text{on } \partial S_n. \end{aligned}$$

The eigenvalue μ_n is simple and $\mu_n \downarrow \mu$ as $n \to \infty$, where μ is given in (1.8).

(*b*) *There exists an unbounded, connected set $\mathscr{C}_n \subset \mathbb{R} \times C^{2,\beta}(\overline{S}_n)$, $\beta \in (0, 1)$, with $(\mu_n, 0)$ in its closure, of non-trivial solutions (λ, u) of* (2.1)-(2.2) *satisfying* (P_n+).

PROOF. (*a*) The fact that the maximum is taken in (2.10) is a consequence of the compact embedding of $H_0^1(S_n)$ into $L^2(S_n)$ and the positivity of a on $[0, 1]$. For $f \in C_0^{\beta}(\overline{S}_n)$, let $u = G_n(f) \in C_0^{2,\beta}(\overline{S}_n)$ denote the unique solution of $-\nabla \cdot (a(y)\nabla u) = f$ on S_n with $u = 0$ on ∂S_n. Set $E = C_0^{2,\beta}(\overline{S}_n)$ and let $E+ = \{u \in E : u > 0$ on S_n, $\partial u/\partial \boldsymbol{n} < 0$ on ∂S_n, $u_{xy}(-n, -1)$, $u_{xy}(n, 1) > 0$, and $u_{xy}(n, -1)$, $u_{xy}(-n, 1) < 0\}$, where $\boldsymbol{n}$ denotes the outward normal to ∂S_n away from the four corners. The set $E+$ is a cone with interior in E. The maximum principle plus Lemma 2 in [38] ensures that G_n maps $\overline{E+} \to \overline{E+}$ and G_n maps non-trivial elements of the boundary of $\overline{E+}$ into $E+$. Hence, G_n is a strongly positive operator in the sense of Krein-Rutman [39], and so μ_n is a simple eigenvalue.

If $u \in C_0^{\infty}(S_n)$, then (1.8) gives

$$\iint_{S_n} bu^2 \leqslant \frac{1}{\mu} \iint_{S_n} a(u_y)^2 < \frac{1}{\mu} \iint_{S_n} a|\nabla u|^2,$$

and so $\mu_n > \mu$. Define $v_n(x, y) = \cos(\pi x/2n) w(y) \in H_0^1(S_n)$, where w denotes the positive eigenfunction corresponding to (1.8). A calculation yields

$$\frac{1}{\mu} > \frac{1}{\mu_n} \geqslant \frac{\iint_{S_n} bv_n^2}{\iint_{S_n} a|\nabla v_n|^2} = \frac{1}{\mu}\{1 + O(1/n^2)\},$$

whence $\mu_n \to \mu$ as $n \to \infty$. Since $S_m \subset S_n$ if $n > m$, equation (2.10) ensures that the μ_n are monotone decreasing.

(*b*) This is an immediate consequence of [1] and Lemma 2.1. q.e.d.

Since $\lambda_n \to \mu$ and there is a global branch $\mathscr{C}_n$ bifurcating from each $(\mu_n, 0)$, one's intuition suggests there might be a limiting branch $\mathscr{C}$ bifurcating from $(\mu, 0)$. We shall show that this is so, but one must be careful. For example, the same arguments for Theorem 2.2 give the existence of an unbounded branch of solutions (λ, u) bifurcating from $(\mu_n, 0)$ and satisfying (2.1)-(2.2) and (P_n-). Theorem 1.1(*d*) shows that these sets do not converge to a branch of solutions of (1.1)-(1.3) satisfying $(P-)$. Another reason to be careful is the following observation: the condition $\tilde{A} > 0$ ensures that $\mathscr{C}_n$ initially branches to the left of $\lambda = \mu_n$, but it is not obvious that the branch crosses the line $\lambda = \mu < \mu_n$. Indeed, for many interesting applications (Theorem 7.2) one has an *a priori* bound on $|u|_{C_0(\bar{S}_n)}$, independently of n, for elements of $\mathscr{C}_n$, and $\mathscr{C}_n$ is unbounded in the positive λ-direction.

A final problem is the topology in which the limit of $\mathscr{C}_n$ to $\mathscr{C}$ is to be taken. In some spaces, $\mathscr{C}$ is an unbounded, connected set containing $(\mu, 0)$ in its closure, while viewed in other spaces the point $\lambda = \mu$ is an asymptotic bifurcation point. In certain physical problems (Theorem 7.2) the branch $\mathscr{C}$ is not even unbounded when viewed as a subset of $\mathbb{R} \times C_0(\bar{S})$.

Although the results of sections 3 and 4 would allow us to take the limit of the $\mathscr{C}_n$ in $\mathbb{R} \times \big(L^p(S) \cap C_0(\bar{S})\big)$ with $p \in [1, \infty) \cap (\sigma/2, \infty)$, we shall think of the limit being taken in $\mathbb{R} \times \big(L^{1+\sigma}(S) \cap C_0(\bar{S})\big)$ for definiteness.

Let U be a bounded open set in $\mathbb{R} \times \big(L^{1+\sigma}(S) \cap C_0(\bar{S})\big)$ with $(\mu, 0) \in \bar{U}$. After extending elements of $\mathscr{C}_n$ as zero outside S_n, we have $u \in L^{1+\sigma}(S) \cap C_0(\bar{S})$. Since $\mu_n \to \mu$ and the $\mathscr{C}_n$ are unbounded and connected, there exist $(\lambda_n, u_n) \in \mathscr{C}_n \cap \partial U$ for all large n. Without loss of generality, we may assume that $\lambda_n \to \lambda$ and $u_n \rightharpoonup u$ in $L^{1+\sigma}(S)$ as $n \to \infty$. (Here $\rightharpoonup$ denotes weak convergence.) The following theorem shows that λ does not exceed μ, and proves strong convergence when $\lambda < \mu$. Before proving it, we need some auxilliary functions: for each $\lambda \in \mathbb{R}$, let

$$\alpha(\lambda) = \max_{u \in H_0^1(0,1)} \left\{ \frac{\lambda \int_0^1 bu^2 - \int_0^1 a(u')^2}{\int_0^1 au^2} \right\}.$$

Clearly, $\alpha(0) < 0$, α is monotone increasing, and $\alpha(\mu) = 0$ by (1.8). Let

$w(y;\lambda)$ denote the positive maximizer normalized by $\int_0^1 a(y)w^2(y;\lambda)dy = 1$. The function $w(y)$, introduced after (1.8), is merely $w(y;\mu)$ in our new notation. Clearly, $w(\cdot\,;\lambda)$ satisfies the equation

$$\alpha(\lambda)a(y)w(y;\lambda) = \lambda b(y)w(y;\lambda) + \frac{d}{dy}\left(a(y)\frac{d}{dy}w(y;\lambda)\right), \quad y\in(0,1), \tag{2.11}$$

$$w(0;\lambda) = w(1;\lambda) = 0. \tag{2.12}$$

THEOREM 2.3. *Assume that* (λ_n, u_n) *satisfy* (2.1)-(2.2) *and* (P_n+) *with* $|u_n|_{C_0(\bar S)} + |u_n|_{L^q(S)} \leqslant$ const., *for some* $q\in[1,\infty)$. *If* $\lambda_n\to\lambda$, *then*

(*a*) $\lambda\leqslant\mu$.

(*b*) *If* $\lambda<\mu$, *then there exists* $\beta\in(0,1)$ *and* $D>0$, *both depending on* λ *but not* n, *such that* $|u_n(x,y)|\leqslant D\exp(-\beta|x|)$, $(x,y)\in S_n$. *If* $u_n\rightharpoonup u$ *in* $L^q(S)$, *then* $u_n\to u$ *in* $H_0^1(S)\cap L^p(S)\cap C_0(\bar S)$ *for all* $p\in[1,\infty)$. *The pair* (λ,u) *satisfies* (1.1)-(1.3) *and* $(P+)$.

(*c*) *If* (λ,u) *satisfies* (1.1)-(1.3) *and* $(P+)$ *or* $(P-)$, *then* $\lambda\leqslant\mu$. *If* $\lambda<\mu$, *then* $u\in H_0^1(S)\cap L^p(S)$, $p\in[1,\infty)$, *and* $|u(x,y)|\leqslant D\exp(-\beta|x|)$, *where* D *and* β *depend on* λ.

PROOF. (*a*) Assume that $\lambda>\mu$. Set $u_n(x,y) = v_n(x)w(y;\lambda_n) + z_n(x,y)$ where v_n is defined by

$$v_n(x) = \int_0^1 a(y)u_n(x,y)w(y;\lambda_n)dy.$$

Note that v_n is positive on $(0,n)$, $v_n'(x)<0$ on $(0,n)$, and $\int_0^1 a(y)z_n(x,y)\cdot$ $\cdot w(y;\lambda_n)dy = 0$ for all $x\in[-n,n]$. If we multiply (2.1) by $w(y;\lambda_n)$, integrate over $(0,1)$, and use (2.11)-(2.12), there results

$$-v_n''(x) = \alpha(\lambda_n)v_n(x) + \int_0^1 F(y,u_n(x,y),\lambda_n)w(y;\lambda_n)dy, \quad x\in(-n,n). \tag{2.13}$$

Since the u_n satisfy (P_n+) and are bounded in $L^q(S)$, we have

$$\text{const}\geqslant\int_0^n\int_0^1 u_n(x,y)^q dy\,dx \geqslant x\int_0^1 u_n(x,y)^q dy,$$

whence

$$(2.14)\qquad \int_0^1 u_n(x,y)^q\,dy \leqslant \text{const}/x\,, \qquad x\in(0,n)\,,$$

where the constant is independent of n. Now Lemma 2.1 ensures that $|u_n|_{C^{1,\beta}(\overline{S}_n)} \leqslant \text{const}$, and so $|\partial u_n/\partial y|_{L_\infty(S_n)} \leqslant \text{const}$. Combining this with (2.14) yields

$$(2.15)\qquad |u_n(x,y)| \leqslant \text{const}/x^\delta\,, \qquad (x,y)\in(0,n)\times(0,1)\,,$$

where $1/\delta = 2q$ and the constant is independent of n. Now $\lambda_n \to \lambda > \mu$ by hypothesis, and so we may assume that $\alpha(\lambda_n) > \alpha\big((\lambda+\mu)/2\big) > 0$ for all large n. If we use (1.4) and (2.15) in (2.13), then there exists X, independent of n, such that

$$(2.16)\qquad -\,v_n''(x) \geqslant \frac{1}{2}\alpha\left(\frac{\lambda+\mu}{2}\right)v_n(x)\,, \qquad x\in(X,n)\,,$$

for all sufficiently large n. However, standard oscillation theorems show that a function positive on $(-\,n,n)$ cannot satisfy (2.16) as $n\to\infty$. Hence, we have a contradiction, and so $\lambda \leqslant \mu$.

(*b*) Since $\lambda_n \to \lambda < \mu$, we shall restrict attention to all large n such that $\lambda_n < (\lambda+\mu)/2 < \mu$. Equations (1.4) and (2.15) ensure the existence of X, independent of n, such that

$$|F(y, u_n(x,y), \lambda_n)| \leqslant \text{const}\, u_n(x,y)^{1+\sigma}\,, \qquad (x,y)\in(X,n)\times(0,1)\,,$$

and the constant is independent of n. Since u_n is bounded in $C_0(\overline{S}_n)$, the quantity $|F(y, u_n, \lambda_n)|$ is bounded on $[-\,X, X]\times[0,1]$, independently of n. If we multiply (2.1) by u_n, integrate over $(x,n)\times(0,1)$, and then use (1.8), there results

$$(2.17)\qquad \left(1-\frac{\lambda_n}{\mu}\right)\int_x^n\!\!\int_0^1 a|\nabla u_n|^2 \leqslant \int_x^n\!\!\int_0^1 \{a|\nabla u_n|^2 - \lambda_n b u_n^2\}$$
$$\leqslant -\int_0^1 a(y)\,u_n(x,y)(u_n)_x(x,y)\,dy + \int_x^n\!\!\int_0^1 u_n|F(y,u_n,\lambda_n)|$$
$$\leqslant -\int_0^1 a(y)\,u_n(x,y)(u_n)_x(x,y)\,dy + \text{const}\max_{t\geqslant X}|u_n(t,y)|^\sigma \int_x^n\!\!\int_0^1 a u_n^2$$

for all $x \in (X, n)$. Since $\lambda_n \to \lambda < \mu$ by hypothesis, the left-hand side of (2.17) is bounded below by a constant C, independent of n, times $\int_x^n \int_0^1 a u_n^2$. We may use (2.15) and restrict attention to such large X that

$$\text{const} \max_{t \geqslant X} |u_n(t, y)|^\sigma \int_x^n \int_0^1 a u_n^2 \leqslant \frac{C}{2} \int_x^n \int_0^1 a u_n^2$$

for all $x \in (X, n)$. It follows that

$$\frac{C}{2} \int_x^n \int_0^1 a u_n^2 + \int_0^1 a(y) u_n(x, y) (u_n)_x(x, y)\, dy \leqslant 0\,, \qquad x \in (X, n)\,,$$

or, equivalently,

$$C T_n(x) - T_n''(x) \leqslant 0\,, \qquad x \in (X, n)\,,$$

where

$$T_n(x) = \int_x^n \int_0^1 a u_n^2.$$

If we multiply this inequality by $-T_n'(x) \geqslant 0$ and integrate over (x, n), $x \in (X, n)$, then $T_n'(x) + \sqrt{C}\, T_n(x) \leqslant 0$. This implies the bound

$$\int_x^n \int_0^1 a u_n^2 = T_n(x) \leqslant \text{const} \exp\left(-\sqrt{C} x\right), \qquad x \in (X, n)\,.$$

Since

$$\int_0^1 u_n^2(x, y)\, dy \leqslant \int_{x-1}^x \int_0^1 u_n^2 \leqslant \text{const} \exp\left(-\sqrt{C} x\right), \qquad x \in (X + 1, n)$$

and $|\nabla u_n|_{L^\infty(S_n)} \leqslant \text{const}$ by Lemma 2.1, it follows that

$$|u_n(x, y)| \leqslant \text{const} \exp\left(-\beta |x|\right), \qquad (x, y) \in \bar{S}_n\,, \tag{2.18}$$

where $\beta = \sqrt{C}/2$.

Assume that $u_n \rightharpoonup u$ in $L^q(S)$ and $\lambda_n \to \lambda < \mu$. Lemma 2.1 and equation (2.15) together give $u_n \to u$ in $C_0(\bar{S})$. The use of equation (2.18) shows that $u_n \to u$ in $L^p(S)$ for all $p \geqslant 1$. To prove that $u_n \to u$ in $H_0^1(S)$, we begin with (2.8) of Lemma 2.1 which gives convergence on bounded sets. For the convergence in a neighborhood of infinity, we use (2.17):

$$\left(1-\frac{\lambda_n}{\mu}\right)\int_X^n\int_0^1 a|\nabla u_n|^2 \leqslant -\int_0^1 a(y)u_n(X,y)(u_n)_x(X,y)\,dy + \text{const}\int_X^n\int_0^1 u_n^{2+\sigma},$$

and note that the right-hand side may be made arbitrarily small by taking X large and using (2.18).

It is clear that $(\lambda, u) \in \mathbb{R}\times C_0(\bar{S})$ is a weak solution of (2.1), and so $u \in C^{2,\beta}(\bar{S})$ by Lemma 2.1. We now show that u is not identically zero, whence (λ, u) satisfies $(P+)$ by Lemma 2.1. Assume the contrary, so that $u \equiv 0$ and $u_n \to 0$ in $C_0(\bar{S})$. If we take $x = -n$ in (2.17), then

$$C\iint_{S_n} a u_n^2 \leqslant \left(1-\frac{\lambda_n}{\mu}\right)\iint_{S_n} a|\nabla u_n|^2 \leqslant \iint_{S_n} u_n|F(y, u_n, \lambda_n)| \leqslant \text{const}\iint_{S_n} u_n^{2+\sigma}$$

by (1.4). Since $u_n \to 0$ in $C_0(\bar{S})$ as $n \to \infty$, this relation is clearly impossible, and so we have a contradiction.

(*c*) The proof of (*c*) follows that for (*a*) and (*b*). q.e.d.

REMARK. The assumption that $|u_n|_{L^q(S)} \leqslant \text{const}$, for some $q \in [1, \infty)$, was only needed to show that $u_n \to 0$ uniformly at infinity (cf. (2.15)). If one drops the bound in $L^q(S)$ and assumes instead that $u_n \to 0$ uniformly at infinity, then parts (*a*) and (*b*) hold. However, if one merely assumes u_n bounded in $C_0(\bar{S})$, then the proof of (*a*) does not hold. It is this observation that forces us to consider solutions in $\mathbb{R}\times(L^q(S) \cap C_0(\bar{S}))$ instead of $\mathbb{R}\times C_0(\bar{S})$.

3. – The case $\lambda = \mu$.

We now consider the case that $\lambda_n \to \mu$, $u_n \rightharpoonup u$ in $L^p(S)$ for some $p \in (1, \infty)$, and u_n bounded in $C_0(\bar{S})$. It follows from Lemma 2.1 and (2.15) that $u_n \to u$ in $C_0(\bar{S})$ and that $u \in L^p(S) \cap C_0(\bar{S})$ satisfies (1.1)-(1.3), u is an even function of x, $u \geqslant 0$ on S, and $u_x(x, y) \leqslant 0$, $(x, y) \in (0, \infty)\times(0, 1)$. In Theorem 3.2 we show that such a u is identically zero. (The basis of our proof is the estimate (3.5) which is a forerunner of certain results in section 4.) It then follows that $u_n \rightharpoonup 0$ in $L^p(S)$ and $u_n \to 0$ in $C_0(\bar{S})$, and section 4 is devoted to showing that this convergence in L^p is strong if p is sufficiently large: $p \in (1, \infty) \cap (\sigma/2, \infty)$.

For each $\lambda \in \mathbb{R}$, let $w(\cdot\,; \lambda)$ be the positive eigenfunction of (2.11)-(2.12) normalized by $\int_0^1 a(y)w(y;\lambda)^2 dy = 1$. Set

$$\frac{1}{\tau(\lambda)} = \max_{\substack{z\in H_0^1(0,1) \\ \int_0^1 a(y)z(y)w(y;\lambda)dy=0}} \frac{\int_0^1 bz^2}{\int_0^1 a(z')^2}\,.$$

Clearly τ is a continuous function of λ and $\tau(\lambda)\geqslant\mu$, $\lambda\in\mathbb{R}$, by (1.8). The following lemma will be needed in the proof of Theorem 3.2 and in section 4.

