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# Singular Perturbation Problems in the Calculus of Variations (\*).

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## 1. - Introduction.

In this paper we study the following singular perturbation problem in the Calculus of Variations; given an integral functional of the form

$$F(u) = \int_{\Omega} f(x, u, Du, D^2u, \dots, D^m u) dx;$$

determine the asymptotic behaviour (as  $\varepsilon \rightarrow 0^+$ ) of the infima of the functionals

$$F_{\varepsilon}(u) = \int_{\Omega} f(x, u, \varepsilon Du, \varepsilon^2 D^2u, \dots, \varepsilon^m D^m u) dx$$

(here  $D^k u$  denotes the vector  $(D^k u)_{|\alpha|=k}$  of all  $k$ -th order partial derivatives of  $u$ ).

By means of the  $\Gamma$ -convergence theory we prove that, under suitable assumptions on the integrand  $f$ , there exists a convex integrand  $\psi: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  such that for every  $\varphi \in L^q(\Omega)$

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0^+} \left\{ F_{\varepsilon}(u) + \int_{\Omega} \varphi u dx : u \in W^{m,r}(\Omega) \cap L^p(\Omega) \right\} \\ = \liminf_{\varepsilon \rightarrow 0^+} \left\{ F_{\varepsilon}(u) + \int_{\Omega} \varphi u dx : u \in W_0^{m,r}(\Omega) \cap L^p(\Omega) \right\} \\ = \min \left\{ \int_{\Omega} [\psi(x, u) + \varphi u] dx : u \in L^p(\Omega) \right\}, \end{aligned}$$

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where the exponents  $r$  and  $p$  are related to the behaviour of the integrand  $f$  and  $1/p + 1/q = 1$ . Moreover a formula for the function  $\psi$  is given.

There is an intimate relationship between this kind of problems and some singular perturbation problems in Optimal Control Theory. Consider for example a control problem with a cost functional of the form

$$J(u, v) = \int_{\Omega} [N|v(x)|^2 + |u(x) - b(x)|^p] dx$$

and with a singularity perturbed state equation of the form

$$(E_{\varepsilon}) \begin{cases} \varepsilon^2 \Delta u + g(u) = v \\ u \in H_0^1(\Omega). \end{cases}$$

( $N > 0$ ,  $b \in L^p(\Omega)$ , and  $g: \mathbb{R} \rightarrow \mathbb{R}$  are given;  $u$  and  $v$  are respectively the state variable and the control variable). Problems of this kind have been studied by J. L. Lions in his courses at the Collège de France in 1981-82 and 1982-83, and by A. Bensoussan [2], A. Haraux and F. Murat [11], [12], and V. Komornik [13]. By substituting  $v = \varepsilon^2 \Delta u + g(u)$  in the cost functional, the study of the asymptotic behaviour (as  $\varepsilon \rightarrow 0^+$ ) of

$$\inf \{J(u, v): (u, v) \text{ is a solution of } (E_{\varepsilon})\}$$

is reduced to the study of

$$\inf \left\{ \int_{\Omega} [N|\varepsilon^2 \Delta u + g(u)|^2 + |u - b(x)|^p] dx: u \in H_{\text{loc}}^2(\Omega) \cap H_0^1(\Omega) \right\},$$

which is the problem considered in Section 5.

Some of the results proved in this paper were announced without proof in [4].

We wish to thank Prof. E. De Giorgi for many helpful discussions on this subject.

## 2. - $\Gamma$ -convergence.

In this section we collect some known results of  $\Gamma$ -convergence theory that are used in the sequel. For a general exposition of this subject we refer to [6] and [7].

Let  $A, X$  be two topological spaces (we consider  $A$  as a space of parameters, in general  $A = \bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$  or  $A = \mathbb{R}$ ); let  $A_0 \subseteq A$  and  $X_0 \subseteq X$

with  $X_0$  dense in  $X$ ; for every  $\lambda \in \mathcal{A}_0$  let  $F_\lambda$  be a function from  $X_0$  into  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ ; let  $\lambda_0 \in \mathcal{A}$ ,  $x \in X$  with  $\lambda_0 \in \overline{\mathcal{A}}_0$ ; following [8] we define

$$(2.1) \quad \Gamma(\mathcal{A}^-, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y) = \sup_{U \in \mathcal{J}(x)} \liminf_{\substack{\lambda \rightarrow \lambda_0 \\ \lambda \in \mathcal{A}_0}} \inf_{y \in U \cap X_0} F_\lambda(y),$$

$$(2.2) \quad \Gamma(\mathcal{A}^+, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y) = \sup_{U \in \mathcal{J}(x)} \limsup_{\substack{\lambda \rightarrow \lambda_0 \\ \lambda \in \mathcal{A}_0}} \inf_{y \in U \cap X_0} F_\lambda(y),$$

where  $\mathcal{J}(x)$  denotes the family of all neighbourhoods of  $x$  in the space  $X$ . When the  $\Gamma$ -limits (2.1) and (2.2) coincide, their common value is indicated by

$$\Gamma(\mathcal{A}, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y).$$

The main properties of  $\Gamma$ -limits are given by the following propositions, proved in [3] and [9].

PROPOSITION 2.1. *For every  $x \in X$  define*

$$F^-(x) = \Gamma(\mathcal{A}^-, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y)$$

$$F^+(x) = \Gamma(\mathcal{A}^+, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y).$$

*The functions  $F^-: X \rightarrow \overline{\mathbb{R}}$  and  $F^+: X \rightarrow \overline{\mathbb{R}}$  are lower semicontinuous on  $X$ .*

PROPOSITION 2.2. *Suppose that  $X$  has a countable base for the open sets. For every sequence  $(F_n)$  of functions from  $X_0$  into  $\overline{\mathbb{R}}$ , there exists a subsequence  $(F_{n_k})$  and a function  $F: X \rightarrow \overline{\mathbb{R}}$  such that*

$$F(x) = \Gamma(\overline{\mathcal{N}}, X^-) \lim_{\substack{k \rightarrow \infty \\ y \rightarrow x}} F_{n_k}(y)$$

*for every  $x \in X$ .*

PROPOSITION 2.3. *If  $G: X \rightarrow \mathbb{R}$  is lower semicontinuous at the point  $x \in X$ , then*

$$\Gamma(\mathcal{A}^-, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} [G + F_\lambda](y) \geq G(x) + \Gamma(\mathcal{A}^-, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y)$$

$$\Gamma(\mathcal{A}^+, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} [G + F_\lambda](y) \geq G(x) + \Gamma(\mathcal{A}^+, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y);$$

if in addition  $G$  is continuous at the point  $x$ , then the above inequalities are equalities.

**PROPOSITION 2.4.** *Suppose that there exists  $F: X \rightarrow \overline{\mathbb{R}}$  such that*

$$F(x) = \Gamma(\mathcal{A}, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y)$$

for every  $x \in X$ . Assume further that the functions  $F_\lambda$  are equicoercive on  $X$ , i.e. for every  $s \in \mathbb{R}$  there exists a compact subset  $K_s$  of  $X$  (independent of  $\lambda$ ) such that  $\{x \in X_0: F_\lambda(x) \leq s\} \subseteq K_s$  for every  $\lambda \in \mathcal{A}_0$ .

Then we have

$$\min_X F = \lim_{\lambda \rightarrow \lambda_0} \left[ \inf_{X_0} F_\lambda \right].$$

Moreover, if  $(x_\lambda)_{\lambda \in \mathcal{A}_0}$  is a family of elements of  $X_0$  such that  $\varinjlim_{\lambda \rightarrow \lambda_0} \lambda_\lambda = x$  and  $\varinjlim_{\lambda \rightarrow \lambda_0} [F_\lambda(x_\lambda) - \inf_{X_0} F_\lambda] = 0$ , then  $x$  is a minimum point of  $F$  in  $X$ .

Let  $\mathcal{S}_0(\lambda_0)$  be the set of all sequences in  $\mathcal{A}_0$  converging to  $\lambda_0$  in  $\mathcal{A}$ , and let  $\mathcal{S}(x)$  be the set of all sequences in  $X_0$  converging to  $x$ ; we define (the subscript seq stands for sequential)

$$(2.3) \quad \Gamma_{\text{seq}}(\mathcal{A}^-, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y) = \inf_{(\lambda_h) \in \mathcal{S}_0(\lambda_0)} \inf_{(x_h) \in \mathcal{S}(x)} \liminf_{h \rightarrow \infty} F_{\lambda_h}(x_h)$$

$$(2.4) \quad \Gamma_{\text{seq}}(\mathcal{A}^+, X^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y) = \sup_{(\lambda_h) \in \mathcal{S}_0(\lambda_0)} \inf_{(x_h) \in \mathcal{S}(x)} \limsup_{h \rightarrow \infty} F_{\lambda_h}(x_h).$$

**REMARK 2.5.** If the spaces  $\mathcal{A}$  and  $X$  satisfy the first axiom of countability it is possible to prove (see [3]) that the  $\Gamma_{\text{seq}}$ -limits (2.3) and (2.4) coincide respectively with the  $\Gamma$ -limits (2.1) and (2.2).

**REMARK 2.6.** It is not difficult to see that in the case  $\mathcal{A} = \overline{\mathbb{N}}$ ,  $\mathcal{A}_0 = \mathbb{N}$ ,  $\lambda_0 = \infty$ , the  $\Gamma_{\text{seq}}$ -limits (2.3) and (2.4) of a sequence  $(F_h)_{h \in \mathbb{N}}$  of functions reduce respectively to

$$\inf_{(x_h) \in \mathcal{S}(x)} \liminf_{h \rightarrow \infty} F_h(x_h) \quad \text{and} \quad \inf_{(x_h) \in \mathcal{S}(x)} \limsup_{h \rightarrow \infty} F_h(x_h).$$

Suppose that  $X$  is a reflexive separable Banach space with dual  $X'$ . Let  $(x'_h)$  be a sequence dense in the unit ball of  $X'$ ; we introduce the metric  $\delta$

on  $X$  defined by

$$\delta(x, y) = \sum_{h=1}^{\infty} 2^{-h} |\langle x'_h, x - y \rangle|.$$

It is known that the metric space  $(X, \delta)$  is separable.

Let us denote by  $w$  the weak topology of  $X$ .

We shall use the following proposition proved in [1].

**PROPOSITION 2.7.** *Assume that  $X$  is a reflexive Banach space, that  $\lambda_0$  has a countable neighbourhood base in  $\Lambda$ , and that there exist two constants  $c_1, c_2 \in \mathbb{R}$ , with  $c_2 > 0$ , such that*

$$F_\lambda(x) \geq c_1 + c_2 \|x\|$$

for every  $\lambda \in \Lambda_0, x \in X_0$ .

Then for every  $x \in X$

$$\Gamma_{\text{seq}}(\Lambda^-, w^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y) = \Gamma(\Lambda^-, w^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y) = \Gamma(\Lambda^-, \delta^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y)$$

$$\Gamma_{\text{seq}}(\Lambda^+, w^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y) = \Gamma(\Lambda^+, w^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y) = \Gamma(\Lambda^+, \delta^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y).$$

Using Proposition 2.3 and some general properties of  $\Gamma$ -limits (see [3], [8]) it is easy to obtain the following proposition.

**PROPOSITION 2.8.** *Under the hypotheses of Proposition 2.7, for every  $x \in X, s \in \mathbb{R}$  the following conditions are equivalent:*

i)  $\Gamma(\Lambda, w^-) \lim_{\substack{\lambda \rightarrow \lambda_0 \\ y \rightarrow x}} F_\lambda(y) = s$

ii) *for every sequence  $(\lambda_n)$  in  $\Lambda_0$  converging to  $\lambda_0$  in  $\Lambda$  there exists a subsequence  $(\lambda_{n_k})$  such that*

$$\Gamma(\overline{\mathbb{N}}, w^-) \lim_{\substack{k \rightarrow \infty \\ y \rightarrow x}} F_{\lambda_{n_k}}(y) = s.$$

### 3. - Statement of the result.

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$ , let  $m \geq 1$  be an integer, and let  $p, r$  be two real numbers with  $p > 1, 1 \leq r \leq p$ .

We indicate by  $d = d(n, m)$  the number of multi-indices  $\alpha \in \mathbb{N}^n$  such that  $1 \leq |\alpha| \leq m$ , by  $\mathcal{A}(\mathbb{R}^n)$  the family of all bounded open subsets of  $\mathbb{R}^n$ , and by  $\mathcal{A} = \mathcal{A}(\Omega)$  the family of all open subsets of  $\Omega$ .

For every  $k = 1, 2, \dots, m$  and every  $u \in W_{\text{loc}}^{m,r}(\mathcal{A})$ , with  $\mathcal{A} \in \mathcal{A}(\mathbb{R}^n)$ , we denote by  $D^k u$  the vector  $(D^\alpha u)_{|\alpha|=k}$  of all  $k$ -th order partial derivatives of  $u$ .

