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# On the comparative theory of primes

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# On the Comparative Theory of Primes.

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**1.** Knapowski and Turàn investigated in a series of papers ([3]) sign changes of the functions (in the case of $i = 1, 2, 4$)

$$(1.1)\qquad \begin{cases} \Delta_1(x, q, l_1, l_2) = \sum_{p\leqslant x} \varepsilon(p, q, l_1, l_2) \\ \Delta_2(x, q, l_1, l_2) = \sum_{n\leqslant x} \varepsilon(n, q, l_1, l_2)\dfrac{\Lambda(n)}{\log n} \\ \Delta_3(x, q, l_1, l_2) = \sum_{p\leqslant x} \varepsilon(p, q, l_1, l_2)\log p \\ \Delta_4(x, q, l_1, l_2) = \sum_{n\leqslant x} \varepsilon(m, q, l_1, l_2)\Lambda(n) \end{cases}$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^n \\ 0 & \text{otherwise} \end{cases}$$

$$(1.2)\qquad \varepsilon(n) = \varepsilon_1(n) - \varepsilon_2(n)\,, \qquad \varepsilon_i(n) = \begin{cases} 1 & \text{if } n \equiv l_i(q) \\ 0 & \text{otherwise} \end{cases}$$

and (what we shall always assume without mentioning)

$$(1.3)\qquad (l_1, q) = (l_2, q) = 1\,, \qquad l_1 \not\equiv l_2 (\bmod q)\,.$$

For a general modulus $q$ Knapowski and Turàn needed always the so called Haselgrove condition (H), that the $L$-functions have no real non-trivial zeros or, in an explicit formulation, they assumed the existence of

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a number $A(q)$, $0 < A(q) < 1$, such that (for $s = \sigma + it$)

$$L(s, \chi, q) \neq 0 \quad \text{for} \quad |t| \leqslant A(q) . \tag{1.4}$$

It is easy to see that in the case of the existence of a real zero, being on the rigth from all complex zeros of all $L(s, \chi, q)$ functions, the functions $\Delta_i(x)$ are of constant sign for suitable pairs $l_1, l_2$ for $x > x_0$.

Thus, at the present stage of analytic number theory, a hypothesis of type (1.4) is necessary in all investigations.

We note that (1.4) has been verified by Spira [6] for all $q < 25$. The other assumption, the so called finite Riemann-Piltz conjecture (FR-P),

$$L(s, \chi, q) \neq 0 \quad \text{for } \sigma \neq \tfrac{1}{2} , \qquad |t| \leqslant D = cq^{10} , \tag{1.5}$$

used in many investigations of Knapowski and Turàn, is of more technical nature. The aim of Knapowski and Turàn was to achieve effective results, i.e. explicitly dependent only on $q$, $A(q)$ (and in given cases on $D$). In the case of $l_1 = 1$ or $l_2 = 1$ they were able to find, using only Haselgrove condition, infinitely many sign changes of $\Delta_1(x)$ for $1 \leqslant i \leqslant 4$. (Although they did not treat the case $i = 3$, we mention the corresponding results for $i = 3$ too, if it is possible to obtain it, using the method applied by them for the other cases). Also, they could give a lower estimation of the number $V_i(Y, q, l_1, l_2)$ of sign changes of $\Delta_i(x)$ in the interval $(2, Y)$ and an explicit upper bound for the first sign change (cf. parts I-III of [3]).

For general $l_1$ and $l_2$, besides (H), they had to assume also the finite Riemann-Piltz conjecture and even this led to results only for $i = 2$ and 4 (with other numerical estimates, naturally than in the earlier mentioned case). In the case $i = 1$ and 3 they needed the additional assumption that $l_1$ and $l_2$ would be both quadratic residues or both non-residues, besides (FR-P) and (H), (cf. parts V and VI of [3]). In the case of $i = 4$ they succeeded in showing for general $l_1$ and $l_2$ the above mentioned results in a quantitatively much weaker but (as all the earlier mentioned results) also effective form, without supposing (FR-P), *i.e.* only assuming (H) (cf. part VII of [3]).

We note that the most important open problem of the comparative prime number theory is to assure infinitely many sign changes of $\Delta_1(x, q, l_1, l_2)$ for all pairs $l_1, l_2$ with (1.3). This problem seems to be hopeless at present, even supposing the (infinite) Riemann-Pilitz conjecture, besides (H) naturally. (One has nearly the same difficulties for $\Delta_3(x)$, but this is not so important from an arithmetic point of view).

In this work (assuming (H) and (FR-P) in a slightly weaker form) we shall treat the case $i = 4$, for general $l_1, l_2$, (and the case $i = 3$ if $l_1$ and $l_2$ have the same quadratic character) improving earlier results of Knapowski and Turàn, which we summarize now (in some cases with slight changes) as follows:

We shall use the notations $\exp_{\nu+1}(x) = \exp(\exp_\nu(x))$, $\exp_1(x) = \exp(x)$, $\log_{\nu+1} x = \log\log_\nu x$, $\log_1 x = \log x$.

