On the isoperimetric inequality for minimal surfaces


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On the Isoperimetric Inequality for Minimal Surfaces.

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For any compact minimal submanifold of dimension \( k \) in \( \mathbb{R}^n \), it is known that there exists a constant \( \bar{C}_k \) depending only on \( k \), such that

\[
V(\partial M)^{k/(k-1)} > \bar{C}_k V(M),
\]

where \( V(\partial M) \) and \( V(M) \) are the \((k-1)\)-dimensional and \( k\)-dimensional volumes of \( \partial M \) and \( M \) respectively. We refer to [6] for a more detailed reference on the inequality. An open question [6] is to determine the best possible value of \( \bar{C}_k \). When \( M \) is a bounded domain in \( \mathbb{R}^k \subseteq \mathbb{R}^n \), the sharp constant is given by

\[
C_k = \frac{V(\partial D)^{k/(k-1)}}{V(D)},
\]

where \( D \) is the unit disk in \( \mathbb{R}^k \). One speculates that \( C_k \) is indeed the sharp constant for general minimal submanifolds in \( \mathbb{R}^n \).

In the case \( k = 2 \), \( C_2 = 4\pi \), it was proved [1] (see [7]) that if \( \Sigma \) is a simply connected minimal surface in \( \mathbb{R}^n \), then

\[
\ell(\partial \Sigma)^2 > 4\pi A(\Sigma),
\]

where \( \ell(\partial \Sigma) \) and \( A(\Sigma) \) denote the length of \( \partial \Sigma \) and the area of \( \Sigma \) respectively. In 1975, Osserman-Schiffer [5] showed that (2) is valid with a strict inequality for doubly-connected minimal surfaces in \( \mathbb{R}^3 \). Feinberg [2] later generalized this to doubly-connected minimal surfaces in \( \mathbb{R}^n \) for all \( n \). So far, the sharp constant, (1), has been established for minimal surfaces with topological restrictions.

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The purpose of this article is to prove the isoperimetric inequality (2) for those minimal surfaces in $\mathbb{R}^n$ whose boundaries satisfy some connectedness assumption (see Theorem 1). This has the advantage that the topology of the minimal surface itself can be arbitrary. An immediate consequence of Theorem 1 is a generalization of the theorem of Osserman-Schiffer. In fact, Theorem 2 states that any minimal surface (not necessarily doubly-connected) in $\mathbb{R}^3$ whose boundary has at most two connected components must satisfy inequality (2).

Finally, in Theorem 3, we also generalize the non-existence theorem of Hildebrandt [3], Osserman [4], and Osserman-Schiffer [5] to higher codimension.

1. – Isoperimetric inequality.

DEFINITION. The boundary $\partial \Sigma$ of a surface $\Sigma$ in $\mathbb{R}^n$ is weakly connected if there exists a rectangular coordinate system $\{x^a\}_{a=1}^n$ of $\mathbb{R}^n$, such that, for every affine hypersurface $H^{n-1} = \{x^a = \text{const.}\}$ in $\mathbb{R}^n$, $H$ does not separate $\partial \Sigma$. This means, if $H \cap \partial \Sigma = \emptyset$, then $\partial \Sigma$ must lie on one side of $H$.

In particular, if $\partial \Sigma$ is a connected set, then $\partial \Sigma$ is weakly connected.

THEOREM 1. Let $\Sigma$ be a compact minimal surface in $\mathbb{R}^n$. If $\partial \Sigma$ is weakly connected, then

$$l(\partial \Sigma)^2 > 4\pi A(\Sigma).$$

Moreover, equality holds iff $\Sigma$ is a flat disk in some affine 2-plane of $\mathbb{R}^n$.

PROOF. Let us first prove the case when $\partial \Sigma$ is connected. By translation, we may assume that the center of mass of $\partial \Sigma$ is at the origin, i.e.,

$$\int_{\partial \Sigma} x^a = 0, \quad \text{for all } 1 \leq a \leq n.$$  \hfill (3)

By the assumption on the connectedness of $\partial \Sigma$, any coordinate system $\{x^a\}_{a=1}^n$ satisfies the definition of weakly connectedness.

