

ANNALI DELLA SCUOLA NORMALE SUPERIORE DI PISA *Classe di Scienze*

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 11, n° 1 (1984), p. 57-85

http://www.numdam.org/item?id=ASNSP_1984_4_11_1_57_0

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The Continuity of the Rearrangement in $W^{1,p}(\mathbb{R})$.

J. M. CORON

1. – Introduction.

Let, in the following, p be a real number such that $1 < p < +\infty$. Let u be a nonnegative function of $W^{1,p}(\mathbb{R})$. Let u^* be the rearrangement of u , that is the unique function u^* which is even, nonincreasing on $[0, +\infty]$ and such that:

for all $y \in \mathbb{R}$ $\text{meas} \{x | u^*(x) \geq y\} = \text{meas} \{x | u(x) \geq y\}$ ($\text{meas } A$ stands for the Lebesgue measure of A).

We know (see, for example [1] appendix 1, [2], [3], [4] p. 154, [5], [6], [7] and [8]) that u^* is in $W^{1,p}(\mathbb{R})$ and:

$$(1) \quad \int_{\mathbb{R}} \left| \frac{du^*}{dx} \right|^p dx \leq \int_{\mathbb{R}} \left| \frac{du}{dx} \right|^p dx.$$

Let $W_+^{1,p}(\mathbb{R})$ be the set of nonnegative functions of $W^{1,p}(\mathbb{R})$; the weak and the strong topologies of $W^{1,p}(\mathbb{R})$ induce two topologies on $W_+^{1,p}(\mathbb{R})$; we shall also call them weak and strong topologies respectively.

Let c be a positive real number and let:

$$\Phi_c(u) = \int_{\mathbb{R}} \left| \frac{du}{dx} \right|^p dx - c \int_{\mathbb{R}} \left| \frac{du^*}{dx} \right|^p dx, \quad u \in W_+^{1,p}(\mathbb{R}).$$

The purpose of this article is to prove the following theorem:

THEOREM. Φ_c is weakly l.s.c. if and only if $c \leq 1/2^p$.

COROLLARY. The rearrangement is a continuous mapping from $W_+^{1,p}(\mathbb{R})$ into $W_+^{1,p}(\mathbb{R})$ for the strong topologies.

Pervenuto alla Redazione il 27 Dicembre 1982 ed in forma definitiva il 27 Aprile 1983.

PROOF OF COROLLARY. Let $u_n \in W_+^{1,p}(\mathbf{R})$, $u_n \rightarrow u$ in $W^{1,p}(\mathbf{R})$.

Since the rearrangement is a continuous mapping from the set of non-negative functions of $L^p(\mathbf{R})$ into $L^p(\mathbf{R})$ (see appendix 0) we have:

$$u_n^* \rightarrow u^* \quad \text{in } L^p(\mathbf{R}).$$

Therefore, using (1), we have $u_n^* \rightharpoonup u^*$ in $W^{1,p}(\mathbf{R})$ weakly. Let $c \in (0, 1/2^p]$.

$$\Phi_c(u) \leq \liminf \Phi_c(u_n).$$

But

$$\int_{\mathbf{R}} \left| \frac{du_n}{dx} \right|^p dx \rightarrow \int_{\mathbf{R}} \left| \frac{du}{dx} \right|^p dx$$

hence

$$\overline{\lim} \int_{\mathbf{R}} \left| \frac{du_n^*}{dx} \right|^p dx \leq \int_{\mathbf{R}} \left| \frac{du^*}{dx} \right|^p dx$$

and therefore (since $1 < p < +\infty$ and $u_n^* \rightharpoonup u^*$ in $W^{1,p}(\mathbf{R})$)

$$u_n^* \rightarrow u^* \quad \text{in } W^{1,p}(\mathbf{R}).$$

The proof of the theorem will be divided in two parts.

In part *A* we assume that $c \leq 1/2^p$ and we prove that Φ_c is weakly l.s.c.. In part *B* we assume that $c > 1/2^p$ and we construct a sequence u_n such that $u_n \rightharpoonup u$ in $W_+^{1,p}(\mathbf{R})$ and $\Phi_c(u) > \lim \Phi_c(u_n)$.

I thank H. Brezis, T. Gallouet, E. Lieb and L. Nirenberg who initiate this work.

2. – Proof of the theorem.

Part A. Here we assume that $c \leq 1/2^p$ and we prove that Φ_c is weakly l.s.c. Let $f \in W^{1,p}(\mathbf{R})$, we shall use the following notation

$$|f| = \left(\int_{\mathbf{R}} \left| \frac{df}{dx} \right|^p dx \right)^{1/p}.$$

Let u_n be a sequence of functions in $W_+^{1,p}(\mathbf{R})$ such that

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\mathbf{R}) \text{ when } n \rightarrow +\infty.$$

If $u = 0$, we have:

$$\Phi_c(u) \leq \liminf \Phi_c(u_n) \quad \text{since } \Phi_c \geq 0.$$

Therefore we may assume that $u \neq 0$.

Let v be in $W^{1,p}(\mathbf{R})$ and let:

$$V(v) = \{y \in \mathbf{R} \mid \text{there exists } x \text{ in } u^{-1}(y) \text{ such that either } v \text{ is not differentiable in } x \text{ or } v \text{ is derivable in } x \text{ and } v'(x) = 0\}.$$

One can prove (see appendix 1) that $V(v)$ is negligible for the Lebesgue measure (this is a little modification of Sard's theorem). Let $\eta > 0$; since $V(u)$ is negligible, there exist m and M , real numbers, such that

$$(2) \quad m \notin V(u), \quad M \notin V(u), \quad 0 < m < M$$

$$(3) \quad M < \max_{x \in \mathbf{R}} u(x)$$

and if

$$g(x) = \min(u(x), m)$$

$$f(x) = \max(u(x), M) - M$$

we have:

$$(4) \quad |g|^p \leq \eta, \quad |f|^p \leq \eta.$$

Let:

$$g_n(x) = \min(u_n(x), m)$$

$$f_n(x) = \max(u_n(x), M) - M$$

$$\bar{u}(x) = \max(\min(u(x), M), m) - m$$

$$\bar{u}_n(x) = \max(\min(u_n(x), M), m) - m.$$

$\bar{u}$ and $\bar{u}_n$ are in $W_+^{1,p}(\mathbf{R})$ and:

$$\bar{u}_n \rightharpoonup \bar{u} \quad \text{in } W^{1,p}(\mathbf{R}) \text{ when } n \rightarrow +\infty.$$

For the moment being let us assume that:

$$(5) \quad \Phi_c(\bar{u}) \leq \liminf_{n \rightarrow +\infty} \Phi_c(\bar{u}_n);$$

we have:

$$\Phi_c(u) = \Phi_c(\bar{u}) + \Phi_c(g) + \Phi_c(f)$$

$$\Phi_c(u_n) = \Phi_c(\bar{u}_n) + \Phi_c(g_n) + \Phi_c(f_n).$$

Using (4), (1) and (5), this yields

$$\Phi_\varepsilon(u) \leq \lim_{n \rightarrow +\infty} \Phi_\varepsilon(u_n) + 2\eta$$

and the theorem is proved.

