

ANNALI DELLA SCUOLA NORMALE SUPERIORE DI PISA *Classe di Scienze*

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 11, n° 2 (1984), p. 327-341

http://www.numdam.org/item?id=ASNSP_1984_4_11_2_327_0

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On Relatively Bounded Perturbations of Linear C_0 -Semigroups.

W. DESCH - W. SCHAPPACHER (*)

In recent years we see an increasing interest and literature devoted to various system-theoretical investigations of systems of the form

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + Du(t), \quad t > 0, \\ x(0) &= x_0.\end{aligned}$$

Here A is the infinitesimal generator of a linear C_0 -semigroup $T(\cdot)$ on a Banach space X and D is a continuous linear operator from the Banach-space of control parameters into X . Of particular interest are problems concerning controllability, observability, boundary control etc. If the control is implemented through a feedback relation and we deal with the realistic case of having only a finite number of controls available, we face the following problem raised for instance in [11], p. 105:

Let A be the infinitesimal generator of a C_0 -semigroup $T(\cdot)$ on X and let B be a linear operator in X satisfying

- (i) Range (B) is finite-dimensional and
- (ii) B is A -bounded, i.e. $D(B) \supset D(A)$ and there are nonnegative constants a and b such that $\|Bx\| \leq a\|Ax\| + b\|x\|$ for all $x \in D(A)$.

Under which assumptions is $(A + B)$ the infinitesimal generator of a C_0 -semigroup on X ?

If X is reflexive, then Hess proved in [6] that (i)-(ii) imply that the A -bound of B is zero and hence we can apply a general perturbation result

(*) This author was supported in part by the Fonds zur Forderung der Wissenschaftlichen Forschung, Austria, No. P 4534.

Pervenuto alla Redazione il 6 Giugno 1983 ed in forma definitiva il 23 Gennaio 1984.

(Kato [8], p. 499) to conclude that if A generates an analytic semigroup so does $(A + B)$. (See also Zabczyk [12]).

A similar problem arises in the context of the semigroup approach to functional differential equations in the state space

$$X = \mathbb{R}^n \times L^p(-r, 0; \mathbb{R}^n) \quad \text{with } 0 < r < \infty \text{ and } 1 \leq p < \infty.$$

Given a linear map $L: X \supset D(L) \rightarrow \mathbb{R}^n$ we consider the Cauchy-problem

$$(1) \quad \begin{aligned} \frac{d}{dt} x(t) &= L(x(t), x_t), \quad t > 0, \\ x(0) &= \eta, \quad x(s) = \phi(s) \quad \text{a.e. on } -r \leq s < 0. \end{aligned}$$

The « history » x_t is given by $x_t(s) = x(s + t)$, $t \geq 0$, $s \in [-r, 0]$.

If $x(t, \eta, \phi)$ denotes the solution of (1) we define the associated solution semigroup $T(\cdot)$ by $T(t)(\eta, \phi) = (x(t), x_t)$. In [7] it was shown that the infinitesimal generator A of this semigroup is given by

$$\begin{aligned} D(A) &= \{(\phi(0), \phi) \mid \phi \in W^{1,p}(-r, 0, \mathbb{R}^n)\}, \\ A(\phi(0), \phi) &= (L(\phi(0), \phi), \dot{\phi}). \end{aligned}$$

Obviously, A can be split up as $A = A_0 + B$ where

$$A_0(\phi(0), \phi) = (0, \dot{\phi}), \quad B(\phi(0), \phi) = (L(\phi(0), \phi), 0).$$

Delfour [5] proved that this operator A is the infinitesimal generator of a C_0 -semigroup iff L is a continuous map $D(A_0) \rightarrow \mathbb{R}^n$, i.e. (i) and (ii) hold. Thus it seems to be attractive to conjecture that (i) and (ii) imply that $(A + B)$ is the infinitesimal generator also for non-analytic semigroups.

It is the objective of this paper to show that the above conjecture is false even if the unperturbed semigroup is ultimately compact, differentiable or a C_0 -group in a Hilbert-space!

On the other hand, we verify that if B satisfies an additional continuity assumption then $(A + B)$ is the infinitesimal generator of a C_0 -semigroup on X without any restriction on the range of B .