Let $X = (x^1, \ldots, x^n)$ be the position vector, then $|X|^2 = \sum_{a=1}^n (x^a)^2$ must satisfy

$$\Delta(|X|^2) = 4,$$  \hfill (4)

due to the minimality assumption on $\Sigma$. Here $\Delta$ denotes the Laplacian on $\Sigma$ with respect to the induced metric from $\mathbb{R}^n$. Integrating (4) over $\Sigma$, and
applying the divergence theorem, we have

\[ 4A(\Sigma) = 2 \int_{\partial \Sigma} |X| \frac{\partial |X|}{\partial \nu} , \]

where $\partial / \partial \nu$ is the outward unit normal vector to $\partial \Sigma$ on $\Sigma$. Since $\partial |X|/\partial \nu < 1$, we have

\[ 2A(\Sigma) < \int_{\partial \Sigma} |X| < (\partial \Sigma)^{\frac{1}{4}} \left( \int_{\partial \Sigma} (|X|^{2})^{\frac{1}{4}} \right) . \]

In order to estimate the right hand side of (6), we will estimate $\int (x^\alpha)^{2}$ for each $1 < \alpha < n$. By (3), the Poincaré inequality implies that

\[ \int_{\partial \Sigma} (x^\alpha)^{2} \leq \frac{l(\partial \Sigma)^{2}}{4\pi^{2}} \int_{\partial \Sigma} \left( \frac{dx^\alpha}{ds} \right)^{2} , \]

where $d/ds$ is differentiation with respect to arc-length. Combining with (6) yields

\[ 4\pi A(\Sigma) \leq l(\partial \Sigma)^{\frac{1}{4}} \left( \int_{\partial \Sigma} \left| \frac{dx}{ds} \right|^{2} \right)^{\frac{1}{4}} = l(\partial \Sigma)^{\frac{1}{2}} , \]

because $\left( dX/ds \right)$ is just the unit tangent vector to $\partial \Sigma$.

Equality holds at (8), implies

\[ \frac{\partial |X|}{\partial \nu} = 1 \]

\[ |X| = \text{constant} = R \]

and equality at (7). The latter implies that

\[ \alpha^\alpha = a^\alpha \sin \frac{2\pi s}{l(\partial \Sigma)} + b^\alpha \cos \frac{2\pi s}{l(\partial \Sigma)} \]

where $a^\alpha$ and $b^\alpha$’s are constants for all $1 < \alpha < n$. By rotation, we may assume that

\[ \begin{align*}
X(0) &= (R, 0, 0, \ldots, 0) \\
\frac{dX}{ds}(0) &= (0, 1, 0, \ldots, 0),
\end{align*} \]
because (10) implies that $\partial \Sigma$ lies on the sphere of radius $R$. Evaluating (11) at $s = 0$, we deduce that
\[ b_1 = R, \quad b_\alpha = 0 \quad \text{for } 2 < \alpha < n \]
(13) and
\[ a_\alpha = \frac{l(\partial \Sigma)}{2\pi}, \quad a_\alpha = 0 \quad \text{for } \alpha \neq 2. \]

On the other hand, summing over $1 < \alpha < n$ on (7), we derive
\[ R^2 l(\partial \Sigma) = \int_{\delta\Sigma} |X|^2 = \left( \frac{l(\partial \Sigma)}{2\pi} \right)^2 l(\partial \Sigma), \]
(14)
Hence
\[ R = \frac{l(\partial \Sigma)}{2\pi}. \]

Combining with (13), (11) becomes

\[
\begin{cases}
  x^1 = R \cos \left( \frac{s}{R} \right) \\
  x^2 = R \sin \left( \frac{s}{R} \right)
\end{cases}
\]
(15)
and
\[ x^\alpha = 0 \quad \text{for } 3 < \alpha < n. \]

This implies $\partial \Sigma$ is a circle on the $x^1x^2$-plane centered at the origin of radius $R$. Equation (9) shows that $\Sigma$ is tangent to the $x^1x^2$-plane along $\partial \Sigma$. By the Hopf boundary lemma, this proves that $\Sigma$ must be the disk spanning $\partial \Sigma$.