The integrands we shall consider are Borel functions  $f: \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow [0, +\infty[$  which satisfy the following properties:

(3.1) *there exist  $c \geq 1$  and  $a \in L^1(\Omega)$  such that*

$$-a(x) + |s|^p \leq f(x, s, z) \leq a(x) + c[|s|^p + |z|^r]$$

*for every  $x \in \Omega$ ,  $s \in \mathbb{R}$ ,  $z \in \mathbb{R}^d$ ;*

(3.2) *there exist  $a \in L^1(\Omega)$ , an increasing continuous function  $\sigma: [0, +\infty[ \rightarrow [0, +\infty[$  with  $\sigma(0) = 0$ , and a Borel function  $\omega: \Omega \times \mathbb{R}^n \rightarrow [0, +\infty[$  with*

$$\lim_{v \rightarrow 0} \int_{\Omega} \omega(x, y) dx = \int_{\Omega} \omega(x, 0) dx = 0,$$

*such that*

$$|f(y, t, w) - f(x, s, z)| \leq \omega(x, y - x) + \sigma(|y - x| + |t - s| + |w - z|)(a(x) + f(x, s, z))$$

*for every  $x \in \Omega$ ,  $s \in \mathbb{R}$ ,  $z \in \mathbb{R}^d$ ;*

(3.3) *there exists  $a \in L^1(\Omega)$ , a Borel function  $\gamma: \mathbb{R} \times \mathbb{R}^d \rightarrow [0, +\infty[$ , and a function  $\lambda: \mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n) \rightarrow [0, +\infty[$  such that*

(i) *for every  $x \in \Omega$ ,  $s \in \mathbb{R}$ ,  $z \in \mathbb{R}^d$*

$$\gamma(s, z) \leq f(x, s, z) + |s|^p + a(x)$$

(ii) *for every pair  $A, A' \in \mathcal{A}(\mathbb{R}^n)$  with  $A \subset\subset A'$  and for every  $u \in W^{m,r}(A')$*

$$\int_A \sum_{|\alpha| \leq m} |D^\alpha u|^\tau dx \leq \lambda(A, A') \int_{A'} \gamma(u, Du, D^2 u, \dots, D^m u) dx$$

(iii) *for every pair  $A, A' \in \mathcal{A}(\mathbb{R}^n)$  with  $A \subset\subset A'$*

$$\limsup_{t \rightarrow +\infty} \lambda(tA, tA') < +\infty.$$

For every  $\varepsilon > 0$  we consider the functional  $F_\varepsilon(u, A)$  defined for every  $A \in \mathcal{A}$  and for every  $u \in W_{\text{loc}}^{m,r}(A)$  by

$$(3.4) \quad F_\varepsilon(u, A) = \int_A f(x, u, \varepsilon Du, \varepsilon^2 D^2 u, \dots, \varepsilon^m D^m u) dx.$$

It is possible to verify (see section 6) that hypotheses (3.1), (3.2), (3.3) are fulfilled, for example, by the functionals

$$F_\varepsilon(u, A) = \int_A [(\varepsilon|Du| + P_k(u) + a(x))^2 + |u - b(x)|^{2k}] dx,$$

$$F_\varepsilon(u, A) = \int_A [|\varepsilon^2 \Delta u + P_k(u) + a(x)|^2 + |u - b(x)|^{2k}] dx,$$

$$F_\varepsilon(u, A) = \int_A [\varphi(x, u, \varepsilon Du, \varepsilon^2 D^2 u) |\varepsilon^2 \Delta u + P_k(u) + a(x)|^2 + |u - b(x)|^{2k}] dx,$$

where  $k \geq 1$  is an integer,  $P_k$  is a polynomial of degree less than or equal to  $k$ ,  $a \in L^2(\Omega)$ ,  $b \in L^{2k}(\Omega)$ , and  $\varphi: \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is uniformly continuous and satisfies  $0 < \inf \varphi \leq \sup \varphi < +\infty$ .

Other examples of functionals verifying hypotheses (3.1), (3.2), (3.3) can be found in Section 5.

Define now for every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$

$$(3.5) \quad T(u, A) = \begin{cases} 0 & \text{if } u \in W_0^{m,r}(A) \\ +\infty & \text{otherwise.} \end{cases}$$

Let us denote by  $w - L^p(A)$  the weak topology of  $L^p(A)$ . The main result we prove in this paper is the following.

**THEOREM 3.1.** *Let  $f: \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow [0, +\infty[$  be a Borel function satisfying hypotheses (3.1), (3.2), (3.3), and let  $F_\varepsilon$  be the functionals defined by (3.4). Then there exists a Borel function  $\psi: \Omega \times \mathbb{R} \rightarrow [0, +\infty[$  such that*

(i) *for every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$ ,  $w_0 \in W^{m,r}(A) \cap L^p(A)$*

$$\begin{aligned} \int_A \psi(x, u) dx &= \Gamma(\mathbb{R}, w - L^p(A)^-) \lim_{\substack{\varepsilon \rightarrow 0^+ \\ v \rightarrow u}} F_\varepsilon(v, A) \\ &= \Gamma(\mathbb{R}, w - L^p(A)^-) \lim_{\substack{\varepsilon \rightarrow 0^+ \\ v \rightarrow u}} [F_\varepsilon(v, A) + T(v - w_0, A)]; \end{aligned}$$

(ii) *for every  $x \in \Omega$  the function  $s \rightarrow \psi(x, s)$  is convex on  $\mathbb{R}$ ;*

(iii) *for every  $(x, s) \in \Omega \times \mathbb{R}$*

$$f^-(x, s, 0) \leq \psi(x, s) \leq f^+(x, s, 0)$$

where  $f^+(x, s, z)$  is the greatest function convex in  $s$  which is less than or equal to  $f(x, s, z)$  and  $f^-(x, s, z)$  is the greatest function convex in  $(s, z)$  which is less than or equal to  $f(x, s, z)$ .

Moreover the following representation formulae for  $\psi$  hold for a.a.  $x \in \Omega$  and all  $s \in \mathbb{R}$ :

$$\begin{aligned} \psi(x, s) &= \liminf_{\varepsilon \rightarrow 0^+} \left\{ F_\varepsilon(x, u) : u \in W^{m,r}(Y) \cap L^p(Y), \int_Y u \, dy = s \right\} \\ &= \liminf_{\varepsilon \rightarrow 0^+} \left\{ F_\varepsilon(x, u) : u - s \in W_0^{m,r}(Y) \cap L^p(Y), \int_Y u \, dy = s \right\} \\ &= \inf \left\{ F_\varepsilon(x, u) : \varepsilon > 0, u - s \in W_0^{m,r}(Y) \cap L^p(Y), \int_Y u \, dy = s \right\} \\ &= \inf \left\{ F_\varepsilon(x, u) : \varepsilon > 0, u \in W_\#^{m,r}(Y) \cap L^p(Y), \int_Y u \, dy = s \right\}, \end{aligned}$$

where  $Y$  denotes the unit cube  $]0, 1[^n$ ,  $W_\#^{m,r}(Y)$  denotes the space of all  $Y$ -periodic functions of  $W_{loc}^{m,r}(\mathbb{R}^n)$ , and

$$F_\varepsilon(x, u) = \int_Y f(x, u(y), \varepsilon Du(y), \varepsilon^2 D^2 u(y), \dots, \varepsilon^m D^m u(y)) \, dy.$$

**COROLLARY 3.2.** *Let  $w_0 \in W^{m,r}(\Omega) \cap L^p(\Omega)$ , let  $W(w_0) = \{u \in L^p(\Omega) : u - w_0 \in W_0^{m,r}(\Omega)\}$ , and let  $V$  be a set such that  $W(w_0) \subseteq V \subseteq W_{loc}^{m,r}(\Omega) \cap L^p(\Omega)$ . Then we have*

$$\begin{aligned} (3.6) \quad \liminf_{\varepsilon \rightarrow 0^+} \left\{ \int_\Omega F_\varepsilon(u, \Omega) \, dx + \int_\Omega g u \, dx : u \in V \right\} \\ = \min \left\{ \int_\Omega \psi(x, u) \, dx + \int_\Omega g u \, dx : u \in L^p(\Omega) \right\} \end{aligned}$$

for every  $g \in L^q(\Omega) (1/p + 1/q = 1)$ .

**PROOF.** It follows from Theorem 3.1, Proposition 2.3 and Proposition 2.4 that

$$\begin{aligned} \min \left\{ \int_\Omega \psi(x, u) \, dx + \int_\Omega g u \, dx : u \in L^p(\Omega) \right\} \\ = \liminf_{\varepsilon \rightarrow 0^+} \left\{ F_\varepsilon(u, \Omega) + \int_\Omega g u \, dx : u \in W_{loc}^{m,r}(\Omega) \cap L^p(\Omega) \right\} \\ = \liminf_{\varepsilon \rightarrow 0^+} \left\{ F_\varepsilon(u, \Omega) + \int_\Omega g u \, dx + T(u - w_0, \Omega) : u \in W_{loc}^{m,r}(\Omega) \cap L^p(\Omega) \right\} \\ = \liminf_{\varepsilon \rightarrow 0^+} \left\{ F_\varepsilon(u, \Omega) + \int_\Omega g u \, dx : u \in W(w_0) \right\}. \end{aligned}$$

Since  $W(w_0) \subseteq V \subseteq W_{loc}^{m,r}(\Omega) \cap L^p(\Omega)$  we obtain (3.6). ■

**4. – Proof of the result.**

In this section we prove Theorem 3.1.

The function  $f$  and the functionals  $F_\varepsilon$  are supposed to satisfy the hypotheses of the theorem. In what follows we shall write briefly  $f(x, u, \varepsilon^k D^k u)$  instead of  $f(x, u, \varepsilon Du, \varepsilon^2 D^2 u, \dots, \varepsilon^m D^m u)$ . Let  $(\varepsilon_n)$  be a sequence in  $]0, +\infty[$  converging to 0. For every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$  set

$$F^+(u, A) = I(\overline{\mathbb{N}}^+, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} F_{\varepsilon_h}(u, A).$$

LEMMA 4.1. *For every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$  we have*

$$F^+(u, A) \leq \int_A f(x, u, 0) dx.$$

PROOF. Let  $A \in \mathcal{A}$ ,  $u \in L^p(A)$ . Let  $\varrho$  be a non-negative function in  $C_0^\infty(\mathbb{R}^n)$  such that  $\int \varrho dx = 1$ , let  $\theta = 1/(n + m + 1)$ , let  $\varrho_h(x) = \varepsilon_h^{-n\theta} \varrho(\varepsilon_h^{-\theta} x)$ , and let  $u_h = \varrho_h * u$ . We have

$$F_{\varepsilon_h}(u_h, A) = \int_A f(x, \varrho_h * u, \varepsilon_h^k D^k \varrho_h * u) dx.$$

It is easy to see that  $(\varrho_h * u)_h$  converges to  $u$  in  $L^p(A)$  and that  $(\varepsilon_h^k D^k \varrho_h * u)_h$  converges to 0 in  $L^p(A)$  (hence in  $L^r(A)$ ) for  $k = 1, 2, \dots, m$ . Since  $f(x, s, z)$  is continuous in  $(s, z)$ , inequalities (3.1) ensure that

$$\int_A f(x, u, 0) dx = \lim_{h \rightarrow \infty} \int_A f(x, \varrho_h * u, \varepsilon_h^k D^k \varrho_h * u) dx.$$

By Remark 2.6 and Proposition 2.7 we have

$$F^+(u, A) \leq \limsup_{h \rightarrow \infty} F_{\varepsilon_h}(u_h, A) = \int_A f(x, u, 0) dx$$

and the lemma is proved. ■

LEMMA 4.2. *Let  $A, B, C \in \mathcal{A}$  with  $C \subset\subset A \cup B$ . For every  $u \in L^p(A \cup B)$  we have*

$$F^+(u, C) \leq F^+(u, A) + F^+(u, B).$$

PROOF. Let  $K = \bar{C} - B$  and let  $A_0, B_0$  be two open sets, with  $\text{meas}(\partial A_0) = \text{meas}(\partial B_0) = 0$ , such that  $K \subseteq A_0 \subset\subset B_0 \subset\subset A$ . Fix an integer  $\nu$  and a family  $(A_i)_{1 \leq i \leq \nu}$  of open sets, with  $\text{meas}(\partial A_i) = 0$ , such that  $A_0 \subset\subset A_1 \subset\subset \dots \subset\subset A_\nu \subset\subset B_0$ . Define  $S_i = C \cap (A_i - \bar{A}_{i-1})$  and  $S = C \cap (B_0 - A_0)$ . For every  $i = 1, 2, \dots, \nu$  there exists  $\varphi_i \in C_0^\infty(A_i)$  such that  $0 \leq \varphi_i \leq 1$  and  $\varphi_i = 1$  on  $A_{i-1}$ .