THEOREM A. Assume (H) and (FR-P) (cf. (1.4)-(1,5)). Then for

$$Y > \max\left\{\exp_2(cq^{20}),\ \exp_2\left(\frac{c}{A^3(q)}\right)\right\} \tag{1.6}$$

one has for $i = 2, 4$

$$\max_{Y^{2/5} \leqslant x \leqslant Y} \Delta_i(x, q, l_1, l_2) > \sqrt{Y}\exp\left(-44\frac{\log Y \log_3 Y}{\log_2 Y}\right) \tag{1.7}$$

for all pairs $l_1$ and $l_2$ with (1.3), where, as always in the following, the generic symbol $c$ replaces an explicitly calculable positive absolute constant, which might have different values at various appearances.

Since the opposite inequality clearly holds, by changing the role of $l_1$ and $l_2$, one obtains the following

COROLLARY. On the above conditions, for $i = 2, 4$ one has for the number of sign changes of $\Delta_i(x)$ the lower estimate

$$V_i(Y, q, l_1, l_2) > \log_2 Y\,. \tag{1.8}$$

THEOREM B. Assume (H) (cf. (1.4)). Then for

$$Y > \max\left(\exp_2(q^c),\ \exp_2\left(\frac{c}{A^3(q)}\right)\right) \tag{1.9}$$

all functions $\Delta_4(x, q, l_1, l_2)$ with (1.3) change their sign in the interval

$$[\log^2 Y/4,\ Y]\,. \tag{1.10}$$

For the proofs of Theorem A and B see parts V and VII of [3], respectively.

In the present work we shall show (roughly speaking) that in the case of $i = 4$ (1.7) can be improved to

$$\max_{x \in [Y\exp(-c(q)\log^{\frac{1}{2}} Y),\, Y]} \Delta_4(x) > \sqrt{Y}\exp\left(-\,c(q)\log^{\frac{1}{2}} Y\right) \tag{1.11}$$

thereby obtaining

$$V_4(Y) > c(q)\log^{\frac{1}{4}} Y . \tag{1.12}$$

Furthermore, we get for the first sign change of $\Delta_4(x)$ the upper bound

$$Y_0 = \exp\left(\frac{q^4}{A(q)}\right), \tag{1.13}$$

which improves (1.6). (In the above formulas we neglected some $\log q$, $\log\big(1/A(q)\big)$ and $\log\log Y$ factors).

Due to the lower estimate (1.11) by the strong form of the prime number theorem of arithmetic progressions, we obtain the same results in the case of $i = 3$, if $l_1$ and $l_2$ are both quadratic residues or both quadratic non-residues. (In Knapowski-Turàn's proof this needs relatively many additional efforts, as can be seen from part VI of [3]).

Finally we remark that by ineffective methods (essentially due to Lan dau [5, § 197], Grosswald [2] and Anderson-Stark [1]) one obtains that if for a non-trivial zero $\varrho_0 = \beta_0 + i\gamma_0$ of an $L(s, \chi, q)$, function one has $\gamma_0 \neq 0$, $\beta_0 \geqslant \frac{1}{2}$ and

$$a(\varrho_0) = \sum_{\substack{\chi(\text{mod } q)\\ L(\varrho_0,\chi)=0}} \big(\bar\chi(l_2) - \bar\chi(l_1)\big) m_\chi(\varrho_0) \neq 0 \tag{1.14}$$

(where $m_\chi(\varrho_0)$ denotes the multiplicity of $\varrho_0$ as a zero of $L(s, \chi)$), then for every $l_1, l_2$ with (1.3)

$$\varlimsup_{x\to\infty} \frac{\Delta_4(x)}{x^{\beta_0}} \geqslant \left|\frac{a(\varrho_0)}{\varrho_0}\right|, \quad \varliminf_{x\to\infty} \frac{\Delta_4(x)}{\chi^{\beta_0}} \leqslant -\left|\frac{a(\varrho_0)}{\varrho_0}\right| . \tag{1.15}$$

This is valid for $\Delta_3(x)$ too, if $l_1$ and $l_2$ have the same quadratic character, or in case of $\beta_0 > \frac{1}{2}$ for every pair $l_1, l_2$ with (1.3). However, this theorem does not yield any localisation of sign changes or lower estimation of $V_i(Y, q, l_1, l_2)$.

We further note that the method of Knapowski and Turàn, (also in the present refined form) does not furnish better lower estimates than (essentially) $\sqrt{Y}$, even assuming the existence of a zero $\varrho_0$ with $\beta_0 > \frac{1}{2}$, $\gamma_0 \neq 0$ and $a(\varrho_0) \neq 0$.