Some counterexamples.

To begin with, we provide some counterexamples to the above mentioned conjecture. Let $X = l^2$. Given a sequence (λ_n) of complex numbers so that $\operatorname{Re} \lambda_n \leq 0$ for all n it is clear that the linear operator $A = \operatorname{diag}(\lambda_n)$ is the infinitesimal generator of a C_0 -semigroup $S(\cdot)$ given by $S(t) = \operatorname{diag}(\exp[\lambda_n t])$.

Next, define a linear operator B in X by

$$(Bx)_n = \alpha_n \sum_{j=1}^{\infty} \alpha_j \lambda_j x_j,$$

with $\alpha = (\alpha_j) \in l^2(\mathbf{R})$ is chosen so that B is A -bounded and $\limsup_{n \rightarrow \infty} \alpha_n^2 |\lambda_n| \cdot \exp [\operatorname{Re} \lambda_n] = \infty$. The specific choice of (λ_n) and (α_n) is still at our disposal.

We claim that the operator $\mathcal{A} = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$ cannot be the infinitesimal generator of a C_0 -semigroup on $X \times X$. Infact, if $\mathcal{A}$ were the infinitesimal generator of a C_0 -semigroup $\mathfrak{T}(\cdot)$ on $X \times X$ we consider the elements $y_m = (\delta_{m,k})_{k=1, \dots} \in X$. As $y_m \in D(A^\infty)$ we infer that $t \rightarrow S(t)y_m \in C^\infty(0, \infty; D(A))$ and hence $BS(t)y_m \in C^\infty(0, \infty; X)$.

Consequently,

$$\tilde{x}(t) = \begin{pmatrix} \int_0^t S(t-s) BS(s) y_m ds \\ S(t) y_m \end{pmatrix}$$

would be a strong solution of the Cauchy-problem $(d/dt)\tilde{x}(t) = \mathcal{A}\tilde{x}(t)$.

Being a C_0 -semigroup there must exist a constant M so that $\|\mathfrak{T}(t)\| \leq M$ for $0 \leq t \leq 1$. In particular we thus would expect that $\left\| \mathfrak{T}(1) \begin{pmatrix} 0 \\ y_m \end{pmatrix} \right\| \leq M$, and in particular

$$(2) \quad \left| \left(\int_0^1 S(1-s) BS(s) y_m ds \right)_m \right| \leq M.$$

The left hand side of this inequality can be rewritten as

$$\int_0^1 \exp [(1-s)\lambda_m] \alpha_m \lambda_m \alpha_m \exp [s\lambda_m] ds = \exp [\lambda_m] \alpha_m^2 \lambda_m$$

and as by assumption $\limsup_{m \rightarrow \infty} [\operatorname{Re} \lambda_m] \alpha_m^2 |\lambda_m| = \infty$ we see that we cannot have an estimate of the form (2), i.e. $\mathfrak{T}(\cdot)$ is not a C_0 -semigroup.

We now specify the (λ_m) and (α_m) :

1. The case of C_0 -group.

Let $\lambda_m = im$, $m = 1, 2, \dots$. Then $A = -A^*$ and so $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ generate a

C_0 -group on X . Putting

$$\alpha_m = \begin{cases} m^{-\frac{1}{2}} & \text{if } m^{\frac{1}{2}} \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

we see that $\alpha = (\alpha_m) \in l^2(\mathbb{R})$.

Moreover,

$$\alpha_m^2 |\lambda_m| \exp [\operatorname{Re} \lambda_m] = \begin{cases} m^{\frac{1}{2}} & \text{if } m^{\frac{1}{2}} \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

and hence $\limsup_{m \rightarrow \infty} \alpha_m^2 |\lambda_m| \exp [\operatorname{Re} \lambda_m] = \infty$.

The associated operator B clearly satisfies (i), (ii) but according to the above consideration $\mathcal{A} + B$ is not an infinitesimal generator.