LEMMA 3.1. *$\tau(\lambda)>\mu$ for all $\lambda\in\mathbb{R}$.*

PROOF. The maximizing function z satisfies

$$-\frac{d}{dy}\left(a(y)\frac{dz}{dy}\right) = \tau(\lambda)b(y)z(y) + \beta a(y)w(y;\lambda)\,, \quad y\in(0,1)\,,$$

where β is a parameter. If $\tau(\lambda)=\mu$, then multiplying this equation by $w(y;\mu)$ and integrating over $(0,1)$ yields

$$\beta\int_0^1 a(y)w(y;\lambda)w(y;\mu)dy = 0\,,$$

whence $\beta=0$ since the integrand is strictly positive on (0,1). However, $\tau(\lambda)=\mu$ and $\beta=0$ in the differential equation imply that $z(y)=\text{const}\,w(y;\mu)$, so that

$$0=\int_0^1 a(y)w(y;\lambda)z(y)dy = \text{const}\int_0^1 a(y)w(y;\lambda)w(y;\mu)\,dy\,.$$

This shows that $z\equiv 0$ which is a contradiction. q.e.d.

THEOREM 3.2. *Let $u\in C_0(\bar{S})\cap C^2(\bar{S})$ be a solution of* (1.1)-(1.3) *with $\lambda=\mu$, $u\geqslant 0$ on S, and $u_x(x,y)\leqslant 0$ for $(x,y)\in(0,\infty)\times(0,1)$. Then $u\equiv 0$ in S.*

PROOF. We assume that u is not identically zero so that (λ, u) satisfies $(P+)$ by Lemma 2.1(*c*). We shall derive a contradiction from this assumption. Set $u(x,y)=v(x)w(y)+z(x,y)$, where $w(y)=w(y;\mu)$ has been normalized by $\int_0^1 aw^2=1$, and $v(x)=\int_0^1 a(y)u(x,y)w(y)dy$. Note that $v>0$ on $(-\infty,\infty)$, $v'<0$ on $(0,\infty)$, and $\int_0^1 a(y)z(x,y)w(y)dy=0$ for all x. Equation (2.13) gives

$$(3.1)\qquad -v''(x)=\int_0^1 F(y,u,\mu)w(y)\,dy=\int_0^1 A(y,\mu)w(y)\big(v(x)w(y)\big)^{1+\sigma}dy$$
$$+\int_0^1 A(y,\mu)w(y)\{u(x,y)^{1+\sigma}-\big(v(x)w(y)\big)^{1+\sigma}\}\,dy$$
$$+\int_0^1 w(y)\{F(y,u,\mu)-A(y,\mu)u^{1+\sigma}\}\,dy\,.$$

The first term on the right of (3.1) is $\tilde{A}v(x)^{1+\sigma}$, where $\tilde{A}>0$ by (2.4). Since $u(x,y)\to 0$ as $|x|\to\infty$ and $F(y,u,\mu)\sim A(y,\mu)u^{1+\sigma}$ for such values, we have

$$(3.2)\qquad -v''(x)> cv(x)^{1+\sigma}-d\int_0^1 w(y)|z(x,y)|^{1+\sigma}dy\,,\qquad x\in(X,\infty)$$

where $c=\tilde{A}/2$, $d>0$, and X is sufficiently large.

We shall restrict attention in the remainder of this proof to $x\geqslant X$. If we multiply (3.2) by $-v'(x)>0$ and integrate over (s,t), $X\leqslant s<t<\infty$, there results

$$(3.3)\qquad \frac{c}{2+\sigma}v(s)^{2+\sigma}+\frac{1}{2}\big(v'(s)\big)^2\leqslant\frac{c}{2+\sigma}v(t)^{2+\sigma}$$
$$+\frac{1}{2}\big(v'(t)\big)^2-d\int_s^t v'(x)\int_0^1 w(y)|z(x,y)|^{1+\sigma}dy\,dx\,.$$

Similarly, integrating (3.2) over (s,t) yields

$$(3.4)\qquad c\int_s^t v^{1+\sigma}\leqslant d\int_s^t\int_0^1 w|z|^{1+\sigma}-v'(t)\,.$$

We shall prove that

$$(3.5)\qquad \frac{\int_0^1|z(x,y)|^{1+\sigma}dy}{v(x)^{1+\sigma}}\to 0\qquad\text{as } x\to\infty\,.$$

If we assume this for the moment and use it in (3.3), then

$$v(s)^{2+\sigma}\leqslant-\varepsilon\int_s^t v'(x)v(x)^{1+\sigma}dx+\text{const}\big(v(t)^{2+\sigma}+(v'(t))^2\big)$$
$$=\frac{\varepsilon}{2+\sigma}v(s)^{2+\sigma}+\text{const}\,\big(v(t)^{2+\sigma}+v'(t)^2\big)$$

where $\varepsilon \to 0$ as $s \to \infty$. If we let $t \to \infty$, then $v(s)^{2+\sigma} \leqslant \big(\varepsilon/(2+\sigma)\big) v(s)^{2+\sigma}$, whence $v(s) \equiv 0$ for large s, and this is a contradiction.

The rest of this proof is devoted to proving (3.5), and we begin by showing

$$\frac{\int_0^1 z^2(x, y)\, dy}{v^2(x)} \to 0 \qquad \text{as } x \to \infty .$$

Define $W(x, y) = z(x, y)/v(x)$ and set $M(x) = \int_0^1 a(y) W^2(x, y)\, dy$. Equation (4.9) shows that either $M(x) \to 0$ as $x \to \infty$ (which we want) or

$$M'(x) > 0 \qquad \text{for all large } x, \tag{3.6}$$

so that $M(x) \to M \in (0, \infty]$ as $x \to \infty$. We shall assume (3.6) and derive a contradiction. Note that

$$\lim_{x\to\infty} \frac{v^2(x)}{\int_0^1 a(y)\, z^2(x, y)\, dy} \to \frac{1}{M} \in [0, \infty) . \tag{3.7}$$

From (1.1) and the representation $u(x, y) = v(x) w(y) + z(x, y)$, there results

$$\begin{aligned} -\nabla\cdot(a\nabla z) - \mu b z &= \nabla\cdot(a\nabla v w) + \mu b v w + F(y, u, \mu) \\ &= a w v'' + v(a w')' + \mu b v w + F(y, u, \mu) \\ &= -\, a w \int_0^1 F(y, u, \mu) w + F(y, u, \mu) , \qquad (x, y) \in S \end{aligned} \tag{3.8}$$

where we have used the fact that $(aw')' + \mu b w = 0$ on $(0, 1)$ by (2.11), and we have used (3.1) to replace v''. Multiplying this equation by z and integrating over $(s, t)\times(0, 1)$, $0 < s < t < \infty$, gives

$$\begin{aligned} \left(1 - \frac{\mu}{\tau(\mu)}\right) \iint_{s<x<t} a|\nabla z|^2 &\leqslant \iint_{s<x<t} \{a|\nabla z|^2 - \mu b z^2\} = \int_0^1 a(y)\, z(t, y)\, z_x(t, y)\, dy \\ &\quad - \int_0^1 a(y)\, z(s, y)\, z_x(s, y)\, dy + \iint_{s<x<t} z F(y, u, \mu) \\ &\leqslant \int_0^1 a(y)\, z(t, y)\, z_x(t, y)\, dy - \int_0^1 a(y)\, z(s, y)\, z_x(s, y)\, dy + \text{const} \iint_{s<x<t} \{|z| v^{1+\sigma} + |z|^{2+\sigma}\} \end{aligned} \tag{3.9}$$

for all large s, say $s \in (X, \infty)$.

If $\sigma \in (0,1)$, then

$$\iint\limits_{s<x<t} |z| v^{1+\sigma} \leqslant \Big(\iint\limits_{s<x<t} |z|^{2/(1-\sigma)}\Big)^{(1-\sigma)/2} \Big(\int_s^t v^2\Big)^{(1+\sigma)/2} \leqslant \text{const} \max_{x\geqslant s} |z(x,y)|^\sigma \iint\limits_{s<x<t} z^2$$

where we have used (3.7). If $\sigma \geqslant 1$, then

$$\iint\limits_{s<x<t} |z| v^{1+\sigma} \leqslant \text{const} \max_{x\geqslant s} |z(x,y)| \int_s^t v^2 \leqslant \text{const} \max_{x\geqslant s} |z(x,y)| \iint\limits_{s<x<t} z^2 .$$

If we use these estimates in (3.9), restrict s to be large, and let $t \to \infty$, there results

$$\frac{C}{2}\int_s^\infty\int_0^1 a(y)\, z^2(x,y)\, dy\, dx \leqslant \Big(1 - \frac{\mu}{\tau(\mu)}\Big)\int_s^\infty\int_0^1 a(y) |\nabla z(x,y)|^2\, dy\, dx$$
$$\leqslant -\int_0^1 a(y)\, z(s,y)\, z_x(s,y)\, dy, \qquad s \in (X, \infty),$$

where $C > 0$ is independent of s. Define $N(s) = \int_s^\infty \int_0^1 a z^2$ so that $CN(s) - N''(s) \leqslant 0$, $s \in (X, \infty)$. This differential inequality yields

$$\int_s^\infty\int_0^1 a(y) z^2(x,y)\, dy\, dx \leqslant \text{const} \exp(-\sqrt{C} s), \quad s \in (X, \infty). \tag{3.10}$$

If we combine this with (3.7), then

$$\int_s^\infty v^2(x)\, dx \leqslant \text{const} \exp(-\sqrt{C} s), \qquad s \in (X, \infty). \tag{3.11}$$

Now $0 < u(x,y) = v(x)w(y) + z(x,y)$ by hypothesis. If we set $z = z^+ - z^-$ where $z^+(x,y) = \max\,(0, z(x,y))$ and $z^-(x,y) = \max\,(0, -z(x,y))$, then $z^-(x,y) \leqslant v(x)w(y)$. Since $\int_0^1 a(y) z(x,y) w(y) = 0$, it follows that

$$\int_0^1 a(y) z^+(x,y) w(y)\, dy = \int_0^1 a(y) z^-(x,y) w(y)\, dy \leqslant \int_0^1 a(y) w^2(y) v(x)\, dy = v(x).$$

Hence

$$\int_0^1 |z(x,y)|w(y)\,dy \leqslant \frac{1}{\min\limits_{y\in[0,1]} a(y)} \int_0^1 a(y)\bigl(z^+(x,y)+z^-(x,y)\bigr)\,w(y)\,dy \leqslant \frac{2v(x)}{\min a} \tag{3.12}$$

for all $x \in (-\infty, \infty)$. If we use this in (3.3), then

$$\frac{1}{2}\,(v'(s))^2 \leqslant \text{const}\,\bigl(v(t)^{2+\sigma} + (v'(t))^2\bigr) - d\,\max_{x\geqslant s} |z(x,y)|^\sigma \int_s^t v'(x)\left(\frac{2v(x)}{\min a}\right) dx$$

$$\leqslant \text{const}\,\bigl(v(t)^{2+\sigma} + v'(t))^2\bigr) + \frac{\varepsilon^2}{2}\,v^2(s)\,,$$

where $\varepsilon \to 0$ as $s \to \infty$. If we let $t \to \infty$, then

$$-\,v'(s) \leqslant \varepsilon v(s)\,, \qquad s \in (X, \infty) \tag{3.13}$$

whence

$$v(x) \geqslant v(X)\exp\{\varepsilon(X-x)\}\,, \qquad x \in (X, \infty)\,.$$

If we use this in (3.11), an obvious contradiction arises, and so (3.6) is false. Hence

$$\lim_{x\to\infty} M(x) = \lim_{x\to\infty} \frac{\int_0^1 z^2(x,y)\,dy}{v^2(x)} = 0\,, \quad \liminf_{x\to\infty} M'(x) \leqslant 0\,. \tag{3.14}$$

Equation (3.14) proves (3.5) for $\sigma \in (0, 1]$, and so we restrict attention to proving (3.5) for $\sigma > 1$. Equation (4.7) gives

$$C \iint\limits_{s<x<t} |\nabla W|^2 \leqslant \frac{1}{2} M'(t) - \frac{1}{2} M'(s) + \iint\limits_{s<x<t} F(y,u,\mu)\,W/v + 2\iint\limits_{s<x<t} \frac{av'}{v}\,W W_x + \iint\limits_{s<x<t} \frac{av''}{v}\,W^2 \tag{3.15}$$

for $0 < s < t < \infty$. Now

$$\iint\limits_{s<x<t} \left|\frac{F}{v}\right| |W| \leqslant \text{const} \iint\limits_{s<x<t} \{v^\sigma |W| + |W|^2 |z|^\sigma\} \leqslant \text{const}\int_s^t v^{2\sigma} + \frac{C}{4} \iint\limits_{s<x<t} |\nabla W|^2 \tag{3.16}$$

for large s. Equation (3.13) gives $|v'(x)/v(x)| \to 0$ as $x \to \infty$, whence

$$2 \iint\limits_{s<x<t} a \left| \frac{v'}{v} W W_x \right| \leqslant \frac{C}{4} \iint\limits_{s<x<t} |\nabla W|^2 \tag{3.17}$$

for large s. Equation (3.1) gives $|v''(x)/v(x)| \to 0$ as $x \to \infty$, whence

$$\iint\limits_{s<x<t} \left| \frac{av''}{v} \right| W^2 \leqslant \frac{C}{4} \iint\limits_{s<x<t} |\nabla W|^2 \tag{3.18}$$

for all large s. If we use (3.16)-(3.18) in (3.15), there results

$$\begin{aligned} \frac{C}{4} \iint\limits_{s<x<t} |\nabla W|^2 &\leqslant \frac{1}{2} M'(t) - \frac{1}{2} M'(s) + \text{const} \int_s^t v^{2\sigma} \\ &\leqslant \frac{1}{2} M'(t) - \frac{1}{2} M'(s) + \text{const} \int_s^t v^{1+\sigma} \\ &\leqslant \frac{1}{2} M'(t) - \frac{1}{2} M'(s) + \text{const} \iint\limits_{s<x<t} |z|^{1+\sigma} + \text{const}\, |v'(t)| \\ &\leqslant \frac{1}{2} M'(t) - \frac{1}{2} M'(s) + \text{const} \iint\limits_{s<x<t} |z|^2 + \text{const}\, |v'(t)| \end{aligned} \tag{3.19}$$

where we have used (3.4) and the fact that $\sigma > 1$. We wish to let $t \to \infty$ in (3.19) and must first show that $z \in L^2(S)$. Equation (3.4) gives

$$\iint\limits_{s<x<t} |z| v^{1+\sigma} \leqslant \max_{x \geqslant s} |z(x,y)| \int_s^t v^{1+\sigma} \leqslant \varepsilon \iint\limits_{s<x<t} |z|^2 + \varepsilon |v'(t)|$$

where $\varepsilon \to 0$ as $s \to \infty$. The use of this in (3.9) yields (3.10), and so $z \in L^2(S)$. If we let $t \to \infty$ by using the second part of (3.14) in (3.19), then $\nabla W \in L^2(S)$. Standard theory then gives

$$\int_0^1 W(x, y)^p dy \to 0 \qquad \text{as } x \to \infty$$

for $p \in [2, \infty)$, and this proves (3.5) when $\sigma > 1$. q.e.d.

4. – The Fourier decomposition of solutions.

In this section, we assume that (λ_n, u_n) satisfies (2.1)-(2.2) and (P_n+), $\lambda_n \to \mu$, and $|u_n|_{C_0(\bar{S})} + |u_n|_{L^q(S)} \leqslant \text{const}$, for some $q \in [1, \infty)$. The use of Lemma 2.1 and (2.15) allows us to assume that $u_n \to u$ in $C_0(\bar{S})$. Theorem 3.2 shows that $u \equiv 0$, and so $(\lambda_n, u_n) \to (\mu, 0)$ in $\mathbb{R} \times C_0(\bar{S})$. Our intention is to prove in Theorem 4.4 that $u_n \to 0$ in $L^p(S)$ for suitable p. We shall use this result and Theorem 2.3 in section 5 to prove Theorem 1.1.

We write $u_n(x, y) = v_n(x) w(y; \lambda_n) + z_n(x, y)$, where w satisfies (2.11)-(2.12) and is normalized by $\int_0^1 a(y) w^2(y; \lambda_n)\, dy = 1$. Here

$$v_n(x) = \int_0^1 a(y) w(y; \lambda_n) u_n(x, y)\, dy\,, \qquad x \in [-n, n]\,,$$

and we note that v_n is an even function of x with $v_n > 0$ on $(-n, n)$ and $v_n' < 0$ on $(0, n]$. We also have

$$\int_0^1 a(y) z_n(x, y) w(y; \lambda_n)\, dy = 0\,, \qquad x \in [-n, n]\,, \tag{4.1}$$

so that Lemma 3.1 may be applied to $z_n(x, \cdot)$.

If we multiply (2.1) by $w(y; \lambda_n)$ and integrate over $(0, 1)$, there results

$$\begin{aligned} -v_n''(x) &= \alpha(\lambda_n) v_n(x) + \int_0^1 F(y, u_n(x, y), \lambda_n) w(y; \lambda_n)\, dy \\ &= \alpha(\lambda_n) v_n(x) + \int_0^1 A(y, \lambda_n) w(y; \lambda_n) u_n(x, y)^{1+\sigma}\, dy \\ &\quad + \int_0^1 \{F(y, u_n, \lambda_n) - A(y, \lambda_n) u_n^{1+\sigma}\} w(y; \lambda_n)\, dy\,. \end{aligned} \tag{4.2}$$

Since $u_n \to 0$ in $C_0(\bar{S})$ as $n \to \infty$, the final term in (4.2) may be made less than

$$\varepsilon \int_0^1 u_n(x, y)^{1+\sigma} w(y; \lambda_n)\, dy\,, \qquad x \in [-n, n]\,, \tag{4.3}$$

where $\varepsilon \to 0$ as $n \to \infty$. In Lemmas 4.2 and 4.3, we show that z_n is small, in a certain sense, compared to v_n; more precisely

$$(4.4)\qquad \sup_{x\in(-n,n)} \frac{\int_0^1 |z_n(x,y)|^p\,dy}{v_n(x)^p} \to 0 \quad \text{as } n\to\infty$$

for any $p\in[1,\infty)$. Equation (4.4) implies that

$$\frac{\int_0^1 A(y,\lambda_n)w(y;\lambda_n)u_n(x,y)^{1+\sigma}\,dy}{v_n(x)^{1+\sigma}\int_0^1 A(y,\mu)w(y;\mu)^{2+\sigma}\,dy} \to 1\,,$$

uniformly on $(-n,n)$, as $n\to\infty$. The use of this estimate with (2.4) and (4.3) in (4.2) yields

$$(4.5)\qquad \alpha(\lambda_n)v_n(x) + (\tilde{A}-\varepsilon)v_n(x)^{1+\sigma} \leqslant -v_n''(x) \leqslant \alpha(\lambda_n)v_n(x) + (\tilde{A}+\varepsilon)v_n(x)^{1+\sigma}\,,$$

where $x\in(-n,n)$, $\tilde{A}>0$ and $\varepsilon\to 0$ as $n\to\infty$. These differential inequalities will be the basis of our proof in Theorem 4.4 that $u_n\to 0$ in $L^p(S)$ as $n\to\infty$ for suitable p. They will also be used in section 6 for considering bifurcation and asymptotic bifurcation. We begin with some technical results needed in Lemmas 4.2 and 4.3.

