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The $\oplus_c$-Topology is Not Completable.

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1. – Introduction.

G. D'Este [5] introduced and studied an interesting and difficult functorial topology defined on the category of abelian groups: Let $\oplus_c$ be the class of all direct sums of cyclic p-groups. For each group A let $\mathfrak{U}_A = \{U \leqslant A: A/U \in \oplus_c\}$. Then $\mathfrak{U}_A$ is a neighborhood basis at 0, a «local basis» for short, for some topology on A which makes A into a topological group. We write $A[\mathfrak{U}_A] = A[\oplus_c]$ for this topological group. Every homomorphism $f: A \to B$ is then a continuous map $f: A[\oplus_c] \to B[\oplus_c]$. In the terminology of Boyer-Mader [2], $\oplus_c$ is a discrete class and $A \to A[\oplus_c]$, $f \to f$ is the corresponding minimal functorial topology. This minimal functorial topology as well as the associated topology on an individual group is called the *$\oplus_c$-topology*. Every group $A[\oplus_c]$ has a (Hausdorff) completion $\check{A}$ and if the completion topology of $\check{A}$ is the $\oplus_c$-topology then A is called *completable*; if every A is completable then the $\oplus_c$-topology is *completable*. A crucial result in [5], Theorem 1.4, states that the $\oplus_c$-topology is indeed completable. In this note we disprove this claim. This is achieved by noting that separable $p^{\omega+1}$-projective p-groups are either $\oplus_c$-complete or not completable. We then construct such groups which are $\oplus_c$-incomplete as well as some which are $\oplus_c$-complete. Unfortunately, the error invalidates most of D'Este's results, and as it stands very little is known about the $\oplus_c$-topology.

In Section 2 we summarize what is known about the $\oplus_c$-topology. Section 3 contains our examples.

All groups in this paper are abelian. The notation is standard and follows Fuchs [6]. The background on linear functorial topologies can be found in Mader [9]. Unless indicated otherwise a topological group carries the $\oplus_c$-topology. $\check{A}$ denotes the $\oplus_c$-completion of A, and $\hat{A}$ the p-adic

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completion. The explicit construction of the completion of a group with linear topology can be found in [6; Vol. I, pp. 68/69] as well as the definition of the appropriate topology which is called the completion topology. Suppose T is a functorial topology so that, for every abelian group A, we obtain the topological group TA with A as the underlying group. Every subgroup of A then has two topologies: its own functorial topology and the topology induced by the topology of TA. The subgroup is called T-concordant if these two topologies coincide. Maps are written on the right.

I owe thanks to Ray Mines with whom I began studying the paper of G. D'Este and who first noted the likely errors.

3. – Properties of the $\oplus_c$-topology.

Most of the results in this section are due to D'Este [5]. We indicate how the results follow from the general considerations of Mader [9].

The first observation follows from the fact that the class $\oplus_c$ is closed under arbitrary direct sums ([9; 3.21 and 4.1c]).

(2.1) $(\oplus_i A_i)^{\vee} = \oplus_i \check{A}_i$. *In particular,* ([5; 2.1]) *any direct sum of $\oplus_c$-complete groups is $\oplus_c$-complete.* □

The following fact is true for any minimal functorial topology and follows from [9; 3.21 and 4.1c].

(2.2) ([5; *Lemma* 1.3]). *A direct summand of a $\oplus_c$-complete group is $\oplus_c$-complete.* □

The next result essentially follows from the fact that an extension of a direct sum of cyclic groups by a bounded group is a direct sum of cyclic groups ([6; 18.3]).

(2.3) ([5; *Lemma* 2.3]). *If B is a subgroup of A such that A/B is a bounded p-group then B is a $\oplus_c$-concordant open and hence closed subgroup of A. Thus A is $\oplus_c$-complete if and only if B is $\oplus_c$-complete.* □

It is helpful to compare the $\oplus_c$-topology with better understood topologies. If A/U is a bounded p-group then $A/U \in \oplus_c$. Hence the p-adic topology is weaker than the $\oplus_c$-topology. On the other hand it is easy to see that each subgroup U with $A/U \in \oplus_c$ is closed in the p-adic topology. Hence [9; 4.11] applies:

(2.4) *The natural map* $\check{A} \to \hat{A}$, *where* $\hat{A}$ *denotes the p-adic completion of* A, *is injective.* □

If A is a p-group and U is a large subgroup of A then $A/U \in \oplus_c$ by [6; 67.4]. Hence the large subgroup or inductive topology is weaker than the $\oplus_c$-topology. It is well-known ([3; 3.9] or [4; 2.8]) that the completion of a p-group A in the large subgroup topology is its torsion-completion $\bar{A}$, i.e. the maximal torsion subgroup of $\hat{A}$. Also the p-adic topology is weaker than the large subgroup topology. Hence there are natural maps $\check{A} \to \bar{A} \to \hat{A}$. The following fact now follows from (2.4).

(2.5) ([5; *Lemma* 1.2]). *For a p-group A the* $\oplus_c$*-completion* $\check{A}$ *is naturally imbedded in the torsion-completion* $\bar{A}$. *In particular,* $\check{A}$ *is again p-primary.* □

If $\check{A}$ is purely imbedded in $\bar{A}$ then $\bar{A}$ is also the torsion completion of $\check{A}$ and by (2.5) we have $(A^{\vee})^{\vee}$ imbedded in $\bar{A}$. Thus, if $\check{A}$ *were always* purely imbedded in $\bar{A}$ then $(A^{\vee})^{\vee}$ and all transfinitely iterated $\oplus_c$-completions would be contained in $\bar{A}$ and hence the chain of iterated completions would have to become stationary. It will be shown below that for minimal functorial topologies in general, the chain of iterated completions of a group A becomes stationary if and only if it is constant, i.e. A is completable.