2. The case of an ultimately compact and differentiable semigroup.

Let $\lambda_m = -m + i \exp [4m]$ and set $\alpha_m = \exp [-m]$. Then we obtain $\alpha_m^2 |\lambda_m| \exp [\operatorname{Re} \lambda_m] \geq \exp [m] \rightarrow \infty$ as $m \rightarrow \infty$ and so again the associated operator $\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$ cannot be an infinitesimal generator. In order to show that $S(\cdot)$ is differentiable for $t \geq 4$ it is sufficient to verify that $\lambda_m \exp [\lambda_m t]$ is bounded for $t \geq 4$. This follows from

$$|\lambda_m \exp [\lambda_m t]| = |-m + i \exp [4m]| \exp [-mt] \leq m \exp [-mt] + \exp [m(4-t)].$$

It is also obvious that $S(\cdot)$ is compact for $t \geq 4$.

Some generation results.

As already pointed out in the introduction, we present a general generation result that seems to be very useful in applications. Throughout this section, we assume that $(X, \|\cdot\|)$ is a Banach-space. If A is a closed linear operator in X we let X_A stand for the Banach space $(D(A), |\cdot|_A)$, with $|x|_A = \|x\| + \|Ax\|$.

THEOREM. Let A be the infinitesimal generator of a C_0 -semigroup $T(\cdot)$ on X . Let $(Z, |\cdot|_Z)$ be a Banach space such that

(Z1) Z is continuously embedded in X ,

(Z2) there is a $t_0 > 0$ so that for all continuous functions

$$\phi: [0, t_0] \rightarrow Z \quad \text{we have} \quad \int_0^t T(t-s)\phi(s)ds \in D(A) \quad \text{for all } t \in [0, t_0]$$

(Z3) there is an increasing continuous function $\gamma: [0, t_0] \rightarrow [0, \infty)$ satisfying $\gamma(0) = 0$ and

$$\left| \int_0^t T(t-s)\phi(s)ds \right|_A \leq \gamma(t) \sup_{0 \leq s \leq t} |\phi(s)|_Z.$$

Then for any continuous linear operator $B: X_A \rightarrow Z$, $(A + B)$ is the infinitesimal generator of a C_0 -semigroup on X .

PROOF. To begin with, we verify that under the above assumptions the map $t \rightarrow \int_0^t T(t-s)\phi(s)ds$ is continuous from $[0, t_0]$ into X_A : In fact, given any continuous $\phi: [0, t_0] \rightarrow Z$ and $t \in [0, t_0]$, we define for all $0 \leq h \leq t$

$$\phi_h(s) = \begin{cases} \phi(s-h) & \text{for } h \leq s \leq t+h \\ 0 & \text{otherwise.} \end{cases}$$

Then we obtain for all $t \in [0, t_0 - h]$

$$\begin{aligned} & \left| \int_0^{t+h} T(t+h-s)\phi(s)ds - \int_0^t T(t-s)\phi(s)ds \right|_A \\ &= \left| \int_0^{t+h} T(t+h-s)(\phi(s) - \phi_h(s))ds \right|_A \\ &\leq \left| \int_h^{t+h} T(t+h-s)(\phi(s) - \phi_h(s))ds \right|_A + \left| \int_0^h T(t+h-s)(\phi(s) - \phi_h(s))ds \right|_A \\ &\leq \gamma(t) \sup |\phi(s+h) - \phi(s)|_Z + |T(t)|_A \gamma(h) \sup |\phi(s) - \phi_h(s)|_A. \end{aligned}$$

As the right side converges to 0 for $h \rightarrow 0$, the claim follows.

In the next step we verify that $(A + B)$ is a closed linear operator. To this end, we first estimate $(\lambda I - A)^{-1}B$ as an operator from X_A into X_A . Let N and ω be constants such that

$$\|T(t)\| \leq N \exp[\omega t] \quad \text{for all } t \geq 0.$$

Then we obtain for all sufficiently large λ and all $x \in D(A)$

$$\begin{aligned} & |(\lambda I - A)^{-1} Bx|_A \\ &= \left| \int_0^t \exp[-\lambda s] T(s) Bx ds + \exp[-\lambda t] T(t) \int_0^\infty \exp[-\lambda s] T(s) Bx ds \right|_A \\ &\leq \exp[-\lambda t] \left| \int_0^t T(t-s) (\exp[\lambda s] Bx) ds \right|_A + N \exp[(\omega - \lambda)t] |(\lambda I - A)^{-1} Bx|_A \\ &\leq \gamma(t) \sup |\exp[\lambda s] Bx|_Z + N \exp[(\omega - \lambda)t] |(\lambda I - A)^{-1} Bx|_A. \end{aligned}$$