**2.** In our results, we shall always assume that the Haselgrove condition (H) and the fiuite Riemann-Piltz conjecture (FR-P) hold, the second one up to a level $D$ with

$$D = c_0 q^2 \log^6 q \tag{2.1}$$

where $c_0$ is a sufficiently large positive absolute constant.

Here and in the sequel, we shall denote by $c_i$, $i = 0, 1, \ldots$ explicitly calculable positive absolute constants; moreover, the generic simbol $c$, and the signs $\gg$, $\ll$, $0$, replace such constants; $\exp(x) = e^x$, $\log_2 Y = \log\log Y$.

We shall also assume without any further mention the trivial condition (1.3) on $l_1, l_2$.

Our results are the following:

THEOREM 1. *Assume* (H), (FR-P) *and let be such that*

$$Y > Y_0 = \exp\left(c \frac{D^5}{A(q)} \log \frac{1}{A(q)}\right) \tag{2.2}$$

*Then, there exists $x$ with*

$$x \in \left[Y \exp\left(-\frac{Cq}{\sqrt{A(q)}} (\log Y)^{\frac{1}{2}} (\log_2 Y)^{\frac{3}{2}}\right), Y\right] \tag{2.3}$$

*such that*

$$\Delta_4(x, q, l_1, l_2) > \sqrt{Y} \exp\left(-\frac{Cq}{\sqrt{A(q)}} (\log Y)^{\frac{1}{2}} (\log_2 Y)^{\frac{3}{2}}\right). \tag{2.4}$$

COROLLARY 1. *Under the assumptions of Theorem* 1, *for $Y$ verifying* (2.2) *we have at least one sign change of $\Delta_4(x)$ for $x$ belonging to the interval* (2.3).

*Hence, we obtain*

$$V_4(Y, q, l_1, l_2) \gg \frac{\sqrt{A(q)}}{q} \frac{\log^{\frac{1}{2}} Y}{\log^{\frac{3}{2}} Y} \qquad \text{for } Y > \exp\left(cD^5 A^{-1}(q) \log \frac{1}{A(q)}\right). \tag{2.5}$$

THEOREM 2. *Assume* (H), (FR-P) *and let*

$$Y > \exp\left(\frac{cD^2}{A(q)} \log^3 \frac{D}{A(q)}\right). \tag{2.6}$$

*Then, there exists $x$ with*

$$x \in \left[Y \exp\left(-\frac{c \log^{\frac{2}{3}} Y \log_2 Y}{(DA(q))^{\frac{1}{3}}}\right), Y\right) \tag{2.7}$$

*such that*

$$\Delta_4(x) > \sqrt{x} \exp\left(-\frac{cq^{8/3} \log^4 q}{A^{\frac{3}{2}}(q)} (\log Y)^{\frac{1}{3}} (\log_2 Y)^2\right). \tag{2.8}$$

Using the fact that the error term given by the Prime Number Theorem for aritmetic progression (see for instance Prachar [5, pp. 297-298]) is smaller than the lower bounds (2.4), (2.7), we obtain

COROLLARY 2. *The statements of Theorem* 1, *Corollary* 1 *and Theorem* 2 *continue to be true without any change also for* $\Delta_3(x)$, *if* $l_1$ *and* $l_2$ *are both quadratic non-residues or if they are both quadratic residues* (mod $q$).

Actually one can assure (2.2)-(2.4) also for $i = 3$ if $l_1$ is a quadratic non-residue and $l_2$ a residue. However, in this case we canuot guarantee the opposite inequality, which follows in the earlier cases just by changing the role of $l_1$ and $l_2$.

**3.** We introduce

$$F(s) = \sum_{n=1}^{\infty} \frac{\varepsilon(n)\Lambda(n)}{n}. \tag{3.1}$$

We have

$$F(s) = \frac{1}{\varphi(q)} \sum_{\chi(\text{mod } q)} (\bar{\chi}(l_2) - \bar{\chi}(l_1)) \frac{L'}{L}(s, \chi) = \frac{f_{-1}}{s} + f_0 + \dots \tag{3.2}$$

where (see [5], 7.4.21 and 7.4.46)

$$|f_{-1}| = \left| \frac{1}{\varphi(q)} \sum_{\substack{\chi(\text{mod } q)\\ \chi(-1)=1}} (\bar{\chi}(l_2) - \bar{\chi}(l_1)) \right| \leqslant 2, \tag{3.3}$$

$$f_0 = O(\log^2 q) \tag{3.4}$$

if there exists no Siegel-zero, as assured by (H).

Furthermore, let

$$a_\varrho = \frac{1}{\varphi(q)} \sum_{\substack{\chi(\text{mod } q)\\ L(\varrho,\chi)=0}} (\bar{\chi}(l_2) - \bar{\chi}(l_1))\, m(\varrho) = \operatorname{Res}_{s=\varrho} F(s) \tag{3.5}$$

where $m_\chi(\varrho)$ denotes the multiplicity of $\varrho$ as a zero of $L(s, \chi)$.