In what follows the letter  $c$  will denote various positive constants (independent of  $h, i, \nu$ ), whose value can change from one line to the next.

Fix  $u \in L^p(A \cup B)$  and  $\eta > 0$ ; there exists a sequence  $(u_h)$  in  $W_{\text{loc}}^{m,r}(A) \cap L^p(A)$ , converging to  $u$  weakly in  $L^p(A)$  and a sequence  $(v_h)$  in  $W_{\text{loc}}^{m,r}(B) \cap L^p(B)$  converging to  $u$  weakly in  $L^p(B)$  such that

$$F^+(u, A) + \eta \geq \limsup_{h \rightarrow \infty} F_{\varepsilon_h}(u_h, A) \quad \text{and} \quad F^+(u, B) + \eta \geq \limsup_{h \rightarrow \infty} F_{\varepsilon_h}(v_h, B).$$

For every  $i = 1, 2, \dots, \nu$  and for every  $h \in \mathbb{N}$  set

$$w_{i,h} = \varphi_i u_h + (1 - \varphi_i) v_h.$$

Using (3.1) we obtain

$$\begin{aligned} F_{\varepsilon_h}(w_{i,h}, C) &\leq F_{\varepsilon_h}(u_h, C \cap A_{i-1}) + F_{\varepsilon_h}(v_h, C - \bar{A}_i) \\ &+ c \int_{S_i} \left[ |a(x) + |w_{i,h}|^p + \sum_{k=1}^m |\varepsilon_h^k D^k w_{i,h}|^r \right] dx \\ &\leq F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) + c \int_{S_i} \left\{ |a(x) + |u_h|^p + |v_h|^p + \sum_{k=1}^m [|\varepsilon_h^k D^k u_h|^r + |\varepsilon_h^k D^k v_h|^r] \right. \\ &\left. + c_\nu \sum_{k=1}^m \varepsilon_h^{kr} \sum_{j=0}^{k-1} [ |D^j u_h|^r + |D^j v_h|^r ] \right\} dx, \end{aligned}$$

where  $c_\nu$  depends on  $\sup |D^\alpha \varphi_i|$  for  $i = 1, 2, \dots, \nu$  and  $|\alpha| \leq m$ . Since the strips  $S_i$  are pairwise disjoint, for every  $h \in \mathbb{N}$  there exists an index  $i_h \in \{1, 2, \dots, \nu\}$  such that

$$\int_{S_{i_h}} \{ \dots \} dx \leq \frac{1}{\nu} \int_S \{ \dots \} dx.$$

Define  $w_h = w_{i_h, h}$ . Then

$$\begin{aligned} F_{\varepsilon_h}(w_h, C) &\leq F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) + \frac{c}{\nu} \int_S \left\{ |a(x) + |u_h|^p + |v_h|^p \right. \\ &\left. + \sum_{k=1}^m [ |\varepsilon_h^k D^k u_h|^r + |\varepsilon_h^k D^k v_h|^r ] + c_\nu \sum_{k=1}^m \varepsilon_h^{kr} \sum_{j=0}^{k-1} [ |D^j u_h|^r + |D^j v_h|^r ] \right\} dx. \end{aligned}$$

Let  $E = A \cap B$ . Since  $S \subset\subset E$ , there exists  $S' \in \mathcal{A}$  such that  $S \subset\subset S' \subset\subset E$ . Since  $(u_h)$  and  $(v_h)$  are bounded in  $L^p(S')$ , using inequalities as

$$\int_S |D^k w|^r dx \leq \sigma \int_{S'} |D^m w|^r dx + c_\sigma \int_{S'} |w|^r dx$$

(which hold for  $1 \leq k \leq m$  and for every  $\sigma > 0$ ) we get

$$(4.1) \quad F_{\varepsilon_h}(w_h, C) \leq F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) + \frac{c}{\nu}(1 + \varepsilon_h c_{\nu, \sigma}) + \frac{c}{\nu}(1 + \sigma c_\nu) \int_{S'} \sum_{k=1}^m [|\varepsilon_h^k D^k u_h|^r + |\varepsilon_h^k D^k v_h|^r] dx.$$

Define now  $U_h(x) = u_h(\varepsilon_h x)$  and  $V_h(x) = v_h(\varepsilon_h x)$ ; then, using (3.3), we get

$$(4.2) \quad \int_{S'} \sum_{k=1}^m [|\varepsilon_h^k D^k u_h|^r + |\varepsilon_h^k D^k v_h|^r] dx = \varepsilon_h^n \int_{\varepsilon_h^{-1} S'} \sum_{k=1}^m [ |D^k U_h|^r + |D^k V_h|^r ] dx \leq \lambda(\varepsilon_h^{-1} S', \varepsilon_h^{-1} E) \varepsilon_h^n \int_{\varepsilon_h^{-1} E} [\gamma(U_h, D^k U_h) + \gamma(V_h, D^k V_h)] dx = \lambda(\varepsilon_h^{-1} S', \varepsilon_h^{-1} E) \int_E [\gamma(u_h, \varepsilon_h^k D^k u_h) + \gamma(v_h, \varepsilon_h^k D^k v_h)] dx \leq \lambda(\varepsilon_h^{-1} S', \varepsilon_h^{-1} E) [c + F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B)].$$

Since the sequences  $(w_{i,k})$  converge to  $u$  weakly in  $L^p(C)$ , it is easy to see that the sequence  $(w_h)$  converges to  $u$  weakly in  $L^p(C)$ . Therefore, passing to the limit in (4.1) as  $h \rightarrow \infty$ , and using (4.2) and (3.3) (iii) we get

$$F^+(u, C) \leq F^+(u, A) + F^+(u, B) + 2\eta + \frac{c}{\nu} + \frac{c}{\nu}(1 + \sigma c_\nu) M [c + F^+(u, A) + F^+(u, B) + 2\eta],$$

where  $M = \limsup_{t \rightarrow +\infty} \lambda(tS', tE)$ . Passing to the limit first as  $\sigma \rightarrow 0$ , then as  $\nu \rightarrow +\infty$ , and finally as  $\eta \rightarrow 0$ , we obtain

$$F^+(u, C) \leq F^+(u, A) + F^+(u, B). \quad \blacksquare$$

REMARK 4.3. In the same way we can prove that for every  $A, B \in \mathcal{A}$ ,

with  $B \subset\subset A$ , and for every compact subset  $K$  of  $B$

$$F^+(u, A) \leq F^+(u, B) + F^+(u, A - K)$$

for every  $u \in L^p(A)$ . This fact, combined with Lemma 4.1 and inequalities (3.1), implies that

$$F^+(u, A) = \sup \{F^+(u, B) : B \in \mathcal{A}, B \subset\subset A\}.$$

**LEMMA 4.4.** *There exist a subsequence  $(\varepsilon_{n_k})$  of  $(\varepsilon_n)$  and a functional  $F$  such that*

$$(4.3) \quad F(u, A) = \Gamma(\overline{\mathbb{N}}, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} F_{\varepsilon_{n_k}}(v, A)$$

for every  $A \in \mathcal{A}$  and for every  $u \in L^p(A)$ . Moreover for every  $u \in L^p(\Omega)$  the set function  $A \rightarrow F(u, A)$  is the trace on  $\mathcal{A}$  of a regular Borel measure defined on  $\Omega$ .

**PROOF.** Let  $\mathcal{U}$  be a countable base for the open subsets of  $\Omega$ , closed under finite unions; note that for every  $A, B \in \mathcal{A}$  with  $A \subset\subset B$ , there exists  $U \in \mathcal{U}$  such that  $A \subset\subset U \subset\subset B$ . By the compactness of  $\Gamma$ -convergence (see Propositions 2.2 and 2.7) there exists a subsequence of  $(\varepsilon_n)$  (which we still denote by  $(\varepsilon_n)$ ) such that for every  $B \in \mathcal{U}$ ,  $u \in L^p(B)$  there exists the  $\Gamma$ -limit

$$G(u, B) = \Gamma(\overline{\mathbb{N}}, w - L^p(B)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} F_{\varepsilon_n}(v, B).$$

For every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$  we set

$$F(u, A) = \sup \{G(u, B) : B \in \mathcal{U}, B \subset\subset A\}.$$

It is easy to see that for every  $u \in L^p(\Omega)$  the set function  $A \rightarrow G(u, A)$  is superadditive on  $\mathcal{U}$ , so  $A \rightarrow F(u, A)$  is superadditive on  $\mathcal{A}$ . It follows from Lemma 4.2 that  $A \rightarrow F(u, A)$  is subadditive. So  $A \rightarrow F(u, A)$  is increasing, superadditive, subadditive, and inner regular. By a result of measure theory (see [10] Proposition 5.5 and Theorem 5.6) this implies that  $A \rightarrow F(u, A)$  is the trace on  $\mathcal{A}$  of a regular Borel measure defined on  $\Omega$ . It remains to prove (4.3). Let

$$F^-(u, A) = \Gamma(\overline{\mathbb{N}}^-, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} F_{\varepsilon_n}(v, A)$$

and

$$F^+(u, A) = \Gamma(\overline{\mathbf{N}}^+, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} F_{\varepsilon_h}(v, A).$$

By Remark 4.3 we have

$$\begin{aligned} F^+(u, A) &= \sup \{F^+(u, B) : B \in \mathcal{A}, B \subset\subset A\} \\ &= \sup \{G(u, B) : B \in \mathcal{A}, B \subset\subset A\} = F(u, A) \leq F^-(u, A) \leq F^+(u, A), \end{aligned}$$

which proves (4.3). ■

LEMMA 4.5. *Let  $F$  be the functional introduced in Lemma 4.4. There exists a Borel function  $\psi : \Omega \times \mathbf{R} \rightarrow [0, +\infty[$  such that*

(i) *for every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$*

$$F(u, A) = \int_A \psi(x, u) dx,$$

(ii) *for every  $x \in \Omega$  the function  $s \rightarrow \psi(x, s)$  is convex on  $\mathbf{R}$ ,*

(iii) *for every  $x \in \Omega$ ,  $s \in \mathbf{R}$*

$$-a(x) + |s|^p \leq \psi(x, s) \leq f^+(x, s, 0).$$

PROOF. Let us denote by  $\mathfrak{B} = \mathfrak{B}(\Omega)$  the class of all Borel subsets of  $\Omega$ . For every  $u \in L^p(\Omega)$  we denote by  $\Phi(u, \cdot)$  the measure on  $\mathfrak{B}$  which extends  $F(u, \cdot)$ ; it is easy to see that for every  $B \in \mathfrak{B}$

$$\Phi(u, B) = \inf \{F(u, A) : A \in \mathcal{A}, A \supseteq B\}.$$

First of all we prove that the functional  $\Phi$  is local on  $\mathfrak{B}$ , that is: if  $u = v$  a.e. on a Borel set  $B$ , then  $\Phi(u, B) = \Phi(v, B)$ . Let  $u, v \in L^p(\Omega)$  and let  $B \in \mathfrak{B}$  with  $u = v$  a.e. on  $B$ ; without loss of generality we may suppose that  $u = v$  everywhere on  $B$  and  $u \leq v$  everywhere on  $\Omega$ . By Lusin's theorem, for every  $\varepsilon > 0$  there exists  $A_\varepsilon \in \mathcal{A}$ , with  $\text{meas}(A_\varepsilon) < \varepsilon$ , such that the restrictions  $u|_{\Omega - A_\varepsilon}$  and  $v|_{\Omega - A_\varepsilon}$  are continuous. Then the set  $B_\varepsilon = A_\varepsilon \cup \{x \in \Omega : v(x) < u(x) + \varepsilon\}$  is open; moreover  $B_\varepsilon \supseteq B$ . Define now

$$u_\varepsilon(x) = \begin{cases} v(x) & \text{if } x \in B_\varepsilon \\ u(x) + \varepsilon & \text{if } x \in \Omega - B_\varepsilon; \end{cases}$$

it is easy to see that  $(u_\varepsilon)$  converges to  $u$  strongly in  $L^p(\Omega)$  as  $\varepsilon \searrow 0$ . For every  $\eta > 0$  there exist an open set  $A$  and a compact set  $K$  such that  $K \subseteq B \subseteq A \subseteq \Omega$ ,  $F(u, A) < \Phi(v, B) + \eta$  and  $\int_{A-K} [a(x) + c|u|^p] dx < \eta$ .

Since  $F(\cdot, A)$  is lower semicontinuous with respect to the weak topology of  $L^p(A)$  (see Proposition 2.1) and  $F$  is local on  $\mathcal{A}$ , using Lemma 4.1 and inequalities (3.1) we obtain

$$\begin{aligned} \Phi(u, B) &\leq F(u, A) \leq \liminf_{\varepsilon \rightarrow 0^+} F(u_\varepsilon, A) \leq \liminf_{\varepsilon \rightarrow 0^+} [F(v, A \cap B_\varepsilon) + F(u_\varepsilon, A - K)] \\ &\leq F(v, A) + \liminf_{\varepsilon \rightarrow 0^+} \int_{A-K} [a(x) + c|u_\varepsilon|^p] dx \leq \Phi(v, B) + 2\eta. \end{aligned}$$

Since  $\eta < 0$  was arbitrary, we get

$$\Phi(u, B) \leq \Phi(v, B).$$

The opposite inequality can be proved in a similar way.