Putting

$$t(\lambda) = \frac{\ln N + \ln 2}{\lambda - \omega}$$

we obtain

$$|(\lambda I - A)^{-1} Bx|_A \leq 2\gamma(t(\lambda)) \beta |x|_A,$$

where β denotes the norm of B regarded as an operator from X_A into Z , and as $t(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$ we deduce that

$$K(\lambda) = 2\gamma(t(\lambda)) \quad \text{converges to } 0 \quad \text{as } \lambda \rightarrow \infty.$$

In order to prove that $(A + B)$ is closed, let (x_n) be a sequence in X_A such that $x_n \rightarrow x$ (in X) and $y_n := (A + B)x_n$ converges to $y \in X$. We have to show that (x_n) is a Cauchy sequence in X_A : Fix some $\lambda > 0$ with $K(\lambda) < 1$. Then we have

$$\begin{aligned} |x_n - x_m|_A &= |\lambda(\lambda I - A)^{-1}(x_n - x_m) - (\lambda I - A)^{-1}A(x_n - x_m)|_A \\ &= |\lambda(\lambda I - A)^{-1}(x_n - x_m) - (\lambda I - A)^{-1}(y_n - y_m) + (\lambda I - A)^{-1}B(x_n - x_m)|_A \\ &\leq q\lambda \|x_n - x_m\| + q \|y_n - y_m\| + K(\lambda) |x_n - x_m|_A, \end{aligned}$$

where q denotes the norm of $(\lambda I - A)^{-1}$ regarded as an operator from X into X_A .

Consequently, we obtain

$$|x_n - x_m|_A \leq \frac{1}{1 - K(\lambda)} (\lambda q \|x_n - x_m\| + q' \|y_n - y_m\|).$$

This implies that $y_n = (A + B)x_n$ converges to $(A + B)x$ so that $(A + B)x = y$. The proof that $(A + B)$ is in fact the infinitesimal generator of a C_0 -semi-

group on X is performed by making use of Ball's Theorem ([1]). Roughly speaking, we are to show that for any $x \in X$ the Cauchy problem

$$(3) \quad \begin{aligned} \frac{d}{dt}x(t) &= (A + B)x(t), \quad t > 0 \\ x(0) &= x, \end{aligned}$$

admits a unique weak solution $x(t)$ on $[0, \infty)$, i.e. for all $x^* \in D((A + B)^*)$ and $t \geq 0$

$$\langle x(t), x^* \rangle = \langle x, x^* \rangle + \int_0^t \langle x(s), (A + B)^* x^* \rangle ds.$$

So, fix $x \in X$ and $t > 0$. Given any continuous function $z: [0, t] \rightarrow X_A$ we put

$$(\mathfrak{C}z)(t) = \int_0^t T(t-s)Bz(s)ds + \int_0^t T(t-s)xds, \quad 0 \leq t \leq t.$$

By the above considerations $(\mathfrak{C}z)(\cdot)$ is a continuous function $[0, t] \rightarrow X_A$. If β denotes again the norm of B regarded as an operator $X_A \rightarrow Z$, we obtain for all $0 \leq t \leq t$:

$$|\mathfrak{C}z_1(t) - \mathfrak{C}z_2(t)|_A \leq \beta\gamma(t) \sup |z_1(s) - z_2(s)|_A.$$