LEMMA 1. *For* $\exp[-3\mu] \leqslant K \leqslant \mu/9$, $\mu \geqslant \log q$, *we have*

$$\frac{1}{2\sqrt{\pi K}} \int_{\exp(\mu+3\sqrt{\mu K})}^{\exp(\mu-3\sqrt{\mu K})} \frac{\Delta_4(x)}{x} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx \tag{3.6}$$
$$= \sum_{\varrho} \frac{a_\varrho}{\varrho} \exp(K\varrho^2 + \mu\varrho) + \mu f_{-1} + f_0 + O(\exp(-\mu/4))$$

*where the sum is performed on the non-trivial zeros* $\varrho$ $(0 < \mathrm{Re}\,\varrho < 1)$ *of L-functions* (mod $q$).

PROOF. We use the following integral formula

$$(3.7)\qquad \frac{1}{2i\pi}\int_{(2)} \exp(Ks^2 + \alpha s)\,ds = \frac{1}{2\sqrt{\pi K}}\exp\left(-\frac{\alpha^2}{4K}\right), \quad K \in \mathbf{R}^+, \quad \alpha \in \mathbf{C}$$

and the fact that, by partial summation

$$(3.8)\qquad F(s) = \sum_{n=1}^{\infty}\frac{\varepsilon(n)\Lambda(n)}{n^s} = s\int_1^{\infty}\frac{\Delta_4(x)}{x^{s+1}}\,dx\,.$$

Then, we have

$$(3.9)\qquad \frac{1}{2\sqrt{\pi K}}\int_1^{\infty}\frac{\Delta_4(x)}{x}\exp\left(-\frac{(\mu-\log x)^2}{4K}\right)dx$$
$$= \frac{1}{2\pi i}\int_1^{\infty}\frac{\Delta_4(x)}{x}\int_{(2)}\exp[Ks^2 + (\mu - \log x)s]\,ds\,dx$$
$$= \frac{1}{2\pi i}\int_{(2)}\frac{F(s)}{s}\exp[Ks^2 + \mu s]\,ds\,.$$

Since the following well-known estimate holds:

$$(3.10)\qquad \frac{L'}{L}(s, \chi) \ll \log(q(|t| + 2)) \ll (\log q)\log(|t| + 2) \qquad \text{for } s = -\frac{1}{2} + it$$

we obtain

$$(3.11)\qquad R = \frac{1}{2\pi i}\int_{(-\frac12)}\frac{F(s)}{s}\exp(Ks^2 + \mu s)\,ds \ll (\log q)\exp\left[\frac{K}{4} - \frac{\mu}{2}\right]$$
$$\cdot\left\{\int_0^{2+1/K}\frac{\log(|t| + 2)}{|t| + \frac12}\,dt + \int_{2+1/K}^{\infty}\exp[-Kt^2]\,dt\right\}$$
$$\ll \mu\exp[-\mu/3]\left\{\log^2\left(2 + \frac{1}{K}\right) + \frac{1}{\sqrt{K}}\exp\left[-K\left(2 + \frac{1}{K}\right)^2\right]\right\} \ll \exp[-\mu/4]\,.$$

Now, by Cauchy's residues theorem, we get

$$(3.12)\qquad \frac{1}{2\pi i}\int_{(2)}\frac{F(s)}{s}\exp[Ks^2 + \mu s]\,ds = \sum_{\varrho}\frac{a_\varrho}{\varrho}\exp[K\varrho^2 + \mu\varrho] + \mu f_{-1} + f_0 + R$$

and, in view of (3.11),

$$\frac{1}{2\pi i}\int_{(2)}\frac{F(s)}{s}\exp(Ks^2+\mu s)\,ds=\sum_{\varrho}\frac{a_\varrho}{\varrho}\exp[K\varrho^2+\mu\varrho) + \mu f_{-1}+f_0+O(\exp(-\mu/4)). \tag{3.13}$$

Moreover, using the trivial estimate $\Delta_4(x)\ll x$, we obtain

$$\frac{1}{2\sqrt{\pi K}}\left|\int_{\exp(\mu+3\sqrt{\mu K})}^{\infty}\frac{\Delta_4(x)}{x}\exp\left(-\frac{(\mu-\log x)^2}{4K}\right)dx\right| \tag{3.14}$$

$$\leqslant\left|\int_{\frac{3\sqrt{\mu}}{2}}^{\infty}\exp\left(-(y-\sqrt{K})^2+\mu+K\right)dy\right|\ll\exp\left(-\left(\frac{3\sqrt{\mu}}{2}-\sqrt{K}\right)^2+\mu+K\right) \leqslant\exp(-\mu/4]$$

where we have introduced in the integral the variable $y=\log x-\mu/2\sqrt{K}$ and we have used $3\sqrt{\mu}/2>\sqrt{K}$, $\mu>9K$.