LEMMA 4.1. *Assume that* (λ_n, u_n) *satisfies* (2.1)-(2.2) *and* (P_n+) *with* $\lambda_n\to\mu$ *and* $u_n\to 0$ *in* $C_0(\bar{S})$ *as* $n\to\infty$. *Then*

$$(a)\qquad \max_{y\in[0,1]}\left|\frac{z_n(0,y)}{v_n(0)}\right| \to 0 \quad \textit{as } n\to\infty \quad \textit{and}$$

$$\max_{y\in[0,1]}\left|\frac{z_n(0,y)}{v_n(0)}\right| \leqslant \text{const}\, v_n(0)^{\sigma/2} \quad \textit{if } \lambda_n\leqslant\mu\,.$$

$$(b)\qquad \sup_{x\in(-n,n)}\left|\frac{v_n''(x)}{v_n(x)}\right| \to 0 \quad \textit{as } n\to\infty\,.$$

PROOF. (a) We shall drop the subscript n in this proof except where it is necessary. Since $u_x(0,y)=0$ for $y\in[0,1]$ and $u_x(x,y)<0$, $(x,y)\in$ $\in(0,n)\times(0,1)$, it follows that $u_{xx}(0,y)\leqslant 0$, $y\in[0,1]$. (In fact, one has a strict inequality on $(0,1)$ by the strong maximum principle applied to the equation for u_x.) The use of this in (2.1) gives

$$0\leqslant \frac{d}{dy}\left(a(y)\frac{\partial u}{\partial y}(0,y)\right) + \lambda_n b(y)u(0,y) + F(y,u(0,y),\lambda_n)\,,\qquad y\in(0,1)\,,$$

and multiplying by $u(0, y)$ and integrating over $(0,1)$ yields

$$\int_0^1 \{a(y)(z_y(0, y))^2 - \lambda_n b(y) z(0, y)^2\} dy = \alpha(\lambda_n) v(0)^2 + \int_0^1 u(0, y) F(y, u(0, y), \lambda_n) dy$$
$$< \alpha(\lambda_n) v(0)^2 + \text{const}\left(v(0)^{2+\sigma} + \int_0^1 |z(0, y)|^{2+\sigma} dy\right),$$

where we have used the representation $u(0, y) = v(0) w(y; \lambda_n) + z(0, y)$. If we use Lemma 3.1, then

$$\text{const} \max_{y\in[0,1]} |z(0, y)|^2 \leqslant \text{const} \int_0^1 (z_y(0, y))^2 dy \leqslant \alpha(\lambda_n) v(0)^2 + \text{const}\, v(0)^{2+\sigma},$$

and the result follows from $\alpha(\lambda_n) \to \alpha(\mu) = 0$ as $n \to \infty$. Recall that $\lambda_n \leqslant \mu$ gives $\alpha(\lambda_n) \leqslant 0$.

(*b*) Equation (4.2) gives

$$\left|\frac{v''(x)}{v(x)}\right| \leqslant |\alpha(\lambda_n)| + \text{const}\, |u|^\sigma_{C_0(\bar{S})} \int_0^1 \left\{1 + \frac{w(y; \lambda_n)|z(x, y)|}{v(x)}\right\} dy$$
$$\leqslant |\alpha(\lambda_n)| + \text{const}\, |u|^\sigma_{C_0(\bar{S})}$$

by (3.12). q.e.d.

Lemma 4.2. *If* (λ_n, u_n) *are as in Lemma* 4.1, *then*

$$\sup_{x\in(-n,n)} \frac{\int_0^1 z_n(x, y)^2 dy}{v_n(x)^2} \to 0 \qquad \textit{as } n \to \infty.$$

Proof. We shall drop the n subscript on all quantities except λ_n. If we set $W(x, y) = z(x, y)/v(x)$, then a calculation using (2.1) and (4.2) gives

$$(4.6) \qquad -\nabla\cdot(a(y)\nabla W(x, y)) - \lambda_n b(y) W(x, y)$$
$$= -\frac{a(y) w(y; \lambda_n)}{v(x)} \int_0^1 F(y, u, \lambda_n) w(y; \lambda_n) dy + \frac{F(y, u, \lambda_n)}{v(x)}$$
$$+ \frac{2a(y) v'(x)}{v(x)} W_x(x, y) + \frac{a(y) v''(x)}{v(x)} W(x, y), \qquad (x, y) \in S_n.$$

Define $M(x) = \int_0^1 a(y) W(x, y)^2 dy$. If we multiply (4.6) by W and integrate over $(s, t) \times (0, 1)$, where $0 < s < t < n$, there results

$$(4.7) \qquad \left(1 - \frac{\lambda_n}{\tau(\lambda_n)}\right) \int_s^t \int_0^1 a|\nabla W|^2 \leqslant \int_s^t \int_0^1 \{a|\nabla W|^2 - \lambda_n b W^2\}$$

$$= \frac{1}{2} M'(t) - \frac{1}{2} M'(s) + \int_s^t \int_0^1 \frac{F(y, u, \lambda_n)}{v} W + \int_s^t \frac{v'}{v} M' + \int_s^t \int_0^1 \frac{av''}{v} W^2 ,$$

where we have used Lemma 3.1 and the obvious fact that

$$\int_0^1 a(y) W(x, y) w(y; \lambda_n) = 0 , \qquad x \in [-n, n] .$$

Since $\lambda_n \to \mu$, $\tau(\lambda_n) \to \tau(\mu) > \mu$, and $|v''(x)/v(x)|$ is uniformly small on $(-n, n)$ by Lemma 4.1, we have

$$C \int_s^t \int_0^1 a W^2 = C \int_s^t M \leqslant \frac{1}{2} M'(t) - \frac{1}{2} M'(s) + \int_s^t \frac{v'}{v} M' + \int_s^t \int_0^1 \frac{|F(y, u, \lambda_n)|}{v} |W| ,$$

for all n sufficiently large, where C is a constant independent of s, t, and n. Now $|F(y, u_n, \lambda_n| \leqslant \text{const}\, u_n^{1+\sigma}$ by (1.4) since $u_n \to 0$ in $C_0(\bar{S})$ as $n \to \infty$. This allows us to estimate as follows:

$$\int_s^t \int_0^1 \frac{|F(y, u, \lambda_n)|}{v} |W| \leqslant \text{const} \int_s^t \int_0^1 |W| \{v^\sigma + |z|^\sigma |W|\} \leqslant \frac{C}{2} \int_s^t \int_0^1 a W^2 + \text{const} \int_s^t v^{2\sigma}$$

for all large n. The use of this estimate gives

$$(4.8) \qquad C \int_s^t M \leqslant M'(t) - M'(s) + 2 \int_s^t \frac{v'}{v} M' + \text{const} \int_s^t v^{2\sigma}$$

for all (s, t) with $0 < s < t < n$ and n sufficiently large. We now prove the following result:

(4.9) if $M'(x) \geqslant 0$, $x \in (S, t) \subset (0, n)$ and $M'(t) = 0$, then

$$M(x) \leqslant M(t) \leqslant \text{const}\, v(t)^{2\sigma} \leqslant \text{const}\, v(0)^{2\sigma}, \; x \in (S, t) .$$

If we restrict s in (4.8) to lie in (S, t), then

$$C\int_s^t M \leqslant \text{const}\int_s^t v^{2\sigma}$$

since $M' \geqslant 0$ on (S, t) by hypothesis and $v' \leqslant 0$ on $(0, n)$. Dividing each side of this inequality by $t - s$ and letting $s \to t$ gives (4.9).

If M attains its supremum on $[0, n)$, then (4.9) and Lemma 4.1 give $\sup_{x\in(-n,n)} M(x) \to 0$ as $n \to \infty$ since $u_n \to 0$ in $C_0(\bar{S})$. Since this is the desired result, *we shall assume for the remainder of this proof that* $M'(x) > 0$ *on* $(L(n), n)$ *for some* $L(n) \in (0, n)$. Since $u \in C^{2,\beta}(\bar{S}_n) \cap C_0(\bar{S})$ and $v(x) \sim$ const $(n - x)$ for $x \sim n$, the quantity $|u(x, y)/v(x)|$ is bounded for all x near to n, independently of y. In particular, $M(x)$ is bounded as $x \to n$. A simple calculation shows that $W_x(x, y) \to 0$ as $x \to n$, whence $M'(x) \to 0$ as $x \to n$. If we restrict s to $(L(n), n)$ and let $t \to n$ in (4.8), then

$$C\int_s^n M \leqslant \text{const}\int_s^n v^{2\sigma}, \qquad s \in (L(n), n).$$

If we divide this expression by $(n - s)$ and let $s \to n$, there results $M(x) \to M(n) = 0$, which contradicts the assumption that $M'(x) > 0$ on $(L(n), n)$. q.e.d.

Lemma 4.2 proves equation (4.4) for all $p \in [1, 2]$, and we now proceed to the general case.

LEMMA 4.3. *If* (λ_n, u_n) *are as in Lemma* 4.1, *then*

$$\sup_{x\in(-n,n)} \frac{\int_0^1 |z_n(x, y)|^p\, dy}{v_n(x)^p} \to 0 \qquad \text{as } n \to \infty \qquad \text{for any } p \in [1, \infty).$$

PROOF. We shall restrict attention to $p \in (2, \infty)$ and drop the n subscript when convenient. We begin by deriving an estimate (4.10) which will be used later in the proof. If $\gamma \in C^1([0, 1]\times[0, 1])$ with $\gamma(x, 0) = \gamma(x, 1) = 0$, $x \in [0, 1]$, then standard theory gives

$$\int_0^1\int_0^1 |\nabla\gamma|^2 \geqslant \text{const}\left(\int_0^1\int_0^1 |\gamma|^{2+r}\right)^{2/(2+r)}$$

where $r \geqslant 0$, and the constant depends on r, but is independent of γ. By rescaling, we have

$$(4.10)\qquad \int_s^t\int_0^1 |\nabla\gamma|^2 \geqslant \mathrm{const}\,(t-s)^{r/(2+r)}\left(\int_s^t\int_0^1 |\gamma|^{2+r}\right)^{2/(2+r)}$$

if $|t-s|\leqslant 1$, and $\gamma\in C^1([s,t)\times[0,1])$ satisfies $\gamma(x,0)=\gamma(x,1)=0$, $x\in[s,t]$.

Let m be a non-negative integer, and set $\tilde{M}(x)=\int_0^1 a(y)\,W(x,y)^{2m+2}dy$, where $W(x,y)=z(x,y)/v(x)$. If we multiply (4.6) by W^{2m+1} and integrate over $(s,t)\times(0,1)$, where $0<s<t<n$ and $t-s\leqslant 1$, there results

$$(4.11)\qquad \frac{2m+1}{(m+1)^2}\int_s^t\int_0^1 a|\nabla W^{m+1}|^2 = \lambda_n\int_s^t\int_0^1 bW^{2m+2}+\frac{1}{2m+2}\tilde{M}'(t)-\frac{1}{2m+2}\tilde{M}'(s)$$
$$-\int_s^t\int_0^1 \frac{a(y)\,w(y;\lambda_n)\,W(x,y)^{2m+1}}{v(x)}\left(\int_0^1 F(p,u(x,p),\lambda_n)\,w(p;\lambda_n)\,dp\right)dy\,dx$$
$$+\int_s^t\int_0^1 \frac{F(y,u,\lambda_n)}{v}W^{2m+1}+\int_s^t\int_0^1 \frac{av''}{v}W^{2m+2}+\frac{1}{m+1}\int_s^t\int_0^1 \frac{v'}{v}\tilde{M}'$$
$$=\lambda_n\int_s^t\int_0^1 bW^{2m+2}+\frac{1}{2m+2}\tilde{M}'(t)-\frac{1}{2m+2}\tilde{M}(s)+C+D+E+F,$$

where C, D, E, and F denote the last four terms on the right of (4.11). Recall that Lemma 4.1(a) gives $\tilde{M}(0)\to 0$ as $n\to\infty$. Assume that $\tilde{M}'(x)\geqslant 0$, $x\in(S,t)$ and $\tilde{M}'(t)=0$; we shall use (4.11) to show that $\tilde{M}(t)$ must be small.

Choosing $r=2m/(m+1)$ in (4.10) yields

$$(4.12)\qquad \frac{2m+1}{(m+1)^2}\int_s^t\int_0^1 a|\nabla W^{m+1}|^2\geqslant \mathrm{const}\,(t-s)^{m/(2m+1)}\left(\int_s^t\int_0^1 W^{4m+2}\right)^{(m+1)/(2m+1)}.$$

If we apply Hölder's and Young's inequalities to the integral for bW^{2m+2}, then

$$(4.13)\qquad \lambda_n\int_s^t\int_0^1 bW^{2m+2}\leqslant\frac{\mathrm{const}}{\varepsilon}(t-s)^{-m}\left(\int_s^t\int_0^1 W^2\right)^{m+1}$$
$$+\varepsilon(t-s)^{m/(2m+1)}\left(\int_s^t\int_0^1 W^{4m+2}\right)^{(m+1)/(2m+1)}$$

for any $\varepsilon > 0$. If $s \in (S, t)$ and $\tilde{M}'(t) = 0$, then $\tilde{M}'(t) - \tilde{M}'(s) \leqslant 0$ in (4.11), and so we conclude by estimating the terms C, D, E, and F.

To estimate C and D, we note that

$$\frac{|F(y, u, \lambda_n)|}{v} \leqslant \text{const}\,(v^\sigma + |z|^\sigma |W|)\,.$$

By using Hölder's and Young's inequality, one easily proves the following estimate:

$$|C|,\ |D| \leqslant \text{const} \int_s^t \int_0^1 W^{2m+2} + \text{const} \int_s^t v^{\sigma(2m+2)}\,. \tag{4.14}$$

Lemma 4.1(*b*) ensures that

$$|E| \leqslant \int_s^t \int_0^1 \left|\frac{av''}{v}\right| W^{2m+2} \leqslant \text{const}\,\varepsilon \int_s^t \int_0^1 W^{2m+2} \tag{4.15}$$

where $\varepsilon \to 0$ as $n \to \infty$. Finally, the assumption that $\tilde{M}'(x) \geqslant 0$, $x \in (S, t)$, gives $F \leqslant 0$. The use of this with (4.12)-(4.15) in (4.11) yields

$$(t-s)^{m/(2m+1)} \left(\int_s^t \int_0^1 W^{4m+2}\right)^{(m+1)/(2m+1)} \leqslant \text{const}\,(t-s)^{-m} \left(\int_s^t \int_0^1 W^2\right)^{m+1} + \text{const} \int_s^t v^{\sigma(2m+2)},$$

or, after dividing both sides by $t - s$,

$$\frac{\int_s^t \tilde{M}(x)\,dx}{t-s} \leqslant \text{const} \left\{\frac{\int_s^t \int_0^1 W^{4m+2}}{t-s}\right\}^{(m+1)/(2m+1)} \leqslant \text{const} \left\{\frac{\int_s^t \int_0^1 W^2}{t-s}\right\}^{m+1} + \text{const} \left\{\frac{\int_s^t v^{\sigma(2m+2)}}{t-s}\right\} \tag{4.16a}$$

for all $s \in (S, t)$ with $t - s \leqslant 1$. Letting $s \to t$ gives

$$\tilde{M}(t) \leqslant \text{const} \left(\int_0^1 W(t, y)^2\,dy\right)^{m+1} + \text{const}\, v(t)^{\sigma(2m+2)} \tag{4.16b}$$

which becomes arbitrarily small as $n \to \infty$ by Lemma 4.2 and $u_n \to 0$ in $C_0(\bar{S})$ as $n \to \infty$. Hence, we may assume that $\tilde{M}'(t) > 0$ for all t sufficiently near to $t = n$, say $t \in (L(n), n)$. The argument in Lemma 4.2 shows that $\tilde{M}'(t) \to 0$ as $t \to n$, so letting $t \to n$ in (4.11) and restricting s to $(L(n), n) \cap (n-1, n)$ yields (4.16a) with t replaced by n. If we let $s \to n$, then

$$\tilde{M}(n) \leqslant \text{const}\left(\int_0^1 W(n, y)^2\, dy\right)^{m+1}$$

which is arbitrarily small as $n \to \infty$ by Lemma 4.2. q.e.d.

REMARK. It is quite possible that the dependence of the various constants on p may allow

$$\sup_{(x,y)\in S_n}\left|\frac{z_n(x, y)}{v_n(x)}\right| \to 0 \qquad \text{as } n \to \infty$$

which is similar to Theorem 3.2 of [20] and Theorem 6.2 of this paper. Fortunately, we only need (4.4) for our results. The proof of the following theorem is almost identical to arguments in [13] and [20].

THEOREM 4.4. *Assume that* (λ_n, u_n) *satisfies* (2.1)-(2.2) *and* (P_n+) *with* $\lambda_n \to \mu$ *and* $u_n \to 0$ *in* $C_0(\bar{S})$ *as* $n \to \infty$. *Then*

(*a*) $|u_n|^p_{L^p(S)} \leqslant \text{const}\, v_n(0)^{p-\sigma/2}$,

(*b*) $u_n \to 0$ *in* $L^p(S)$ *if* $p \in [1, \infty] \cap (\sigma/2, \infty]$.

PROOF. Lemma 4.3 shows that

$$\frac{|u_n|^p_{L^p(S)}}{|v_n|^p_{L^p(-n,n)}} \to \int_0^1 w(y, \mu)^p\, dy \qquad \text{as } \lambda_n \to \mu\,, \tag{4.17}$$

and so it suffices to estimate v_n in $L^p(-n, n)$. The use of Lemma 4.3 with (2.4) and (4.3) in (4.2) yields

$$\alpha(\lambda_n)v_n(x) + (\tilde{A} - \varepsilon)v_n(x)^{1+\sigma} \leqslant -v_n''(x) \leqslant \alpha(\lambda_n)v_n(x) + (\tilde{A} + \varepsilon)v_n(x)^{1+\sigma}\,, \quad x \in [0, n]\,, \tag{4.18}$$

where $\varepsilon \to 0$ as $n \to \infty$. Since v_n is an even function of x, we shall restrict attention to $x \geqslant 0$. If we multiply (4.18) by $-v_n'(x) \geqslant 0$ and integrate over

(x, n), there results

$$(4.19)\qquad 0 < v_n'(n)^2 \leqslant v_n'(x)^2 + \alpha(\lambda_n)\, v_n(x)^2 + \frac{2(\tilde{A}+\varepsilon)}{2+\sigma}\, v_n(x)^{2+\sigma}\,, \qquad x \in [0, n].$$

By evaluating the right-hand side at $x = 0$, it follows that

$$\alpha(\lambda_n) + \frac{2(\tilde{A}+\varepsilon)}{2+\sigma} v_n(x)^\sigma$$

is positive at $x = 0$ and is monotone decreasing to $\alpha(\lambda_n)$ at $x = n$.