So the functional  $\Phi: L^p(\Omega) \times \mathcal{B} \rightarrow [0, +\infty[$  is local on  $B$ , for every  $u \in L^p(\Omega)$  the set function  $\Phi(u, \cdot)$  is a measure, and the function  $\Phi(\cdot, \Omega)$  is lower semicontinuous in the weak topology of  $L^p(\Omega)$ . This implies (see [5]) that there exists a non-negative Borel function  $\psi(x, s)$ , convex in  $s$ , such that

$$\Phi(u, B) = \int_B \psi(x, u) dx$$

for every  $u \in L^p(\Omega)$ ,  $B \in \mathcal{B}$ . Since  $\Phi(u, A) = F(u, A)$  for every  $A \in \mathcal{A}$ , we obtain (i) and (ii). Finally, (iii) follows from inequalities (3.1) and from Lemma 4.1. ■

LEMMA 4.6. *For every  $A \in \mathcal{A}$  and for every  $u \in W^{m,r}(A) \cap L^p(A)$  we have*

$$F^+(u, A) \geq \Gamma(\bar{\mathbf{N}}^+, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} [F_{\varepsilon_h}(v, A) + T(v - u, A)]$$

where  $T$  is the functional defined by (3.5).

PROOF. Let  $A \in \mathcal{A}$ ,  $u \in W^{m,r}(A) \cap L^p(A)$ , and  $\eta > 0$ . There exists a sequence  $(u_n)$  in  $W^{m,r}_{loc}(A) \cap L^p(A)$  converging to  $u$  weakly in  $L^p(A)$  such that

$$F^+(u, A) + \eta \geq \limsup_{h \rightarrow \infty} F_{\varepsilon_h}(u_n, A).$$

Let  $A_0, B_0$  be two open sets with  $A_0 \subset\subset B_0 \subset\subset A$  and  $\text{meas}(\partial A_0) = \text{meas}(\partial B_0) = 0$ . Fix an integer  $\nu$  and, for  $i = 1, 2, \dots, \nu$ , define  $A_i$  and  $\varphi_i$  as in Lemma 4.2. Set

$$w_{i,h} = \varphi_i u_h + (1 - \varphi_i) u;$$

we have  $T(w_{i,h} - u, A) = 0$ . With the same argument used in the proof of Lemma 4.2 we get

$$\begin{aligned} F_{\varepsilon_h}(w_{i,h}, A) &\leq F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(u, A - \bar{A}_0) + \frac{c}{\nu}(1 + \varepsilon_h c_{\nu, \sigma}) \\ &\quad + \frac{c}{\nu}(1 + \sigma c_\nu) \lambda(\varepsilon_h^{-1} S', \varepsilon_h^{-1} A) [c + F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(u, A - \bar{A}_0)], \end{aligned}$$

where  $B_0 - \bar{A}_0 \subset\subset S' \subset\subset A$ . Since  $(w_{i,h})$  converges to  $u$  weakly in  $L^p(A)$  we have

$$\begin{aligned} \inf \left\{ \limsup_{h \rightarrow \infty} [F_{\varepsilon_h}(v_h, A) + T(v_h - u, A)]: v_h \rightarrow u \text{ in } w - L^p(A) \right\} \\ \leq F^+(u, A) + \eta + \int_{A - \bar{A}_0} [a(x) + c|u|^p] dx + \frac{c}{\nu} \\ + \frac{c}{\nu}(1 + \sigma c_\nu) M \left\{ c + F^+(u, A) + \eta + \int_{A - \bar{A}_0} [a(x) + c|u|^p] dx \right\}, \end{aligned}$$

where  $M = \limsup_{t \rightarrow +\infty} \lambda(tS', tA)$ . Passing to the limit first as  $\sigma \rightarrow 0$ , next as  $\nu \rightarrow +\infty$ , then as  $\eta \rightarrow 0$ , and finally as  $A_0 \uparrow A$ , we get the thesis. ■

LEMMA 4.7. Assume that

$$\int_A \psi(x, u) dx = I(\bar{N}, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} F_{\varepsilon_h}(v, A)$$

for every  $A \in \mathcal{A}$  and for every  $u \in L^p(A)$ . Then

$$\int_A \psi(x, u) dx = I(\bar{N}, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} [F_{\varepsilon_h}(v, A) + T(v - w_0, A)]$$

for every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$ ,  $w_0 \in W^{m, \tau}(A) \cap L^p(A)$ .

PROOF. Let  $A \in \mathcal{A}$ ,  $u \in L^p(A)$ ,  $w_0 \in W^{m, \tau}(A) \cap L^p(A)$ . There exists a sequence  $(u_k)$  in  $W^{m, \tau}(A) \cap L^p(A)$  converging to  $u$  strongly in  $L^p(A)$  such that

$u_k - w_0 \in W_0^{m,r}(A)$ . Using Lemma 4.6 we obtain for every  $k \in \mathbb{N}$

$$\begin{aligned} \int_A \psi(x, u_k) dx &\geq \Gamma(\overline{\mathbb{N}}^+, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u_k}} [F_{\varepsilon_h}(v, A) + T(v - u_k, A)] \\ &= \Gamma(\overline{\mathbb{N}}^+, w - L^p(A)) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u_k}} [F_{\varepsilon_h}(v, A) + T(v - w_0, A)]. \end{aligned}$$

Since  $\Gamma$ -limits are lower semicontinuous (see Proposition 2.1) and  $\int_A \psi(x, v) dx$  is continuous in  $L^p(A)$  (see Lemma 4.5), passing to the limit as  $k \rightarrow +\infty$  we obtain

$$\begin{aligned} \int_A \psi(x, u) dx &\geq \Gamma(\overline{\mathbb{N}}^+, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} [F_{\varepsilon_h}(v, A) + T(v - w_0, A)] \\ &\geq \Gamma(\overline{\mathbb{N}}^-, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} [F_{\varepsilon_h}(v, A) + T(v - w_0, A)] \geq \int_A \psi(x, u) dx. \quad \blacksquare \end{aligned}$$

Let  $Y = ]0, 1[{}^n$  and let  $W_{\#}^{m,r}(Y)$  be the space of all  $Y$ -periodic functions of  $W_{\text{loc}}^{m,r}(\mathbb{R}^n)$ ; for every  $\varepsilon > 0$ ,  $x \in \Omega$ ,  $s \in \mathbb{R}$  we set

$$(4.4) \quad \left\{ \begin{array}{l} W(s) = \left\{ u \in W^{m,r}(Y) \cap L^p(Y) : \int_Y u(y) dy = s \right\} \\ W_0(s) = \left\{ u \in W^{m,r}(Y) \cap L^p(Y) : \int_Y u(y) dy = s, u - s \in W_0^{m,r}(Y) \right\} \\ W_{\#}(s) = \left\{ u \in W_{\#}^{m,r}(Y) \cap L^p(Y) : \int_Y u(y) dy = s \right\} \\ m^{\varepsilon}(x, s) = \inf \left\{ \int_Y f(x, u(y), \varepsilon^k D^k u(y)) dy : u \in W(s) \right\} \\ m_0^{\varepsilon}(x, s) = \inf \left\{ \int_Y f(x, u(y), \varepsilon^k D^k u(y)) dy : u \in W_0(s) \right\} \\ m_{\#}^{\varepsilon}(x, s) = \inf \left\{ \int_Y f(x, u(y), \varepsilon^k D^k u(y)) dy : u \in W_{\#}(s) \right\} \\ m_0(x, s) = \inf \{ m_0^{\varepsilon}(x, s) : \varepsilon > 0 \} \\ m_{\#}(x, s) = \inf \{ m_{\#}^{\varepsilon}(x, s) : \varepsilon > 0 \}. \end{array} \right.$$

LEMMA 4.8. *For every  $x \in \Omega$ ,  $s \in \mathbb{R}$*

$$m_0(x, s) = \lim_{\varepsilon \rightarrow 0^+} m_0^{\varepsilon}(x, s).$$

PROOF. Let  $x \in \Omega$ ,  $s \in \mathbf{R}$ ,  $u \in W_0(s)$ ,  $\varepsilon, \eta \in \mathbf{R}$  with  $0 < \eta \leq \varepsilon$ . Let  $v$  be the  $Y$ -periodic extension of  $u$ , that is the function which satisfies  $v(x + y) = v(x)$  for every  $x \in \mathbf{R}^n$ ,  $y \in \mathbf{Z}^n$  and  $v(x) = u(x)$  for every  $x \in Y$ . There exist  $N \in \mathbf{N}$  and  $\delta \in [0, 1[$  such that  $\varepsilon = (N + \delta)\eta$ . Define for every  $y \in Y$

$$w(y) = \begin{cases} v\left(\frac{\varepsilon}{\eta}y\right) & \text{if } y \in N\frac{\eta}{\varepsilon}Y \\ s & \text{otherwise.} \end{cases}$$

Then  $w \in W_0(s)$  and

$$\begin{aligned} \int_Y f(x, w(y), \eta^k D^k w(y)) dy &\leq \left(N\frac{\eta}{\varepsilon}\right)^n \int_Y f(x, u(y), \varepsilon^k D^k u(y)) dy + n\frac{\delta\eta}{\varepsilon} f(x, s, 0) \\ &\leq \int_Y f(x, u(y), \varepsilon^k D^k u(y)) dy + n\frac{\eta}{\varepsilon} f(x, s, 0). \end{aligned}$$

This implies that for every  $\varepsilon, \eta \in \mathbf{R}$ , with  $0 < \eta \leq \varepsilon$

$$m_0^\eta(x, s) \leq m_0^\varepsilon(x, s) + n\frac{\eta}{\varepsilon} f(x, s, 0),$$

and from this inequality it follows that

$$\inf_{\varepsilon > 0} m_0^\varepsilon(x, s) = \lim_{\varepsilon \rightarrow 0^+} m_0^\varepsilon(x, s). \quad \blacksquare$$

LEMMA 4.9. Suppose that the function  $f$  does not depend on the variable  $x$  and that

$$\int_A \psi(u) dx = \Gamma(\bar{\mathbf{N}}, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} F_{\varepsilon_h}(v, A)$$

for every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$ . Then  $m^\varepsilon$ ,  $m_0^\varepsilon$  and  $m_0$  do not depend on  $x$  and

$$\lim_{h \rightarrow \infty} m^{\varepsilon_h}(s) = m_0(s) = \psi(s)$$

for every  $s \in \mathbf{R}$ .

PROOF. Let  $s \in \mathbf{R}$  and let  $(u_h)$  be a sequence converging to  $s$  weakly in  $L^p(Y)$  such that  $u_h - s \in W_0^{m,r}(Y)$ ; let  $\varphi \in C_0^\infty(Y)$  with  $\int \varphi dx = 1$ ; there exists a sequence  $(\eta_h)$  converging to 0 in  $\mathbf{R}$  such that  $\int_Y [u_h(y) + \eta_h \varphi(y)] dy = s$  for every  $h \in \mathbf{N}$ . Then by hypothesis (3.2) we have

$$m_0^{\varepsilon_h}(s) \leq F_{\varepsilon_h}(u_h + \eta_h \varphi, Y) \leq F_{\varepsilon_h}(u_h, Y) + \sigma(\eta_h M) \left[ \int_Y a(x) dx + F_{\varepsilon_h}(u_h, Y) \right],$$

where  $M = \sup \sum_{|\alpha| \leq m} |D^\alpha \varphi|$ . Passing to the limit as  $h \rightarrow +\infty$  we obtain

$$m_0(s) \leq \liminf_{h \rightarrow \infty} F_{\varepsilon_h}(u_h, Y).$$

Since  $(u_h)$  is arbitrary, by Lemma 4.7 we get

$$(4.5) \quad m_0(s) \leq I(\overline{\mathbf{N}}, w - L^p(A)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow s}} [F_{\varepsilon_h}(v, A) + T(v - s, A)] = \psi(s).$$