It remains to prove (5); without any restriction we may assume that

$$\text{Max}_{x \in \mathbf{R}} u_n(x) > M. \text{ Let } \bar{M} = M - m.$$

Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ be a sequence of r strictly positive numbers (r depends on ε) such that:

$$\sum_{i=1}^r \varepsilon_i = \bar{M}$$

Let

$$A(\varepsilon) = \left\{ \sum_{i=1}^k \varepsilon_i \mid 1 \leq k \leq r-1 \right\}$$

$$\tilde{A}(\varepsilon) = A(\varepsilon) \cup \{0, \bar{M}\}.$$

We are going to define by induction a finite sequence of real numbers. Let

$$a_1 = \inf \{x \mid \bar{u}(x) \neq 0\}$$

(it is easy to see that a_1 exists). Assume that a_{i-1} is defined. Either:

$$\{x \mid \bar{u}(x) \in \tilde{A}(\varepsilon) - \{\bar{u}(a_{i-1})\}\} \cap [a_{i-1}, +\infty) = \emptyset$$

then we stop here the sequence a_j ; we have $\bar{u}(a_{i-1}) = 0$ and:

$$\bar{u}(x) < \varepsilon_1 \quad \forall x \in [a_{i-1}, +\infty)$$

or:

$$\{x \mid \bar{u}(x) \in \tilde{A}(\varepsilon) - \{\bar{u}(a_{i-1})\}\} \cap [a_{i-1}, +\infty) \neq \emptyset,$$

then we let:

$$a_i = \min \{x \mid \bar{u}(x) \in \tilde{A}(\varepsilon) - \{\bar{u}(a_{i-1})\} \text{ and } x \geq a_{i-1}\}.$$

We are going to prove that the sequence a_i has only a finite number of terms.

Let

$$\mu = \min_{1 \leq j \leq r} \varepsilon_j; \quad \mu > 0.$$

We have

$$\mu \leq |\bar{u}(a_{i+1}) - \bar{u}(a_i)|$$

but

$$|\bar{u}(a_{i+1}) - \bar{u}(a_i)| \leq \int_{a_i}^{a_{i+1}} |\bar{u}'(\tau)| d\tau \leq |\bar{u}|(a_{i+1} - a_i)^{1/q}$$

with

$$\frac{1}{p} + \frac{1}{q} = 1$$

therefore:

$$(6) \quad \mu \leq (a_{i+1} - a_i)^{1/q} |\bar{u}|.$$

Let $b = \sup \{x | \bar{u}(x) \neq 0\}$; $b < +\infty$ and

$$(7) \quad \forall i \quad a_i \leq b$$

then using (6) and (7) we see that the sequence (a_i) has only a finite number of terms. Let l be the number of terms of the sequence a_i . With $\bar{u}$ and the sequence a_i we are going to define a new function in $W_+^{1,p}(\mathbb{R})$ $P_\varepsilon \bar{u}$ as follows:

when $x \geq a_l$ let $(P_\varepsilon \bar{u})(x) = 0$

when $x \leq a_1$ let $(P_\varepsilon \bar{u})(x) = 0$

when $a_i < x \leq a_{i+1}$:

— either $\bar{u}(a_i) < \bar{u}(a_{i+1})$ then we let:

$$(P_\varepsilon \bar{u})(x) = \max_{y \in [a_i, x]} \bar{u}(y)$$

— or $\bar{u}(a_i) > \bar{u}(a_{i+1})$ then we let:

$$(P_\varepsilon \bar{u})(x) = \min(\bar{u}(a_i), \max_{y \in [x, a_{i+1}]} \bar{u}(y)).$$

It is easy to see that $P_\varepsilon \bar{u}$ is a continuous function; using appendix 2 we see that $P_\varepsilon \bar{u}|_{[a_i, a_{i+1}]} \in W^{1,p}((a_i, a_{i+1}))$ and

$$\int_{a_i}^{a_{i+1}} |(P_\varepsilon \bar{u})'|^p dx = \int_{a_i}^{a_{i+1}} |(P_\varepsilon \bar{u})'| |\bar{u}'|^{p-1} dx.$$

Thus $P_\varepsilon \bar{u} \in W_+^{1,p}(\mathbb{R})$ and

$$(8) \quad |P_\varepsilon \bar{u}|^p = \int_{\mathbb{R}} |(P_\varepsilon \bar{u})'| |\bar{u}'|^{p-1} dx .$$

We are now going to define a_i^n and $P_\varepsilon \bar{u}_n$;

Let δ_0 be such that

$$u(a_1 - \delta_0) < m$$

$$u(a_l + \delta_0) < m$$

let

$$a_1^n = \inf \{x | \bar{u}_n(x) \neq 0 \text{ and } a_1 - \delta_0 \leq x \leq a_l + \delta_0\}$$

a_1^n exists for n large enough and, always for n large enough,

$$\bar{u}_n(a_1^n) = 0 .$$

Let us assume that a_{i-1}^n is defined.

Either:

$$\{x | \bar{u}_n(x) \in \tilde{A}(\varepsilon) - \{\bar{u}_n(a_{i-1})\}\} \cap [a_{i-1}^n, a_l + \delta_0] \neq \emptyset$$

then we stop here the sequence a_i^n we have $a_{i+1}^n \leq a_l + \delta_0$ and for n large enough (i.e. if $u_n(a_1 + \delta_0) < m$):

$$\bar{u}_n(a_{i-1}^n) = 0 ,$$

or:

$$\{x | \bar{u}_n(x) \in \tilde{A}(\varepsilon) - \{u_n(a_{i-1}^n)\}\} \cap [a_{i-1}^n, a_l + \delta_0] \neq \emptyset$$

and then we set

$$a_i^n = \min \{x | \bar{u}_n(x) \in \tilde{A}(\varepsilon) - \{\bar{u}_n(a_{i-1})\} \text{ and } x \in [a_{i-1}^{i-1}, a_l + \delta_0]\} .$$

In the same way as for the sequence a_i , one can prove that the sequence a_i^n has only a finite number of terms and we define $P_\varepsilon \bar{u}$ from $(a_i^n)_i$ and $\bar{u}_n$ in the same way we have defined $P_\varepsilon \bar{u}$ from $(a_i)_i$ and $\bar{u}$. Let us remark that:

$$P_\varepsilon \bar{u}_n \in W^{1,p}(\mathbb{R})$$

and

$$\text{Supp } P_\varepsilon \bar{u}_n \subset [a_1 - \delta_0, a_l + \delta_0] .$$

We are going to prove:

$$(9) \quad P_\varepsilon \bar{u} \rightarrow \bar{u} \quad \text{in } W^{1,p}(\mathbf{R}) \quad \text{when } |\varepsilon| \rightarrow 0$$

$$(10) \quad (P_\varepsilon \bar{u})^* \rightarrow (\bar{u})^* \quad \text{in } W^{1,p}(\mathbf{R}) \quad \text{when } |\varepsilon| \rightarrow 0$$

$$(11) \quad \Phi_c(P_\varepsilon \bar{u}_n) \leq \Phi_c(\bar{u}_n)$$

$$(12) \quad \text{If } A(\varepsilon) \cap V(\bar{u}) = \emptyset \text{ then:}$$

$$\Phi_c(P_\varepsilon \bar{u}) \leq \lim_{n \rightarrow +\infty} \Phi_c(P_\varepsilon \bar{u}_n).$$

Before proving (9), (10), (11) and (12) we are going to explain how from (9), (10), (11) and (12) we can deduce (5). Let $\gamma > 0$; since $V(u)$ is negligible, from (9) and (10) we deduce that there exists a sequence $\varepsilon = (\varepsilon_i)_{1 \leq i \leq r}$ of strictly positive numbers with $\sum_{i=1}^r \varepsilon_i = \bar{M}$ such that

$$A(\varepsilon) \cap V(\bar{u}) = \emptyset$$

and:

$$(13) \quad \Phi_c(P_\varepsilon \bar{u}) \geq \Phi_c(\bar{u}) - \gamma.$$

Using (11) and (12) we have:

$$(14) \quad \Phi_c(P_\varepsilon \bar{u}) \leq \lim_{n \rightarrow +\infty} \Phi_c(\bar{u}_n).$$

We use (13) and (14); we obtain

$$\Phi_c(\bar{u}) - \gamma \leq \lim_{n \rightarrow +\infty} \Phi_c(\bar{u}_n) \quad \forall \gamma > 0$$

which establishes (5).

It remains to prove (9), (10), (11), (12).