(i) We begin with the case $\alpha(\lambda_n) \geqslant 0$. Multiplying the left-hand side of (4.18) by $-v_n'$ and integrating from 0 to x yields

$$(4.20)\qquad \begin{aligned} v_n'(x)^2 &\geqslant \alpha(\lambda_n)\{v_n(0)^2 - v_n(x)^2\} + \frac{2(\tilde{A}-\varepsilon)}{2+\sigma}\{v_n(0)^{2+\sigma} - v_n(x)^{2+\sigma}\} \\ &\geqslant \frac{2(\tilde{A}-\varepsilon)}{2+\sigma}\{v_n(0)^{2+\sigma} - v_n(x)^{2+\sigma}\}\,. \end{aligned}$$

The change of variables $t = v_n(x)$ gives

$$(4.21)\qquad \begin{aligned} \int_0^n v_n(x)^p\,dx &= \int_0^{v_n(0)} \frac{t^p}{-v_n'(x)}\,dt \\ &\leqslant \sqrt{\frac{2+\sigma}{2(\tilde{A}-\varepsilon)}} \int_0^{v_n(0)} \frac{t^p}{\sqrt{v_n(0)^{2+\sigma} - t^{2+\sigma}}} = \text{const}\, v_n(0)^{p-\sigma/2}. \end{aligned}$$

(ii) If $\alpha(\lambda_n) < 0$, then

$$\alpha(\lambda_n) + \frac{2(\tilde{A}+\varepsilon)}{2+\sigma}\, v_n(x)^\sigma$$

is monotone decreasing on $(0, n)$ from a positive value at $x = 0$ to a negative value $\alpha(\lambda_n)$ at $x = n$. In particular, there is a unique $X = X(n) \in (0, n)$ such that

$$-\alpha(\lambda_n) = \frac{2(\tilde{A}+\varepsilon)}{2+\sigma}\, v_n(X)^\sigma\,.$$

If we restrict $x \in (X, n)$ in (4.19), then

$$(4.22)\qquad \begin{aligned} v_n'(x)^2 &\geqslant v_n(x)^2 \left\{-\alpha(\lambda_n) - \frac{2(\tilde{A}+\varepsilon)}{2+\sigma}\, v_n(x)^\sigma\right\} \\ &= \frac{2(\tilde{A}+\varepsilon)}{2+\sigma}\, v_n(x)^2\{v_n(X)^\sigma - v_n(x)^\sigma\}\,, \qquad x \in (X, n)\,. \end{aligned}$$

Hence

$$(4.23)\qquad \int_X^n v_n(x)^p\,dx=\int_0^{v_n(X)}\frac{t^p}{-v_n'(x)}\,dt\leqslant \text{const}\int_0^{v_n(X)}\frac{t^{p-1}}{\sqrt{v_n(X)^\sigma-t^\sigma}}\,dt$$
$$=\text{const}\,v_n(X)^{p-\sigma/2}\leqslant \text{const}\,v_n(0)^{p-\sigma/2}.$$

For the remaining integral over $(0, X)$, we return to (4.18) and restrict $x\in(0, X)$:

$$-v_n''(x)\geqslant\alpha(\lambda_n)v_n(x)+(\tilde{A}-\varepsilon)v_n(x)^{1+\sigma}\geqslant-\frac{2(\tilde{A}+\varepsilon)}{2+\sigma}v_n(x)^{1+\sigma}$$
$$+(\tilde{A}-\varepsilon)v_n(x)^{1+\sigma}=\left\{\frac{\sigma\tilde{A}-4\varepsilon-\varepsilon\sigma}{2+\sigma}\right\}v_n(x)^{1+\sigma},\qquad x\in(0,X).$$

If we multiply this inequality by $-v_n'(x)$ and integrate over 0 to x, there results

$$v_n'(x)^2\geqslant\left\{\frac{\sigma\tilde{A}-4\varepsilon-\varepsilon\sigma}{(2+\sigma)^2}\right\}\left(v_n(0)^{2+\sigma}-v_n(x)^{2+\sigma}\right),\qquad x\in(0,X).$$

This estimate may be used with a change of variables in the remaining integral:

$$(4.24)\qquad \int_0^X v_n(x)^p\,dx\leqslant\int_0^{v_n(0)}\frac{t^p}{-v_n'(x)}\,dt\leqslant \text{const}\int_0^{v_n(0)}\frac{t^p}{\sqrt{v_n(0)^{2+\sigma}-t^{2+\sigma}}}\,dt$$
$$\leqslant \text{const}\,v_n(0)^{p-\sigma/2}.$$

Equations (4.17), (4.21), (4.23), and (4.24) prove part (*a*), and (*b*) follows immediately since $v_n(0)\to0$ as $n\to\infty$. q.e.d.

5. – The proof of Theorem 1.1.

(*a*) Let $\mathscr{S}=\{(\lambda,u)\in\mathbb{R}\times(L^{1+\sigma}(S)\cap C_0(\bar{S})):(\lambda,u)$ satisfies (1.1)-(1.3) and $(P+)\}\cup\{(\mu,0)\}$. Theorems 2.3 and 3.2 show that $(\lambda,u)\in\mathscr{S}\setminus\{(\mu,0)\}$ implies $\lambda<\mu$. Let U denote a bounded, open set in $\mathbb{R}\times(L^{1+\sigma}(S)\cap C_0(\bar{S}))$ with $(\mu,0)$ in its interior. Theorem 2.2 shows that $\partial U\cap\mathscr{C}_n\neq\emptyset$ for all large n, so there exist $(\lambda_n,u_n)\in\partial U\cap\mathscr{C}_n$ for such n. Without loss of generality, we may assume that $\lambda_n\to\lambda\leqslant\mu$ and $u_n\rightharpoonup u$ in $L^{1+\sigma}(S)$. If $\lambda<\mu$, then $u_n\to u$ in $L^{1+\sigma}(S)\cap C_0(\bar{S})$ by Theorem 2.3. If $\lambda=\mu$, then Lemma 2.1 and Theorem 3.2 give $u_n\rightharpoonup0$ in $L^{1+\sigma}(S)$ and $u_n\to0$ in $C_0(\bar{S})$. Theorem 4.4 shows that $u_n\to0$ in $L^{1+\sigma}(S)$, whence $(\lambda_n,u_n)\in\partial U$ converges in $\mathbb{R}\times(L^{1+\sigma}(S)$

$\cap\, C_0(\bar{S})\big)$ to $(\mu, 0) \in U$. This is impossible, and so $(\lambda_n, u_n) \to (\lambda, u) \in \mathbb{R} \times \big(L^{1+\sigma}(S) \cap C_0(\bar{S})\big)$ with $\lambda < \mu$. Theorem 2.3 shows that $(\lambda, u) \in \mathscr{S} \setminus \{(\mu, 0)\}$, and so $\partial U \cap (\mathscr{S} \setminus (\mu, 0))\} \neq \emptyset$ for all bounded, open sets U in $\mathbb{R} \times \big(L^{1+\sigma}(S) \cap C_0(\bar{S})\big)$ with $(\mu, 0) \in U$.

Let $\mathscr{C}$ denote the maximal connected subset of $\mathscr{S} \setminus \{(\mu, 0)\}$ containing $(\mu, 0)$ in its closure. In order to show that $\mathscr{C}$ is unbounded, it suffices by Theorem A.6 of [5] to show that $\mathscr{S}$ is closed, and bounded subsets are relatively compact. We begin by showing that $\mathscr{S}$ is closed. Let $(\lambda_n, u_n) \in \mathscr{S}$ converge in $\mathbb{R} \times \big(L^{1+\sigma}(S) \cap C_0(\bar{S})\big)$ to (λ, u). If $\lambda < \mu$, then the arguments for Theorem 2.3(*b*) ensure that $(\lambda, u) \in \mathscr{S} \setminus \{(\mu, 0)\}$. If $\lambda_n \to \mu$, then $u_n \to u \equiv 0$ by Theorem 3.2, whence $(\lambda_n, u_n) \to (\mu, 0) \in \mathscr{S}$. Hence, $\mathscr{S}$ is closed.

In order to show that bounded subsets of $\mathscr{S}$ are relatively compact, let $(\lambda_n, u_n) \in \mathscr{S}$ with $\lambda_n \to \lambda \leqslant \mu$, $u_n \rightharpoonup u$ in $L^{1+\sigma}(S)$ and $|u_n|_{C_0(\bar{S})} \leqslant \text{const.}$ If $\lambda < \mu$, then the arguments for Theorem 2.3(*b*) give $u_n \to u$ in $L^{1+\sigma}(S) \cap C_0(\bar{S})$. If $\lambda_n \to \mu$, then Theorem 3.2 gives $u_n \rightharpoonup 0$ in $L^{1+\sigma}(S)$ and $u_n \to 0$ in $C_0(\bar{S})$. In order to show that $u_n \to 0$ in $L^{1+\sigma}(S)$, it suffices to show that

$$\sup_{x \in (-\infty, \infty)} \frac{\int_0^1 |z_n(x, y)|^{1+\sigma}}{v_n(x)^{1+\sigma}} \to 0 \quad \text{as } n \to \infty, \tag{5.1}$$

where we use the usual decomposition $u_n(x, y) = v_n(x) w(y; \lambda_n) + z_n(x, y)$. Indeed, if (5.1) holds, then we are led to (4.18) on $(-\infty, \infty)$, and the proof of Theorem 4.4 then gives $u_n \to 0$ in $L^{1+\sigma}(S)$. Hence, if $\lambda_n \to \mu$, and (5.1) holds, then $(\lambda_n, u_n) \to (\mu, 0)$ in $\mathbb{R} \times \big(L^{1+\sigma}(S) \cap C_0(\bar{S})\big)$. Since $(\mu, 0) \in \mathscr{S}$, this shows that bounded subsets of $\mathscr{S}$ are relatively compact.

To prove (5.1), we set

$$W_n(x, y) = z_n(x, y)/v_n(x) \qquad \text{and} \qquad M_n(x) = \int_0^1 a(y)\, W_n(x, y)^2\, dy\,.$$

Lemma 4.1(*a*) gives $M_n(0) \to 0$ as $n \to \infty$ while (4.9) shows that $M_n(x_n) \to 0$ as $n \to \infty$ if M_n has a local maximum at x_n. In order to show that

$$\sup_{x \in (-\infty, \infty)} M_n(x) \to 0 \quad \text{as } n \to \infty, \tag{5.2}$$

it suffices to show that

$$M_n(x) \to 0 \quad \text{as } x \to \infty \qquad \text{for all large } n\,. \tag{5.3}$$

If we assume this is false, then (4.9) yields

$$M_n'(x) > 0 \quad \text{on } \big(L(n), \infty\big) \qquad \text{for all large } n. \tag{5.4}$$

Equation (4.7) gives

$$\left(1-\frac{\lambda_n}{\tau(\lambda_n)}\right)\int_s^t\int_0^1 a|\nabla W_n|^2 \leqslant \int_s^t\int_0^1 \{a|\nabla W_n|^2 - \lambda_n b W_n^2\} = \frac{1}{2}M_n'(t) - \frac{1}{2}M_n'(s) \tag{5.5}$$
$$+\int_s^t\int_0^1 \frac{F(y, u_n, \lambda_n)}{v_n} W_n + \int_s^t \frac{v_n'}{v_n} M_n' + \int_s^t\int_0^1 \frac{a v_n''}{v_n} W_n^2 .$$

Equation (4.2) and the fact that $u_n \to 0$ in $C_0(\bar{S})$ together gives

$$\left|-\frac{v_n''(x)}{v_n(x)} - \alpha(\lambda_n)\right| \leqslant \frac{1}{v_n(x)}\int_0^1 |F(y, u_n(x, y), \lambda_n)|\, w(y;\lambda_n)\, dy$$
$$\leqslant \text{const}\,\frac{|u_n(x;\cdot)|^\sigma_{C_0([0,1])}}{v_n(x)}\int_0^1 \{v_n(x) + |z_n(x, y)|\, w(y;\lambda_n)\}\, dy$$
$$\leqslant \text{const}\,|u_n(x,\cdot)|^\sigma_{C_0([0,1])}$$

by (3.12). Hence,

$$\frac{-v_n''(x)}{v_n(x)} \to \alpha(\lambda_n) < 0 \quad \text{and} \quad \frac{-v_n'(x)}{v_n(x)} \to \sqrt{-\alpha(\lambda_n)} \quad \text{as } x \to \infty \tag{5.6}$$

and

$$v_n(x) \leqslant \text{const}\exp(-\alpha_n x) \quad \text{for all large } x, \tag{5.7}$$

where α_n denotes any element of $\big(0, \sqrt{-\alpha(\lambda_n)}\big)$.

Since $\alpha(\lambda_n) \to \alpha(\mu) = 0$, the use of (5.6) yields

$$\int_s^t\int_0^1 a\left|\frac{v_n''}{v_n}\right| W_n^2 \leqslant \varepsilon\int_s^t\int_0^1 a|\nabla W_n|^2$$

for all sufficiently large s and n. Since $u_n \to 0$ in $C_0(\bar{S})$, we have

$$\int_s^t\int_0^1 \frac{|F(y, u_n, \lambda_n)|}{v_n}|W_n| \leqslant \text{const}\int_s^t\int_0^1 \{v_n^\sigma|W_n| + |z_n|^\sigma W_n^2\} \leqslant \varepsilon\int_s^t\int_0^1 a|\nabla W_n|^2 + \text{const}\int_s^t v_n^{2\sigma}$$

where $\varepsilon \to 0$ as $s \to \infty$. The use of these estimates in (5.5) gives

$$(5.8)\qquad C\int_s^t\int_0^1 aW_n^2 = C\int_s^t M_n \leqslant \frac{1}{2}M_n'(t) - \frac{1}{2}M_n'(s) + \text{const}\int_s^t v_n^{2\sigma} + \int_s^t \frac{v_n'}{v_n}M_n' \leqslant \frac{1}{2}M_n'(t) + \int_s^t \frac{v_n'}{v_n}M_n' + D$$

for all large s since we are assuming (5.4). Here C is a positive constant and D is a bound for const $|v_n^{2\sigma}|_{L^1(\mathbf{R})}$, which is finite by (5.7). Assume $s \in (L(n), \infty)$ and define

$$H_n(t) = -\int_s^t \frac{v_n'}{v_n}M_n', \qquad t > s > L(n).$$

Note that $H_n > 0$ by (5.4) and since $v_n' < 0$ on $(0, \infty)$. Equation (5.8) yields $H_n(t) \leqslant \frac{1}{2}M_n'(t) + D$, or, equivalently

$$(5.9)\qquad 0 \leqslant \frac{d}{dt}\{v_n(t)^2 H_n(t) - Dv_n(t)^2\}, \qquad t > s > L(n).$$

We now show that $v_n(t)^2 H_n(t) \to 0$ as $t \to \infty$. An integration by parts yields

$$(5.10)\qquad H_n(t) + \int_s^t M_n\left(\frac{v_n'}{v_n}\right)^2 = \frac{-v_n'(t)}{v_n(t)}M_n(t) + \frac{v_n'(s)}{v_n(s)}M_n(s) + \int_s^t \frac{v_n''}{v_n}M_n.$$

Note that

$$(5.11)\qquad v_n(t)^2\left|\frac{v_n'(t)}{v_n(t)}\right| M_n(t) = \frac{|v_n'(t)|}{v_n(t)}\int_0^1 a(y)\, z_n(t, y)^2\, dy$$
$$= \frac{|v_n'(t)|}{v_n(t)}\int_0^1 a(y)\frac{z_n(t, y)^2}{w(y; \lambda_n)}\, w(y; \lambda_n)\, dy$$
$$\leqslant \text{const}\,\frac{|v_n'(t)|}{v_n(t)}\,|\nabla z_n(t, \cdot)|_{L_\infty(0,1)}\int_0^1 |z_n(t, y)|\, w(y; \lambda_n)\, dy$$
$$\leqslant \text{const}\,|v_n'(t)|\,|\nabla z_n(t, \cdot)|_{L_\infty(0,1)} \to 0 \qquad \text{as } t \to \infty,$$

where we have used (3.12). Equation (5.6) leads to the estimate

$$v_n(t)^2\int_s^t \frac{|v_n''|}{v_n} M_n \leqslant \text{const}\, v_n(t)^2 \int_s^t \frac{\int_0^1 a(y)\, z_n^2(x,y)\,dy}{v_n(x)^2}\,dx$$

$$\leqslant \text{const}\, v_n(t)^2 \int_s^T \frac{\int_0^1 a z_n^2}{v_n^2} + \text{const}\int_T^t\int_0^1 z_n^2,$$

where we have used the fact that $v_n(t)/v_n(x) < 1$ for $s < x < t$. Hence,

$$\limsup_{t\to\infty} v_n(t)^2 \int_s^t \frac{|v_n''|}{v_n} M_n \leqslant \text{const}\int_T^\infty\int_0^1 z_n^2 \to 0 \qquad \text{as } T\to\infty$$

since $z_n \in L^2(S)$ by Lemma 2.3. The use of this with (5.11) gives $v_n(t)^2 H_n(t) \to 0$ as $t\to\infty$, and so (5.9) yields

$$-\int_s^t \frac{v_n'(x)}{v_n(x)} M_n'(x)\,dx \leqslant D \qquad \text{for all } t > s > L(n)\,. \tag{5.12}$$

Now $v_n'(x)/v_n(x) \to -\sqrt{-\alpha(\lambda_n)}$ as $x\to\infty$ by (5.6) while $M_n'(x) > 0$ for $x > s$ by our hypothesis (5.4). Since the bound in (5.12) is independent of t, it follows that $M_n' \in L^1(\mathbb{R})$, so that

$$\liminf_{x\to\infty} M_n'(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} M_n(x) = M \in (0,\infty)\,. \tag{5.13}$$

Equation (5.8) gives

$$C\int_s^t M_n(x)\,dx \leqslant \tfrac{1}{2} M_n'(t) + D\,,$$

which is incompatible with (5.13) for large t. Hence, we have shown that (5.4) is false, so

$$M_n(x) = \frac{\int_0^1 a(y)\, z_n(x,y)^2\,dy}{v_n(x)^2} \to 0 \qquad \text{as } x\to\infty \tag{5.3}$$

and this gives (5.2) as noted before.

We now prove (5.1). Equation (5.5) and the arguments after it give

$$E\int_s^t\int_0^1 a|\nabla W_n|^2 \leqslant \frac{1}{2}M_n'(t)-\frac{1}{2}M_n'(s)+\text{const}\int_s^t v_n^{2\sigma}+2\int_s^t\int_0^1 a\frac{v_n'}{v_n}W_n(W_n)_x$$
$$\leqslant \frac{1}{2}M_n'(t)-\frac{1}{2}M_n'(s)+\text{const}\int_s^t v_n^{2\sigma}+\text{const}\sqrt{-\alpha(\lambda_n)}\int_s^t\int_0^1 a|\nabla W_n|^2$$

where E is a constant, independent of large s and t. Since $\alpha(\lambda_n)\to 0$ as $\lambda_n\to\mu$, we may bound $\text{const}\sqrt{-\alpha(\lambda_n)}$ by $E/2$ for large n, whence

$$\frac{E}{2}\int_s^t\int_0^1 a|\nabla W_n|^2 \leqslant \frac{1}{2}M_n'(t)-\frac{1}{2}M_n'(s)+\text{const}\int_s^t v_n^{2\sigma}. \tag{5.14}$$

Since (5.4) is false, we have $\liminf_{t\to\infty} M_n'(t)\leqslant 0$, and the use of this with (5.7) allows us to let $t\to\infty$ in (5.14):

$$\iint_S |\nabla W_n|^2<\infty, \tag{5.15}$$

whence

$$\lim_{x\to\infty}\int_0^1 |W_n(x,y)|^p\,dy=0 \tag{5.16}$$

for $p\in[2,\infty)$. In the proof of Lemma 4.3 we showed that $\int_0^1 |W_n(x_n,y)|^p\,dy\to 0$ as $n\to\infty$ if the function $\int_0^1 |W_n(\cdot,\lambda)|^p dy$, $p\in[2,\infty)$, has local maxima at x_n. Combining this with (5.16) gives

$$\sup_{x\in(-\infty,\infty)}\frac{\int_0^1 |z_n(x,y)|^p\,dy}{v_n(x)^p}\to 0 \quad \text{as } n\to\infty$$

$p\in[2,\infty)$. The use of this estimate with (5.2) proves (5.1).