For the general case when $\partial \Sigma$ is not connected. Let $\partial \Sigma = \bigcup_{i=1}^p \sigma_i$, where $\sigma_i$'s are connected closed curves. By the assumption on weakly connectedness, we may choose $\{x^3\}_{i=1}^n$ to be the appropriate coordinate system. For any fixed $1 < \alpha < n$, we claim that there exist translations $A_i^\sigma$, $2 < i < p$, generated by vectors $v_i^\sigma$ perpendicular to $\partial \sigma$, such that the union of the set of translated curves $\{A_i^\sigma\}_{i=1}^p$ together with $\sigma_1$ form a connected set. We prove the claim by induction on the number of curves, $p$. When $p = 2$, we observe that since no planes of the form $x^2 = \text{constant}$ separates $\sigma_1$ and $\sigma_2$, this is equivalent to the fact that there exists a number $x$, such that the plane $H = \{x^2 = x\}$ must intersect both $\sigma_1$ and $\sigma_2$. Let $q_1$ and $q_2$ be the points of intersection between $H$ with $\sigma_1$ and $\sigma_2$ respectively.
Clearly one can translate $q_2$ along $H$ to $q_1$. Denote this by $A_2^a$, and $\sigma_1 \cup A_2^a \sigma_2$ is connected now. For general $p$, we consider the set of numbers defined by

$$y_i = \max \{|x^a|_{a_i}\}.$$  

Without loss of generality, we may assume $y_1 < y_2 < \ldots < y_p$: Now we claim that the set $\bigcup_{i=2}^{p} \sigma_i$ cannot be separated by hyperspaces of the form $H = \{x^a = \text{constant}\}$. If so, say $H = \{x^a = x\}$ separates $\bigcup_{i=1}^{p} \sigma_i$, then $x$ must be in the range of $|x^a|_{a_i}$. This is because $\bigcup_{i=1}^{p} \sigma_i$ cannot be separated hence $H \cap \sigma_1 \neq \emptyset$. On the other hand, since $H$ separates $\bigcup_{i=2}^{p} \sigma_i$, this means there exists some $\sigma_i$, $2 < i < p$, lying on the left of $H$, hence $y_i < x < y_1$, for some $2 < i < p$, which is a contradiction. By induction, there exist translations, $A_i^a$, $3 < i < p$, perpendicular to $\partial/\partial x^a$ such that $\sigma = \sigma_1 \cup \bigcup_{i=3}^{p} A_i^a \sigma_i$ is connected. However, $\bigcup_{i=1}^{p} \sigma_i$ is non-separable by $H = \{x^a = \text{constant}\}$ implies $\sigma_1 \cup \sigma$ is non-separable also. Hence, there exists a translation $A^a$ perpendicular to $\partial/\partial x^a$, such that $\sigma_1 \cup A^a \sigma_2$ is connected. The set $A = A_2^a, A^a_3, A^a_4, \ldots, A^a_p$ gives the desired translations. Notice that since all translations are perpendicular to $\partial/\partial x^a$, then

$$|x^a|_{a_i} = |x^a|_{A^a \sigma_i}, \quad \text{for all } i.$$  

By the connectedness of $\sigma = \sigma_1 \cup A^a_2 \sigma_2 \cup \ldots \cup A^a_p \sigma_p$: we can view $\sigma^a$ as a Lipschitz curve in $\mathbb{R}^n$. Clearly

$$\int_{\sigma} x^a = \sum_{i=1}^{p} \int_{\sigma_i} x^a = 0,$$

hence the Poincaré inequality can be applied to yield

$$\sum_{i=1}^{p} \int_{\sigma_i} (x^a)^2 = \int_{\sigma} (x^a)^2 \leq \frac{l(\partial \Sigma)^2}{4 \pi^2} \int_{\sigma^a} \left(\frac{dx^a}{ds}\right)^2 = \frac{l(\partial \Sigma)^2}{4 \pi^2} \sum_{i=1}^{p} \int_{\sigma_i} \left(\frac{dx^a}{ds}\right)^2.$$  

Summing over all $1 < a < n$ and proceeding as the connected case we derived the inequality (8).

When equality occurs, we will show that $\partial \Sigma$ is actually connected, and hence by the previous argument it must be a circle and $\Sigma$ must be a disk. To see this, we observe that (10) still holds on $\partial \Sigma$. In particular, we may
assume that \( X(0) \) is a point on \( \sigma_1 \), and (12) is valid. However, Poincaré
inequality is now applied on \( \sigma^x \) instead of \( \partial \Sigma \), therefore equation (11) only
applies to the curve \( \sigma^x \). On the other hand, since \( X(0) \in \sigma_1 \), and \( \sigma^x = \sigma_1 \)
\( \cup \left\{ \bigcup_{i=2}^p A_i^+ \sigma_i \right\} \), the argument concerning the coefficients \( a_i \) and \( b_i \)'s is still
valid. Equations (15) can still be concluded on each \( \sigma^x \), hence on \( \partial \Sigma \), by (17).
This implies \( \partial \Sigma \) is a circle, and the Theorem is proved.