PROOF OF (9). (8) yields:

$$(15) \quad |P_\varepsilon \bar{u}| < |\bar{u}|.$$

But there exists α in $\mathbf{R}$ such that

$$\text{Supp } \bar{u} \subset [-\alpha, \alpha].$$

Then we have:

$$(16) \quad \text{Supp } P_\varepsilon \bar{u} \subset [-\alpha, \alpha].$$

From (15) and (16) it follows that $P_\varepsilon \bar{u}$ is bounded in $W^{1,p}(\mathbf{R})$. But it is easy to see that:

$$\|P_\varepsilon \bar{u} - \bar{u}\|_\infty \leq 2\varepsilon.$$

Then using (15) we have (9).

PROOF of (10). Since the rearrangement is a continuous mapping from the set of nonnegative functions of $L^p(\mathbf{R})$ into $L^p(\mathbf{R})$ it follows from (9) and (1) that (since $\exists c|\text{Supp } P_\varepsilon \bar{u} \subset [-c, c]$):

$$(17) \quad (P_\varepsilon \bar{u})^* \rightharpoonup \bar{u}^* \quad \text{in } W^{1,p}(\mathbf{R}) \quad \text{when } |\varepsilon| \rightarrow 0.$$

We are going to prove that:

$$(18) \quad \lim_{|\varepsilon| \rightarrow 0} |(P_\varepsilon \bar{u})^*| = |\bar{u}^*|.$$

Clearly (10) follows from (17) and (18).

Let ε^k with $|\varepsilon^k| \rightarrow 0$ when $k \rightarrow +\infty$.

Let

$$\bar{u}^k = P_{\varepsilon^k} \bar{u}$$

$$v^k(y) = -\text{meas} \{x | \bar{u}^k(x) \geq y\}$$

$$v(y) = -\text{meas} \{x | \bar{u}(x) \geq y\}.$$

We have (see appendix 3):

$$(19) \quad |(\bar{u}^k)^*|^p = 2^p \int_0^{\bar{M}} \frac{1}{[(v^k)'(y)]^{p-1}} dy$$

$$(20) \quad |\bar{u}^*|^p = 2^p \int_0^{\bar{M}} \frac{1}{(v'(y))^{p-1}} dy.$$

We are going to prove:

$$(21) \quad \text{there exists a function } h \text{ of } L^1((0, \bar{M})) \text{ such that}$$

$$\frac{1}{[(v^k)'(y)]^{p-1}} \leq h(y) \quad \text{a.e. } y \in (0, \bar{M})$$

$$(22) \quad (v^k)'(y) \xrightarrow{(k \rightarrow +\infty)} v'(y) \quad \text{a.e. } y \in (0, \bar{M}).$$

Clearly (18) follows from (19), (20), (21) and (22).

PROOF OF (21) AND (22). Let

$$C =]0, \bar{M}[- \left(\bigcup_{k \in \mathbb{N}} V(\bar{u}^k) \cup V(\bar{u}) \bigcup_{k \in \mathbb{N}} A(\varepsilon^k) \right).$$

$[0, \bar{M}] - C$ is negligible. Let $y \in C$. Using appendix 4 we see that v^k is differentiable in y and:

$$(v^k)'(y) = \sum_{x \in (\bar{u}^k)^{-1}(y)} \frac{1}{|(\bar{u}^k)'(x)|}$$

(remark: since $y \in C$, $(\bar{u}^k)^{-1}(y)$ is a finite set)

Then, using the convexity of t^{1-p} we have

$$(23) \quad \frac{1}{[(v^k)'(y)]^{p-1}} \leq \sum_{x \in (\bar{u}^k)^{-1}(y)} |(\bar{u}^k)'(x)|^{p-1}.$$

Let

$$h^k(y) = \sum_{x \in \bar{u}^{k-1}(y)} |(\bar{u}^k)'(x)|^{p-1}$$

On $[a_i, a_{i+1}]$ $\bar{u}^k$ is monotone; let θ_i^k be the unique function from $\bar{u}^k([a_i, a_{i+1}]) \cap C$ into $[a_i, a_{i+1}]$ such that:

$$\bar{u}^k \circ \theta_i^k = Id_{C \cap \bar{u}^k([a_i, a_{i+1}])}.$$

We have:

$$\int_a^{a_{i+1}} |(\bar{u}^k)'(x)|^p dx = \int_{\bar{u}^k([a_i, a_{i+1}]) \cap C} |(\bar{u}^k)'(\theta_i^k(y))|^{p-1} dy.$$

Then it is easy to see that h^k is a measurable function and that

$$\int_0^{\bar{M}} h^k(y) dy = |\bar{u}^k|^p$$

but $(\bar{u}^k)' \rightarrow \bar{u}'$ in $L^p(\mathbb{R})$ when $k \rightarrow +\infty$, and thus

$$\int_0^{\bar{M}} h^k(y) dy \rightarrow |\bar{u}|^p \quad (k \rightarrow +\infty).$$

Using Fatou's lemma we obtain

$$(24) \quad \int_0^{\bar{M}} \liminf_k h^k(y) dy \leq |\bar{u}|^p.$$

Let

$$h(y) = \sum_{x \in \bar{u}^{-1}(y)} |\bar{u}'(x)|^{p-1}.$$

We are going to prove that

$$(25) \quad \text{if } y \in C, (\bar{u}^k)^{-1}(y) \subset \bar{u}^{-1}(y)$$

and if $x \in (\bar{u}^k)^{-1}(y)$ then $\bar{u}'(x) = (\bar{u}^k)'(x)$

$$(26) \quad \text{if } y \in C, \text{ for } k \text{ sufficiently large we have}$$

$$(\bar{u}^k)^{-1}(y) = \bar{u}^{-1}(y).$$

Before proving (25) and (26) we are going to deduce (21) and (22) from (25) and (26).

Using (25) we have:

$$h^k(y) \leq h(y)$$

Using (25) and (26) $h^k(y) \rightarrow h(y)$ ($k \rightarrow +\infty$) $\forall y \in C$.

Using (24)

$$\int_0^{\bar{M}} h(y) dy \leq |\bar{u}|^p$$

which gives (20).

(22) follows from (25), (26) and appendix 4.

PROOF OF (25). Let x be in $(\bar{u}^k)^{-1}(y)$, $a_i < x < a_{i+1}$; let us assume that, for example, $\bar{u}(a_i) < \bar{u}(a_{i+1})$ (the proof in the case $\bar{u}(a_i) > \bar{u}(a_{i+1})$ would be nearly the same).

Let z be in $[a_i, a_{i+1}]$

$$\bar{u}^k(z) = \text{Max}_{y \in [a_i, z]} \bar{u}(y).$$

We have $\bar{u}^k(x) \geq \bar{u}(x)$; but if $\bar{u}^k(x) > \bar{u}(x)$ it is easy to see that $(\bar{u}^k)'(x) = 0$ in contradiction with $y \in C$ therefore $\bar{u}^k(x) = \bar{u}(x)$. We recall that $\bar{u}$ and $\bar{u}^k$ are differentiable in x (since $y \in C$). Let $\tau > 0$ with $x + \tau < a_{i+1}$

$$\frac{\bar{u}(x + \tau) - \bar{u}(x)}{\tau} \leq \frac{\bar{u}^k(x + \tau) - \bar{u}^k(x)}{\tau} \rightarrow (\bar{u}^k)'(x)$$

therefore

$$(27) \quad \bar{u}'(x) \leq (\bar{u}^k)'(x).$$

Let

$$\begin{aligned} \tau_n \rightarrow 0 \quad \tau_n > 0 \quad \text{with } x + \tau_n < a_{i+1} \\ \bar{u}^k(x + \tau_n) = \bar{u}(x + \bar{\tau}_n) \quad \text{with } 0 \leq \bar{\tau}_n \leq \tau_n \\ 0 \leq \frac{\bar{u}^k(x_n + \tau) - \bar{u}^k(x)}{\tau_n} = \frac{\bar{u}(x + \bar{\tau}_n) - \bar{u}(x)}{\tau \bar{\tau}_n} \cdot \frac{\bar{\tau}_n}{\tau_n} \\ \frac{\bar{u}^k(x + \tau_n) - \bar{u}^k(x)}{\tau_n} \rightarrow (\bar{u}^k)'(x) > 0 . \end{aligned}$$

Hence:

$$(28) \quad (\bar{u}^k)'(x) \leq \bar{u}'(x) .$$

From (27) and (28) we deduce

$$(\bar{u}^k)'(x) = \bar{u}'(x) .$$

Thus (25) is proved.