Since we have shown that $\mathscr{S}$ is closed in $\mathbb{R}\times(L^{1+\sigma}(S)\cap C_0(\bar S))$, bounded subsets of $\mathscr{S}$ are relatively compact, and $\mathscr{S}\cap\partial U\neq\emptyset$ for all bounded, open sets U with $(\mu,0)\in U$, the hypotheses of Theorem A.6 of [5] are satisfied, and so there exists an unbounded, connected set $\mathscr{C}\subset\mathbb{R}\times(L^{1+\sigma}(S)\cap C_0(\bar S))$ of solutions (λ,u) of (1.1)-(1.3) satisfying $(P+)$. In addition, $(\mu,0)$ is in the closure of $\mathscr{C}$.

Theorem 2.3 shows that $\mathscr{C} \subset \mathbf{R}\times(H_0^1(S) \cap C_0(\bar{S}))$ and $\mathscr{C} \subset \mathbf{R}\times(L^p(S) \cap C_0(\bar{S}))$ for all $p \in [1, \infty)$, and we now show that $\mathscr{C}$ is unbounded and connected in these topologies. If $\mathscr{C}$ were disconnected in $\mathbf{R}\times(H_0^1(S) \cap C_0(\bar{S}))$, then there would exist closed, disjoint sets $F_i \subset \mathbf{R}\times(H_0^1(S) \cap C_0(\bar{S}))$, $i = 1, 2$, such that $\mathscr{C} \subset F_1 \cup F_2$ and $\mathscr{C} \cap F_i \neq \emptyset$, $i = 1, 2$. Let $(\lambda_n, u_n) \in \mathscr{C} \cap F_i$ with $(\lambda_n, u_n) \to (\lambda, u) \in \mathscr{C}$ in $\mathbf{R}\times(L^{1+\sigma}(S) \cap C_0(\bar{S}))$. Since $\lambda < \mu$ for elements of $\mathscr{C}$, the proof of Theorem 2.3(*b*) gives $(\lambda_n, u_n) \to (\lambda, u)$ in $\mathbf{R}\times(H_0^1(S) \cap C_0(\bar{S}))$, whence each $\mathscr{C} \cap F_i$ is closed as a subset of $\mathscr{C}$ with the topology induced from $\mathbf{R}\times(L^{1+\sigma}(S) \cap C_0(\bar{S}))$. However, this contradicts the connectedness of $\mathscr{C}$ in $\mathbf{R}\times(L^{1+\sigma}(S) \cap C_0(\bar{S}))$, and so $\mathscr{C}$ is connected in $\mathbf{R}\times(H_0^1(S) \cap C_0(\bar{S}))$. A similar argument shows that $\mathscr{C}$ is connected in $\mathbf{R}\times(L^p(S) \cap C_0(\bar{S}))$.

To complete the proof of (*a*), we must show that $\mathscr{C}$ is unbounded in $\mathbf{R}\times(H_0^1(S) \cap C_0(\bar{S}))$ and in $\mathbf{R}\times(L^p(S) \cap C_0(\bar{S}))$, $p \in [1, \infty)$. Assume that $(\lambda_n, u_n) \in \mathscr{C}$ have $\lambda_n \to \lambda$ and $|u_n|_{L^{1+\sigma}(S)} + |u_n|_{C_0(\bar{S})} \to \infty$ as $n \to \infty$, while $|u_n|_{L^p(S)} + |u_n|_{C_0(\bar{S})} \leqslant \text{const}$ for some $p \in [1, \infty)$. If $\lambda < \mu$, then the proof of Theorem 2.3 give $|u_n|_{L^{1+\sigma}(S)} \leqslant \text{const}$, which is a contradiction. If $\lambda = \mu$, then $u_n \to 0$ in $C_0(\bar{S})$ by Theorem 3.2 and the proof of Theorem 4.4 then gives $u_n \to 0$ in $L^{1+\sigma}(S)$, which is a contradiction. Since $H_0^1(S)$ is embedded into $L^2(S)$, it follows that $\mathscr{C}$ is unbounded in $\mathbf{R}\times(H_0^1(S) \cap C_0(\bar{S}))$.

(*b*) Theorems 2.3(*b*) and 3.2 prove (*b*).

(*c*) Let $(\lambda_n, u_n) \in \mathscr{C}$ converge to $(\mu, 0)$ in $\mathbf{R}\times(L^{1+\sigma}(S) \cap C_0(\bar{S}))$. Since $|u_n|_{C_0(\bar{S})} \to 0$, it follows that $\lambda = \mu$ is a bifurcation point in $L^\infty(S)$, while the case $p \in [1, \infty) \cap (\sigma/2, \infty)$ is identical to the proof of Theorem 4.4. The proof of asymptotic bifurcation and the case $p = \sigma/2 \geqslant 1$ appear in Theorem 6.1 and the remark thereafter, respectively.

(*d*) Assume that there is a solution (λ, u) of (1.1)-(1.3) satisfying ($P-$) with $|\lambda - \mu| + |u|_{C_0(\bar{S})}$ sufficiently small. Multiply equation (2.1) by $u(x, y)$, integrate over $(-X, X)\times(0, 1)$, and use the decomposition $u(x, y) = v(x)w(y; \lambda) + z(x, y)$, where w satisfies (2.11):

$$(5.17) \qquad \iint\limits_{|x|\leqslant X} \{a|\nabla z|^2 - \lambda b z^2\} + \int_{-X}^{X} \{(v')^2 - \alpha(\lambda)v^2\} = 2\int_0^1 a(y) z(X, y) z_x(X, y)\, dy + 2v(X)v'(X) + \iint\limits_{|x|\leqslant X} uF(y, u, \lambda).$$

Since $\lambda \leqslant \mu$ by Theorem 2.3, we have $-\alpha(\lambda) > 0$. Since $|u|_{C_0(\bar{S})} + |\lambda - \mu|$

is assumed to be small, we have

$$\iint\limits_{|x|\leqslant X} |uF(y, u, \lambda) - A(y, \mu)u|u|^{1+\sigma}| \leqslant \varepsilon \iint\limits_{|x|\leqslant X} |u|^{2+\sigma}, \tag{5.18}$$

where $\varepsilon \to 0$ as $|\lambda - \mu| + |u|_{C_0(\bar{S})} \to 0$. Since $u \leqslant 0$ on S by hypothesis, we have

$$\iint\limits_{|x|\leqslant X} A(y, \mu)u|u|^{1+\sigma} = -\iint\limits_{|x|\leqslant X} A(y, \mu)|u|^{2+\sigma} \leqslant -\frac{\tilde{A}}{2}\int\limits_{-X}^{X} v^{2+\sigma} + d\iint\limits_{|x|\leqslant X} |z|^{2+\sigma} \tag{5.19}$$

as for (3.2). The use of (5.18)-(5.19) with Lemma 3.1 in (5.17) yields

$$\begin{aligned}\int\limits_{-X}^{X} (v')^2 + C\iint\limits_{|x|\leqslant X} z^2 &\leqslant 2\int\limits_{0}^{1} a(y)\, z(X, y)\, z_x(X, y)\, dy + 2v(X)v'(X) + \text{const}\iint\limits_{|x|\leqslant X} |z|^{2+\sigma} \\ &\leqslant 2\int\limits_{0}^{1} a(y)\, z(X, y)\, z_x(X, y)\, dy + 2v(X)v'(X) + \frac{C}{2}\iint\limits_{|x|\leqslant X} z^2\end{aligned}$$

upon taking $|u|_{C_0(\bar{S})}$ sufficiently small. If we let $X \to \infty$, there follows that $u \equiv 0$ in S. This is a contradiction, and so there are no solutions of (1.1)-(1.3) satisfying $(P-)$ if $|\lambda - \mu| + |u|_{C_0(\bar{S})}$ is sufficiently small.

(*e*) All the arguments hold for $\tilde{A} < 0$ if the rôles of $(P+)$ and $(P-)$ are reversed. q.e.d.

6. – Bifurcation and asymptotic bifurcation.

Let $(\lambda_n, u_n) \in \mathscr{C}$ with $\lambda_n \to \mu$ and $u_n \to 0$ in $C_0(\bar{S})$. One of the key steps in the proof of Theorem 1.1 was (5.2) and (5.16):

$$\sup_{x\in(-\infty,\infty)} \frac{\int_0^1 |z_n(x, y)|^p\, dy}{v_n(x)^p} \to 0 \quad \text{as } n \to \infty \tag{6.1}$$

for any $p \in [1, \infty)$. This gave equation (4.18) on the whole line

$$\alpha(\lambda_n)v_n(x) + (\tilde{A} - \varepsilon)v_n(x)^{1+\sigma} \leqslant -v_n''(x) \leqslant \alpha(\lambda_n)v_n(x) + (\tilde{A} + \varepsilon)v_n(x)^{1+\sigma}, \quad x \in (-\infty, \infty), \tag{6.2}$$

where $\tilde{A} > 0$ by hypothesis. If we multiply this equation by $-v_n'(x) > 0$ and integrate over $(0, \infty)$, there results

$$(6.3)\quad \alpha(\lambda_n)v_n(0)^2 + \frac{2(\tilde{A}-\varepsilon)}{2+\sigma}v_n(0)^{2+\sigma} \leqslant 0 \leqslant \alpha(\lambda_n)v_n(0)^2 + \frac{2(\tilde{A}+\varepsilon)}{2+\sigma}v_n(0)^{2+\sigma},$$

whence

$$\lim_{n\to\infty}\frac{-\alpha(\lambda_n)}{v_n(0)^\sigma} = \frac{2}{2+\sigma}\tilde{A} = \frac{2}{2+\sigma}\int_0^1 A(y, \mu)w(y)^{2+\sigma}\,dy\,.$$

The proof of Theorem 4.4 holds for (6.2), and so

$$(6.4)\qquad |u_n|^p_{L^p(S)} \leqslant \mathrm{const}\,(-\alpha(\lambda_n))^{(2p-\sigma)/2\sigma}$$

for all $p \in [1, \infty)$. Since $\alpha(\lambda_n) \to 0$ as $\lambda_n \to \mu$, equation (6.4) gives bifurcation when $p > \sigma/2$. For the case of asymptotic bifurcation, we must prove the reverse inequality in (6.4).

THEOREM 6.1. *Let $(\lambda_n, u_n) \in \mathscr{C}$ satisfy $\lambda_n \to \mu$ and $u_n \to 0$ in $C_0(\bar{S})$ as $n \to \infty$. Then*

$$(6.5)\quad (a)\ |u_n|^p_{L^p(S)} \geqslant \mathrm{const}\,(-\alpha(\lambda_n))^{(2p-\sigma)/2\sigma}, \qquad p \in [1, \infty),$$

and the constant is independent of n.

(*b*) *The point $\lambda = \mu$ is an asymptotic bifurcation point if $p \in [1, \infty) \cap \cap (0, \sigma/2)$.*

PROOF. (*a*) For large n, equation (6.2) gives $-v_n''(x) \geqslant \alpha(\lambda_n)v_n(x)$, $x \in [0, \infty)$, whence

$$-v_n'(x) \leqslant \sqrt{-\alpha(\lambda_n)}\,v_n(x)\,, \qquad x \in [0, \infty)\,.$$

It follows that

$$\int_0^\infty v_n(x)^p\,dx = \int_0^{v_n(0)} \frac{t^p}{-v_n'(x)}\,dt \geqslant \frac{v_n(0)^p}{p\sqrt{-\alpha(\lambda_n)}},$$

and the use of (6.3) gives

$$\int_0^\infty v_n(x)^p \geqslant \mathrm{const}\,(-\alpha(\lambda_n))^{(2p-\sigma)/2\sigma}.$$

If we combine this inequality with (6.1), then (*a*) follows.

(*b*) If $p < \sigma/2$, then the right-hand side of (6.5) is unbounded as $\lambda_n \to \mu$. q.e.d.

Equations (6.4) and (6.5) give

$$\text{(6.6)} \qquad \text{const}\,(-\alpha(\lambda_n))^{(2p-\sigma)/2\sigma} \leqslant |u_n|^p_{L^p(S)} \leqslant \text{const}\,(-\alpha(\lambda_n))^{(2p-\sigma)/2\sigma}.$$

If $p = \sigma/2$, then neither bifurcation nor asymptotic bifurcation occurs.

We conclude this section with the case $p = \infty$ for (6.1); a similar result was obtained in Theorem 3.2 of [20] for an easier problem.

THEOREM 6.2. *Let $(\lambda_n, u_n) \in \mathscr{C}$ satisfy $\lambda_n \to \mu$ and $u_n \to 0$ in $C_0(\bar{S})$ as $n \to \infty$. Then*

$$\sup_{(x,y)\in S} \left| \frac{z_n(x, y)}{v_n(x)} \right| \leqslant \text{const}\,\sqrt{-\alpha(\lambda_n)} \simeq \text{const}\, v_n(0)^{\sigma/2},$$

and the constant is independent of n.

PROOF. We drop the n subscript in this proof. Set $W(x, y) = z(x, y)/v(x)$, so that W satisfies the equation (cf. (4.6))

$$\text{(6.7)} \qquad -\nabla\cdot(a\nabla W) = \lambda b W - \frac{a(y)w(y;\lambda)}{v}\int_0^1 F(y, u, \lambda)\, w(y;\lambda)\, dy$$
$$+ \frac{F(y, u, \lambda)}{v} + \frac{2av'}{v} W_x + \frac{av''}{v} W \equiv C + D + D + F + G \quad \text{in } S.$$

For any $r \in \mathbf{R}$, set $Q_r = (r, r+1)\times(0, 1)$ and $R_r = (r-1, r+2)\times(0, 1)$. Equation (6.2) shows that

$$\sup_{x\in(-\infty,\infty)} \left(\left| \frac{v'(x)}{v(x)} \right| + \left| \frac{v''(x)}{v(x)} \right| \right) \to 0$$

as $\lambda \to \mu$. If $\varphi \in C_0^\infty(R_r)$, then simple calculations give the following:

$$\left| \iint_{R_r} C\varphi \right| \leqslant \text{const}\, |W|_{L^2(R_r)} |\varphi|_{L^2(R_r)},$$

$$\left| \iint_{R_r} D\varphi \right| \leqslant \text{const}\, \{|v^\sigma|_{L^2(R_r)} + |W|_{L^2(R_r)}\} |\varphi|_{L^2(R_r)},$$

$$\left| \iint_{R_r} E\varphi \right| \leqslant \text{const}\, \{|v^\sigma|_{L^2(R_r)} + |W|_{L^2(R_r)}\} |\varphi|_{L^2(R_r)},$$

$$\left|\iint_{R_r} F\varphi\right| = 2\left|\iint_{R_r} a\frac{v'}{v}W_x\varphi\right| = 2\left|\iint_{R_r} aW\left[\varphi\left\{\frac{v''}{v}-\left(\frac{v'}{v}\right)^2\right\}+\frac{v'}{v}\varphi_x\right]\right|$$

$$\leqslant \text{const}\,|W|_{L^2(R_r)}|\varphi|_{L^2(R_r)} + \text{const}\,|W|_{L^4(R_r)}|\varphi_x|_{L^{4/3}(R_r)},$$

$$\left|\iint_{R_r} G\varphi\right| \leqslant \text{const}\,|W|_{L^2(R_r)}|\varphi|_{L^2(R_r)}.$$

If we apply these estimates to (6.7) along with the arguments of Agmon [37] and standard embedding theorems, then

$$\max_{(x,y)\in Q_r}\left|\frac{z(x,y)}{v(x)}\right| \leqslant \text{const}\left\{|v^\sigma|_{L^2(R_r)} + |W|_{L^2(R_r)} + |W|_{L^4(R_r)}\right\}. \tag{6.8}$$

The function $M(x) = \int_0^1 W(x,y)^2 dy$ goes to zero as x goes to infinity (cf. (5.2)), while (4.9) shows that $M(\tilde{x}) < \text{const}\, v(0)^{2\sigma}$ if $\tilde{x} > 0$ is a local maximizer of M. To estimate $M(0)$, we use Lemma 4.1(a): $M(0) \leqslant \text{const}\, v(0)^\sigma$ since $\lambda_n < \mu$. It follows that $M(x) \leqslant \text{const}\, v(0)^\sigma$, and so

$$|W|_{L^2(R_r)} \leqslant \text{const}\, v(0)^{\sigma/2}.$$

The function $\tilde{M}(x) = \int_0^1 W(x,y)^4 dy$ goes to zero as x goes to infinity (cf. (5.16)) while the proof of Lemma 4.3 (equation (4.16b) with $m = 1$) yields

$$\tilde{M}(\tilde{x}) \leqslant \text{const}\left(\int_0^1 W(\tilde{x},y)^2\,dy\right)^2 + \text{const}\, v(\tilde{x})^{4\sigma} \leqslant \text{const}\, v(0)^{2\sigma}$$

at a local maximizer $\tilde{x} > 0$ of $\tilde{M}$. Lemma 4.1 gives $\tilde{M}(0) \leqslant \text{const}\, v(0)^{2\sigma}$, and so $|\tilde{M}(x)| \leqslant \text{const}\, v(0)^{2\sigma}$ for all $x \in \mathbb{R}$, whence $|W|_{L^4(R_r)} \leqslant \text{const}\, v(0)^{\sigma/2}$.

If we use all of these estimates in (6.8), then

$$\max_{(x,y)\in Q_r}\left|\frac{z(x,y)}{v(x)}\right| \leqslant \text{const}\, v(0)^{\sigma/2} \leqslant \text{const}\sqrt{-\alpha(\lambda_n)}$$

by (6.3). Since $r \in (-\infty, \infty)$ was arbitrary, the proof is complete. q.e.d.

Theorem 6.2 shows that $v(x)w(y;\lambda)$ is the dominant term in the decomposition $u(x,y) = v(x)w(y;\lambda) + z(x,y)$ when $\lambda \to \mu$. In part (a) of the following theorem, we give a precise description of the function v for λ near to μ. Part (*b*) is really just a restatement of the previous theorem, and is added so that the results are in the form of Theorem 3.2 for [2].