**Theorem 2.** Let \( \Sigma \) be a compact minimal surface in \( \mathbb{R}^3 \). If \( \partial \Sigma \) consists
of at most two components, then

\[
l(\partial \Sigma)^2 > 4\pi A(\Sigma).
\]

Moreover, equality holds iff \( \Sigma \) is a flat disk in some affine 2-plane of \( \mathbb{R}^3 \).

**Proof.** In view of Theorem 1, it suffices to prove that when \( \partial \Sigma = \sigma_1 \cup \sigma_2 \)
has exactly two connected components and is not weakly connected, \( \Sigma \) must
be disconnected into two components \( \Sigma_1 \) and \( \Sigma_2 \) with \( \partial \Sigma_1 = \sigma_1 \) and \( \partial \Sigma_2 = \sigma_2 \).
Indeed, if this is the case, we simply apply Theorem 1 to \( \Sigma_1 \) and \( \Sigma_2 \) sepa-
rately and derive

\[
l(\partial \Sigma)^2 = l(\sigma_1)^2 + l(\sigma_2)^2
\geq 4\pi A(\Sigma_1) + A(\Sigma_2)
= 4\pi A(\Sigma).
\]

In this case, equality will never be achieved for (2).

To prove the above assertion, we assume that \( \partial \Sigma = \sigma_1 \cup \sigma_2 \) is not weakly
connected. This implies, there exists an affine plane \( P_1' \) in \( \mathbb{R}^3 \) separating \( \sigma_1 \)
and \( \sigma_2 \). For any oriented affine 2-plane in \( \mathbb{R}^3 \) must be divided into two open
half-spaces. Defining the sign of these half-spaces in the manner correspond-
ing to the orientation of the 2-plane, we consider the sets \( S_i^+ \) (or \( S_i^- \)) as fol-
lows: a 2-plane \( P \) is said to be in \( S_i^+ \) (or \( S_i^- \)) for \( i = 1 \) or \( 2 \), if \( \sigma_i \) is contained
in the positive (or negative) open half-space defined by \( P \). Obviously,
\( P_1' \in S_1^+ \cap S_2^- \) for a fixed orientation of \( P_1' \). However, by the compactness
of \( \partial \Sigma = \sigma_1 \cup \sigma_2 \), \( S_1^+ \cap S_2^- \neq \emptyset \) and \( S_1^- \cap S_2^+ \neq \emptyset \). Hence \( \partial S_1^+ \cap \partial S_2^- \neq \emptyset \), by
virtue of the fact that both \( S_1^+ \) and \( S_2^- \) are connected sets. This gives a 2-plane
in \( \mathbb{R}^3 \), \( P_1 \), which has the property that \( \sigma_1 \) (and \( \sigma_2 \)) lies in the closed positive
(respectively negative) half-space defined by \( P_1 \). Moreover, both the sets
\( \sigma_1 \cap P_1 \) and \( \sigma_2 \cap P_1 \) are nonempty.

By the assumption that \( \partial \Sigma \) is not weakly connected and since \( P_1 \) does
not separate \( \sigma_1 \) and \( \sigma_2 \), there exists an affine 2-plane in \( \mathbb{R}^3 \), \( P_2' \), which is per-
pendicular to \( P_1 \) and separating \( \sigma_1 \) and \( \sigma_2 \). Let us define \( \overline{S} \) to be the set of
oriented affine 2-planes in \( \mathbb{R}^n \) which are perpendicular to \( P_i \). Setting \( \overline{S}_i^+ \) (or \( \overline{S}_i^- \)) to be \( S_i^+ \cap \overline{S} \) (or \( S_i^- \cap \overline{S} \)), and as before, we conclude that \( \partial \overline{S}_i^+ \cap \partial \overline{S}_i^- \neq \emptyset \). Hence, there exists an affine 2-plane, \( P_2 \), perpendicular to \( P_1 \), and having the property that \( \sigma_i \) (and \( \sigma_a \)) lie in the closed positive (respectively negative) half-space defined by \( P_2 \) and both sets \( \sigma_i \cap P_2 \) and \( \sigma_a \cap P_2 \) are nonempty.