Choosing t sufficiently small we conclude that there exists a unique fixed point of $\mathfrak{C}$. As $\beta\gamma(t)$ does not depend on x , we may continue this procedure and obtain a continuous function $y: [0, \infty) \rightarrow X_A$ satisfying

$$y(t) = \int_0^t T(t-s)By(s)ds + \int_0^t T(t-s)xds.$$

Putting $x(t) = (A + B)y(t) + x$, it is clear that $x(0) = x$ and x is continuous $[0, \infty) \rightarrow X$. Moreover, we have for all $h > 0$

$$\begin{aligned} \frac{1}{h}(y(t+h) - y(t)) &= \frac{1}{h} \left(\int_0^{t+h} T(t+h-s)(x + By(s))ds - \int_0^t T(t-s)(x + By(s))ds \right) \\ &= \frac{1}{h}(T(h) - I) \int_0^t T(t-s)(x + By(s))ds + \frac{1}{h} \int_t^{t+h} T(t+h-s)(x + By(s))ds. \end{aligned}$$

and as $y(t) \in D(A)$, the right hand side converges to $Ay(t) + x + By(t)$ as $h \rightarrow 0^+$. Therefore, we have

$$\frac{d^+}{dt} y(t) = (A + B)y(t) + x = x(t).$$

For any $x^* \in D((A + B)^*)$ we obtain

$$\langle x(t), x^* \rangle = \langle x, x^* \rangle + \langle y(t), (A + B)^* x^* \rangle$$

and hence $x(t)$ is a weak solution of (3).

In order to verify uniqueness of this weak solution, let $x(\cdot)$ be any weak solution of (3) with $x(0) = 0$. Putting

$$y(t) = \int_0^t x(s) ds,$$

we get for all $x^* \in D((A + B)^*)$

$$\langle x(t), x^* \rangle = \int_0^t \langle x(s), (A + B)^* x^* \rangle ds = \langle y(t), (A + B)^* x^* \rangle.$$

As $(A + B)$ is closed, this implies that $y(\cdot) \in D(A)$ and hence

$$x(t) = \frac{d}{dt} y(t) = (A + B)y(t).$$

From the variation of constants formula for $T(\cdot)$ we get

$$y(t) = \int_0^t T(t-s)By(s)ds.$$

The unique solution of this integral equation is $y = 0$ and hence $x = 0$.

Hence (3) admits a unique weak solution for all $x \in X$ showing that $(A + B)$ is the infinitesimal generator of a C_0 -semigroup on X .

The particular choice of Z depends of course, heavily on the problem under consideration and it may vary through a large class of different spaces and is illustrated by the following examples.

1. $Z = X_A$.

A particular interesting case of Z is provided by putting $Z = X_A$. Clearly, assumptions (Z1)-(Z3) are satisfied (with $\gamma(t) = tM \exp[\omega t]$), and hence we deduce that for any continuous linear operator $B: X_A \rightarrow X_A$ the operator $(A + B)$ is the infinitesimal generator of a C_0 -semigroup on X .

2. Delay equations on product spaces.

Let Y be a real Banach space and put $X = Y \times L^p(-r, 0; Y)$, $1 \leq p < \infty$, $0 < r \leq \infty$. Let $T(\cdot)$ denote the solution semigroup of the unperturbed equation given by

$$\begin{aligned} T(t)(\eta, \varphi) &= (\eta, \psi) \\ \psi(s) &= \begin{cases} \eta & \text{if } s + t \geq 0 \\ \varphi(s + t) & \text{if } s + t < 0. \end{cases} \end{aligned}$$

As already mentioned in the introduction its infinitesimal generator A is given by

$$\begin{aligned} D(A) &= \{(u(0), u) \mid u \in W^{1,p}(-r, 0; Y)\}, \\ A(u(0), u) &= (0, \dot{u}). \end{aligned}$$

Then the assumptions of the generation theorem are satisfied with $Z = Y \times \{0\}$. Therefore, for any linear operator B that maps $W^{1,p}(-r, 0; Y)$ continuously into Y the operator $(A + B)$ is an infinitesimal generator of a C_0 -semigroup on X .