THEOREM 6.3. *Let $(\lambda_n, u_n) \in \mathscr{C}$ satisfy $\lambda_n \to \mu$ and $u_n \to 0$ in $C_0(\bar{S})$ as $n \to \infty$. Set $\delta_n = \sqrt{-\alpha(\lambda_n)}$ and write $u_n(x, y) = v_n(x)w(y; \lambda_n) + z_n(x, y)$. Then*

$$(a) \quad \frac{v_n(x/\delta_n)}{\delta_n^{2/\sigma}} \to V(x) \quad \text{as } \delta_n \to 0 \quad \text{uniformly on } (-\infty, \infty).$$

Here V denotes the unique solution of the problem

$$\text{(6.9)} \qquad \left.\begin{aligned} -V''(x) &= -V(x) + \tilde{A}V(x)^{1+\sigma} \\ V(x) &> 0, \end{aligned}\right\} \quad x \in [0, \infty),$$

$$V'(0) = 0 \text{ and } V(x) \to 0 \text{ as } x \to \infty.$$

The initial-value $V(0)$ is given by $V(0) = \big((2 + \sigma)/2\tilde{A}\big)^{1/\sigma}$.

$$(b) \quad \frac{u_n(x, y) - v_n(x)\, w(y; \lambda_n)}{\delta_n^{2/\sigma}} \to 0 \quad \text{uniformly on } S \text{ as } \delta_n \to 0.$$

PROOF. Define $\tilde{v}_n(x) = v_n(x/\delta_n)/\delta_n^{2/\sigma}$, and note that

$$\int_0^\infty \tilde{v}_n(x)^p dx = \big(-\alpha(\lambda_n)\big)^{(\sigma-2p)/2\sigma} \int_0^\infty v_n(x)^p dx \leqslant \text{const}$$

by (6.1) and (6.6), and the constant is independent of n. Since $\tilde{v}_n$ is monotone on $(0, \infty)$, we have $\tilde{v}_n(x) \leqslant \text{const}/|x|^{1/p}$, and so it suffices to show that $\tilde{v}_n$ converges to V on compact subsets of $[0, \infty)$. Equation (6.2) gives

$$\text{(6.10)} \qquad -\tilde{v}_n + (\tilde{A} - \varepsilon)\tilde{v}_n^{1+\sigma} \leqslant -\tilde{v}_n'' \leqslant -\tilde{v}_n + (\tilde{A} + \varepsilon)\tilde{v}_n^{1+\sigma} \quad \text{on } [0, \infty)$$

where $\varepsilon \to 0$ as $n \to \infty$. Now

$$\tilde{v}_n(0) = \left(\frac{v_n(0)^\sigma}{-\alpha(\lambda_n)}\right)^{1/\sigma} \to \left(\frac{2 + \sigma}{2\tilde{A}}\right)^{1/\sigma} \quad \text{as } n \to \infty.$$

Since $\tilde{v}_n$ is decreasing, it follows that $\tilde{v}_n$ is bounded on $[0, \infty)$ independently of n. Equation (6.10) shows the same for $\tilde{v}_n''$ and similarly for $\tilde{v}_n'$. Hence, the $\tilde{v}_n$ converge uniformly on $[0, \infty)$ to a function V which satisfies (6.9).

(*b*) Theorem 6.2 gives

$$|u_n(x, y) - v_n(x)w(y; \lambda_n)| \leqslant \text{const}\sqrt{-\alpha(\lambda_n)}\, v_n(x)$$
$$\leqslant \text{const}\,(-\alpha(\lambda_n))^{(2+\sigma)/2\sigma} = \text{const}\,\delta_n^{(2+\sigma)/\sigma}$$

by (6.3). q.e.d.

7. – Solitary waves in stratified fluids.

7.1 *Derivation of the physical equations.*

Consider a heterogeneous, incompressible fluid moving in the presence of gravity in the region $S = \mathbb{R}\times(0, 1)$, where the lines $y = 0, 1$ are solid boundaries. In addition, the flow is assumed to be inviscid, non-diffusive, and at rest at infinity. We are interested in waves of permanent form which are moving from right to left with speed $c > 0$. After changing to a moving reference frame, we arrive at the steady Euler equations:

$$\left.\begin{aligned} &\varrho(\boldsymbol{q}\cdot\nabla\boldsymbol{q}) = -\nabla p - g\varrho\boldsymbol{k} &&(7.1)\\ &\nabla\cdot\boldsymbol{q} = 0 &&(7.2)\end{aligned}\right\}\ \text{in } S,$$

$$\boldsymbol{q}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial S, \tag{7.3}$$

$$\boldsymbol{q} \to (c, 0) \quad \text{at infinity}, \tag{7.4}$$

where ϱ denotes the density, $\boldsymbol{q}$ is the velocity of the fluid, p is the pressure, g is the gravitational constant, $\boldsymbol{k} = (0, 1)$ is the vertical unit vector, and $\boldsymbol{n}$ is the unit normal to ∂S. The flow is assumed to be symmetric about the line $x = 0$. These equations and some of those to follow may be found in [3]-[4] and [23]-[24]. Additional references for stratified fluids are [25-[35].

The assumption that density does not diffuse means that $\boldsymbol{q}\cdot\nabla\varrho = 0$. Clearly $\nabla\cdot(\sqrt{\varrho}\,\boldsymbol{q}) = 0$, and so $\sqrt{\varrho}\,\boldsymbol{q} = (\psi_y, -\psi_x)$, where ψ is referred to as the pseudo-stream function.

The trivial solution to (7.1)-(7.4) is $\boldsymbol{q} = (c, 0)$ and the corresponding pseudo-streamfunction is

$$\Psi(y) = c\int_0^y \sqrt{\varrho(t)}\,dt.$$

Hence, given any suitably smooth density function ϱ_∞ defined on $[0, 1]$

with $\varrho'_\infty \leqslant 0$ on $[0,1]$ and $\varrho_\infty(0) > \varrho_\infty(1) > 0$, we can find a trivial solution by setting $\Psi(y) = c\tilde{\Psi}(y)$, where

$$\tilde{\Psi}(y) = \int_0^y \sqrt{\varrho_\infty(t)}\,dt\,. \tag{7.5}$$

The corresponding pressure is

$$P(y) = -g\int_0^y \varrho_\infty(t)\,dt\,.$$

Let Y and $\tilde{Y}$ denote the inverses of Ψ and $\tilde{Y}$, respectively, so that

$$Y(\Psi(y)) = \tilde{Y}(\tilde{\Psi}(y)) = y \qquad \text{on } [0,1]\,.$$

We demand (and this is a crucial point to which we return in (7.18)) that each streamline of ψ goes to $\pm\infty$ and that the density ϱ approaches a prescribed density ϱ_∞ as $x \to \pm\infty$. If $(\varrho, \boldsymbol{q})$ satisfy (7.1)-(7.4) for some $c > 0$, then a calculation shows that $\nabla H \cdot \boldsymbol{q} = 0$, where $H = p + \frac{1}{2}\varrho|\boldsymbol{q}|^2 + g\varrho y$ is the total head pressure. Hence, H is a function of only ψ, and there results

$$H(\psi) = P(Y(\psi)) + \tfrac{1}{2}\varrho_\infty(Y(\psi))c^2 + g\varrho_\infty(Y(\psi))Y(\psi). \tag{7.6}$$

From equations (7.1)-(7.2) and the relation $\boldsymbol{q}\cdot\nabla\varrho = 0$, one may derive [3], [22], [24], [25], [27] Yih's equation for $\psi = \psi(x, y)$:

$$\Delta\psi + gy\frac{d}{d\psi}\varrho_\infty(Y(\psi)) = \frac{d}{d\psi}H(\psi) \quad \text{in } S\,,$$

$$\psi(x,0) = 0\,, \quad \psi(x,1) = \Psi(1)\,, \quad x \in \mathbf{R}\,,$$

$$\psi(x,y) \to \Psi(y) \quad \text{as } x \to \pm\infty\,.$$

Substituting (7.6) into this equation yields

$$-\Delta\tilde{\psi} = W(\tilde{\psi})\{\tfrac{1}{2} - \lambda(y - \tilde{Y}(\tilde{\psi}))\} \quad \text{in } S\,, \tag{7.7}$$

$$\tilde{\psi}(x,0) = 0\,, \quad \tilde{\psi}(x,1) = \tilde{\Psi}(1)\,, \quad x \in \mathbf{R}\,, \tag{7.8}$$

$$\tilde{\psi}(x,y) \to \tilde{\Psi}(y) \quad \text{as } x \to \pm\infty\,, \tag{7.9}$$

where $\tilde{\psi} = \psi/c$, $\lambda = g/c^2$, and

$$W(z) = \frac{-\varrho'_\infty(\tilde{Y}(z))}{\sqrt{\varrho_\infty(\tilde{Y}(z))}}\,.$$

Note that the functions $\tilde{\Psi}$, $\tilde{Y}$ and W are known as soon as ϱ_∞ is specified, and *are independent of c*. The function $\tilde{\Psi}$ is the trivial solution of (7.7)-(7.9), and we now consider non-trivial solutions $(\lambda, \tilde{\psi})$. Since the flow $\boldsymbol{q}$ is to be symmetric about $x = 0$, the function $\tilde{\psi}$ will be an even function of x.

We are interested in «waves of elevation» for which $\tilde{\psi}(x, y) < \tilde{\Psi}(y)$ in S and «waves of depression» for which $\tilde{\psi} > \tilde{\Psi}$ in S. We shall restrict out attention for the moment to waves of elevation. If $\tilde{\psi}(x, y) = \tilde{\Psi}(y) - \varphi(x, y)$, then φ satisfies a suitable semilinear equation in S [3], [23] with $\varphi = 0$ on ∂S, $\varphi > 0$ on S, and $\varphi \to 0$ at infinity. Instead of this approach, let us set $\tilde{\psi}(x, y) = \tilde{\Psi}(y) - \sqrt{\varrho_\infty(y)}\,u(x, y)$ so that (7.7)-(7.9) gives

$$(7.10)\quad -\nabla\cdot(\varrho_\infty(y)\nabla u) = \sqrt{\varrho_\infty(y)}\Big[\frac{1}{2}\{W(\tilde{\Psi}) - W(\tilde{\Psi} - \sqrt{\varrho_\infty}u) - \sqrt{\varrho_\infty}\,W'(\tilde{\Psi})u\} + \lambda W(\tilde{\Psi} - \sqrt{\varrho_\infty}u)\{\tilde{Y}(\tilde{\Psi}) - \tilde{Y}(\tilde{\Psi} - \sqrt{\varrho_\infty}u)\}\Big] \quad \text{in } S\,.$$

As $u \to 0$, the linear term on the right-hand side is

$$\lambda\sqrt{\varrho_\infty(y)}\,W(\tilde{\Psi})\tilde{Y}'(\tilde{\Psi})\sqrt{\varrho_\infty(y)}\,u = -\lambda\varrho'_\infty(y)u\,.$$

Hence our equation becomes

$$(7.11)\qquad -\nabla\cdot(\varrho_\infty(y)\nabla u) = -\lambda\varrho'_\infty(y)u + F(y, u, \lambda) \quad \text{in } S\,,$$

$$(7.12)\qquad u(x, 0) = u(x, 1) = 0\,, \quad x \in \mathbb{R}\,,$$

$$(7.13)\qquad u(x, y) \to 0 \quad \text{as } (x, y) \to \infty \qquad \text{in } S\,.$$

If we set $a(y) = \varrho_\infty(y)$, $b(y) = -\varrho'_\infty(y)$, and recall that $\varrho_\infty > 0$, $\varrho'_\infty \leqslant 0$ on $[0, 1]$, then (7.11)-(7.13) are a special case of (1.1)-(1.3). A solution (λ, u) satisfying $(P+)$ will give rise to a solution of $(\lambda, \tilde{\psi})$ of (7.7)-(7.9), where $\tilde{\psi} = \tilde{\Psi} - \sqrt{\varrho_\infty}u$. Before applying our general results, we recall that the function F in (1.1) was assumed to be a smooth function defined on all of $[0, 1]\times\mathbb{R}^2$. However, functions such as $\tilde{Y}(\tilde{\psi})$ and $W(\tilde{\psi})$ are only defined as long as

$$(7.14)\qquad 0 \leqslant \tilde{\psi} \leqslant \tilde{\Psi}(1)\,,$$

and so $F(y, u, \lambda)$ is only defined as long as $0 \leqslant u(x, y) < \tilde{\Psi}(y)/\sqrt{\varrho_\infty(y)}$. (We say that a function $\tilde{\psi}$ satisfying (7.14) has $\tilde{\psi}$ lying in the *physical range.*) This difficulty is surmounted by extending ϱ_∞ to be defined on all of R with $\varrho_\infty > 0$ and $\varrho'_\infty \leqslant 0$ there. The function $\tilde{\Psi}$ is then defined for all

$y \in (-\infty, \infty)$ by (7.5), the inverse function by $\tilde{Y}(\tilde{\Psi}(y)) = y$, $y \in (-\infty, \infty)$, and W as before. If the function ϱ_∞ was suitably smooth on $[0,1]$, then the extension will be correspondingly smooth on $\mathbb{R}$ as will F on $[0,1] \times \mathbb{R}^2$. It follows easily from (7.10) that $F(y, u, \lambda) \equiv A(y, \lambda) u^2$ as $u \to 0$, so that $\sigma = 1$ in (1.4). We note that μ, $A(y, \mu)$, $w(y)$ and therefore the *sign of* $\tilde{A}$ (excluding the case $\tilde{A} = 0$) *are determined from* ϱ_∞ *as defined on* $[0,1]$, *and do not depend on its extension to* $\mathbb{R}$. Indeed, a calculation using (1.9) and (7.10)-(7.11) yields

$$\tilde{A} = \frac{3}{2}\int_0^1 \varrho_\infty(y)(w'(y))^3\,dy \tag{7.15}$$

where w denotes the positive solution of

$$-(\varrho_\infty w')' = -\mu \varrho'_\infty w \quad \text{on } (0,1),$$
$$w(0) = w(1) = 0 .$$

Here w is normalized by $\int_0^1 \varrho_\infty w^2 = 1$.

THEOREM 7.1. *Let* ϱ_∞ *be a suitably smooth function defined on* $[0,1]$ *with* $\varrho'_\infty(y) \leqslant 0$, $y \in [0,1]$, *and* $\varrho_\infty(0) > \varrho_\infty(1) > 0$. *Let* ϱ_∞ *be extended to* $(-\infty, \infty)$ *in any smooth fashion such that* $\varrho'_\infty \leqslant 0$ and $\varrho_\infty > 0$ *there. Define*

$$\frac{1}{\mu} = \max_{u \in H^1_0(0,1)} \frac{-\int_0^1 \varrho'_\infty u^2}{\int_0^1 \varrho_\infty (u')^2},$$

and let $w(y)$ *denote the corresponding positive eigenfunction*:

$$-(\varrho_\infty(y) w'(y))' = -\mu \varrho'_\infty(y) w(y) \quad \text{on } (0,1),$$
$$w(0) = w(1) = 0 .$$

If

$$\tilde{A} = \int_0^1 A(y, \mu) w^3(y)\,dy > 0 ,$$

then there exists an unbounded, connected set $\mathscr{C}$ *in* $\mathbb{R} \times (H^1_0(S) \cap C_0(\bar{S}))$ *and in* $\mathbb{R} \times (L^p(S) \cap C_0(\bar{S}))$, $p \in [1, \infty)$, *of solutions* (λ, u) *of* (7.11)-(7.13) *satisfying*

$(P+)$. *In particular, u is an even function of x, $u > 0$ on S, and $u_x(x, y) < 0$ on $(0, \infty) \times (0, 1)$.*

(*b*) *The point $(\mu, 0)$ belongs to the closure of $\mathscr{C}$ in $\mathbb{R} \times (H_0^1(S) \cap C_0(\bar{S}))$ and in $\mathbb{R} \times (L^p(S) \cap C_0(\bar{S}))$.*

(*c*) $\{\lambda : (\lambda, u) \in \mathscr{C}\} \subset (0, \mu)$ *and* $\mu > 1$.

PROOF. (*a*) This follows from Theorem 1.1(*a*).

(*b*) Since $\sigma = 1$, Theorem 1.1(*c*) shows that $\lambda = \mu$ is a bifurcation point in $L^p(S) \cap C_0(\bar{S})$ for all $p \in [1, \infty)$. In order to prove the result for $\mathbb{R} \times (H_0^1(S) \cap C_0(\bar{S}))$, we begin by considering a sequence $(\lambda_n, u_n) \in \mathscr{C}$ with $\lambda_n \to \mu$ and $u_n \to 0$ in $L^{1+\sigma}(S) \cap C_0(\bar{S})$ as $n \to \infty$. If we multiply (1.1) by $u_n(x, y)$ and integrate over S, then we arrive at (5.17) with $X = \infty$:

$$\iint_S \{a|\nabla z_n|^2 - \lambda_n b z_n^2\} + \int_{-\infty}^{\infty} \{(v_n')^2 - \alpha(\lambda_n) v_n^2\}$$
$$= \iint_S u_n F(y, u_n, \lambda_n) \leqslant \text{const} \iint_S |u_n|^{2+\sigma} \to 0 \qquad \text{as } n \to \infty .$$

Since $\alpha(\lambda_n) < 0$, it follows from Lemma 3.1 that $z_n \to 0$ in $H_0^1(S)$ and $v_n' \to 0$ in $L_2(\mathbb{R})$ as $n \to \infty$. We already known that $u_n \to 0$ in $L^2(S)$, whence $v_n \to 0$ in $L_2(\mathbb{R})$, and so $u_n \to 0$ in $H_0^1(S)$.

(*c*) Theorem 1.1(*b*) gives $\{\lambda : (\lambda, u) \in \mathscr{C}\} \subset (-\infty, \mu)$, and since $\mathscr{C}$ is connected with $(\mu, 0)$ in its closure, it suffices to show that $\lambda \neq 0$. We shall assume that $\lambda = 0$, and derive a contradiction. If we differentiate (7.7) with respect to x, there results a linear equation for $\tilde{\varphi}_x$. Since $\tilde{\varphi}_x(x, y) = -\sqrt{\varrho_\infty}(y) u_x(x, y) > 0$ on $S^+ = (0, \infty) \times (0, 1)$, while $\tilde{\varphi}_x = 0$ on ∂S^+, the strong maximum principle yields

$$(7.16) \qquad \begin{aligned} &\tilde{\varphi}_{xx}(0, y) > 0 , \qquad y \in (0, 1) \qquad \text{and} \\ &\tilde{\varphi}_{xy}(x, 1) < 0 , \qquad \tilde{\varphi}_{xy}(x, 0) > 0 , \qquad\qquad x \in (0, \infty) . \end{aligned}$$

We are now in a position to use the assumption that $\lambda = 0$. Equation (7.7) yields $-\Delta\tilde{\varphi} = \frac{1}{2} W(\tilde{\varphi})$ in S, whence

$$\tilde{\varphi}_{yy}(0, y) = -\tilde{\varphi}_{xx}(0, y) - \tfrac{1}{2} W(\tilde{\varphi}(0, y)) < 0 , \qquad y \in (0, 1) ,$$

since $W \geqslant 0$. The use of the estimates $\tilde{\varphi}_{xy}(x, 1) < 0$, $x > 0$, and $\tilde{\varphi}_y(\infty, 1)$

$= \sqrt{\varrho_\infty(1)} > 0$ gives $\tilde{\psi}_y(0, 1) > 0$, and so

$$\tilde{\psi}_y(0, y) > 0\,, \qquad y \in [0, 1]\,.$$

If we multiply the inequality

$$\tilde{\psi}_{yy}(0, y) < -\frac{1}{2}\, W(\tilde{\psi}(0, y)) = \frac{1}{2}\, \frac{\varrho'_\infty\big(\tilde{Y}(\tilde{\psi}(0, y))\big)}{\sqrt{\varrho_\infty\big(\tilde{Y}(\tilde{\psi}(0, y))\big)}}$$

by $\tilde{\psi}_y(0, y)$ and integrate over $(0, 1)$, then

$$(7.17) \qquad \big(\tilde{\psi}_y(0, 1)\big)^2 - \varrho_\infty(1) \leqslant \big(\tilde{\psi}_y(0, 0)\big)^2 - \varrho_\infty(0)\,.$$

We claim that this contradicts (7.16). To see this, we note that $\tilde{\psi}_{xy}(x, 1) < 0$, $x > 0$, gives $\tilde{\psi}_y(0, 1) > \tilde{\psi}_y(\infty, 1) = \sqrt{\varrho_\infty(1)}$ and similarly $\tilde{\psi}_y(0, 0) < \tilde{\psi}_y(\infty, 0) = \sqrt{\varrho_\infty(0)}$. The use of this in (7.17) yields the inequality $0 < 0$, which is the desired contradiction. Hence, $\lambda > 0$.