Consider now a subsequence  $(\varepsilon_{h_k})$  such that  $\liminf_{h \rightarrow \infty} m^{\varepsilon_h}(s) = \lim_{h \rightarrow \infty} m^{\varepsilon_{h_k}}(s)$ . For every  $k \in \mathbf{N}$  there exists  $w_k \in W(s)$  such that  $F_{\varepsilon_{h_k}}(w_k, Y) \leq m^{\varepsilon_{h_k}}(s) + 1/k$ . By hypothesis (3.1) the sequence  $(w_k)$  is bounded in  $L^p(Y)$ ; thus for a suitable subsequence  $(w_{k_i})$ , we have that  $(w_{k_i})$  converges weakly in  $L^p(Y)$  to a function  $u$  such that  $\int_Y u(y) dy = s$ . Therefore, using Jensen's inequality, Remark 2.6, Lemma 4.8 and inequality (4.5), we get

$$\begin{aligned} m_0(s) \leq \psi(s) &= \psi\left(\int_Y u(y) dy\right) \leq \int_Y \psi(u) dy = I(\overline{\mathbf{N}}, w - L^p(Y)^-) \lim_{\substack{h \rightarrow \infty \\ v \rightarrow u}} F_{\varepsilon_h}(v, Y) \\ &\leq \liminf_{i \rightarrow \infty} F_{\varepsilon_{h_{k_i}}}(w_{k_i}, Y) \leq \lim_{k \rightarrow \infty} m^{\varepsilon_{h_k}}(s) = \liminf_{h \rightarrow \infty} m^{\varepsilon_h}(s) \\ &\leq \limsup_{h \rightarrow \infty} m^{\varepsilon_h}(s) \leq \limsup_{h \rightarrow \infty} m_0^{\varepsilon_h}(s) = m_0(s). \quad \blacksquare \end{aligned}$$

LEMMA 4.10. *Suppose that the function  $f$  does not depend on the variable  $x$ . Then there exists a convex function  $\psi: \mathbf{R} \rightarrow [0, +\infty[$  such that*

$$(4.6) \quad \int_A \psi(u) dx = I(\mathbf{R}, w - L^p(A)^-) \lim_{\substack{\varepsilon \rightarrow 0^+ \\ v \rightarrow u}} F_\varepsilon(v, A) \\ = I(\mathbf{R}, w - L^p(A)^-) \lim_{\substack{\varepsilon \rightarrow 0^+ \\ v \rightarrow u}} [F_\varepsilon(v, A) + T(v - w_0, A)]$$

for every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$ ,  $w_0 \in W^{m,\tau}(A) \cap L^p(A)$ .

Moreover  $m^\varepsilon, m_0^\varepsilon, m_0$  do not depend on  $x$  and

$$\psi(s) = m_0(s) = \lim_{\varepsilon \rightarrow 0^+} m_0^\varepsilon(s) = \lim_{\varepsilon \rightarrow 0^+} m^\varepsilon(s)$$

for every  $s \in \mathbf{R}$ .

PROOF. Let  $(\varepsilon_h)$  be a sequence in  $\mathbf{R}$  converging to 0 such that  $\varepsilon_h > 0$  for every  $h \in \mathbf{N}$ . By Lemmas 4.4, 4.5 and 4.7 there exist a subsequence  $(\varepsilon_{h_k})$

of  $(\varepsilon_n)$  and a Borel function  $\psi(x, s)$ , convex in  $s$ , such that

$$\begin{aligned} \int_A \psi(x, u) dx &= \Gamma(\bar{N}, w - L^p(A)^-) \lim_{\substack{k \rightarrow \infty \\ v \rightarrow u}} F_{\varepsilon_{h_k}}(v, A) \\ &= \Gamma(\bar{N}, w - L^p(A)^-) \lim_{\substack{k \rightarrow \infty \\ v \rightarrow u}} [F_{\varepsilon_{h_k}}(v, A) + T(v - w_0, A)] \end{aligned}$$

for every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$ ,  $w_0 \in W^{m,r}(A) \cap L^p(A)$ . Since  $f$  does not depend on  $x$ , it is easy to see that  $\int_{v+A} \psi(x, u(x-y)) dx = \int_A \psi(x, u(x)) dx$  for every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$  and for every  $y \in \mathbb{R}^n$  such that  $y + A \subset \Omega$ . This implies that  $\psi$  does not depend on  $x$ , that is  $\psi(x, s) = \psi(s)$ .

By Lemma 4.9 we have

$$\psi(s) = m_0(s)$$

for every  $s \in \mathbb{R}$ . So the function  $\psi$  does not depend on the sequence  $(\varepsilon_n)$ . By Proposition 2.8 this implies (4.6).

By Lemma 4.9 we have  $m_0(s) = \lim_{k \rightarrow \infty} m^{\varepsilon_{h_k}}(s)$ . Since the limit does not depend on the sequence  $(\varepsilon_n)$ , we obtain

$$m_0(s) = \lim_{\varepsilon \rightarrow 0^+} m^\varepsilon(s).$$

The equality  $m_0(s) = \lim_{\varepsilon \rightarrow 0^+} m^\varepsilon(s)$  has already been proved in Lemma 4.8. ■

**PROOF OF THEOREM 3.1.** Let  $(\varepsilon_n)$  be a sequence in  $]0, +\infty[$  converging to 0. By Lemmas 4.4, 4.5 and 4.7 there exist a subsequence  $(\varepsilon_{h_k})$  of  $(\varepsilon_n)$  and a Borel function  $\psi: \Omega \times \mathbb{R} \rightarrow [0, +\infty[$ , which satisfies condition (ii) of the theorem, such that

$$\begin{aligned} \int_A \psi(x, u) dx &= \Gamma(\bar{N}, w - L^p(A)^-) \lim_{\substack{k \rightarrow \infty \\ v \rightarrow u}} F_{\varepsilon_{h_k}}(v, A) \\ &= \Gamma(\bar{N}, w - L^p(A)^-) \lim_{\substack{k \rightarrow \infty \\ v \rightarrow u}} [F_{\varepsilon_{h_k}}(v, A) + T(v - w_0, A)] \end{aligned}$$

for every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$ ,  $w_0 \in W^{m,r}(A) \cap L^p(A)$ .

In order to prove (i), by Proposition 2.8 we have only to show that

$$(4.7) \quad \psi(x, s) = m_0(x, s) = \lim_{\varepsilon \rightarrow 0^+} m^\varepsilon(x, s) = m_\sharp(x, s)$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ , where  $m_0$  and  $m^\varepsilon$  are defined by (4.4).

Let  $N \geq 1$  be an integer; for every  $j \in \mathbb{Z}^n$  we set  $Y_N^j = (1/N)(Y + j)$  and  $\Omega_N^j = \Omega \cap Y_N^j$  (here  $Y = ]0, 1[^n$ ). Define  $f_N: \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow [0, +\infty[$  by

$$f_N(x, s, z) = \int_{\Omega_N^j} f(y, s, z) dy \quad \text{for } x \in \Omega_N^j,$$

where  $\int_A$  denotes the average over the set  $A$ . Define

$$F_\varepsilon^N(u, A) = \int_A f_N(x, u, \varepsilon^k D^k u) dx$$

and let  $(m_N)^\varepsilon(x, s)$ ,  $(m_N)_0^\varepsilon(x, s)$ ,  $(m_N)_0(x, s)$  be the functions related to  $f_N$  defined as in (4.4). Since  $f_N$  is piecewise constant with respect to the variable  $x$ , by Lemmas 4.5 and 4.10 there exists a Borel function  $\psi_N(x, s)$ , piecewise constant in  $x$  and convex in  $s$ , such that

$$\int_A \psi_N(x, u) dx = \Gamma(\bar{N}, w - L^p(A)^-) \lim_{\substack{\varepsilon \rightarrow 0^+ \\ v \rightarrow u}} F_\varepsilon^N(v, A)$$

for every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$ ; moreover

$$(4.8) \quad \psi_N(x, s) = (m_N)_0(x, s) = \lim_{\varepsilon \rightarrow 0^+} (m_N)_0^\varepsilon(x, s) = \lim_{\varepsilon \rightarrow 0^+} (m_N)^\varepsilon(x, s)$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ .

Let  $Q_N = ]-1/N, 1/N[^n$ . If  $Y_N^j \subseteq \Omega$ , using condition (3.2) we obtain for every  $x \in Y_N^j$ ,  $s \in \mathbb{N}$ ,  $z \in \mathbb{R}^d$

$$\begin{aligned} |f_N(x, s, z) - f(x, s, z)| &= \left| \int_{Y_N^j - x} [f(x + y, s, z) - f(x, s, z)] dy \right| \\ &\leq 2^n \int_{Q_N} |f(x + y, s, z) - f(x, s, z)| dy \leq 2^n \int_{Q_N} \{\omega(x, y) + \sigma(|y|)[a(x) + f(x, s, z)]\} dy \\ &\leq 2^n \int_{Q_N} \omega(x, y) dy + 2^n \sigma\left(\frac{\sqrt{n}}{N}\right) [a(x) + f(x, s, z)]. \end{aligned}$$

This implies that

$$(4.9) \quad \begin{aligned} &\left[1 - 2^n \sigma\left(\frac{\sqrt{n}}{N}\right)\right] f(x, s, z) - 2^n \sigma\left(\frac{\sqrt{n}}{N}\right) a(x) - 2^n \int_{Q_N} \omega(x, y) dy \\ &\leq f_N(x, s, z) \leq \left[1 + 2^n \sigma\left(\frac{\sqrt{n}}{N}\right)\right] f(x, s, z) + 2^n \sigma\left(\frac{\sqrt{n}}{N}\right) a(x) + 2^n \int_{Q_N} \omega(x, y) dy \end{aligned}$$

for every  $s \in \mathbf{R}$ ,  $z \in \mathbf{R}^d$  and for every  $x \in \Omega$  such that  $\text{dist}(x, \mathbf{R}^n - \Omega) > \sqrt{n}/N$ . Passing to the  $\Gamma$ -limit along the sequence  $(\varepsilon_{h_k})$  we obtain

$$\begin{aligned}
 (4.10) \quad & \left[1 - 2^n \sigma \left(\frac{\sqrt{n}}{N}\right)\right] \int_A \psi(x, u) dx - 2^n \sigma \left(\frac{\sqrt{n}}{N}\right) \int_A a(x) dx - 2^n \int_A dx \int_{Q_N} \omega(x, y) dy \\
 & \leq \int_A \psi_N(x, u) dx \leq \left[1 + 2^n \sigma \left(\frac{\sqrt{n}}{N}\right)\right] \int_A \psi(x, u) dx \\
 & \quad + 2^n \sigma \left(\frac{\sqrt{n}}{N}\right) \int_A a(x) dx + 2^n \int_A dx \int_{Q_N} \omega(x, y) dy
 \end{aligned}$$

for every  $A \in \mathcal{A}$  with  $d(A, \mathbf{R}^n - \Omega) > \sqrt{n}/N$  and for every  $u \in L^p(A)$ . By (3.2) we have

$$(4.11) \quad \lim_{N \rightarrow \infty} \int_A dx \int_{Q_N} \omega(x, y) dy = \lim_{N \rightarrow \infty} \int_{Q_N} dy \int_A \omega(x, y) dx = 0$$

for every  $A \in \mathcal{A}$  with  $A \subset\subset \Omega$ . Thus, passing to the limit in (4.10) as  $N \rightarrow +\infty$  we get

$$(4.12) \quad \int_A \psi(x, u) dx = \lim_{N \rightarrow \infty} \int_A \psi_N(x, u) dx$$

for every  $A \in \mathcal{A}$  with  $A \subset\subset \Omega$  and for every  $u \in L^p(A)$ .

Using the definitions of  $m^\varepsilon$  and  $(m_N)^\varepsilon$ , from (4.9) we obtain that

$$\begin{aligned}
 & \left[1 - 2^n \sigma \left(\frac{\sqrt{n}}{N}\right)\right] m^\varepsilon(x, s) - 2^n \sigma \left(\frac{\sqrt{n}}{N}\right) a(x) - 2^n \int_{Q_N} \omega(x, y) dy \\
 & \leq (m_N)^\varepsilon(x, s) \leq \left[1 + 2^n \sigma \left(\frac{\sqrt{n}}{N}\right)\right] m^\varepsilon(x, s) + 2^n \sigma \left(\frac{\sqrt{n}}{N}\right) a(x) + 2^n \int_{Q_N} \omega(x, y) dy
 \end{aligned}$$

for every  $x \in \Omega$  with  $\text{dist}(x, \mathbf{R}^n - \Omega) > \sqrt{n}/N$  and for every  $s \in \mathbf{R}$ . Letting  $\varepsilon \rightarrow 0^+$  and using (4.8) we get

$$\begin{aligned}
 (4.13) \quad & \left[1 - 2^n \sigma \left(\frac{\sqrt{n}}{N}\right)\right] \limsup_{\varepsilon \rightarrow 0^+} m^\varepsilon(x, s) - 2^n \sigma \left(\frac{\sqrt{n}}{N}\right) a(x) - 2^n \int_{Q_N} \omega(x, y) dy \\
 & \leq \lim_{\varepsilon \rightarrow 0^+} (m_N)^\varepsilon(x, s) = \psi_N(x, s) \leq \left[1 + 2^n \sigma \left(\frac{\sqrt{n}}{N}\right)\right] \liminf_{\varepsilon \rightarrow 0^+} m^\varepsilon(x, s) \\
 & \quad + 2^n \sigma \left(\frac{\sqrt{n}}{N}\right) a(x) + 2^n \int_{Q_N} \omega(x, y) dy.
 \end{aligned}$$

Equality (4.11) implies that there exists an increasing sequence of integers  $(N_k)$  such that  $\lim_{k \rightarrow \infty} \int_{Q_{N_k}} \omega(x, y) dy = 0$  for a.a.  $x \in \Omega$ . Letting  $N \rightarrow +\infty$  in (4.13) along the sequence  $(N_k)$ , we get that there exists

$$\lim_{\varepsilon \rightarrow 0^+} m^\varepsilon(x, s) = m(x, s)$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ , and that

$$m(x, s) = \lim_{k \rightarrow \infty} \psi_{N_k}(x, s)$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ . In the same way we prove that

$$m_0(x, s) = \lim_{k \rightarrow \infty} \psi_{N_k}(x, s).$$

Using (4.12) we obtain

$$\int_A m(x, s) dx = \int_A m_0(x, s) dx = \lim_{k \rightarrow \infty} \int_A \psi_{N_k}(x, s) dx = \int_A \psi(x, s) dx$$

for every  $A \in \mathcal{A}$  with  $A \subset \subset \Omega$  and for every  $s \in \mathbb{R}$ .