PROOF OF (26). Let $y \in C$ and $x \in \bar{u}^{-1}(y)$; we are going to prove that if k is sufficiently large then $x \in (\bar{u}^k)^{-1}(y)$. Since $\bar{u}^{-1}(y)$ is a finite set this will prove (26). u is derivable in x and $\bar{u}'(x) \neq 0$ (since $y \in C$). Let us assume that for example $\bar{u}'(x) > 0$ (the proof in the case $\bar{u}'(x) < 0$ would be nearly the same). Let $\eta > 0$ such that:

$$\begin{aligned} z \in [x - \eta, x) &\Rightarrow \bar{u}(z) < \bar{u}(x) \\ z \in (x, x + \eta] &\Rightarrow \bar{u}(z) > \bar{u}(x) . \end{aligned}$$

Let

$$\delta = \text{Min} (\bar{u}(x + \eta) - \bar{u}(x), \bar{u}(x) - \bar{u}(x - \eta)) .$$

Let us assume that

$$(29) \quad |\varepsilon^k| < \frac{\delta}{2} .$$

Let a_i^k be the sequence used for definition of $\bar{u}^k$ (see above definition of a_i). It is easy to see, using (29), that if

$$a_i^k < x < a_{i+1}^k$$

then

$$x - \eta < a_i^k < a_{i+1}^k < x + \eta .$$

Then

$$\bar{u}^k(a_i^k) < \bar{u}(x) < \bar{u}^k(a_{i+1}^k)$$

and

$$\bar{u}^k(x) = \operatorname{Max}_{v \in [a_i^k, x]} \bar{u}(x).$$

(26) is proved, and so (10) is proved.

PROOF OF (11). Now ε is fixed.

Using (15) with $\bar{u}_n$ instead of $\bar{u}$ we have

$$|P_\varepsilon \bar{u}_n| \leq |\bar{u}_n|.$$

Let

$$\begin{aligned} v_n(y) &= - \operatorname{meas} \{x | \bar{u}_n(x) \geq y\} \\ w_n(y) &= - \operatorname{meas} \{x | P_\varepsilon \bar{u}_n(x) \geq y\}. \end{aligned}$$

Let $D =]0, m[- (V(\bar{u}_n) \cup V(P_\varepsilon \bar{u}_n) \cup A(\varepsilon))$; $[0, m] \setminus D$ is negligible. Using Appendix 3, we know that, if $y \in D$, then v_n and w_n are differentiable in y and:

$$\begin{aligned} v'_n(y) &= \sum_{x \in \bar{u}_n^{-1}(y)} \frac{1}{|(\bar{u}_n)'(x)|} \\ w'_n(y) &= \sum_{x \in (P_\varepsilon \bar{u}_n)^{-1}(y)} \frac{1}{|(P_\varepsilon \bar{u}_n)'(x)|}. \end{aligned}$$

But (see the proof of (25))

$$(P_\varepsilon \bar{u}_n)^{-1}(y) \subset \bar{u}_n^{-1}(y)$$

and if $x \in (P_\varepsilon \bar{u}_n)^{-1}(y)$, we have $(\bar{u}_n)'(x) = (P_\varepsilon \bar{u}_n)'(x)$ therefore

$$(30) \quad w'_n(y) \leq v'_n(y).$$

But (see appendix 3):

$$|\bar{u}_n^*| = \int_0^M \frac{2^p}{(v'_n(y))^{p-1}} dy$$

and

$$|(P_\varepsilon \bar{u}_n)^*| = \int_0^M \frac{2^p}{(w'_n(y))^{p-1}} dy.$$

Then (11) follows from (30).

PROOF OF (12). First we show that:

$$(31) \quad \lim_{n \rightarrow +\infty} a_1^n = a_1.$$

PROOF OF (31). We have $y_n(a_1^n) = m$ and

$$a_1 - \delta_0 \leq a_1^n \leq a_1 + \delta_0.$$

We extract from the sequence a_1^n a convergent subsequence, (we shall also note a_1^n) such that:

$$a_1^n \rightarrow b \quad \text{when } n \rightarrow +\infty.$$

We have $u(b) = m$.

Since $m \notin V(u)$, $\forall \delta > 0$ there exists x such that

$$u(x) > m \quad \text{and} \quad |b - x| < \delta.$$

Hence

$$(32) \quad a_1 \leq b.$$

But $u(a_1) = m$ and $m \notin V(u)$ then, $\forall \delta > 0$, there exists x' such that:

$$u(x') > m \quad \text{and} \quad |a_1 - x'| < \delta.$$

We have:

$$\lim_{n \rightarrow +\infty} u_n(x') = u(x').$$

Thus for n sufficiently large

$$u_n(x') > m$$

and therefore (if $\delta < \delta_0$):

$$a_1^n \leq x' \leq a_1 + \delta.$$

Then:

$$(33) \quad b \leq a_1.$$

Clearly (31) follows from (32) and (33).

Let l_n be the number of terms of sequence a_i^n .

We assume that:

$$A(\varepsilon) \cap V(\bar{u}) = \emptyset.$$

Using the arguments of the Proof of (32) it is easy to prove that there exists n_0

such that

$$n \geq n_0 \Rightarrow l_n = l$$

and

$$\lim_{n \rightarrow +\infty} a_i^n = a_i$$

and, then, there exists n_1 such that:

$$n \geq n_1 \Rightarrow l_n = 1 \quad \text{and} \quad \bar{u}_n(a_i^n) = \bar{u}(a_i) \quad \forall i \in \varepsilon[1, l].$$

Let x be a real number with $a_i < x < a_{i+1}$; for n sufficiently large, $a_i^n < x < a_{i+1}^n$,

$$\bar{u}_n(a_i^n) = \bar{u}(a_i) \quad \text{and} \quad \bar{u}_n(a_{i+1}^n) = \bar{u}(a_{i+1}).$$

Now using the definitions of $P_\varepsilon \bar{u}_n$ and $P_\varepsilon \bar{u}$ it is easy to see that:

$$P_\varepsilon \bar{u}_n(x) \rightarrow P_\varepsilon \bar{u}(x)$$

and the same method yields: if $x > a_1$ or $x < a_1$ then:

$$P_\varepsilon \bar{u}_n(x) = 0 = P_\varepsilon \bar{u}(x)$$

for n sufficiently large but (see (15) with $\bar{u}_n$ instead of $\bar{u}$) $P_\varepsilon \bar{u}_n'$ is bounded in $W^{1,p}(\mathbf{R})$. (Let us recall that $\|P_\varepsilon \bar{u}_n\|_\infty \leq \bar{M}$ and $\text{Supp } P_\varepsilon u_n \subset [a_1 - \delta_0, a_l + \delta_0]$).