We now show that $\mu > 1$, and shall do so by assuming that $\mu \in (0, 1]$ and then deriving a contradiction. Set $\tau(y) = -\mu\varrho'_\infty(y)$ so that $(\varrho_\infty w')' + \tau w = 0$ on $(0, 1)$. Set $v(y) = y$ and $\beta(y) = -\varrho'_\infty(y)/y$ so that $(\varrho_\infty v')' + \beta v = 0$ on $(0, 1)$. Define

$$V(y) = \varrho_\infty(y)v(y)w'(y) - \varrho_\infty(y)v'(y)w(y)$$

and note that $V(0) = 0$ and $V(1) = \varrho_\infty(1)w'(1) < 0$. A simple calculation yields

$$V'(y) = -\varrho'_\infty wv\left(\frac{1}{y} - \mu\right) \geqslant 0 \qquad \text{on } (0, 1)$$

since $\mu \leqslant 1$. It follows that $V(1) \geqslant 0$, and this is a contradiction. q.e.d.

REMARKS. (i) A similar theorem holds for the case $\tilde{A} < 0$ of waves of depression; *throughout the remainder of this paper, we assume that* $\tilde{A} > 0$ (waves of elevation).

(ii) Since λ arose as g/c^2 in (7.7)-(7.9), the estimate $\lambda > 0$ in (*c*) is as expected.

(iii) Since $\mu > 1$, it follows that the speed $c = \sqrt{g/\mu}$ of infinitesimal internal solitary waves is always strictly less than $\sqrt{g}$, the corresponding speed of infinitesimal solitary waves with a free surface and with asymptotic height $h = 1$ at infinity. Equation (7.24) shows that the estimate $\mu > 1$

is best possible. Indeed, let $\varrho_\infty(y; \delta, h)$ denote densities with $\varrho_\infty(y) = \varrho_0$, $y \in [0, h - \delta]$, $\varrho_\infty(y) = \varrho_1$, $y \in [h + \delta, 1]$, and ϱ_∞ monotone decreasing and smooth on $[h - \delta, h + \delta]$. Here $h \in (0, 1)$, $\delta > 0$ is sufficiently small, and $\varrho_0 > \varrho_1 > 0$. Equation (7.24) shows that $\mu(\delta, h) \to 1$ on $\delta \to 0$, $\varrho_1 \to 0$, and $h \to 1$.

(iv) A variant of the proof that $\mu > 1$ yields the better estimate

$$\mu > \max_{\alpha \geqslant 0} g(\alpha)$$

where

$$g(\alpha) = \frac{(1+\alpha)\big(\varrho_\infty(1)\big)^\alpha}{\int_0^1 \varrho_\infty(t)^\alpha\, dt}.$$

This is accomplished by setting $v(y) = \int_0^y \varrho_\infty(t)^\alpha\, dt$ and

$$\beta(y) = -\frac{(1+\alpha)\varrho_\infty(y)^\alpha \varrho'_\infty(y)}{\int_0^y \varrho_\infty(t)^\alpha\, dt}.$$

7.2 *Physically relevant solutions.*

Theorem 7.1 gives a global branch of solutions (λ, u) satisfying (7.11)-(7.13) and $(P+)$, and one can show that elements of $\mathscr{C}$ give rise to solutions of (7.1)-(7.4). To see this, define $c = \sqrt{g/\lambda}$, $\varrho(x, y) = \varrho_\infty\big(\tilde{Y}(\tilde{\psi}(x, y))\big)$, $\boldsymbol{q} = (c/\sqrt{\varrho})(\tilde{\psi}_y, -\tilde{\psi}_x)$ and $p(x, y) = H\big(c\tilde{\psi}(x, y)\big) - \frac{1}{2}\varrho(x, y)\,|\boldsymbol{q}|^2 - gy\varrho(x, y)$ where H is given in (7.6). A direct calculation shows that (7.1)-(7.4) is satisfied. A natural question to ask is whether the converse holds—if we have a suitable solution of (7.1)-(7.4) and define $\lambda = g/c^2$, $\varrho_\infty(y) = \lim_{x\to\infty} \varrho(x, y)$ and $\tilde{\psi}$ from the relation $\boldsymbol{q} = (c/\sqrt{\varrho})(\tilde{\psi}_y, -\tilde{\psi}_x)$ with $\tilde{\psi}(x, 0) = 0$, $x \in (-\infty, \infty)$, then does $(\lambda, \tilde{\psi})$ satisfy (7.7)-(7.9)? By « suitable » we mean $\boldsymbol{q}$ is symmetric about $x = 0$ and the vertical component of $\boldsymbol{q}$ is negative in the right half of the strip. In terms of $\tilde{\psi}$, this means that $\tilde{\psi}$ is an even function of x and $\tilde{\psi}_x(x, y) < 0$, $(x, y) \in (0, \infty) \times (0, 1)$. If one looks at the formal derivation of (7.7)-(7.9) from (7.1)-(7.4), then what is necessary is for all pseudo-streamlines to connect from $-\infty$ to ∞:

(7.18) for each $(\tilde{x}, \tilde{y}) \in S$, the level set $\{(x, y) : \tilde{\psi}(x, y) = \tilde{\psi}(\tilde{x}, \tilde{y})\}$ is an unbounded set going from $x = -\infty$ to $x = \infty$.

Since we are assuming that our « suitable » solution has $\tilde{\psi}_x < 0$ on the right-half strip and $\tilde{\psi}(x, y) \to \tilde{\Psi}(y)$ as $x \to \infty$, it is easy to see that (7.18) is equivalent to

$$(7.19) \qquad \tilde{\psi}_y(0, y) \geqslant 0 \quad \textit{for all } y \in [0, 1] .$$

This merely says that the horizontal component of $\boldsymbol{q}$ is non-negative on the line of symmetry. Hence, if a suitable solution of (7.1)-(7.4) satisfies (7.19), the corresponding $(\lambda, \tilde{\psi})$ satisfies (7.7)-(7.9), and so we can unequivocally identify the physical problem with the mathematical one (7.11)-(7.13). When we speak of (λ, u) satisfying (7.19), we shall mean that $\tilde{\psi} = \tilde{\Psi} - \sqrt{\varrho_\infty} u$ satisfies it.

We now consider further implications of (7.19). If it holds, then $0 \leqslant \tilde{\psi}(0, y) \leqslant \tilde{\psi}(0, 1) = \tilde{\Psi}(1)$ on $[0, 1]$. Since $\tilde{\psi}_x(x, y) < 0$, $(x, y) \in (0, \infty) \times \times (0, 1)$ and $\tilde{\psi}(x, y) \to \tilde{\Psi}(y)$ as $x \to \infty$, it follows that $0 \leqslant \tilde{\psi}(x, y) \leqslant \tilde{\Psi}(1)$ in S. This means that $\tilde{\psi}$ satisfies (7.14) and so u is in the range for which $F(y, u, \lambda)$ was originally defined before extending ϱ_∞. Assume we have a solution of (7.7)-(7.9) for which (7.19) fails to hold so that $\tilde{\psi}_y(0, \tilde{y}) < 0$ for some $\tilde{y} \in (0, 1)$. It follows that $\tilde{\psi}$ will have an infinite number of closed streamlines centered about the point where $\tilde{\psi}(0, \cdot)$ takes its local minimum, and there will be places where fluid of higher density is above that of lower density. This configuration is presumably unstable and unlikely to be observed. With the arguments above in mind, we say a solution (λ, u) of (7.11)-(7.13) is *physically relevant* if (7.19) holds. As noted before, a physically relevant solution gives rise to a pseudo-stream function $\tilde{\psi}$ lying in the physical range.

Let $\mathcal{D}$ denote the maximal connected subset of $\mathcal{C}$ in $\mathbb{R} \times (H_0^1(S) \cap C_0(\bar{S}))$ which contains $(\mu, 0)$ in its closure and such that $(\lambda, u) \in \mathcal{D}$ satisfies (7.19). *This set $\mathcal{D}$ is the maximal branch of physically relevant solutions bifurcating from the trivial solution. The set $\mathcal{D}$ is independent of the extension of ϱ_∞ outside* $[0, 1]$. In general, $\mathcal{D}$ is a bounded subset of $\mathcal{C}$; indeed, in Theorem 7.4, we give a condition which will ensure this. The proof consists of showing that $(\lambda_n, u_n) \in \mathcal{C}$ with $\lambda_n \to \lambda$ and $|u_n|_{H_0^1(S)} \to \infty$ implies $|u_n|_{C_0(\bar{S})} \to \infty$. Since (λ, u) satisfying (7.19) implies

$$0 < u(x, y) \leqslant \tilde{\Psi}(y) / \sqrt{\varrho_\infty(y)} \quad \text{in } S,$$

it follows that large amplitude elements of $\mathcal{C}$ do not satisfy (7.19). On the other hand, there are densities ϱ_∞ for which all elements of $\mathcal{C}$ satisfy (7.14) so that $\tilde{\psi}$ always lies in the desired physical range. However, it is not clear that (7.14) implies (7.19), and so the following arguments do not prove that $\mathcal{D}$ is identical to $\mathcal{C}$. The proof of this theorem depends crucially on the connectedness of $\mathcal{C}$.

THEOREM 7.2. *Let ϱ_∞ be a suitably smooth function defined on* $[0, 1]$ *with* $\varrho'_\infty \leqslant 0$ *on* $[0, 1]$ *and* $\varrho_\infty(0) > \varrho_\infty(1) > 0$. *Assume that* $\tilde{A} > 0$ *and* $\varrho'_\infty(0) = 0$. *Then all elements of* $\mathscr{C}$ *satisfy*

$$0 < \tilde{\psi}(x, y) < \tilde{\Psi}(y) < \tilde{\Psi}(1) \quad \text{on } S. \tag{7.20}$$

PROOF. Let ϱ_∞ be extended to $(-\infty, \infty)$ in any manner such that $\varrho'_\infty \leqslant 0$ and $\varrho_\infty > 0$ there. Let $\mathscr{C}$ denote the branch of Theorem 7.1. If we can show that (7.20) holds, then u always lies in the range for which $F(y, u, \lambda)$ is defined before extending ϱ_∞ outside $[0, 1]$. In particular, if (7.20) holds, then $\mathscr{C}$ is independent of ϱ_∞.

Let $\mathscr{E}$ denote the maximal connected subset of $\mathscr{C}$ in $\mathbb{R} \times (H_0^1(S) \cap C_0(\bar{S}))$ containing $(\mu, 0)$ in its closure and such that $(\lambda, u) \in \mathscr{E}$ has $\tilde{\psi} = \tilde{\Psi} - \sqrt{\varrho_\infty}\, u$ satisfying (7.20). Since $u > 0$ on S, $\mathscr{E}$ consists of solutions for which $\tilde{\psi} > 0$ on S. We shall prove that the assumption (7.20) forces $\mathscr{E}$ to be open and closed in $\mathscr{C}$, whence $\mathscr{C} = \mathscr{E}$ by connectedness.

We begin by claiming that all elements of $\mathscr{C}$ sufficiently near to $(\mu, 0)$ satisfy (7.20). Indeed, if $|\lambda - \mu|$ and $|u|_{C_0(\bar{S})}$ are so small that

$$\tfrac{1}{2} - \lambda(y - \tilde{Y}(\tilde{\psi})) \geqslant 0 \quad \text{in } S,$$

then $-\Delta\tilde{\psi} \geqslant 0$ in S by (7.7). The boundary conditions (7.8)-(7.9) then give $\tilde{\psi} > 0$ in S. We now show that $\mathscr{E}$ is closed in $\mathscr{C}$. If $(\lambda_n, u_n) \in \mathscr{E}$ converge to $(\lambda, u) \in \mathscr{C}$, then $\tilde{\psi} \geqslant 0$ on S and we must prove that the inequality is sharp. Assume the contrary so that $\tilde{\psi}(0, \tilde{y}) = 0$ for some $\tilde{y} \in (0, 1)$. Since $\tilde{Y}(0) = 0$ by construction, it follows that $\varrho'_\infty\big(\tilde{Y}(\tilde{\psi}(0, \tilde{y}))\big) = 0$, whence $W(\tilde{\psi}(0, \tilde{y})) = 0$. The use of this in (7.7) yields $\tilde{\psi}_{yy}(0, \tilde{y}) = -\tilde{\psi}_{xx}(0, \tilde{y}) < 0$ by (7.16). This is a contradiction, and so $\tilde{\psi} \geqslant 0$ implies that $\tilde{\psi} > 0$ on S.

To show that $\mathscr{E}$ is open in $\mathscr{C}$, we assume the contrary and derive a contradiction. Let $(\lambda, u) \in \mathscr{E}$ and assume that $(\lambda_n, u_n) \in \mathscr{C} \setminus \mathscr{E}$ converge to (λ, u) in $\mathscr{E} \times (H_0^1(S) \cap C_0(\bar{S}))$ as $n \to \infty$. Let $y_n \in (0, 1)$ be such that $0 > \tilde{\psi}_n(0, y_n) = \min_{y \in [0,1]} \tilde{\psi}_n(0, y)$, and assume without loss of generality that $y_n \to \tilde{y} \in [0, 1]$ as $n \to \infty$. Now $\tilde{\psi}(0, \tilde{y}) \leqslant 0$, and so $\tilde{y} = 0$ since $(\lambda, u) \in \mathscr{E}$ ensures that $\tilde{\psi} > 0$ on S. Clearly $(\tilde{\psi}_n)_y(0, y_n) = 0$ and this leads to

$$\tilde{\psi}_y(0, 0) = 0\,. \tag{7.21}$$

However, equation (7.7) shows that $-\Delta\tilde{\psi} \geqslant 0$ on $(-\infty, \infty) \times [0, 1/2\lambda]$ and since $\tilde{\psi}$ takes its minimum zero on this set along the line $y = 0$, the strong maximum principle gives $\tilde{\psi}_y(x, 0) > 0$, $x \in \mathbb{R}$. In particular, $\tilde{\psi}_y(0, 0) > 0$ and this contradicts (7.21). q.e.d.

REMARK. If $\tilde{A} < 0$ and $\varrho'_\infty(1) = 0$, then one can show that elements of $\mathscr{C}$ satisfy $0 < \tilde{\Psi}(y) < \tilde{\varphi}(x, y) < \tilde{\Psi}(1)$. The proof is similar to that for Theorem 7.2, but a more delicate argument is needed to show that $\mathscr{E}$ is open.

It must be emphasized that the condition $\varrho'_\infty(0) = 0$ implies (7.14) along the branch $\mathscr{C}$, but it is not known if it implies the physical condition (7.19). On the other hand, Theorem 7.2 gives a bound on $|u|_{C_0(\bar{S})}$ for all $(\lambda, u) \in \mathscr{C}$:

$$0 < u(x, y) < \tilde{\Psi}(y)/\sqrt{\varrho_\infty(y)} \quad \text{on } S. \tag{7.22}$$

It follows that $\mathscr{C}$ is bounded as a subset of $\mathbb{R} \times C_0(\bar{S})$. Since $\mathscr{C}$ is unbounded in $\mathbb{R} \times (H_0^1(S) \cap C_0(\bar{S}))$ and $\lambda \in (0, \mu)$, we know there are elements for which $\lambda_n \to \lambda$ and $|u_n|_{H_0^1(S)} \to \infty$ as $n \to \infty$. Equation (7.22) shows that u_n is « blowing up » in $H_0^1(S) \cap C_0(\bar{S})$ not in a pointwise sense, but by losing its decay rate at infinity.

THEOREM 7.3. *Let (λ_n, u_n) be $(P+)$ solutions of* (7.11)-(7.13) *with $\lambda_n \to \lambda$ and $|u_n|_{H_0^1(S)} \to \infty$ as $n \to \infty$. Assume that $|u_n|_{C_0(\bar{S})} \leqslant C$, independently of n. Then a subsequence of the $\{u_n\}$ converge uniformly on compact sets to a nontrivial solution (λ, u) of* (7.11)-(7.12). *In addition, $u > 0$ on S, $u_x \leqslant 0$ on $(0, \infty) \times (0, 1)$, and $u(x, y) \to m(y)$ as $x \to +\infty$, uniformly for $y \in [0, 1]$, where m is a positive solution of* (1.6)-(1.7). *This function m satisfies the identity*

$$\int_0^1 a(m')^2 - \lambda \int_0^1 bm^2 = \int_0^1 F(y, m, \lambda) m = 2\int_0^1 H(y, m, \lambda) \tag{7.23}$$

where

$$H(y, u, \lambda) = \int_0^u F(y, t, \lambda)\, dt, \qquad u \in [0, C].$$

PROOF. Since each u_n satisfies (7.11)-(7.12) and $|u_n| < C$, one may obtain a priori bounds, independent of n, for arbitrary derivatives of u_n. It follows that some subsequence, which we still call $\{u_n\}$, converges (with its derivatives) uniformly on compact sets to a solution (λ, u) of (7.11)-(7.12).

Now $\partial u_n/\partial x < 0$ on $(0, \infty) \times (0, 1)$ and

$$-\int_0^\infty dx \int_0^1 \frac{\partial u_n}{\partial x}\, dy = \int_0^1 u_n(0, y)\, dy \leqslant \text{const},$$

whence $u_x \leqslant 0$ on $(0, \infty) \times (0, 1)$ and $u_x \in L^1(S)$. If we combine this with the usual elliptic estimates, then $u(x, y) \to m(y)$ as $x \to \pm\infty$, uniformly for $y \in (0, 1)$, where m is a solution of (1.6)-(1.7).