PROOF. For any $t > 0$ let φ be a continuous function $[0, t] \rightarrow Y$. Define ψ by

$$\psi(s)(\theta) = \begin{cases} \varphi(s) & \text{if } \theta \geq s - t, \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $\theta \in [-r, 0]$ we have

$$\begin{aligned} \int_0^t \psi(s)(\theta) ds &= \int_0^{t+\theta} \varphi(s) ds, \\ \left(\int_0^t \psi(s) ds, \int_0^t \varphi(s) ds \right) &= \int_0^t T(t-s)(\psi(s), \varphi(s)) ds \in D(A) \end{aligned}$$

and since

$$\left(\int_{-r}^0 |\tilde{\varphi}(s+t)|^p ds \right)^{1/p} \leq t^{1/p} \sup |\varphi(\tau)|$$

where

$$\tilde{\varphi}(s) = \begin{cases} \varphi(s) & \text{for } s \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

also (Z3) is satisfied.

Thus it is not the finite-dimensional range property of the perturbation that ensures the generation of a C_0 -semigroup. Of course, the above condition is too restrictive for partial-differential equations involving delay terms, although they are not far away from being necessary ([9]).

We next turn to partial differential equations: To begin with we provide

3. The Favard-class of $T(\cdot)$.

Let A be the infinitesimal generator of a C_0 -semigroup $T(\cdot)$ on X . Then the assumptions of the theorem are satisfied for Z being the Favard class of $T(\cdot)$, i.e.

$$Z = \left\{ x \in X \mid \limsup_{t \rightarrow 0^+} \frac{1}{t} \|T(t)x - x\| \text{ is finite} \right\},$$

$$|x|_Z = \|x\| + \limsup_{t \rightarrow 0^+} \frac{1}{t} \|T(t)x - x\|.$$

PROOF. Let φ be a continuous function $[0, t] \rightarrow Z$. Then there is a sequence (φ_n) of continuously differentiable functions such that $\varphi_n \rightarrow \varphi$ as $n \rightarrow \infty$. Clearly, $\int_0^t T(t-s)\varphi_n(s)ds \in D(A)$, and

$$\begin{aligned} \left\| A \int_0^t T(t-s)\varphi_n(s)ds \right\| &\leq \limsup_{h \rightarrow 0^+} \int_0^t \frac{1}{h} \|(T(h) - I)T(t-s)\varphi_n(s)ds\| \\ &\leq M \exp[\omega t] \int_0^t |\varphi_n(s)|_Z ds \leq tM \exp[\omega t] \sup |\varphi_n|_Z. \end{aligned}$$

(Here M and ω are constants such that $\|T(t)\| \leq M \exp[\omega t]$ for all $t \geq 0$). A closedness argument now shows that the same estimate is also valid for φ .

An interesting application of this result is

4. Integrodifferential equations.

Let Y be a Banach space and consider the Cauchy problem

$$(4) \quad \begin{aligned} \frac{d}{dt} u(t) &= Lu(t) + \int_0^t C(t-s)u(s)ds + f(t), \quad t \geq 0, \\ u(0) &= u_0. \end{aligned}$$

Here L is the infinitesimal generator of a C_0 -semigroup $\mathcal{S}(\cdot)$ on Y and $\{C(t); t \geq 0\}$ is a family of continuous linear operators $X_L \rightarrow Y$ such that for each $x \in X_L$ the map Cx given by $(Cx)(t) = C(t)x$ belongs to $L^1(0, \infty; Y)$.

Following the notation used in [3], we say that (4) is uniformly well posed if for each $u_0 \in D(L)$ and each $f \in W^{1,1}(0, \infty; Y)$ then exists a unique strong solution $u(t, u_0; f)$ of (4) which depends continuously on u_0 (with respect to the Y -norm) and f (with respect to the L^1 -norm), uniformly for t in compact intervals.

There is a large number of papers in which uniform well-posedness of (4) is proven under various additional smoothness assumptions on $C(\cdot)$. Our approach allows the following very general result:

THEOREM. (4) is uniformly well-posed if Cx is of bounded variation for each $x \in D(L)$.

The underlying basic idea that was introduced in [10] and carried out in a more general framework in [3] is to associate to (4) a differential equation in a larger Banach space.

To this end, let $T(\cdot)$ denote the shift semigroup on $L^1(0, \infty; Y)$ defined by $(T(t)\phi)(s) = \phi(s+t)$, $s \geq 0$, $t \geq 0$. Moreover, let D_s denote its infinitesimal generator and let δ be the operator $W^{1,1}(0, \infty; Y) \rightarrow Y$ given by $(\delta\phi = \phi(0))$.