Then:

$$P_\varepsilon u \xrightarrow{n \rightarrow +\infty} P_\varepsilon \bar{u} \quad \text{in } W^{1,p}(\mathbf{R}).$$

For $i \in [1, l]$ and γ in $W^{1,p}(\mathbf{R})$, let $F_i(\gamma)$ be the function of $W^{1,p}(\mathbf{R})$ defined by:

$$F_i(\gamma)(x) = \text{Max} \left(\text{Min} \left(\gamma(x), \sum_{j=0}^i \varepsilon_j \right), \sum_{j=0}^{i-1} \varepsilon_j \right) - \sum_{j=0}^{i-1} \varepsilon_j,$$

with the convention $\varepsilon_0 = 0$. We have:

$$\Phi_c(P_\varepsilon \bar{u}_n) = \sum_{i=1}^l \Phi_c(F_i(P_\varepsilon \bar{u}_n)).$$

and

$$F_i(P_\varepsilon \bar{u}_n) \rightharpoonup F_i(P_\varepsilon \bar{u}) \quad \text{in } W^{1,p}(\mathbf{R}).$$

Then using appendix 3 we see that (19) follows from the following lemma:

LEMMA. Let T and L be two positive real numbers; let k be a positive integer and $(\alpha_n^1, \alpha_n^2, \dots, \alpha_n^k)$ be a sequence of elements in $(W^{1,p}((0, T)))^k$ such that for each i in $[1, k]$:

α_n^i is nondecreasing

$$\alpha_n^i(0) = 0 \quad \alpha_n^i(T) = L$$

$$\alpha_n^i \xrightarrow{n \rightarrow +\infty} \alpha^i \quad \text{in } W^{1,p}((0, T)) .$$

Let

$$\beta_n^i(y) = - \text{meas} \{x \in [0, T] | \alpha_n^i(x) \geq y\} ,$$

$$\beta^i(y) = - \text{meas} \{x \in [0, T] | \alpha^i(x) \geq y\} .$$

Then

$$\begin{aligned} \sum_{i=1}^k \int_0^L \frac{dy}{(\beta_i^i(y))^{p-1}} - c \int_0^L \frac{2^p}{\left(\sum_{i=1}^k \beta_i^i(y)\right)^{p-1}} dy \\ \leq \lim_{n \rightarrow +\infty} \left(\sum_{i=1}^k \int_0^L \frac{dy}{((\beta_i^n)'(y))^{p-1}} - c \int_0^L \frac{2^p dy}{\left(\sum_{i=1}^k (\beta_i^n)'(y)\right)^{p-1}} \right) . \end{aligned}$$

PROOF OF THE LEMMA. Let m_n^i be the unique positive Radon measure on $[0, L]$ such that:

$$0 \leq y < y' \leq L \Rightarrow m_n^i([y, y'[) = \beta_n^i(y') - \beta_n^i(y) , \quad m_n^i([0, L]) = T .$$

Let m^i be the unique positive Radon measure on $[0, L]$ such that:

$$0 \leq y < y' \leq L \Rightarrow m^i([y, y'[) = \beta^i(y') - \beta^i(y) , \quad m^i([0, L]) = T .$$

Let φ be a continuous function from $[0, L]$ into R ; we have:

$$\begin{aligned} \int_{[0, L]} \varphi(y) dm_n^i(y) &= \int_0^T \varphi(\alpha_n^i(x)) dx \\ \int_{[0, L]} \varphi(y) dm^i(y) &= \int_0^T \varphi(\alpha^i(x)) dx . \end{aligned}$$

Since $\alpha_n^i \rightarrow \alpha$ in $W^{1,p}((0, T))$, $\alpha_n^i(x) \rightarrow \alpha^i(x)$, $\forall x \in [0, T]$.

Hence

$$\int_{[0,L]} \varphi(y) dm_n^i(y) \rightarrow \int_0^T \varphi(\alpha^i(x)) dx$$

and:

$$\lim_{n \rightarrow +\infty} \int_{[0,L]} \varphi(y) dm_n^i(y) = \int_{[0,L]} \varphi(y) dm^i(y).$$

But

$$m_n^i = (\beta_n^i)'(y) dy + \nu_n^i \quad m^i = (\beta^i)'(y) dy + \nu^i$$

where ν_n^i and dy are mutually singular, and, ν^i and dy are mutually singular. Therefore the lemma follows from appendix 6.

Part B. Here we assume that $c > 1/2^p$ and we construct a sequence u_n such that $u_n \rightharpoonup u$ in $W_+^{1,p}(\mathbf{R})$ and $\Phi_c(u) > \lim \Phi_c(u_n)$.

It follows from appendix 5 that there exist four real numbers t_1, t_2, s_1, s_2 such that:

$$0 < t_1, \quad 0 < t_2, \quad 0 < s_1, \quad 0 < s_2$$

and:

$$(34) \quad \frac{1}{[(t_1 + t_2)/2]^{p-1}} + \frac{1}{[(s_1 + s_2)/2]^{p-1}} - \frac{2^p c}{[(s_1 + t_1 + s_2 + t_2)/2]^{p-1}} \\ > \frac{1}{2} \left(\frac{1}{t_1^{p-1}} + \frac{1}{s_1^{p-1}} - \frac{2^p c}{(t_1 + s_1)^{p-1}} + \frac{1}{t_2^{p-1}} + \frac{1}{s_2^{p-1}} - \frac{2^p c}{(t_2 + s_2)^{p-1}} \right)$$

Let d_n and e_n be the functions from $]0, 1]$ into $\mathbf{R}$ defined by:

for x in $]0, 1]$ with $k/2^n < x \leq (k+1)/2^n$ where k is an integer we set:

— when k is odd: $d_n(y) = s_1, e_n(y) = -t_1$

— when k is even: $d_n(y) = s_2, e_n(y) = -t_2$.

Let

$$D_n(y) = \int_y^1 d_n(\tau) d\tau \quad \text{for } y \in [0, 1]$$

$$E_n(y) = \int_y^1 e_n(\tau) d\tau \quad \text{for } y \in [0, 1].$$

We have

$$D_n(0) = \frac{s_1 + s_2}{2} \quad E_n(0) = -\frac{t_1 + t_2}{2}$$

and

$$(35) \quad \begin{cases} \lim_{n \rightarrow +\infty} D_n(y) = \frac{s_1 + s_2}{2} (1 - y) & \forall y \in [0, 1] \\ \lim_{n \rightarrow +\infty} E_n(y) = -\frac{t_1 + t_2}{2} (1 - y) & \forall y \in [0, 1]. \end{cases}$$

We are going to define u_n :

when $x \geq (s_1 + s_2)/2$ let $u_n(x) = 0$

when $0 \leq x < (s_1 + s_2)/2$ let $u_n(x)$ be the only real number such that

$$D_n(u_n(x)) = x$$

when $-(t_1 + t_2)/2 < x < 0$ let $u_n(x)$ be the only real number such that

$$E_n(u_n(x)) = x$$

when $x < -(t_1 + t_2)/2$ let $u_n(x) = 0$.

It is easy, using (35), to prove that:

$$(36) \quad \lim_{n \rightarrow +\infty} u_n(x) = u(x)$$

with

$$\begin{aligned} u(x) &= 1 - \frac{2}{s_1 + s_2} x & \text{when } 0 \leq x \leq \frac{s_1 + s_2}{2} \\ u(x) &= 1 + \frac{2}{t_1 + t_2} x & \text{when } -\frac{t_1 + t_2}{2} \leq x < 0 \\ u(x) &= 0 & \text{when } x > \frac{s_1 + s_2}{2} \quad \text{or } x < -\frac{t_1 + t_2}{2}. \end{aligned}$$

We have

$$(37) \quad |u_n|^p = \frac{1}{2} \left\{ \left(\frac{1}{s_1^{p-1}} + \frac{1}{s_2^{p-1}} \right) + \left(\frac{1}{t_1^{p-1}} + \frac{1}{t_2^{p-1}} \right) \right\}.$$

Then u_n is bounded in $W^{1,p}(\mathbf{R})$ and using (36)

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\mathbf{R}) \quad \text{when } n \rightarrow +\infty.$$

An easy computation gives:

$$(38) \quad |u_n^*|^p = \frac{1}{2} \left\{ \frac{2^p}{(s_1 + t_1)^{p-1}} + \frac{2^p}{(s_2 + t_2)^{p-1}} \right\}$$

$$(39) \quad |u|^p = \frac{1}{[(s_1 + s_2)/2]^{p-1}} + \frac{1}{[(t_1 + t_2)/2]^{p-1}}$$

$$(40) \quad |u^*|^p = \frac{1}{((s_1 + s_2)/2 + (t_1 + t_2)/2)^{p-1}}.$$

Using (34), (37), (38), (39) and (40) we have

$$\Phi_c(u) > \lim_{n \rightarrow +\infty} \Phi_c(u_n).$$

Appendix 0.