We claim that $m(y) \neq 0$ on $(0, 1)$. Indeed, since $u_n > 0$ on S, we have $m \geqslant 0$, and as a solution of (1.6)-(1.7) either $m > 0$ on $(0, 1)$ or $m \equiv 0$. If $m \equiv 0$, then $u \to 0$ as $x \to \pm\infty$, so that $u \in C_0(\bar{S})$. Since $u_n \in C_0(\bar{S})$, $(u_n)_x < 0$ on $(0, \infty) \times (0, 1)$, and $u_n \to u$ uniformly on compact sets, it follows that $u_n \to u$ in $C_0(\bar{S})$ as $n \to \infty$. If $\lambda_n \to \lambda < \mu$, then $\{u_n\}$ is bounded in $H_0^1(S)$ by (2.17) with $x = 0$ and $n = \infty$ in the integrals. This contradicts the assumption that the sequence is unbounded in $H_0^1(S)$. If $\lambda_n \to \mu$, then Theorem 3.2 gives $u \equiv 0$, whence $u_n \to 0$ in $C_0(\bar{S})$. Theorem 4.4 gives $\{u_n\}$ bounded in $H_0^1(S)$ which is a contradiction as before. It follows that $m(y) \neq 0$ on $(0, 1)$.

We now prove (7.23). If we multiply (7.11) by $(\partial/\partial x)u_n$ and integrate over $(0, 1)$, then

$$\frac{d}{dx}\left\{\int_0^1 a\left(\frac{\partial u_n}{\partial y}\right)^2 - \int_0^1 a\left(\frac{\partial u_n}{\partial x}\right)^2 - \lambda_n\int_0^1 bu_n^2 - 2\int_0^1 H(y, u_n, \lambda_n)\right\} = 0\,, \qquad x \in (-\infty, \infty)\,.$$

The term in brackets must be constant, and evaluating it at ∞ shows the constant is zero. If we let $n \to \infty$, then

$$-\int_0^1 au_x^2 + \int_0^1 au_y^2 - \lambda\int_0^1 bu^2 - 2\int_0^1 H(y, u, \lambda) = 0\,, \qquad x \in (-\infty, \infty)\,,$$

and letting $x \to \infty$ yields

$$\int_0^1 a(m')^2 - \lambda\int_0^1 bm^2 = 2\int_0^1 H(y, m, \lambda)\,.$$

If we multiply (1.6) by m and integrate over $(0, 1)$, then we arrive at (7.23). q.e.d.

REMARKS. (i) Since $u_x \leqslant 0$ on $S^+ = (0, \infty) \times (0, 1)$, $u_x = 0$ on ∂S^+, and u_x satisfies a linear elliptic equation, either $u_x \equiv 0$ on S or $u_x < 0$ on S^+. The former would give $u(x, y) = m(y)$ on S, whence u would be the « conjugate flow » solution of [27], while the latter would give rise to a solution which behaves like a conjugate flow only at infinity.

(ii) We remind the reader that the conditions $\varrho_\infty'(0) = 0$ and $\tilde{A} > 0$ ensure that the hypotheses of Theorem 7.3 are satisfied.

(iii) If $(P+)$ is replaced by $(P-)$, then the same conclusion holds except that m is now a negative solution.

Another use of the bound (7.22) when $\varrho'_\infty(0) = 0$ is to a problem studied recently by Turner [4]. Let h, ϱ_0, $\varrho_1 > 0$ with $\varrho_0 > \varrho_1$, and $h \in (0, 1)$ and for all small $\delta \geqslant 0$, let $\varrho_\infty(y; \delta) = \varrho_0$ on $[0, h - \delta]$ and $\varrho_\infty(y; \delta) = \varrho_1$ on $[h + \delta, 1]$. On the interval $[h - \delta, h + \delta]$, $\delta > 0$, we assume that $\varrho_\infty(\cdot\,; \delta)$ is smooth and decreasing. Since Theorems 7.1 and 7.2 are applicable to $\varrho_\infty(\cdot\,; \delta)$ for $\delta > 0$, we have the existence of unbounded, connected sets $\mathscr{C}_\delta$ of solutions (λ, u) satisfying (7.22). In [4], Turner used a variational formulation for these densities, and showed that certain small-amplitude solutions converged to a solution of the physical problem corresponding to the discontinuous, piecewise constant density $\varrho_\infty(\cdot\,; 0)$. The natural question is what happens to the $\mathscr{C}_\delta$ as $\delta \to 0$—do they converge to a global branch of solutions? This problem appears quite intractable until one notes that (7.22) gives a bound on $|u|_{C_0(\bar{S})}$, independent of (λ, u) and of δ. In [8], we hope to give a rigorous proof of this method. We also remark that if one calculates μ_δ as in (1.8), then

$$\mu_\delta \to \mu_0 = \frac{1}{\varrho_0 - \varrho_1}\left(\frac{\varrho_0}{h} + \frac{\varrho_1}{1-h}\right) \quad \text{as } \delta \to 0 \tag{7.24}$$

while (7.15) yields

$$\tilde{A}_\delta \to \tilde{A}_0 = \frac{3}{2}\left(\frac{3}{\varrho_0 h + \varrho_1(1-h)}\right)^{3/2}\left\{\frac{\varrho_0}{h^2} - \frac{\varrho_1}{(1-h)^2}\right\} \text{ as } \delta \to 0\,. \tag{7.25}$$

This number is positive if and only if $\varrho_0/h^2 - \varrho_1/(1-h)^2 > 0$ which is the classical condition one finds for the existence of solitary waves of elevation in a two-liquid stratified fluid.

7.3 *Extreme waves.*

If we revert to the hypotheses of Theorem 7.1 and no longer demand that $\varrho'_\infty(0) = 0$, then the connectedness, of $\mathscr{C}$ can be used to determine the existence of «extreme waves» in $\mathscr{D}$. More precisely, assume that $\mathscr{D}$ is bounded in $\mathscr{C}$ so that $\partial\mathscr{D} \neq \emptyset$. If $(\lambda, u) \in \partial\mathscr{D}$, then (i) there exist $(\lambda_n, u_n) \in \in \mathscr{C}\setminus\mathscr{D}$ which converge to (λ, u) and (ii) equality holds for u in (7.19) for at least one point $\tilde{y} \in (0, 1)$. Since $(\tilde{\psi}_n)_y(0, y_n) < 0$ for some $y_n \in (0, 1)$ while $\tilde{\psi}_n$ is even in x and $(\tilde{\psi}_n)_x(x, y) > 0$ on $(0, \infty) \times (0, 1)$, there is a closed Jordan curve, symmetric about $x = 0$, on which $\tilde{\psi}_n$ is constant. The interior of this curve might be considered to be a closed eddy, but we emphasize that these $\tilde{\psi}_n$ are not physically relevant solutions as (7.18) is violated. However, in a formal sense, we may think of (λ, u) as belonging to the boundary of $\mathscr{D}$ when «continuing» past (λ, u) leads to solutions with eddies. Solutions to (7.7)-(7.9) arising from elements of $\partial\mathscr{D}$ will be referred to as *extreme waves*.

Condition (ii) ensures that $\tilde{\psi}_y(0, \tilde{y}) = 0$ for some $\tilde{y} \in (0, 1)$ and since $\tilde{\psi}_x(0, y) = 0$, $y \in [0, 1]$, it follows that $(0, \tilde{y})$ is a stagnation point in the flow (there are no stagnation points in $x < 0$ and $x > 0$ since $\tilde{\psi}_x < 0$ and $\tilde{\psi}_x > 0$ in these regions, respectively). As noted in (7.16), $\tilde{\psi}_{xx}(0, y) > 0$, $y \in (0, 1)$, by the strong maximum principle. If we combine these facts with (7.7), there results that the pseudo-streamline through $(0, \tilde{y})$ is not smooth at this point [36]. For example, if the density ϱ_∞ is real-analytic on $[0, 1]$ (which includes the case in [3]), then a cusp is formed at $(0, \tilde{y})$. More precisely, the level curve behaves like $y = \tilde{y} - B|x|^{2/2m+1}$, where $B > 0$ and m is a positive integer ($m \geqslant 1$). If ϱ_∞ is merely smooth on $[0, 1]$, then similar results hold for the level curves through $(0, \tilde{y})$. Full details of these results appear in [36].

We now give sufficient conditions on the terms in (7.11) to ensure that $\mathscr{D}$ is bounded in $\mathscr{C}$.

THEOREM 7.4. *Let (λ_n, u_n) be $(P+)$ solutions of (7.11)-(7.13) with $u_n(x, y) \leqslant g(y)$, $x \in (\infty, -\infty)$, for some continuous function g on $[0, 1]$. Define*

$$H(y, u, \lambda) = \int_0^u F(y, t, \lambda)\, dt , \qquad u \in [0, g(y)]$$

and assume that there exists $\theta \in (0, \frac{1}{2})$ such that

$$H(y, u, \lambda) \leqslant \theta F(y, u, \lambda) u \tag{7.26}$$

for all $y \in [0, 1]$, $\lambda < \mu$, and $u \in [0, g(y)]$. Then, either

(*a*) $\overline{\lim_{n\to\infty}} |u_n|_{H_0^1(S)} < \infty$

or

(*b*) *$\lambda_n \to \mu$ as $n \to \infty$ and there exists a constant $D > 0$ such that $F(y, Dw(y), \mu) = H(y, Dw(y), \mu) = 0$, $y \in [0, 1]$. Here $w(y)$ is the positive solution of*

$$\begin{aligned} -(\varrho_\infty w') &= -\mu \varrho'_\infty w \quad \textit{on } (0,1), \\ w(0) = w(1) &= 0 . \end{aligned} \tag{7.27}$$

REMARKS. (i) The condition (b) is very unusual and seems unlikely to ever hold. It says that the x-independent problem (1.6)-(1.7) has a solution with $\lambda = \mu$ and $m(y) = Dw(y)$.

(ii) If ($P+$) is replaced by ($P-$), then an analogous theorem holds.

PROOF OF THEOREM 7.4. Assume that (a) is false. We can apply Theorem 7.3 and use (7.23) and (7.26):

$$\int_0^1 a(m')^2 - \lambda\int_0^1 bm^2 = \int_0^1 F(y, m, \lambda)m \leqslant 2\theta\int_0^1 F(y, m, \lambda)m .$$

It follows that

$$\int_0^1 a(m')^2 - \lambda\int_0^1 bm^2 = \int_0^1 F(y, m, \lambda)m \leqslant 0 . \tag{7.28}$$

Since $\lambda \leqslant \mu$, equation (1.8) gives $\lambda = \mu$ and $m(y) = Dw(y)$. Since m satisfies (1.6), we clearly have $F(y, Dw(y), \mu) = 0$, $y \in [0, 1]$ so that $H(y, Dw(y), \mu) \leqslant 0$ $y \in [0, 1]$, by (7.26). Equation (7.23) and (7.28) gives

$$\int_0^1 H(y, Dw(y), \mu)\, dy = 0 ,$$

whence $H(y, Dw(y), \mu) = 0$, $y \in [0, 1]$. q.e.d.

For the problem of stratified fluids, one would try to apply Theorem 7.4 with

$$0 \leqslant u(x, y) \leqslant g(y) \equiv \frac{\tilde{\Psi}(y)}{\sqrt{\varrho_\infty(y)}} = \int_0^y \sqrt{\varrho_\infty(t)}\, dt/\sqrt{\varrho_\infty(y)} .$$

The function F is determined solely from the given function ϱ_∞. Hence, one can check (numerically, if need be) whether (i) (7.26) is valid and whether (ii) the equation $F(y, Dw(y), \mu) = H(y, Dw(y), \mu) = 0$, $y \in [0, 1]$, forces D is to be zero. If (i) and (ii) hold, then there will be extreme waves. Furthermore, there will exist a solution $(\lambda, u) \in \mathscr{C}$ with $g(\tilde{y}) = u(0, \tilde{y})$ for some $\tilde{y} \in (0, 1)$. This follows from the connectedness of $\mathscr{C}$ and its unboundedness in $\mathbb{R} \times (H_0^1(S) \cap C_0(\bar{S}))$. Note that

$$\tilde{\psi}(0, \tilde{y}) = \tilde{\Psi}(\tilde{y}) - \sqrt{\varrho_\infty(\tilde{y})}\, u(0, \tilde{y}) = 0 .$$

If we combine this with the remark after Theorem 7.3, then we see that the condition $\varrho_\infty'(0) = 0$ prevents (i) and (ii) from both holding when $\tilde{A} > 0$. This is not to say that such density functions do not have extreme waves, but rather that a method different from Theorem 7.4 will be needed to find them.

To conclude this section, we apply our theory to a particular example: $\varrho_\infty(y) = \alpha^2 \exp(-\beta y)$, where $\alpha, \beta > 0$. A calculation yields

$$\tilde{\Psi}(y) = \frac{2\alpha}{\beta}\left(1 - \exp(-\beta y/2)\right)$$

and

$$\tilde{Y}(\tilde{\psi}) = \frac{-2}{\beta} \log\left(1 - \frac{\beta\tilde{\psi}}{2\alpha}\right).$$

Equation (7.11) becomes

$$-\nabla\cdot\left(\alpha^2 \exp(-\beta y)\nabla u\right) = \lambda\alpha^2\beta \exp(-\beta y)\, u + F(y, u, \lambda) \tag{7.29}$$

where

$$F(y, u, \lambda) = -\alpha^2\lambda \exp(-\beta y)\left\{\beta u - 2\left(1 + \frac{\beta u}{2}\right)\log\left(1 + \frac{\beta u}{2}\right)\right\}. \tag{7.30}$$

To determine $w(y)$ and μ, we solve the linearized equation

$$-\frac{d}{dy}\left(\alpha^2 \exp(-\beta y)\,\frac{dw}{dy}\right) = \mu\alpha^2\beta \exp(-\beta y) w \qquad \text{on } (0,1),$$

$$w(0) = w(1) = 0.$$

This has the solution $\mu = (1/\beta)(\beta^2/4 + \pi^2)$ and w is a multiple of $\exp(\beta y/2)\sin \pi y$. We can determine the sign of $\tilde{A}$ from (7.15) or we can use (1.9). Clearly, $F(y, u, \lambda) \sim (\alpha^2/4)\lambda \exp(-\beta y)u^2$ as $u \to 0$, and $A(y, \mu) > 0$. Hence, $\tilde{A} > 0$, and so we have waves of elevation. It is an easy calculation to show that (7.26) is satisfied with

$$g(y) = \frac{\beta}{2}\left(\exp(\beta y/2) - 1\right).$$

Since we know $w(y)$ explicitly, one can check that the equation $F(y, Dw(y), \mu) = 0$, $y \in [0, 1]$, has only $D = 0$ as a solution. Theorem 7.4 can be applied to this density, and we conclude that extreme waves exist.

A different way of showing $D = 0$ is as follows. If $D \neq 0$, then we would have a non-trivial positive solution $m(y)$ to (1.6)-(1.7) when $\lambda = \mu$. We claim this is impossible for the present problem. If we multiply (1.6) by $w(y)$ and integrate by parts, then

$$(7.31)\qquad -(\mu-\lambda)\int_0^1 m(y)w(y)\varrho'_\infty(y)\,dy=\int_0^1 w(y)F(y,m(y),\lambda)\,dy\,.$$

It is straightforward to show that F given in (7.30) is positive for all $u>0$. Since $\varrho'_\infty\leqslant 0$, we have $\lambda<\mu$ for any positive x-independent solution of (7.29).

Our final example is for the density function $\varrho_\infty(y)=\cos^2 y$. A calculation yields

$$\tilde{\Psi}(y)=\sin y\,,\qquad \tilde{Y}(\tilde{\psi})=\sin^{-1}\tilde{\psi}$$

and

$$(7.32)\qquad \Delta\tilde{\psi}=\tilde{\psi}\{1-2\lambda(y-\sin^{-1}\tilde{\psi})\}\quad \text{in } S\,,$$

$$(7.33)\qquad \tilde{\psi}(x,0)=0\,,\quad \tilde{\psi}(x,1)=\sin 1\,,\quad x\in(-\infty,\infty)\,,$$

$$(7.34)\qquad \tilde{\psi}(x,y)\to\sin y\quad \text{as } x\to\infty\,.$$

If we set $\tilde{\psi}(x,y)=\sin y-u(x,y)\cos y$, then (7.32) becomes

$$(7.35)\qquad -\nabla\cdot(\cos^2 y\,\nabla u)=(2\lambda\cos y)(\sin y)u+F(y,u,\lambda)\quad \text{in } S\,,$$

where

$$(7.36)\qquad F(y,u,\lambda)$$
$$=2\lambda\cos y[(\sin y-u\cos y)\{y-\sin^{-1}(\sin y-u\cos y)\}-u\sin y]\,.$$

We claim there are no solutions of (7.32)-(7.34) with $0\leqslant\tilde{\psi}(x,y)\leqslant\tilde{\Psi}(1)$ on S which are waves of elevation. Assume the contrary so that $0<u(x,y)\leqslant\tan y$ on S. A calculation shows that $F(y,u,\lambda)\leqslant 0$ for such u, whence

$$\iint_S \varrho_\infty|\nabla u|^2\leqslant-\lambda\iint_S\varrho'_\infty u^2\,.$$

This inequality is impossible by (1.8) since $\lambda<\mu$. Hence, we should have waves of depression; indeed, since $F(y,u,\lambda)\sim-\lambda(1+\cos^2 y)u^2$ as $u\to 0$, equation (1.9) gives $\tilde{A}<0$. Note that Theorem 7.2 is not applicable since $\tilde{A}<0$. We claim that there is an extreme solution, and now prove so by assuming the contrary and deriving a contradiction. If all solutions of (7.32)-(7.34) have $\tilde{\psi}_y(0,y)>0$, $y\in(0,1)$, then $0\leqslant\tilde{\psi}(x,y)=\sin y-u(x,y)$

$\cdot \cos y < \tilde{\Psi}(1) = \sin 1$ on S. Since (λ, u) is a $(P-)$ solution, this gives

$$\frac{\sin y - \sin 1}{\cos y} < u(x, y) < 0 \quad \text{on } S. \tag{7.37}$$

Since we are assuming (7.37) holds along the global branch $\mathscr{C}$, there exist solutions (λ_n, u_n) with $|u_n|_{H_0^1(S)} \to \infty$ as $n \to \infty$. Define

$$H(y, u, \lambda) = \int_0^u F(y, t, \lambda)\, dt, \qquad u \in \left[\frac{\sin y - \sin 1}{\cos y},\ 0\right]$$

where F is given by (7.36). A calculation shows that (7.26) holds for u in this range. Since Theorem 7.4 is applicable with the obvious change from $(P+)$ to $(P-)$ solutions, it follows that (b) is satisfied. In particular, there exists a negative function m such that (μ, m) satisfy (1.6)-(1.7) with $\lambda = \mu$. However, one can show that $F(y, m, \lambda) < 0$ for $\lambda > 0$ and

$$m \in \left(\frac{\sin y - \sin 1}{\cos y},\ 0\right),$$

and so (7.31) ensures that $\lambda < \mu$ for any non-trivial negative solution of (1.6)-(1.7). This is a contradiction, and so (7.37) does not hold on all of $\mathscr{C}$.

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