Since a similar estimate holds, completely trivially, also for $\int_1^{\exp(\mu-3\sqrt{\mu K})}$, we get

$$\frac{1}{2\sqrt{\pi K}}\int_1^{\infty}\frac{\Delta_4(x)}{x}\exp\left(-\frac{(\mu-\log x)^2}{4K}\right)dx \tag{3.15}$$

$$=\frac{1}{2\sqrt{\pi K}}\int_{\exp(\mu-3\sqrt{\mu K})}^{\exp(\mu+3\sqrt{\mu K})}\exp\left(-\frac{(\mu-\log x)^2}{4K}\right)\frac{\Delta_4(x)}{x}\,dx+O(\exp(-\mu/4))\,.$$

Now, our Lemma follows collecting together (3.9), (3.13) and (3.15). ■

Since the main part of (3.6) can be written as a powersum, our problem is reduced to give a good lower bound for it. This is accomplished by means of a « one-sided » powersum theorem of Knapowski and Turàn (see Theorem 4.1 in part III of [3]). In order to obtain sharper estimates, we shall need this result in the following slightly modified form:

LEMMA 2. *Let* $b_j, z_j\in\mathbb{C}$ *for* $j=1,2,\dots,n$, *with*

$$0<\varkappa\leqslant|\arg z_j|\leqslant\pi\qquad\forall_j \tag{3.16}$$

$$|z_1|\geqslant|z_2|\geqslant\dots\geqslant|z_n|\,. \tag{3.17}$$

*Then for any $h$ with $1 \leqslant h \leqslant n$ and for any $m \geqslant 0$, there exists an integer $\nu$ with*

$$\nu \in \left[m, m + n\left(3 + \frac{\pi}{\varkappa}\right)\right] \tag{3.18}$$

*such that*

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu} > \frac{E}{2n+1} \left(\frac{24e(m + n(5 + \pi/\varkappa))}{n}\right)^{-2n} |z|^{\nu} \left|\frac{z_h}{z_1}\right|^{n(5+\pi/\varkappa)} \tag{3.19}$$

*where*

$$E = \min_{l \geqslant h} \left|\operatorname{Re} \sum_{j=1}^{l} b_j\right| . \tag{3.20}$$

PROOF. Following the lines the theorem of [3] quoted above, we obtain the following inequaliry, in the case $|z_1| = 1$,

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu} > \frac{E}{2n+1} \left(\frac{|z_h| - \delta}{48}\right)^{2n} \delta^{m+n(3+\pi/\varkappa)} \tag{3.21}$$

for a suitable $\nu$ verifying (3.18) and for every $\delta$ with

$$0 < \delta < |z_h| \tag{3.22}$$

Then, (3.19) follows by choosing

$$\delta = |z_h| \left(1 - \frac{2n}{m + n(5 + \pi/\varkappa)}\right) \tag{3.23}$$

unlike to Knapowski-Turàn's choice $\delta = |z_h| - 2n/(m + n(3 + \pi/\varkappa))$. ■

According to (3.19), in our applications we shall need a non trivial lower estimate for $E$. This requires a modification of the coefficients in the power-sum, ffurnished by the following Lemma:

LEMMA 3. *There exists a prime $P \equiv l_1 (\operatorname{mod} q)$ with*

$$\frac{D}{2} < P \log^2 P < D \tag{3.24}$$

*such that, for $P_0 = P + \frac{1}{2}$ or $P_0 = P - \frac{1}{2}$ we have*

$$\left|\sum_{\varrho} \frac{a_\varrho}{\varrho} \exp\left(K_0 \varrho^2 + {}_i\mu_0 \varrho\right)\right| > \frac{\log P}{3} \tag{3.25}$$

*where*

$$K_0 = \frac{P_0^2 \log^2 P_0}{1}, \qquad \mu_0 = \log P_0 . \tag{3.26}$$

PROOF. Since the finite Riemann-Piltz conjecture is assumed to be true, the prime number formula of arithmetic progressions, truncated at $D$, assures the existence of a prime $P \equiv l_1 (\mathrm{mod}\, q)$ verifying (3.24). Then, the function $\Delta_4(x)$ has a jump $\log P$ at the point $P$ and we have

$$\Delta_4(P_2) - \Delta_4(P_1) = \log P \tag{3.27}$$

for $P_1 = P - \frac{1}{2}$, $P_2 = P + \frac{1}{2}$.

Now, we use Lemma 1 with $\mu_i = \log P_i$, $K_i = 1/P_i^2 \log^2 P_i$ for $i = 1, 2$. This choice implies

$$[\exp(\mu_i - 3\sqrt{\mu_i K_i}), \exp(\mu_i + 3\sqrt{\mu_i K_i})] \subset \left(P_i - \frac{1}{2}, P_i + \frac{1}{2}\right) \tag{3.28}$$

since $P$ is large enough by (3.24) and (2.1).