In [3] it is shown that (4) is uniformly well-posed if and only if the following abstract Cauchy problem is uniformly well-posed in $X = Y \times Y \times L^1(0, \infty; Y)$

$$(5) \quad \begin{aligned} \frac{d}{dt} x(t) &= \mathcal{A}x(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned}$$

where $\mathcal{A}$ is given by

$$\begin{aligned} D(\mathcal{A}) &= Y \times D(L) \times W^{1,1}(0, \infty; Y), \\ \mathcal{A} &= \begin{pmatrix} 0 & L & 0 \\ 0 & L & \delta \\ 0 & C & D_s \end{pmatrix}. \end{aligned}$$

So, we have to show that $\mathcal{A}$ is the infinitesimal generator of a C_0 -semigroup on X . To prove this claim, we split up $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_0 = \mathcal{B}$$

where

$$\mathcal{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & L & \delta \\ 0 & 0 & D_s \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & C & 0 \end{pmatrix}.$$

It is an elementary calculation to verify that $\mathcal{A}_0$ generates a semigroup $\mathfrak{V}(\cdot)$ given by

$$\mathfrak{V}(t) \begin{pmatrix} x \\ y \\ f \end{pmatrix} = \begin{pmatrix} x \\ S(t)y + \int_0^t S(t-s)f(s)ds \\ T(f)t \end{pmatrix}.$$

The range of $\mathfrak{B}$ consists of all vectors $\begin{pmatrix} x \\ 0 \\ f \end{pmatrix}$ where f is of bounded variation. If we can verify that these vectors belong to the Favard class of $\mathfrak{V}(\cdot)$. Example 2 implies that $\mathcal{A}$ generates a C_0 -semigroup on X and hence (4) is uniformly well-posed.

As

$$\frac{1}{t} \left(\mathfrak{V}(t) \begin{pmatrix} x \\ 0 \\ f \end{pmatrix} - \begin{pmatrix} x \\ 0 \\ f \end{pmatrix} \right) = \left(0, \frac{1}{t} \int_0^t S(t-s)f(s)ds, \frac{1}{t} (T(t)f - f) \right)^x$$

and

$$\frac{1}{t} \left\| \int_0^t S(t-s)f(s)ds \right\| \leq \sup_{0 \leq s \leq t} \|S(t-s)\| \|f(s)\|$$

is bounded as f has bounded variation, and

$$\frac{1}{t} \|T(t)f - f\| = \frac{1}{t} \int_0^\infty \|f(s+t) - f(s)\| ds \leq \text{Var}(f, (0, \infty))$$

([2], Appendix), we conclude that

$$\limsup_{t \rightarrow 0^+} \frac{1}{t} \left\| \mathfrak{V}(t) \begin{pmatrix} x \\ 0 \\ f \end{pmatrix} - \begin{pmatrix} x \\ 0 \\ f \end{pmatrix} \right\| \text{ is finite.}$$

5. Interpolation spaces.

Let A be the infinitesimal generator of an analytic semigroup $T(\cdot)$. Then we can take $Z = (D(A), X)_I$, an interpolation space between $D(A)$ and X .

The proof follows from the general properties of interpolation spaces (see [4]).

If we are dealing with nonlinear perturbations then the situation is much more complicated as is seen by the following example. Roughly speaking, we show that even for a one-dimensional, C^∞ -nonlinear perturbation B that maps X_A into itself the operator $(A + B)$ is not a generator of a nonlinear semigroup on X .

EXAMPLE. Let $X = \mathbb{R} \times C_{u,b}(\mathbb{R}; \mathbb{R})$, where $C_{u,b}(\mathbb{R}; \mathbb{R})$ denotes the usual Banach space of all uniformly continuous, bounded functions $\mathbb{R} \rightarrow \mathbb{R}$.

Let $T(\cdot)$ be the linear semigroup given by $T(t) \begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} x \\ \varphi(t + \cdot) \end{pmatrix}$.