Since  $m$ ,  $m_0$ ,  $\psi$  are continuous in  $s$  (indeed they are convex), this implies that

$$(4.14) \quad m(x, s) = m_0(x, s) = \psi(x, s)$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ .

In order to prove (4.7) it is enough to show that

$$(4.15) \quad \psi(x, s) = m_{\#}(x, s)$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ .

Since  $W_0(s) \subseteq W_{\#}(s) \subseteq W(s)$  we have

$$m^\varepsilon(x, s) \leq m_{\#}^\varepsilon(x, s) \leq m_0^\varepsilon(x, s);$$

thus from (4.14) it follows that

$$(4.16) \quad \psi(x, s) = \lim_{\varepsilon \rightarrow 0^+} m_{\#}^\varepsilon(x, s).$$

By a change of variables, it is easy to verify that  $m_{\#}^{2\varepsilon}(x, s) \geq m_{\#}^\varepsilon(x, s)$  for

every  $\varepsilon > 0$ . Therefore (4.16) yields

$$\psi(x, s) = \lim_{\varepsilon \rightarrow 0^+} m_{\#}^{\varepsilon}(x, s) = \inf_{\varepsilon > 0^+} m_{\#}^{\varepsilon}(x, s).$$

This proves (4.15).

It remains to prove property (iii). The inequality  $\psi(x, s) \leq f^+(x, s, 0)$  follows from Lemma 4.1 and from the convexity of  $\psi(x, \cdot)$ .

Let  $x \in \Omega$ ,  $s \in \mathbb{R}$ ,  $u \in W_0(s)$ ,  $\varepsilon > 0$ ; by Jensen's inequality we have

$$\begin{aligned} f^-(x, s, 0) &= f^-\left(x, \int_Y u(y) dy, \varepsilon^k \int_Y D^k u(y) dy\right) \\ &\leq \int_Y f^-(x, u(y), \varepsilon^k D^k u(y)) dy \leq \int_Y f(x, u(y), \varepsilon^k D^k u(y)) dy. \end{aligned}$$

Thus by the representation formula for  $\psi$  we have

$$f^-(x, s, 0) \leq \psi(x, s). \quad \blacksquare$$

### 5. – Some examples.

In this section we give some examples and applications of Theorem 3.1. In particular we show that the inequalities

$$(5.1) \quad f^-(x, s, 0) \leq \psi(x, s) \leq f^+(x, s, 0)$$

cannot be improved; in fact, there are some examples where  $\psi(x, s) = f^-(x, s, 0)$  (see Proposition 5.9 and Remark 5.10), and some other examples where  $\psi(x, s) = f^+(x, s, 0)$  (see Proposition 5.2). In the case  $f^-(x, s, 0) = f^+(x, s, 0)$  the integrand  $\psi(x, s)$  is determined by the inequalities (5.1); this allows us to generalize some results of A. Bensoussan [2] and V. Kormornik [13] (see Proposition 5.5 and Proposition 5.6).

For every  $p \geq 2$  we denote by  $\mathfrak{G}_p$  the class of functions  $g: \mathbb{R} \rightarrow \mathbb{R}$  such that

$$(5.2) \quad |g(s)| \leq c(1 + |s|^{p/2})$$

$$(5.3) \quad |g(t) - g(s)| \leq \varrho(|t - s|)(1 + |s|^{p/2})$$

for every  $s, t \in \mathbb{R}$ , where  $c$  is a positive constant and  $\varrho: [0, +\infty[ \rightarrow [0, +\infty[$

is an increasing continuous function with  $\varrho(0) = 0$ . Examples of functions of the class  $\mathfrak{G}_p$  are the polynomials of degree less than or equal to  $p/2$ .

Let  $N > 0$ ,  $b \in L^p(\Omega)$ ,  $g \in \mathfrak{G}_p$ ; after some simple calculations (see section 6) one can verify that the functionals

$$F_\varepsilon(u, A) = \int_A [N|\varepsilon^2 \Delta u + g(u)|^2 + |u - b(x)|^p] dx$$

satisfy all hypotheses of Theorem 3.1, with  $m = r = 2$ ,

$$f(x, s, z) = N \left| \sum_{i=1}^n z_{ii} + g(s) \right|^2 + |s - b(x)|^p \quad (\text{here } z = (z_{ij})_{1 \leq i+j \leq 2}),$$

$$\gamma(s, z) = c_1 \left[ \left| \sum_{i=1}^n z_{ii} \right|^2 + s^2 \right],$$

$$\lambda(A', A) = c_2 \max \{1, \text{dist}(A', \mathbf{R}^n - A)^{-4}\},$$

where  $c_1, c_2$  are suitable positive constants.

Let  $\psi(x, s)$  be the function, convex in  $s$ , such that

$$\int_A \psi(x, u) dx = \Gamma(\mathbf{R}, w - L^p(A)^-) \lim_{\substack{\varepsilon \rightarrow 0^+ \\ v \rightarrow u}} F_\varepsilon(v, A)$$

for every  $A \in \mathcal{A}$ ,  $u \in L^p(A)$ .

**PROPOSITION 5.1.** *If  $g$  is an affine function, then*

$$\psi(x, s) = f(x, s, 0) = N|g(s)|^2 + |s - b(x)|^p$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbf{R}$ .

**PROOF.** Since in this case  $f(x, s, z) = f^-(x, s, z) = f^+(x, s, z)$ , the proposition follows from (5.1). ■

In the following proposition we give a new proof of a result due to A. Haraux and F. Murat [11].

**PROPOSITION 5.2.** *Let  $g$  be a decreasing function of the class  $\mathfrak{G}_p$ , let  $b \in L^p(\Omega)$ , and let  $N > 0$ . Then*

$$\psi(x, s) = f^+(x, s, 0)$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbf{R}$ .

PROOF. Let  $x \in \Omega$ ,  $s \in \mathbb{R}$ ,  $\varepsilon > 0$ ,  $u \in W_0(s)$  (see (4.4)). Then

$$\begin{aligned}
 (5.4) \quad & \int_Y [N|\varepsilon^2 \Delta u(y) + g(u(y))|^2 + |u(y) - b(x)|^p] dy \\
 & = \int_Y [N\varepsilon^4 |\Delta u(y)|^2 + N|g(u(y))|^2 + 2N\varepsilon^2 \Delta u(y)g(u(y)) + |u(y) - b(x)|^p] dy.
 \end{aligned}$$

Let us prove that

$$(5.5) \quad \int_Y g(u) \Delta u dy \geq 0.$$

There exists a sequence  $(g_n)$  of decreasing functions of class  $C^1$ , with bounded derivatives, such that  $g(s) = \lim_h g_n(s)$  for every  $s \in \mathbb{R}$ , and  $|g_n(s)| \leq c(1 + |s|^{p/2})$  for every  $h \in \mathbb{N}$ ,  $s \in \mathbb{R}$ .

By the dominated convergence theorem

$$\int_Y g(u) \Delta u dy = \lim_h \int_Y g_n(u) \Delta u dy.$$

Since  $u - s \in W_0^{2,2}(Y)$  we have

$$\int_Y g_n(u) \Delta u dy = - \int_Y g'_n(u) |Du|^2 dy \geq 0,$$

so (5.5) is proved. From (5.4), (5.5) and Jensen's inequality it follows that

$$\begin{aligned}
 & \int_Y [N|\varepsilon^2 \Delta u(y) + g(u(y))|^2 + |u(y) - b(x)|^p] dy \\
 & \geq \int_Y [N|g(u(y))|^2 + |u(y) - b(x)|^p] dy \geq \int_Y f^+(x, u(y), 0) dy \geq f^+(x, s, 0).
 \end{aligned}$$

Since  $\varepsilon > 0$  and  $u \in W_0(s)$  are arbitrary, the representation formula for  $\psi$  implies  $\psi(x, s) \geq f^+(x, s, 0)$ . The opposite inequality follows from (5.1). ■

We construct now an example which shows that the equality  $\psi(x, s) = f^+(x, s, 0)$  does not hold for an arbitrary function  $g \in \mathfrak{G}_p$ .

PROPOSITION 5.3. Let  $n = 1$ ,  $m = p = r = 2$ ,  $\Omega = ]0, 1[$  and let  $g$  be defined by

$$g(s) = \begin{cases} s & \text{if } s < 0 \\ s/4 & \text{if } s \geq 0. \end{cases}$$

If  $N > 6\pi^2 - 16$  and  $b \in L^2(\Omega)$ , then

$$\psi(x, s) < f^+(x, s, 0) = f(x, s, 0) = N|g(s)|^2 + |s - b(x)|^2$$

for a.a.  $x \in \Omega$  and for all  $s > 0$ . If in addition  $b(x) > 0$  for a.a.  $x \in \Omega$ , then

$$\liminf_{\varepsilon \rightarrow 0^+} \{F_\varepsilon(u, \Omega) : u \in W^{2,2}(\Omega)\} < \min \left\{ \int_{\Omega} f^+(x, u, 0) dx : u \in L^2(\Omega) \right\}.$$

PROOF. Define on  $[-\pi, 2\pi]$

$$u(x) = \begin{cases} \frac{k}{2} \sin x & \text{if } x \in [-\pi, 0] \\ k \sin \frac{x}{2} & \text{if } x \in [0, 2\pi] \end{cases}$$

( $k > 0$  is a parameter) and extend  $u$  to  $\mathbb{R}$  by periodicity (the period is  $3\pi$ ). Set  $u_\varepsilon(x) = u(x/\varepsilon)$ ; as  $\varepsilon \rightarrow 0^+$  we have that  $(u_\varepsilon)$  converges to  $k/\pi$  and  $(|u_\varepsilon|^2)$  converges to  $\frac{3}{8}k^2$  weakly in  $L^2(0, 1)$ . Since  $\varepsilon^2 u'' + g(u_\varepsilon) = 0$ , for every  $A \in \mathcal{A}$ ,  $b \in L^2(A)$  we have

$$\begin{aligned} \int_A \psi\left(x, \frac{k}{\pi}\right) dx &\leq \liminf_{\varepsilon \rightarrow 0^+} \int_A [N|\varepsilon^2 u'' + g(u_\varepsilon)|^2 + |u_\varepsilon - b(x)|^2] dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_A [ |u_\varepsilon|^2 - 2u_\varepsilon b(x) + b(x)^2 ] dx = \int_A \left[ \frac{3}{8} k^2 - \frac{2k}{\pi} b(x) + |b(x)|^2 \right] dx. \end{aligned}$$

Therefore, for a.a.  $x \in ]0, 1[$  and for all  $s > 0$ , we have

$$\psi(x, s) \leq \frac{3}{8} \pi^2 s^2 - 2sb(x) + |b(x)|^2.$$

On the other hand

$$\begin{aligned} f^+(x, s, 0) = f(x, s, 0) &= N|g(s)|^2 + |s - b(x)|^2 \\ &= \begin{cases} (N + 1) s^2 - 2sb(x) + |b(x)|^2 & \text{if } s < 0 \\ \left(\frac{N}{16} + 1\right) s^2 - 2sb(x) + |b(x)|^2 & \text{if } s \geq 0. \end{cases} \end{aligned}$$

Therefore, if  $N > 6\pi^2 - 16$ , then  $\psi(x, s) < f^+(x, s, 0)$  for a.a.  $x \in \Omega$  and for

all  $s > 0$ . If in addition  $b(x) > 0$ , we obtain from Corollary 3.2

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0^+} \{F_\varepsilon(u, \Omega) : u \in W^{2,2}(\Omega)\} &= \liminf_{\varepsilon \rightarrow 0^+} \{F_\varepsilon(u, \Omega) : u \in W_0^{2,2}(\Omega)\} \\ &= \min \left\{ \int_\Omega \psi(x, u) dx : u \in L^2(\Omega) \right\} < \left( 1 - \frac{8}{3\pi^2} \right) \int_\Omega |b(x)|^2 dx \\ &< \left( 1 - \frac{16}{N+16} \right) \int_\Omega |b(x)|^2 dx = \min \left\{ \int_\Omega f^+(x, u, 0) dx : u \in L^2(\Omega) \right\}. \quad \blacksquare \end{aligned}$$

We give now another example where  $g$  is a polynomial and the equality  $\psi(x, s) = f^+(x, s, 0)$  is not satisfied.