Let $L_+^p(\mathbf{R})$ be the set of nonnegative functions of $L^p(\mathbf{R})$. Then we have the following (for $1 < p < +\infty$).

PROPOSITION. *The rearrangement is a continuous mapping from $L_+^p(\mathbf{R})$ into $L_+^p(\mathbf{R})$ (for the strong topologies).*

PROOF. First we recall that, if $u \in L_+^p(\mathbf{R})$, $u^* \in L_+^p(\mathbf{R})$ and:

$$\int (u^*)^p dx = \int u^p dx$$

(see [5]).

Let $(u_n)_{n \in \mathbf{N}}$ be a sequence of functions of $L_+^p(\mathbf{R})$ such that

$$u_n \rightarrow u \quad \text{in } L^p(\mathbf{R})$$

We are going to prove that

$$u_n^* \rightarrow u^* \quad \text{in } L^p(\mathbf{R}).$$

Obviously we may assume that

$$u_n(x) \rightarrow u(x) \quad \text{a.e. } x \in \mathbf{R}$$

and

$$\exists h \in L_+^p(\mathbf{R}) \text{ such that } u_n(x) \leq h(x) \text{ a.e. } x \in \mathbf{R}.$$

Let f_n, f and g be the following functions

$$f_n(x) = 1 \quad \text{if } u_n(x) > t$$

$$f_n(x) = 0 \quad \text{if } u_n(x) \leq t$$

$$f(x) = 1 \quad \text{if } u(x) > t$$

$$f(x) = 0 \quad \text{if } u(x) \leq t$$

$$g(x) = 1 \quad \text{if } h(x) > t$$

$$g(x) = 0 \quad \text{if } h(x) \leq t.$$

Then $f_n \rightarrow f$ a.e., $g \in L^1(\mathbf{R})$, $f_n \leq g$ a.e.

Therefore

$$\int f_n \rightarrow \int f.$$

Thus

$$\text{meas } \{x | u_n(x) > t\} \rightarrow \text{meas } \{x | u(x) > t\}.$$

Then the proposition follows easily from the definition of u_n^* and u^* , from:

$$\int (u_n^*)^p dx = \int u_n^p dx \rightarrow \int u^p dx - \int (u^*)^p dx$$

and

$$u_n^* \leq h^*.$$

Appendix 1.

Let u be an absolutely continuous function from $\mathbf{R}$ into $\mathbf{R}$. Let

$$V'(0) = \{y | \text{there exists } x \text{ in } \mathbf{R} \text{ such that } u(x) = y \text{ and either } u \text{ is not derivable in } x \text{ or } u \text{ is derivable in } x \text{ and } u'(x) = 0\}.$$

Then;

$$(41) \quad V(u) \text{ is negligible (for the Lebesgue measure).}$$

PROOF. Let A be a measurable set; we are going to prove that:

$$(42) \quad \lambda^*(u(A)) \leq \int_A |u'(t)| dt$$

where

$$\lambda^*(B) = \text{Inf} \{ \lambda(\Omega) | \Omega \text{ is an open set of } \mathbf{R} \text{ such that } B \subset \Omega \}$$

(λ is the Lebesgue measure).

Property (41) follows easily from (42) by taking

$$A = \{x|u \text{ is not derivable in } x\} \cup \{x|u \text{ is derivable in } x \text{ and } u'(x) = 0\}.$$

Let $\varepsilon > 0$. There exists $\eta > 0$ such that:

$$(43) \quad \text{for any measurable set } E \text{ such that } \lambda(E) < \eta \text{ then } \int_E |u'(\tau)| d\tau < \varepsilon.$$

There exist two sequences of real numbers $(\alpha_i)_{i \in \mathbb{N}}$, $(\beta_i)_{i \in \mathbb{N}}$ such that

$$\alpha_i < \beta_i \quad \forall i \in \mathbb{N}$$

$$]\alpha_i, \beta_i[\cap]\alpha_j, \beta_j[= \emptyset \quad \text{if } i \neq j$$

and:

$$(44) \quad A \subset \Omega \text{ and } \lambda(\Omega - A) < \eta \text{ where } \Omega = \bigcup_{i \in \mathbb{N}}]\alpha_i, \beta_i[.$$

Clearly

$$u(A) \subset \bigcup_{i \in \mathbb{N}} u(]\alpha_i, \beta_i[)$$

$$\lambda^*(u(A)) \leq \sum_{i \in \mathbb{N}} \lambda^*(u(]\alpha_i, \beta_i[))$$

but

$$\lambda^*(u(]\alpha_i, \beta_i[)) = \lambda(u(]\alpha_i, \beta_i[)) \leq \int_{\alpha_i}^{\beta_i} |u'(\tau)| d\tau$$

$$\lambda^*[u(A)] \leq \int_{\Omega} |u'(\tau)| d\tau = \int_A |u'(\tau)| d\tau + \int_{\Omega - A} |u'(\tau)| d\tau$$

we use (43) and (44):

$$\lambda^*(u(A)) \leq \int_A |u'(\tau)| d\tau + \varepsilon.$$

Hence (42) follows.

Appendix 2.

Let u be in $W^{1,p}((0, T))$; let

$$v(x) = \text{Max}_{y \in [0, x]} u(y)$$

then:

$$(45) \quad v \text{ is in } W^{1,p}((0, T)) \text{ and } |v|^p = \int_0^T v'(t) |u'(t)|^{p-1} dt.$$

PROOF OF (45).

(45) is of course true when u is a polynomial function; let u_n be a sequence of polynomial functions such that:

$$u_n \rightarrow u \quad \text{in } W^{1,p}((0, T)).$$

Let

$$v_n(x) = \max_{y \in [0, x]} u_n(y).$$

We have

$$(46) \quad \lim_{n \rightarrow +\infty} v_n(x) = v(x) \quad \forall x \in [0, T].$$

Using (45) for v_n we have

$$|v_n| \leq |u_n|.$$

Then v_n is bounded in $W^{1,p}((0, T))$; using (46) we have:

$$v \in W^{1,p}((0, T)) \text{ and } v_n \rightarrow v \text{ in } W^{1,p}((0, T)) \text{ when } n \rightarrow +\infty.$$

Let x be a point of $(0, T)$ such that v and u are differentiable in x . We are going to prove that:

$$(47) \quad v'(x)^p = v'(x)|u'(x)|^{p-1}.$$

This will prove (45).

Note that since v is nondecreasing, $v'(x) \geq 0$; if $v'(x) = 0$ (47) is of course true. Now let us assume that $v'(x) > 0$. We shall prove that $v(x) = u(x)$. Clearly $v(x) \geq u(x)$. Assume by contradiction that $v(x) > u(x)$; then there exists $\varepsilon > 0$ such that

$$[x, x + \varepsilon] \subset [0, T]$$

and

$$z \in [x, x + \varepsilon] \Rightarrow u(z) < v(x).$$

Therefore

$$z \in [x, x + \varepsilon] \Rightarrow v(z) = v(x)$$

and so $v'(x) = 0$.

A contradiction with $v'(x) > 0$.

We have proved that $v(x) = u(x)$. Since $v \geq u$ and $v(x) = u(x)$, we have (47).

Appendix 3.

This appendix is due to T. Gallouët.