An easy calculation shows that its infinitesimal generator A is given by

$$D(A) = \left\{ \begin{pmatrix} x \\ \varphi \end{pmatrix} \mid \varphi' \in C_{u,b}(\mathbb{R}; \mathbb{R}) \right\}$$

$$A \begin{pmatrix} \varphi \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi' \end{pmatrix}.$$

Let ψ be a C^∞ -function $\mathbb{R} \rightarrow \mathbb{R}$ such that

$$\psi(0) = 0, \quad \psi(s) \geq \inf(|s|, s^2),$$

and ψ is Lipschitzian with constant ≤ 2 .

We define B by $B \begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} \psi(\varphi'(0)) \\ 0 \end{pmatrix}$.

For $\begin{pmatrix} 0 \\ \zeta \end{pmatrix} \in D(A)$, the Cauchy problem

$$\frac{d}{dt} \begin{pmatrix} x \\ \varphi \end{pmatrix} = A \begin{pmatrix} x \\ \varphi \end{pmatrix} + B \begin{pmatrix} x \\ \varphi \end{pmatrix}, \quad \begin{pmatrix} x \\ \varphi \end{pmatrix}(0) = \begin{pmatrix} 0 \\ \zeta \end{pmatrix},$$

has a unique strong solution given by

$$\begin{pmatrix} x \\ \varphi \end{pmatrix}(t) = \left(\int_0^t \psi(\zeta'(s)) ds, \zeta(t + \cdot) \right)$$

Given $t \in (0, 1]$ we choose a $\zeta \in C_{u,b}(\mathbb{R}; \mathbb{R})$ such that the restriction of ζ to $(0, t]$ does not have a bounded variation. Let (ζ_n) be a sequence in $C_{u,b}(\mathbb{R}; \mathbb{R})$ so that $\zeta_n \rightarrow \zeta$ and $(0, \zeta_n)^T \in D(A)$.

If the solutions $\begin{pmatrix} x_n(\cdot) \\ \varphi_n(\cdot) \end{pmatrix}$ with initial values $\begin{pmatrix} 0 \\ \zeta_n \end{pmatrix}$ were convergent to, say $\begin{pmatrix} \eta(t) \\ \omega(t) \end{pmatrix}$, then we clearly have $\omega(t) = \zeta(t + \cdot)$.

Let $0 \leq s_1 \leq s_2 \leq t$. We choose measurable sets E_1 and E_2 such that

$$[s_1, s_2] = E_1 \cup E_2 \quad \text{and} \quad \psi(\zeta'_n(s)) \geq |\zeta'_n(s)| \quad \text{on } E_1$$

and

$$\psi(\zeta'_n(s)) \geq (\zeta'_n(s))^2 \quad \text{on } E_2.$$

Then

$$\int_{s_1}^{s_2} \psi(\zeta'_n(s)) ds \geq \int_{E_1} |\zeta'_n(s)| ds + \int_{E_2} (\zeta'_n(s))^2 ds \geq \int_{E_1} |\zeta'_n(s)| ds + \left(\int_{E_2} |\zeta'_n(s)| ds \right)^2 \cdot \frac{1}{\text{meas}(E_2)}.$$

Consequently,

$$\int_{s_1}^{s_2} |\zeta'_n(s)| ds \leq (s_2 - s_1) + \int_{s_1}^{s_2} \psi(\zeta'_n(s)) ds.$$

Taking the limit $n \rightarrow \infty$ we thus obtain

$$|\zeta(s_2) - \zeta(s_1)| \leq \gamma(s_2) - \gamma(s_1) + s_2 - s_1,$$

where γ is the limit of the sequence of monotone functions $\int_0^t \psi(\zeta'_n(s)) ds$ and hence γ is itself monotone. Therefore ζ must be of bounded variation which contradicts the assumptions. As a consequence we deduce that the solution operators cannot be continuous which is a standing hypothesis for all nonlinear semigroups.

Acknowledgements. This work was heavily motivated by discussions with Prof. R. Vinter during W. S. was visiting Imperial College. It is a pleasure to thank the Royal Society and the Austrian Academy of Sciences for making this visit possible. Moreover, it is a special pleasure to thank Prof. G. Da Prato and I. Lasiecka for several very fruitful discussions and comments.

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