Thus, $\Delta_4(x) = \Delta_4(P_i)$ in the above interval and, using Lemma 1, and setting $x = \exp(\mu + 2\sqrt{K}y)$ we obtain:

$$\begin{aligned} \sum_{\varrho} \frac{a_\varrho}{\varrho} \exp(K_i \varrho^2 + \mu_i \varrho) + f_{-1}\mu_i + f_0 \\ = \frac{\Delta_4(P_i)}{\sqrt{\pi}} \int_{\exp(\mu_i - 3\sqrt{\mu_i K_i})}^{\exp(\mu_i + 3\sqrt{\mu_i K_i})} \exp\left(-\frac{(\mu_i - \log x)^2}{4K_i}\right) \frac{dx}{2\sqrt{K_i}x} + O(\exp(-\mu/4)) \\ = \frac{\Delta_4(P_i)}{\sqrt{\pi}} \int_{-3\sqrt{\mu_i}/2}^{3\sqrt{\mu_i}/2} \exp(-y^2)\, dy + O(\exp(-\mu/4)) = \Delta_4(P_i) + O(P^{-\frac{1}{4}}) . \end{aligned} \tag{3.29}$$

Owing to $|f_{-1}| \leqslant 2$ and (3.27) we get from this

$$O\left(\frac{1}{P}\right) + \sum_{\varrho} \frac{a_\varrho}{\varrho} (\exp(K_2 \varrho^2 + \mu_2 \varrho) - \exp(K_1 \varrho^2 + \mu_1 \varrho)) = \log P + O(P^{-1/4}) \tag{3.30}$$

which implies Lemma 3. ■

In view of the application of Lemma 2 to the power-sum appearing in (2.6), it is also necessary to assure the argument condition (3.16), and this is made by means of

LEMMA 4. *Let $a_j$ be real numbers for $j = 1, \ldots, n$ with $a_j \neq 0$ and*

$$\frac{1}{n} \sum_{j=1}^{n} |a_j|^{-1} \leqslant \eta . \tag{3.31}$$

*Then, for every $H$ there exists an $y_0$ with*

$$y_0 \in [H, H + \eta] \tag{3.32}$$

*such that, for any integer $K$ and for every $j = 1, \ldots, n$*

$$|y_0 a_j - 2K\pi| \geqslant \frac{1}{4n}. \tag{3.33}$$

PROOF. For fixed $j$, (3.33) can be false for $K$ in an interval of length at most $(\eta|a_j| + 1/2n)/2\pi$; for fixed $K$, leaving $j$ fixed, this can happen for $y$ in an interval of length at most $1/2n|a_j|$.

Thus, the total Lebesgue measure of $y$ for which (3.33) is false for fixed $j$ is majorised by

$$\frac{1}{2n|a_j|}\left(\frac{\eta|a_j|}{2} + \frac{3}{2}\right) = \frac{\eta}{4n} + \frac{3}{4}\frac{1}{n|a_j|}. \tag{3.34}$$

Summing over $j = 1, \ldots, n$, we obtain our Lemma. ■

Now, we are in good position to apply Lemma 2.

We introduce the following position:

$$b_\varrho = \frac{a_\varrho}{\varrho} \exp(K_0 \varrho^2 + \mu_0 \varrho) \tag{3.35}$$

for $a_\varrho$ given by (3.5) and $K_0, \mu_0$ furnished by (3.26).

Furthermore, let

$$\lambda > 2D, \quad L > \frac{cq^2}{A(q)} \lambda \log^3 \lambda \tag{3.36}$$

Let $B$ be a real uumber to be chosen later with

$$B \in \left[\frac{c \log \lambda}{A(q)\lambda}, \quad \frac{2c \log \lambda}{A(q)\lambda}\right] \tag{3.37}$$

and let $\nu$ be an integer to be chosen later with

$$\nu \in \left[\frac{L - \mu_0}{B} - cq^2\lambda^2 \log^2 L, \quad \frac{L - \mu_0}{B}\right] \tag{3.38}$$

$$K = K_0 + \frac{B\nu}{\lambda^2} \quad \mu = \mu_0 + B\nu \tag{3.39}$$

$$z_\varrho = \exp\left[\frac{B\varrho^2}{\lambda^2} + B\varrho\right]. \tag{3.40}$$

LEMMA 5. *With the positions* (3.35) *to* (3.40), *we have*

$$\text{Re}\sum_{\varrho}\frac{a_\varrho}{\varrho}\exp(K\varrho^2+\mu\varrho)>\exp\left\{\frac{L}{2}-\frac{LD^2}{\lambda^2}-\frac{cq^2}{A(q)}\lambda\log^3 L\right\} \tag{3.41}$$

*for suitable values of* $\nu$ *and* $B$ *satisfying* (3.38).