Let u be a nondecreasing function in $W^{1,p}((0, T))$ such that $u(0) = 0$ and $u(T) = L$.

Let v the function from $[0, L]$ into $[-T, 0]$ defined by

$$v(y) = - \text{meas} \{x \in [0, T] | u(x) \geq y\};$$

v is a nondecreasing function and then derivable a.e. with $v' \geq 0$. Let $1/v'$ be the function from $[0, L]$ into $\mathbb{R}$ defined by:

$$\begin{aligned} \frac{1}{v'}(y) &= \frac{1}{v'(y)} && \text{if } v \text{ is differentiable in } y \text{ with } v'(y) \neq 0 \\ \frac{1}{v'}(y) &= \alpha && \text{elsewhere } (\alpha \in \mathbb{R}^+ \text{ } \alpha \text{ is fixed}). \end{aligned}$$

Then we have:

$$(48) \quad \int_0^L \left(\frac{1}{v'} \right)^{p-1} dy = |u|^p.$$

PROOF OF (48). We have

$$\{x \in [0, T] | u(x) \geq y\} = [\text{Min } u^{-1}(y), T] \quad \text{for } y \in [0, L].$$

Then

$$(49) \quad v(y) = - (T - \text{Min } u^{-1}(y))$$

and therefore:

$$(50) \quad u(v(y) + T) = y.$$

Since u is absolutely continuous and nondecreasing, we have:

$$(51) \quad \int_0^L \left(\frac{1}{v'}(y) \right)^{p-1} dy = \int_0^T \left(\frac{1}{v'} \right)^{p-1} (u(x)) \cdot u'(x) dx.$$

Let x be in $]0, T[$ such that u is derivable in x with $u'(x) \neq 0$.

We have:

$$x' < x \Rightarrow u(x') < u(x)$$

$$x' > x \Rightarrow u(x') > u(x).$$

Let $y = u(x)$ and h be such that $y + h$ and $y - h$ are in $(0, T)$. Using (50)

we have:

$$\frac{v(y+h) - v(y)}{y+h-y} = \frac{u(v(y+h)) - u(v(y))}{v(h+h) - v(y)};$$

but using (49) and (52) it is easy to see that

$$\lim_{h \rightarrow 0} v(y+h) = v(y).$$

Then v is differentiable in y and $v'(y) = 1/u'(x) \neq 0$. Then using (51) we have (48).

Appendix 4.

Let $u \in W^{1,p}(\mathbf{R})$, $u \geq 0$; let:

$$v(y) = - \text{meas} \{x | u(x) \geq y\}.$$

If $y \notin V(u)$ and $y \in u(\mathbf{R})$ then v is derivable in y and:

$$(53) \quad v'(y) = \sum_{x \in u^{-1}(y)} \frac{1}{|u'(x)|}.$$

PROOF OF (53). First we remark that, since $y \notin V(u)$, $u^{-1}(y)$ has only a finite number of elements. On the other hand the number of elements of $u^{-1}(y)$ is even since $u \rightarrow 0$ at infinity. For simplicity we shall assume that $u^{-1}(y)$ has only two elements x_1, x_2 with $x_1 < x_2$ and we shall prove only the right-differentiability. We have $u'(x_1) > 0$, $u'(x_2) < 0$.

Let $k > 0$ be such that $u^{-1}(y+k) \neq \emptyset$ (if k is sufficiently small $u^{-1}(y+k) \neq \emptyset$).

Let

$$x_1(k) = \text{Min} \{x | u(x) = y+k\}$$

$$x_2(k) = \text{Max} \{x | u(x) = y+k\}.$$

We have

$$\lim_{k \rightarrow 0^+} x_i(k) = x_i \quad \forall i \in \{1, 2\}$$

and

$$u(z) \geq y+k \Rightarrow z \in [x_1(k), x_2(k)].$$

Therefore $\text{meas} \{x | u(x) \geq y+k\} \leq x_2(k) - x_1(k)$.

We have

$$u(x_i(k)) = y+k = u(x_i) + u'(x_i)(x_i(k) - x_i) + (x_i(k) - x_i) \varepsilon_i(k)$$

with

$$\lim_{k \rightarrow 0^+} \varepsilon_i(k) = 0 \quad \text{and} \quad u(x_i) = y.$$

Thus:

$$\lim_{k \rightarrow 0^+} \frac{x_i(k) - x_i}{k} = \frac{1}{u'(x_i)}.$$

Therefore

$$(54) \quad \lim_{k \rightarrow 0^+} \frac{k}{v(y+k) - v(y)} \geq \frac{1}{u'(x_1)} - \frac{1}{u'(x_2)}.$$

Let

$$\begin{aligned} \bar{x}_1(k) &= \text{Max} \left\{ x | u(x) = y + k \text{ et } x \leq \frac{x_1 + x_2}{2} \right\} \\ \bar{x}_2(k) &= \text{Min} \left\{ x | u(x) = y + k \text{ et } x \geq \frac{x_1 + x_2}{2} \right\} \end{aligned}$$

($\bar{x}_i(k)$ is well defined if k is sufficiently small).

We have

$$\lim_{k \rightarrow 0^+} \bar{x}_i(k) = x_i.$$

It is easy to see that if k is sufficiently small,

$$x \in [\bar{x}_1(k), \bar{x}_2(k)] \Rightarrow u(x) \geq y + k.$$

We have

$$\lim_{k \rightarrow 0^+} \bar{x}_i(k) = x_i.$$

as before we prove that

$$\lim_{k \rightarrow 0^+} \frac{\bar{x}_i(k) - x_i}{k} = \frac{1}{u'(x_i)}$$

and we have:

$$\text{meas} \{x | u(x) \geq y + k\} \geq \bar{x}_2(k) - \bar{x}_1(k).$$

Thus we have

$$(55) \quad \lim_{k \rightarrow 0^+} \frac{v(y+k) - v(y)}{k} \leq \frac{1}{u'(x_1)} - \frac{1}{u'(x_2)}.$$

Using (54) and (55) we have

$$\lim_{k \rightarrow 0^+} \frac{v(y+k) - v(y)}{k} = \frac{1}{|u'(x_1)|} + \frac{1}{|u'(x_2)|}.$$

Appendix 5.

Let d be a real number and let

$$\varphi: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$$

$$(x_1, x_2, \dots, x_n) \rightarrow \begin{cases} \sum_{i=1}^n \frac{1}{x_i^{p-1}} - \frac{d}{\left(\sum_{i=1}^n x_i\right)^{p-1}} & \text{if } \forall i \ x_i > 0 \\ +\infty & \text{elsewhere} \end{cases}$$

Then if $d \leq 1$ φ is convex and l.s.c. If $d > 1$ and $n = 2$ φ is not convex on $(\mathbb{R}^+)^n$.

PROOF. 1) $n = 2$.