Arguing once more that \( P_1 \) and \( P_2 \) do not separate the \( \sigma_i \)'s, there must be an affine 2-plane \( P_3 \) perpendicular to both \( P_1 \) and \( P_2 \). Moreover, \( P_3 \) must separate \( \sigma_1 \) and \( \sigma_a \) by the assumption the \( \partial \Sigma \) is not weakly connected. We defined a rectangular coordinate system \( xyz \) such that \( P_1, P_2 \) and \( P_3 \) are the \( xy, yz, \) and \( xz \) planes respectively. Clearly by the properties of the 2-planes \( P_i \)'s, \( \sigma_1 \) and \( \sigma_a \) are contained in the closed octant \( \{x > 0, y > 0, z > 0\} \) and the closed octant \( \{x < 0, y < 0, z < 0\} \) respectively. In particular, \( \sigma_1 \) is contained in the cone defined by \( C_1 = \{ X \in \mathbb{R}^4 | X \cdot V > |X|/\sqrt{3} \} \) and \( \sigma_a \) is contained in the cone \( C_2 = \{ X \in \mathbb{R}^4 | X < -|X|/\sqrt{3} \} \). However, one verifies that the two cones \( C_i, i = 1, 2 \), are contained in the positive and negative cones defined by the catenoid obtained from rotating the catenary along the line given by \( V \). In view of Theorem 6 in [4], the minimal surface \( \Sigma \) must be disconnected. This concludes our proof.

2. Nonexistence.

Let \( (x^1, \ldots, x^n) \) be a rectangular coordinate system in \( \mathbb{R}^n \). We consider the \( (n - 1) \)-dimensional surface of revolution \( S_a \) obtained by rotating the catenary \( x^{n-1} = \cosh (x^n/a) \) around the \( x^n \)-axis. One readily computes that its principal curvatures are

\[
\left( \cosh^{-1}(z/a), -\cosh^{-1}(z/a), -\cosh^{-1}(z/a), \ldots, -\cosh^{-1}(z/a) \right) \\
(n - 2) \text{ copies}
\]

with respect to the inward normal vector (i.e. the normal vector pointing towards the \( x^n \)-axis). The set of hypersurfaces \( \{S_a\}_{a \geq 0} \) defines a cone in \( \mathbb{R}^n \) as in the case when \( n = 3 \) (see [4]). This cone (positive and negative halves) is given by

\[
C = \{ (x^1, \ldots, x^n) \in \mathbb{R}^n | (x^1)^2 + \ldots + (x^{n-1})^2 < (x^n)^2 \sinh^2 \tau \}
\]

where \( \tau \) is the unique positive number satisfying \( \cosh \tau - \tau \sinh \tau = 0 \). If \( \Sigma \) is a compact connected minimal surface in \( \mathbb{R}^n \) with boundary decomposed into \( \partial \Sigma = \sigma_1 \cup \sigma_2 \), where \( \sigma_1 \) and \( \sigma_2 \) (each could have more than one connected component) lie inside the positive and negative part of \( C \) respect-
ively, then arguing as in [5], $\Sigma$ must intersect one of the surfaces $S_a$ tangentially. Moreover, $\Sigma$ must lie in the interior (the part containing the $x^n$-axis) of $S_a$, except at those points of intersection. This violates the maximum principle since $\Sigma$ is minimal and any 2-dimensional subspace of the tangent space of $S_a$ must have nonpositive mean curvature. Hence $\Sigma$ must be disconnected. This gives the following:

**THEOREM 3.** Let $C^+$ and $C^-$ be the positive and negative halves of the cone in $\mathbb{R}^n$ defined by (18). Suppose $\Sigma$ is a minimal surface spanning its boundary $\partial \Sigma = \sigma_1 \cup \sigma_2$. If $\sigma_1 \subset C^+$ and $\sigma_2 \subset C^-$, then $\Sigma$ must be disconnected.

We remark that using similar arguments, one can use surfaces of revolution having principal curvatures of the form $(k\lambda, -\lambda, -\lambda, -\lambda, ..., -\lambda)$ $(n - 2)$ copies as barrier to yield nonexistence type theorems for $(k + 1)$-dimensional minimal submanifolds in $\mathbb{R}^n$.

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