PROOF. By our definitions, we have

$$\sum_{\varrho}\frac{a_\varrho}{\varrho}\exp(K\varrho^2+\mu\varrho)=\sum_{\varrho}b_\varrho z_\varrho^\nu. \tag{3.42}$$

Moreover

$$\left|\sum_{|\varrho|\geqslant 2\lambda}\frac{a_\varrho}{\varrho}\exp(K\varrho^2+\mu\varrho)\right|\ll\sum_{m=[2\lambda]}^{\infty}\exp\big(\mu+K(1-m^2)\big)(\log qm)\ll 1 \tag{3.42}$$

and so we have only to consider

$$\sum_{|\varrho|<2\lambda}b_\varrho z_\varrho^\nu\,. \tag{3.43}$$

Here, the number $n$ of terms is clearly

$$c_1\varphi(q)\,\lambda\log\lambda\leqslant n\leqslant c_2 q\lambda\log\lambda\,. \tag{3.44}$$

Now, we apply Lemma 4, setting $\varrho=\beta+i\gamma$,

$$a_j=\left(\frac{2\beta}{\lambda^2}+1\right)\gamma\,,\qquad H=\frac{c\log\lambda}{A(q)\,\lambda}\,. \tag{3.45}$$

Using Jensen's inequaiity, we have by $A(q)<1$ and $\lambda>q$,

$$\sum_{j=1}^{n}\frac{1}{|a_j|}\ll\sum_{|\varrho|\leqslant 2\lambda}\frac{1}{|\gamma|}<\sum_{A(q)\leqslant|\gamma|\leqslant 1}\frac{1}{|\gamma|}+\sum_{\substack{|\varrho|\leqslant 2\lambda\\ |\gamma|>1}}\frac{1}{|\gamma|}\ll\frac{\varphi(q)\log q}{A(q)}+\varphi(q)\log^2\lambda; \tag{3.46}$$

so, in view of (3.44), condition (3.31) holds with

$$\eta=\frac{c\log\lambda}{A(q)\,\lambda}\,. \tag{3.47}$$

Hence, Lemma 4 says that there exists a $B$ in the interval (3.37) such that, for every $j=1,\dots,n$

$$|Ba_j-2K\pi|\geqslant\frac{1}{4n}=\frac{c}{q\lambda\log\lambda}\,. \tag{3.48}$$

Since in our case $z_{\varrho_j} = |z_{\varrho_j}| \exp(ia_j B)$, this means that

$$(3.49) \qquad |\operatorname{Arg} z_\varrho| > \frac{c}{\eta} = \frac{c}{q\lambda \log \lambda} = \varkappa .$$

Now, we order the numbers $z_\varrho$ according to (3.17) and, in view of (FR-P), we choose $h$ of Lemma 2 as the largest index corresponding to a zero $\varrho$ with $|\varrho_0| \leqslant D$.

Since

$$(3.50) \qquad \left| \sum_{|\varrho|>D} \frac{a_\varrho}{\varrho} \exp(K_0 \varrho^2 + \mu_0 \varrho) \right| \ll \sum_{m=P\log^2 P}^{\infty} \exp\left[\frac{1-m^2}{P^2 \log^2 P}\right] P(\log mq) \ll 1$$

with constants implied by the $\ll$ sign, independent of $c_0$ appearing in (2.1), we have by Lemma 3

$$(3.51) \qquad \left|\sum_{\varrho \in S} b_\varrho\right| > \frac{\log P}{4}$$

for every set $S$ containing all zeros $\varrho$ with $|\varrho| \leqslant D$.

Finally, we have

$$(3.52) \qquad |z_h| \geqslant \exp\left(\frac{B}{2} - \frac{B}{\lambda^2} D^2\right)$$

because $\varrho_0$ is on the critical line, and (by (3.40) and $\lambda > 2D$)

$$(3.52) \qquad \left|\frac{z_h}{z_1}\right| \geqslant \exp\left(\frac{B}{2} - \frac{B}{\lambda^2} D^2 - B - \frac{B}{\lambda^2}\right) > \exp(-B) .$$

Recalling (3.44), (3.49), (3.51), (3.52), (3.53) and choosing $m$ as $m = (L - \mu_0)/B$ in view of (3.38), we obtain by Lemma 2 for suitable $B$ and $\nu$:

$$(3.54) \qquad \operatorname{Re} \sum_{|\varrho|<2\lambda} b_\varrho z_\varrho^\nu > \exp\left\{\frac{B\nu}{2} - B\nu \frac{D^2}{\lambda^2} - c\frac{q^2}{A(q)} \lambda \log^3 \lambda - cq\lambda \log \lambda \log L\right]$$
$$\gg \exp\left\{\frac{L}{2} - \frac{D^2}{\lambda^2} L - \frac{cq^2}{A(q)} \lambda \log^3 L\right\},$$

from which our Lemma immediately follows. ∎

**4.** We formulate the results of Section 3 as:

THEOREM 3. *Assume* $(H)$, $(FR\text{-}P)$ *and*

$$(4.1) \qquad \lambda > 2D, \quad L > \max\left(cq^2 A^{-1}(q) \lambda \log^2 \lambda, \lambda^2\right) .$$