φ is C^∞ on $(\mathbb{R}^+)^2$. Let $x_1 > 0$, $x_2 > 0$ we have:

$$\frac{\partial^2 \varphi}{\partial x_1^2} = p(p-1) \left\{ \frac{1}{x_1^{p+1}} - \frac{d}{(x_1 + x_2)^{p+1}} \right\}$$

$$\frac{\partial^2 \varphi}{\partial x_2^2} = p(p-1) \left\{ \frac{1}{x_2^{p+1}} - \frac{d}{(x_1 + x_2)^{p+1}} \right\}$$

$$\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = -p(p-1) \frac{d}{(x_1 + x_2)^{p+1}}$$

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} = p(p-1) \left\{ \frac{1}{x_1^{p+1}} + \frac{1}{x_2^{p+1}} - \frac{2d}{(x_1 + x_2)^{p+1}} \right\} \geq 0 \quad \text{if } d \leq 1$$

$$\frac{\partial^2 \varphi}{\partial x_1^2} \cdot \frac{\partial^2 \varphi}{\partial x_2^2} - \left(\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \right)^2 = p^2(p-1)^2 \left\{ \frac{(x_1 + x_2)^{p+1} - d(x_1^{p+1} + x_2^{p+1})}{x_1^{p+1} x_2^{p+1} (x_1 + x_2)^{p+1}} \right\} \geq 0 \quad \text{if } d \leq 1.$$

Thus, if $d \leq 1$, φ is convex (and continuous) on $(\mathbb{R}^+)^2$; if $d > 1$ there exists $(x_1, x_2) \in (\mathbb{R}^+)^2$ such that

$$\left(\frac{\partial^2 \varphi}{\partial x_1^2} \frac{\partial^2 \varphi}{\partial x_2^2} - \left(\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \right)^2 \right) (x_1, x_2) < 0$$

and therefore φ is not convex on $(\mathbb{R}^+)^2$. We assume now $d \leq 1$. φ is convex on $(\mathbb{R}^+)^2$ and then φ is convex on $\mathbb{R}^2$. It is easy to see that φ is l.s.c. in (x_1, x_2) if $(x_1, x_2) \neq (0, 0)$. It remains to prove that φ is l.s.c. in $(0, 0)$.

We have

$$\varphi(x_1, x_2) \geq \frac{1}{x_1^{p-1}} \quad \text{if } x_1 > 0$$

$$\varphi(x_1, x_2) = +\infty \quad \text{if } x_1 \leq 0.$$

Thus if $(x_1^n, x_2^n) \rightarrow (0, 0)$ as $n \rightarrow +\infty$ we have

$$\lim_{n \rightarrow +\infty} \varphi(x_1^n, x_2^n) = +\infty = \varphi(0, 0).$$

2) $n \geq 3$; we assume $d \leq 1$.

Since the mapping from $\mathbf{R}^n$ into $R \cup \{+\infty\}$ defined by:

$$(x_1 \dots x_n) \rightarrow \begin{cases} \left\{ \left(\sum_{i=1}^n x_i \right)^{p-1} \right\}^{-1} & \text{if } x_i \geq 0 \sum_{i=1}^n x_i \neq 0 \\ +\infty & \text{elsewhere} \end{cases}$$

is convex l.s.c. We may assume that $d = 1$.

As for $n = 2$ it is easy to prove that φ is l.s.c. We are going to prove that φ is convex on $(\mathbf{R}^{+*})^n$ by induction on n . We shall write φ_n instead of φ ; we assume that φ_{n-1} is convex on $(\mathbf{R}^{+*})^{n-1}$.

Let

$$x = (x_1, x_2, \dots, x_n) \in (\mathbf{R}^{+*})^n$$

$$y = (y_1, y_2, \dots, y_n) \in (\mathbf{R}^{+*})^n$$

Let $t \in [0, 1]$, $\tilde{x} = (x_2, \dots, x_n)$, $\tilde{y} = (y_2, \dots, y_n)$

$$\varphi_n(tx + (1-t)y) = \varphi_2\left(t\left(x_1, \sum_{i=2}^n x_i\right) + (1-t)\left(y_1, \sum_{i=2}^n y_i\right)\right) + \varphi_{n-1}(t\tilde{x} + (1-t)\tilde{y})$$

φ_2 and φ_{n-1} are convex on $(\mathbf{R}^{+*})^2$ and $(\mathbf{R}^{+*})^{n-1}$; therefore

$$\begin{aligned} \varphi_n(tx + (1-t)y) &\leq t\varphi_2\left(x_1, \sum_{i=2}^n x_i\right) + (1-t)\varphi_2\left(y_1, \sum_{i=2}^n y_i\right) + t\varphi_{n-1}(\tilde{x}) + (1-t)\varphi_{n-1}(\tilde{y}) \\ &\leq t\varphi_n(x) + (1-t)\varphi_n(y). \end{aligned}$$

Appendix 6.

Let K be a compact set of $\mathbf{R}$ and $C(K)$ be the set of the continuous functions from K into $\mathbf{R}$; for f in $C(K)$. Let

$$\|f\| = \max_{x \in K} |f(x)|.$$

$\|\cdot\|$ is a norm on $C(K)$; let M be the dual space of $C(K)$.

For m in $\mathcal{M}$ we have the decomposition:

$$m = f dx + \mu, \quad f \in L^1(K), \quad \mu \in \mathcal{M}$$

where $f dx$ and μ are mutually singular. We shall write:

$$f = R(m).$$

Let F be the mapping from $\mathcal{M}^n$ into $w \cup \{+\infty\}$ defined by:

$$F(m_1, m_2, \dots, m_n) = \int_K \varphi(Pm_1, \dots, Pm_n) dx$$

where φ is defined in the appendix 5. We assume (see the definition of φ) that $d \leq 1$.

Let $(m_{i,p})_{1 \leq i \leq n, 0 \leq p}$ be a sequence of elements in $\mathcal{M}^n$ such that:

$$(56) \quad \lim_{p \rightarrow +\infty} \int \theta dm_{i,p} = \int \theta dm_i \quad \forall \theta \in C(K), \quad \forall i \in [1, n]$$

$$\int \theta dm_{i,p} \geq 0 \quad \forall i \in [1, n] \quad \forall p \quad \forall \theta \in C(K) \text{ with } \theta \geq 0.$$

We are going to prove that:

$$(57) \quad F(m_1, \dots, m_n) \leq \varliminf_{p \rightarrow \infty} F(m_{1,p}, \dots, m_{n,p}).$$

Let

$$f_{i,p} = R(m_{i,p}), \quad f_i = R(m_i).$$

Let $r > 0$ and $f_{i,p}^r(x) = \text{Min}(r, f_{i,p}(x))$.

$$\|f_{i,p}^r\|_\infty \leq r.$$

Thus we can extract a subsequence which converges for the topology $\sigma(L^1, L^\infty)$ we shall denote also $f_{i,p}$ such a subsequence:

$$f_{i,p}^r \xrightarrow[p \rightarrow +\infty]{} g_i^r \quad \sigma(L^1, L^\infty).$$

Using appendix 5 we have:

$$(58) \quad \int_K \varphi(g_1^r, \dots, g_n^r) dx \leq \varliminf_{p \rightarrow +\infty} \int_K \varphi(f_{1,p}^r, \dots, f_{n,p}^r) dx.$$

But it is easy to see that:

$$0 \leq \varphi(f_{1,p}^r, \dots, f_{n,p}^r) - \varphi(f_{1,p}, \dots, f_{n,p}) \leq \frac{n}{r^{p-1}}.$$

Thus

$$(59) \quad \int_K \varphi(f_{1,p}^r, \dots, f_{n,p}^r) dx \leq F(m_{1,p}, \dots, m_{n,p}) + \frac{nL}{r^{p-1}}$$

where L is the Lebesgue measure of K .

Let $\theta \in C(K)$ with $\theta \geq 0$ and $i \in [1, n]$.

$$\int_K \theta g_i^r dx = \lim_{r \rightarrow +\infty} \int_K f_{i,p}^r \theta dx \leq \lim_{r \rightarrow +\infty} \int_K \theta dm_{i,p} = \int_K \theta dm_i.$$

Therefore

$$g_i^r \leq f_i.$$

But

$$x_i \leq x'_i \quad \forall i \in [1, n] \Rightarrow 0 \leq \varphi(x'_1, \dots, x'_p) \leq \varphi(x_1, \dots, x_p).$$

Hence

$$(60) \quad \int_K \varphi(f_1, \dots, f_n) dx \leq \int_K \varphi(g_1^r, \dots, g_n^r) dx.$$

Using (58), (59) and (60) we have, for every r in $\mathbb{R}^{+*}$.

$$F(m_1, \dots, m_n) \leq \liminf_r F(m_{1,p}, \dots, m_{n,p}) + \frac{nL}{r^{p+1}}.$$

It gives (57).

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