PROPOSITION 5.4. *Let  $n = 1, m = r = 2, p = 6, \Omega = ]0, 1[$ , and let  $g$  be defined by*

$$g(s) = s^3 + s - \frac{5}{8}.$$

*Then there exist  $s_0 \in ]0, \frac{1}{2}[$  and  $K \in ]0, +\infty[$  with the following property: if  $b \in L^\infty(\Omega)$  and  $N \geq K[1 + \|b\|_{L^\infty(\Omega)}^4]$ , then*

$$\psi(x, s_0) < f^+(x, s_0, 0) = f(x, s_0, 0) = N|g(s_0)|^2 + |s_0 - b(x)|^6$$

for a.a.  $x \in \Omega$ .

PROOF. Let  $u$  be the solution of the Cauchy problem

$$\begin{cases} u'' + u^3 + u - \frac{5}{8} = 0 \\ u(0) = u'(0) = 0. \end{cases}$$

The function  $u$  is periodic with period  $2T$  where

$$T = \int_0^\sigma \left( \frac{5}{4}s - s^2 - \frac{1}{2}s^4 \right)^{-\frac{1}{2}} ds$$

and  $\sigma$  is the unique positive solution of  $\frac{5}{4}s - s^2 - \frac{1}{2}s^4 = 0$ . Let  $s_0$  be defined by

$$s_0 = \frac{1}{2T} \int_0^{2T} u(x) dx = \frac{1}{T} \int_0^T u(x) dx.$$

Since

$$u' = \left( \frac{5}{4}u - u^2 - \frac{1}{2}u^4 \right)^{\frac{1}{3}} \quad \text{in } [0, T]$$

we have

$$\int_0^T u(x) dx = \int_0^\sigma s \left( \frac{5}{4}s - s^2 - \frac{1}{2}s^4 \right)^{-\frac{1}{3}} ds.$$

We prove that  $s_0 < \frac{1}{2}$ ; this is equivalent to show that

$$(5.6) \quad \int_0^\sigma \left( s - \frac{1}{2} \right) \left( \frac{5}{4}s - s^2 - \frac{1}{2}s^4 \right)^{-\frac{1}{3}} ds < 0.$$

Let  $v(s) = \left( \frac{5}{4}s - s^2 - \frac{1}{2}s^4 \right)^{\frac{1}{3}}$ ; the function  $v$  is increasing in  $[0, \frac{1}{2}]$  and decreasing in  $[\frac{1}{2}, \sigma]$ . Let  $v_0 = \sqrt{11/32}$ , let  $w_1: [0, v_0] \rightarrow [0, \frac{1}{2}]$  be the inverse of the function  $v|_{[0, \frac{1}{2}]}$  and let  $w_2: [0, v_0] \rightarrow [\frac{1}{2}, \sigma]$  be the inverse of the function  $v|_{[\frac{1}{2}, \sigma]}$ ; then (5.6) is equivalent to

$$(5.7) \quad \int_0^{v_0} 2 \left( w_1(t) - \frac{1}{2} \right) \left[ \frac{5}{4} - 2w_1(t) - 2(w_1(t))^3 \right]^{-1} dt \\ < \int_0^{v_0} 2 \left( w_2(t) - \frac{1}{2} \right) \left[ \frac{5}{4} - 2w_2(t) - 2(w_2(t))^3 \right]^{-1} dt.$$

Since the function  $(s - \frac{1}{2})(\frac{5}{4} - 2s - 2s^3)^{-1}$  is increasing in  $[0, +\infty[$  and  $0 < w_1(t) < w_2(t)$ , we obtain (5.7). This proves that  $s_0 < \frac{1}{2}$ , hence

$$(s_0^3 + s_0 - \frac{5}{8})^2 > 0.$$

Let  $u_T(x) = u(2Tx)$ ; note that  $u_T$  is 1-periodic and  $s_0 = \int_0^1 u_T(x) dx$ ; by the representation formula for  $\psi$  we get for every  $b \in L^6(\Omega)$

$$(5.8) \quad \psi(x, s_0) \leq \int_0^1 \left[ N \left| \frac{1}{(2T)^2} u_T''(y) + (u_T(y))^3 + u_T(y) - \frac{5}{8} \right|^2 + |u_T(y) - b(x)|^6 \right] dy \\ = \int_0^1 |u_T(y) - b(x)|^6 dy.$$

Using the facts that  $s_0 = \int_0^1 u_x(y) dy$  and that  $0 \leq u_x(y) \leq \sigma < 1$ , we obtain

$$\begin{aligned} \int_0^1 |u_x(y) - b(x)|^6 dy &= |s_0 - b(x)|^6 + \sum_{i=0}^6 \binom{6}{i} (-b(x))^i \left[ \int_0^1 u_x(y)^{6-i} dy - s_0^{6-i} \right] \\ &\leq |s_0 - b(x)|^6 + \sum_{i=0}^4 \binom{6}{i} |b(x)|^i < |s_0 - b(x)|^6 + 56[1 + \|b\|_{L^\infty(\Omega)}^4]. \end{aligned}$$

Let  $K = 56(s_0^3 + s_0 - \frac{5}{8})^{-2}$ ; if  $N \geq K[1 + \|b\|_{L^\infty(\Omega)}]$  we obtain from (5.8)

$$\psi(x, s_0) < |s_0 - b(x)|^6 + N \left( s_0^3 + s_0 - \frac{5}{8} \right)^2 = f^+(x, s_0, 0) = f(x, s_0, 0),$$

and the proposition is proved. ■

**REMARK 5.5.** For every  $N > 0$  let  $b_N = s_0 + [(N/3)(s_0^3 + s_0 - \frac{5}{8})(3s_0^2 + 1)]^{\frac{1}{3}}$ . There exists  $N_0 > 0$  such that for every  $N \geq N_0$  we have  $N \geq K[1 + b_N^4]$ . If in the previous proposition we take  $N \geq N_0$  and  $b(x) = b_N$  for every  $x \in \Omega$ , then we obtain from Corollary 3.2

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \inf \{F_\varepsilon(u, \Omega) : u \in W^{2,2}(\Omega)\} &= \lim_{\varepsilon \rightarrow 0^+} \inf \{F_\varepsilon(u, \Omega) : u \in W_0^{2,2}(\Omega)\} \\ &= \min \left\{ \int_\Omega \psi(x, u) dx : u \in L^6(\Omega) \right\} \leq \int_\Omega \psi(x, s_0) dx < \int_\Omega f(x, s_0, 0) dx \\ &= \min \left\{ \int_\Omega f(x, u, 0) dx : u \in L^6(\Omega) \right\}. \end{aligned}$$

The following proposition generalizes some results proved by V. Komornik in [13].

**PROPOSITION 5.6.** Let  $g$  be a non-negative convex function of the class  $\mathfrak{S}_p$ , let  $b \in L^p(\Omega)$ , and let  $N > 0$ . Then for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$

$$\psi(x, s) = f^-(x, s, 0) = f(x, s, 0) = N|g(s)|^2 + |s - b(x)|^p.$$

**PROOF.** Since  $f^-(x, s, 0) \leq \psi(x, s) \leq f(x, s, 0)$ , it is enough to prove that for a.a.  $x \in \Omega$  and for all  $s_0 \in \mathbb{R}$  we have

$$(5.9) \quad f^-(x, s_0, 0) = f(x, s_0, 0).$$

In order to prove (5.9) we show that

$$(5.10) \quad f(x, s, z) \geq f(x, s_0, 0) + \frac{\partial f}{\partial s}(x, s_0^+, 0)(s - s_0) + \sum_{i=1}^n \frac{\partial f}{\partial z_{ii}}(x, s_0, 0)z_{ii}$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}, s_0 \in \mathbb{R}, z \in \mathbb{R}^d$ . Inequality (5.10) is equivalent to

$$(5.11) \quad N \left( \sum_{i=1}^n z_{ii} \right)^2 + 2N[g(s) - g(s_0)] \sum_{i=1}^n z_{ii} \\ + \{ |s - b(x)|^p + N|g(s)|^2 - |s_0 - b(x)|^p - N|g(s_0)|^2 \\ - [p|s_0 - b(x)|^{p-1} \text{sign}(s_0 - b(x)) + 2Ng(s_0)g'(s_0^+)](s - s_0) \} \geq 0.$$

Since the left hand side of (5.11) is a polynomial of the second order in  $\sum_{i=1}^n z_{ii}$ , inequality (5.11) is equivalent to

$$(5.12) \quad |s - b(x)|^p - p|s_0 - b(x)|^{p-1} \text{sign}(s_0 - b(x))(s - s_0) - |s_0 - b(x)|^p \\ + 2Ng(s_0)[g(s) - g'(s_0^+)(s - s_0) - g(s_0)] \geq 0.$$

Putting  $\varphi(s) = |s - b(x)|^p + 2Ng(s_0)g(s)$ , inequality (5.12) can be written in the form  $\varphi(s) - \varphi'(s_0^+)(s - s_0) - \varphi(s_0) \geq 0$  which is always satisfied because the function  $\varphi$  is convex. ■

The following proposition generalizes some results proved by A. Bensoussan in [2].

**PROPOSITION 5.7.** *Suppose that  $g$  is a function which is convex and non-negative for  $s \geq 0$ , concave and non-positive for  $s \leq 0$ , and which satisfies  $|g(s)| \leq c|s|^{p/2}$  for every  $s \in \mathbb{R}$ . Then there exists  $N_0 > 0$  (depending only on the constants  $p$  and  $c$ ) such that for every  $N \in ]0, N_0]$  and for every  $b \in L^p(\Omega)$  we have*

$$\psi(x, s) = f^-(x, s, 0) = f(x, s, 0) = N|g(s)|^2 + |s - b(x)|^p$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ .

**PROOF.** As in Proposition 5.6 we have only to prove that

$$(5.13) \quad |s - b|^p - p|s_0 - b|^{p-1} \text{sign}(s_0 - b)(s - s_0) - |s_0 - b|^p \\ + 2Ng(s_0)[g(s) - g'(s_0^+)(s - s_0) - g(s_0)] \geq 0$$

for all  $s, s_0, b \in \mathbb{R}$ . Let  $\varphi(s) = |s - b|^p + 2Ng(s_0)g(s)$ ; if  $s_0 \geq 0$  the function  $\varphi$

is convex on  $[0, +\infty[$ ; if  $s_0 \leq 0$  the function  $\varphi$  is convex on  $]-\infty, 0]$ . Therefore, if  $ss_0 \geq 0$

$$(5.14) \quad \varphi(s) - \varphi'(s_0^+)(s - s_0) - \varphi(s_0) \geq 0,$$

hence (5.13) is proved in the case  $ss_0 \geq 0$ . Suppose now  $s_0 > 0$  and  $s < 0$ ; let

$$\begin{aligned} \alpha(s, b) = & |s - b|^p - p|s_0 - b|^{p-1} \text{sign}(s_0 - b)(s - s_0) - |s_0 - b|^p \\ & + 2Ng(s_0)[g(s) - g'(s_0^+)(s - s_0) - g(s_0)]; \end{aligned}$$

we want to prove that  $(\partial\alpha/\partial s)(s^+, b) \leq 0$ . We have

$$\begin{aligned} \frac{\partial\alpha}{\partial s}(s^+, b) = & p|s - b|^{p-1} \text{sign}(s - b) - p|s_0 - b|^{p-1} \text{sign}(s_0 - b) \\ & + 2Ng(s_0)[g'(s^+) - g'(s_0^+)]; \end{aligned}$$

therefore

$$\begin{aligned} \max_{b \in \mathbf{R}} \frac{\partial\alpha}{\partial s}(s^+, b) = & \frac{\partial\alpha}{\partial s}\left(s^+, \frac{s + s_0}{2}\right) = -p2^{2-p}|s - s_0|^{p-1} + 2Ng(s_0)[g'(s^+) - g'(s_0^+)] \\ & \leq -p^{2-p}(s_0 + |s|)^{p-1} + 2NK(c, p) s_0^{p/2} |s|^{-1+p/2} \\ & \leq (-p2^{2-p} + 2NK(c, p))(s_0 + |s|)^{p-1} \end{aligned}$$

where  $K(c, p) = c(p/2)(p/(p-2))^{-1+p/2}$  if  $p > 2$ ,  $K(c, p) = c$  if  $p = 2$ .