*Then there exist*

$$\mu \in \left[L - \frac{cq^2}{A(q)} \lambda \log^3 \lambda, L\right] \tag{4.2}$$

$$K \in \left[\frac{L - \mu_0}{\lambda^2} - \frac{cq^2}{A(q)} \frac{\log^3 \lambda}{\lambda}, K_0 + \frac{L}{\lambda^2}\right] \tag{4.3}$$

*such that*

$$\frac{1}{2\sqrt{\pi K}} \int_{\exp(\mu - 3\sqrt{\mu K})}^{\exp(\mu + 3\sqrt{\mu K})} \frac{\Delta_4(x)}{x} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx > \exp\left\{\frac{L}{2} - \frac{LD^2}{\lambda^2} - \frac{cq^2}{A(q)} \lambda \log^3 L\right\}. \tag{4.4}$$

PROOF. The thorem follows immediately from Lemmas 1 and 5. ∎

PROOF OF THEOREM 2. We set, with the notations of Theorem 3

$$Y = \exp(L_1), \quad L = L_1\left(1 - \frac{7}{2\lambda}\right). \tag{4.5}$$

Then we obtain, by easy calculations from (4.1)-(4.3) $\sqrt{\mu K} \leqslant 1, 1\ L/\lambda$ and so

$$I \subseteq \left[Y \exp\left(-\frac{7L_1}{\lambda} - \frac{cq^2}{A(q)} \lambda \log^3 \lambda\right), Y\right]. \tag{4.6}$$

We set also

$$A = \max_{x \in I} \frac{\Delta_4(x)}{\sqrt{x}}, \qquad I = [\exp(\mu - 3\sqrt{\mu K}), \exp(\mu + 3\sqrt{\mu K})]. \tag{4.7}$$

By Theorem 3, we have for suitable $\mu, K$ verifying (4.2), (4.3),

$$\frac{A}{2\sqrt{\pi K}} \int_{x \in I} \frac{1}{\sqrt{x}} \left(-\frac{(\mu - \log x)^2}{4K}\right) dx > \int_{x \in I} \frac{\Delta_4(x)}{x} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx \cdot \frac{1}{2\sqrt{\pi K}} > \exp\left\{\frac{L}{2} - \frac{LD^2}{\lambda^2} - \frac{cq^2}{A(q)} \lambda \log^3 L\right\}. \tag{4.8}$$

Since, as it is easily verified,

$$\frac{1}{2\sqrt{\pi K}} \int_0^\infty \frac{1}{\sqrt{x}} \exp\left(-\frac{(\mu - \log x)^2}{4K}\right) dx = \exp\left(\frac{\mu}{2} + \frac{k}{4}\right) \tag{4.9}$$

and $\mu \leqslant L$, inequality (4.8) yields

$$A > \exp\left\{-\frac{2L_1D^2}{\lambda^2} - \frac{cq^2}{A(q)}\lambda\log^3 L_1\right\}. \tag{4.10}$$

In order to optimise the lower bound (4.10), we choose

$$\lambda = cA^{\frac{1}{3}}(q)\,q^{\frac{2}{3}}\log^4 q\,\frac{L_1^{\frac{1}{3}}}{\log L_1} \tag{4.11}$$

which satisfies (4.1), in view of (2.6) and (4.5).

Thus, by (2.1), (4.5), (4.6), (4.10) we obtain

$$A > \exp\left(-\frac{cq^{8/3}\log q}{A^{\frac{2}{3}}(q)}(\log Y)^{\frac{1}{3}}(\log_2 Y)^2\right) \tag{4.12}$$

and

$$I \subset \left[Y\exp\left(-\frac{c(\log Y)^{\frac{2}{3}}\log_2 Y}{(DA(q))^{\frac{1}{3}}}\right), Y\right] \tag{4.13}$$

which proves Theorem 2. ■

Proof of Theorem 1. We set now also (4.5) and similarly we obtain (4.6)-(4.10) from (4.1)-(4.3).

In order to optimise localisation in (4.6) we choose

$$\lambda = \frac{cA^{\frac{1}{2}}(q)}{q}\,\frac{L^{\frac{1}{2}}}{(\log L_1)^{\frac{3}{2}}} > D^2\,, \tag{4.14}$$

in view of (2.1)-(2.2), and this proves also (4.1).

In such a way we obtain by (4.10)

$$I \subset \left[Y\exp\left(-\frac{cL_1}{\lambda}\right), Y\right] \tag{4.15}$$

and, for suitable $x$ in $I$, by (4.10), (4.14) and (4.15)

$$\Delta_4(x) = A\sqrt{x} > \exp\left(-\frac{cL_1}{\lambda}\right)\sqrt{Y}\exp\left(-\frac{cL_1}{\lambda}\right), \tag{4.16}$$

which proves Theorem 1, due to the choice of $\lambda$ in (4.14). ■

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