If  $0 < N \leq (p2^{1-p}/K(c, p))$  we have  $(\partial\alpha/\partial s)(s^+, b) \leq 0$  for every  $s < 0, b \in \mathbf{R}$ . This implies that  $\alpha(s, b) \geq \alpha(0, b) = \varphi(0) + \varphi'(s_0^+)s_0 - \varphi(s_0)$ ; therefore by (5.14) we get  $\alpha(s, b) \geq 0$ , hence (5.1) is proved for  $s_0 > 0, s < 0$ . The case  $s_0 < 0, s > 0$  can be proved in the same way. ■

The previous proposition applies for instance to the case  $g(s) = s|s|^{-1+p/2}$  and to the case considered in Proposition 5.3.

If  $b \in L^p(\Omega)$  and if  $g$  satisfies the conditions of Proposition 5.7, it is possible to prove that the set

$$\{N \in ]0, +\infty[: \varphi(x, s) = N|g(s)|^2 + |s - b(x)|^p \text{ for a.a. } x \in \Omega \text{ and for all } s \in \mathbf{R}\}$$

is an interval. In fact the following result holds.

**PROPOSITION 5.8.** *Let  $f(x, s, z)$  be a function satisfying (3.1), (3.2), (3.3) and let  $b \in L^p(\Omega)$ . For every  $\lambda > 0$  let*

$$f_\lambda(x, s, z) = f(x, s, z) + \lambda|s - b(x)|^p,$$

and let  $\psi_\lambda(x, s)$  be the integrand of the  $\Gamma$ -limit associated to  $f_\lambda$  by Theorem 3.1. If there exists  $\lambda_0 > 0$  such that  $\psi_{\lambda_0}(x, s) = f_{\lambda_0}(x, s, 0)$  for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ , then for all  $\lambda \geq \lambda_0$  we have  $\psi_\lambda(x, s) = f_\lambda(x, s, 0)$  for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ .

PROOF. Let  $\lambda > \lambda_0$ ; by Proposition 2.3 and by (5.1)

$$\begin{aligned} f_\lambda(x, s, 0) &= f_{\lambda_0}(x, s, 0) + (\lambda - \lambda_0)|s - b(x)|^p \\ &= \psi_{\lambda_0}(x, s) + (\lambda - \lambda_0)|s - b(x)|^p \leq \psi_\lambda(x, s) \leq f_\lambda(x, s, 0) \end{aligned}$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ . ■

We show now a situation where  $\psi(x, s) = f^-(x, s, 0)$ .

PROPOSITION 5.9. Let  $n = 1$  (hence  $d = m$ ) and let  $f(x, s, z)$  be a function satisfying (3.1), (3.2), (3.3). Suppose that

$$f(x, s, z) = f_1(x, s) + f_2(x, z_m) \quad \text{for all } x \in \Omega, s \in \mathbb{R}, z \in \mathbb{R}^m.$$

Then, if  $\psi(x, s)$  is the integrand of the  $\Gamma$ -limit associated to  $f$  by Theorem 3.1, we have for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$

$$\psi(x, s) = f^-(x, s, 0) = \bar{f}_1(x, s) + \bar{f}_2(x, 0),$$

where  $\bar{f}_1(x, s)$  denotes the greatest function convex in  $s$  which is less than or equal to  $f_1(x, s)$  and  $\bar{f}_2(x, z_m)$  denotes the greatest function convex in  $z_m$  which is less than or equal to  $f_2(x, z_m)$ .

PROOF. By (5.1) it is enough to prove that

$$(5.15) \quad \psi(x, s) \leq f_1(x, s) + \bar{f}_2(x, 0)$$

for a.a.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ . Fix  $x \in \Omega, s \in \mathbb{R}, \eta > 0$ ; there exist  $z > 0, w < 0$ , and  $0 < \lambda < 1$  such that  $\lambda z + (1 - \lambda)w = 0$  and

$$\lambda f_2(x, z) + (1 - \lambda)f_2(x, w) < \eta + \bar{f}_2(x, 0).$$

For every  $h \in \mathbb{N}$  set

$$I_h = \bigcup_{k=-\infty}^{+\infty} \left] \frac{k}{h}, \frac{k + \lambda}{h} \right[ \quad \text{and} \quad J_h = \bigcup_{k=-\infty}^{+\infty} \left] \frac{k + \lambda}{h}, \frac{k + 1}{h} \right[;$$

it is easy to prove that there exists a unique 1-periodic function  $u_n$  such that

$$\int_0^1 u_n(y) dy = s \quad \text{and} \quad u_n^{(m)} = \begin{cases} z & \text{on } I_n \\ w & \text{on } J_n. \end{cases}$$

By the representation formula for  $\psi$  we have

$$\begin{aligned} \psi(x, s) &\leq \int_0^1 [f_1(x, u_n(y)) + f_2(x, u_n^{(m)}(y))] dy \\ &= \int_0^1 f_1(x, u_n(y)) dy + \lambda f_2(x, z) + (1 - \lambda) f_2(x, w). \end{aligned}$$

Since  $(u_n)$  converges to  $s$  uniformly and  $f_1(x, s)$  is continuous in  $s$  we have

$$\psi(x, s) \leq f_1(x, s) + \bar{f}_2(x, 0) + \eta.$$

Since  $\eta$  was arbitrary we obtain (5.15) and so the proposition is proved. ■

**REMARK 5.10.** The previous proposition applies for example to the case

$$F_\varepsilon(u, A) = \int_A [ |(\varepsilon^2 u'' - a(x))^2 + |u - b(x)|^4 ] dx$$

with  $a \in L^2(\Omega)$  and  $b \in L^4(\Omega)$ . In this case we obtain

$$\psi(x, s) = f^-(x, s, 0) = (a(x) \wedge 0)^2 + |s - b(x)|^4$$

while  $f^+(x, s, 0) = |a(x)|^2 + |s - b(x)|^4$ .

### 6. - Appendix.

In this section we prove that the function

$$f(x, s, z) = N \left| \sum_{i=1}^n z_{ii} + g(s) + a(x) \right|^2 + |s - b(x)|^p$$

$(z = (z_{ij})_{1 \leq i+j \leq 2})$  satisfies condition (3.2) whenever  $N > 0$ ,  $p \geq 2$ ,  $g \in \mathfrak{G}_p$ ,  $a \in L^2(\Omega)$ ,  $b \in L^p(\Omega)$ , where  $\mathfrak{G}_p$  is the class of functions defined in section 5. Condition (3.1) is trivial for  $f$  and condition (3.3) follows from well known estimates for the Laplace operator.

First of all we extend the functions  $a$  and  $b$  to all of  $\mathbf{R}^n$ , by setting  $a(x) = b(x) = 0$  for  $x \in \mathbf{R}^n - \Omega$ ; so the function  $f$  is extended to  $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^d$ .

We shall use the following elementary inequalities, which hold for every  $\alpha > 0$ ,  $\beta > 0$ ,  $p > 1$ :

$$\begin{aligned} \alpha\beta &\leq \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q \\ |\alpha^p - \beta^p| &\leq p(1 \vee 2^{p-2})(\alpha^{p-1}|\alpha - \beta| + |\alpha - \beta|^p) \\ (\alpha + \beta)^p &\leq 2^{p-1}\alpha^p + 2^{p-1}\beta^p. \end{aligned}$$

The last inequality implies that

$$|s|^p \leq 2^{p-1}f(x, s, z) + 2^{p-1}|b(x)|^p.$$

In what follows  $q = p/(p-1)$  and  $c_1, c_2, c_3$  are positive constants independent of  $x, y, s, t, z, w$ . Let  $\eta: \mathbf{R}^n \rightarrow [0, +\infty[$  be an arbitrary function with  $\eta(0) = 0$  and  $\eta(y) > 0$  for  $y \neq 0$ , and let  $\eta^*: \mathbf{R}^n \rightarrow [0, +\infty[$  be defined by  $\eta^*(0) = 0$  and  $\eta^*(y) = \eta(y)^{-1}$  for  $y \neq 0$ . For every  $x, y \in \mathbf{R}^n$ ,  $s, t \in \mathbf{R}$ ,  $z, w \in \mathbf{R}^d$  we have

$$\begin{aligned} (6.1) \quad &|f(x+y, s+t, z+w) - f(x, s, z)| \\ &\leq c_1 \left\{ \left| \sum_{i=1}^n z_{ii} + g(s) + a(x) \right| \left[ \sum_{i=1}^n |w_{ii}| + \varrho(|t|)(1 + |s|)^{p/2} + |a(x+y) - a(x)| \right] \right. \\ &+ \left[ \sum_{i=1}^n |w_{ii}| + \varrho(|t|)(1 + |s|)^{p/2} + |a(x+y) - a(x)| \right]^2 \\ &+ |s - b(x)|^{p-1} [ |t| + |b(x+y) - b(x)| ] + [ |t| + |b(x+y) - b(x)|^p ] \left. \right\} \\ &\leq c_2 \left\{ f(x, s, z)^{\frac{1}{2}} \sum_{i=1}^n |w_{ii}| + f(x, s, z)^{\frac{1}{2}} \varrho(|t|) (f(x, s, z) + |b(x)|^p + 1)^{\frac{1}{2}} \right. \\ &+ f(x, s, z)^{\frac{1}{2}} |a(x+y) - a(x)| + \left( \sum_{i=1}^n |w_{ii}| \right)^2 \\ &+ \varrho(|t|)^2 (f(x, s, z) + |b(x)|^p + 1) + |a(x+y) - a(x)|^2 \\ &+ f(x, s, z)^{1/q} |t| + f(x, s, z)^{1/q} |b(x+y) - b(x)| + |t|^p + |b(x+y) - b(x)|^p \left. \right\} \\ &\leq c_3 \left\{ (f(x, s, z) + |b(x)|^p + 1) \left[ \sum_{i=1}^n |w_{ii}| + \varrho(|t|) + \left( \sum_{i=1}^n |w_{ii}| \right)^2 \right] \right. \\ &+ \varrho(|t|)^2 + |t| + |t|^p \left. \right\} + \eta(y) f(x, s, z) \\ &+ \eta^*(y) |a(x+y) - a(x)|^2 + |a(x+y) - a(x)|^2 \\ &+ \eta(y)^{q/p} f(x, s, z) + \eta^*(y) |b(x+y) - b(x)|^p + |b(x+y) - b(x)|^p \left. \right\} \\ &\leq (f(x, s, z) + |b(x)|^p + 1) \lambda(y, t, w) \\ &+ c_3 (1 + \eta^*(y)) [ |a(x+y) - a(x)|^2 + |b(x+y) - b(x)|^p ] \end{aligned}$$

where

$$\lambda(y, t, w) = c_3 \left[ \sum_{i=1}^n |w_{ii}| + \varrho(|t|) + \left( \sum_{i=1}^n |w_{ii}| \right)^2 + \varrho(|t|)^2 + |t| + |t|^p + \eta(y) + \eta(y)^{q/p} \right].$$

Since  $a \in L^2(\mathbb{R}^n)$  and  $b \in L^p(\mathbb{R}^n)$ , we have

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}^n} [|a(x+y) - a(x)|^2 + |b(x+y) - b(x)|^p] dx = 0.$$

Therefore there exists a continuous function  $\eta: \mathbb{R}^n \rightarrow [0, +\infty[$  such that  $\eta(0) = 0$ ,  $\eta(y) > 0$  for  $y \neq 0$ , and

$$\lim_{y \rightarrow 0} (1 + \eta^*(y)) \int_{\mathbb{R}^n} [|a(x+y) - a(x)|^2 + |b(x+y) - b(x)|^p] dx = 0.$$

For every  $x, y \in \mathbb{R}^n$  we set

$$\omega(x, y) = c_3(1 + \eta^*(y)) [|a(x+y) - a(x)|^2 + |b(x+y) - b(x)|^p].$$

Since  $\lambda$  is continuous and  $\lambda(0, 0, 0) = 0$ , there exists an increasing continuous function  $\sigma: [0, +\infty[ \rightarrow [0, +\infty[$ , with  $\sigma(0) = 0$ , such that

$$\lambda(y, t, w) \leq \sigma(|y| + |t| + |w|)$$

for every  $y \in \mathbb{R}^n$ ,  $t \in \mathbb{R}$ ,  $w \in \mathbb{R}^d$ .

Therefore from (6.1) it follows that

$$\begin{aligned} |f(x+y, s+t, z+w) - f(x, s, z)| \\ \leq \sigma(|y| + |t| + |w|)(f(x, s, z) + |b(x)|^p + 1) + \omega(x, y) \end{aligned}$$

for every  $x, y \in \mathbb{R}^n$ ,  $s, t \in \mathbb{R}$ ,  $z, w \in \mathbb{R}^d$ . This shows that condition (3.2) is satisfied.

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