This is D'Este's idea. It fails because $\check{A}$ need not be pure in $\bar{A}$, and the error is made in the middle of page 244 by equating two distinct imbeddings in $\bar{A}$.

(2.6) ITERATED COMPLETIONS. *Let T be a minimal functorial topology on the category of abelian groups. Let LA be the completion of TA as an abstract group and let $\varepsilon_A\colon A \to LA$ be the natural map. For simplicity assume that TA is Hausdorff. Define $L^0A = A$, $L^1A = LA$, $\varepsilon_{01}\colon L^0A \to L^1A\colon \varepsilon_{01} = \varepsilon_A$ and let $\varepsilon_{ii}\colon L^iA \to L^iA\colon \varepsilon_{ii} = 1$. Suppose $L^\alpha A$ and maps $\varepsilon_{\alpha\beta}\colon L^\alpha A \to L^\beta A$ have been defined for $\alpha \leqslant \beta < \lambda$ satisfying $\varepsilon_{\alpha\beta} \circ \varepsilon_{\beta\gamma} = \varepsilon_{\alpha\gamma}$ for $\alpha \leqslant \beta \leqslant \gamma < \lambda$. If $\lambda - 1$ exists let $L^\lambda A = L(L^{\lambda-1}A)$ and $\varepsilon_{\alpha\lambda} = \varepsilon_{\alpha\lambda_{-1}} \circ \varepsilon_{L^{\lambda-1}A}$; if λ is a limit ordinal let $L^\lambda A = \varinjlim \{L^\alpha A\colon \alpha < \lambda\}$ and $\varepsilon_{\alpha\lambda} = \varinjlim \{\varepsilon_{\alpha\beta}\colon \alpha \leqslant \beta < \lambda\}$. In any case let $\varepsilon_{\lambda\lambda} = 1$. Then, clearly, each $\varepsilon_{\alpha\beta}$ is injective and $\varepsilon_{\alpha\beta} \circ \varepsilon_{\beta\gamma} = \varepsilon_{\alpha\gamma}$ for $\alpha \leqslant \beta \leqslant \gamma$. Furthermore, if some $\varepsilon_{\alpha\beta}$ with $\alpha < \beta$ is bijective then A is completable, and if so the whole chain of iterated completions is constant.*

PROOF. Suppose $\varepsilon_{\lambda\beta}$ is bijective for $\lambda < \beta$. Then $\varepsilon_{\lambda\,\lambda+1}\colon L^\lambda A \to L(L^\lambda A)$ is bijective, i.e. $TL^\lambda A$ is complete. We identify all $L^\alpha A$, $\alpha \leqslant \lambda$, with their images in $L^\lambda A$. By [9; 5.7] $L^2A = LA \oplus K_1$. Suppose that K_α, $\alpha < \mu \leqslant \lambda$, has been found such that $L^\alpha A = LA \oplus K_\alpha$ and $K_\alpha \leqslant K_\beta$ for $\alpha \leqslant \beta < \mu$. If

$\mu-1$ exists then $L^{\mu}A = L(L^{\mu-1}A) = L^2A \oplus LK_\alpha = LA \oplus (K_1 \oplus LK_\alpha)$ and we let $K_\mu = K_1 \oplus LK_\alpha$. If μ is a limit ordinal then $L^\mu A = \bigcup_{\alpha<\mu} L^\alpha A = LA \oplus \bigcup_{\alpha<\mu} K_\alpha$ and we let $K_\mu = \bigcup_{\alpha<\mu} K_\alpha$. Hence, by induction, $L^\lambda A = LA \oplus K_\lambda$ and TLA is complete as a direct summand of a complete group. □

Megibben [10] called a p-group A *thick* if $A/U \in \oplus_c$ implies that U contains a large subgroup of A.

(2.7) ([5; 1.1]). *A p-group A is thick if and only if the $\oplus_c$-topology on A coincides with the large subgroup topology. The completion of a thick group A is its torsion-completion $\bar{A}$ and every thick group is $\oplus_c$-completable.*

PROOF. It has been mentioned earlier that $\bar{A}$ is the completion of A in the large subgroup topology. Since $\bar{A}/A$ is divisible, it is $\oplus_c$-indiscrete and it follows from the completability criterion of Mines-Oxford (see [9; 5.10 (6)]) that A is completable. □

The next result follows immediately from (2.7) and (2.1).

(2.8) ([5; 2.2]). *Direct sums of torsion-complete p-groups are $\oplus_c$-complete.* □

A little more can be asserted.

(2.9) *If A is the direct sum of thick groups A_i then $\check{A} = \oplus_i \bar{A}_i$ and A is completable although usually not thick.*

PROOF. $\check{A} = \oplus_i \check{A}_i$ and $\check{A}_i = \bar{A}_i$ since A_i is thick. Completability follows since $\check{A}/A \cong \oplus \bar{A}_i/A_i$ is divisible. If $A = \oplus_i A_i$ and if A is thick then $\bar{A} = \oplus_i \bar{A}_i$. By [6; 71.3] there is m such that $p^m \bar{A}_i = 0$ for almost all i. Hence A is usually not thick. □

(2.10) REMARK. *We just showed: If $A = \oplus_i A_i$ is thick then, for some positive integer m, $p^m A_i = 0$ for almost all i.* □

There is also a large class of groups for which the $\oplus_c$-topology coincides with the p-adic topology. This is trivially the case for torsion-free groups of finite p-rank ($= \dim A/pA$).

(2.11) *The $\oplus_c$-completion of a direct sum of torsion-free groups of finite rank is the free p-adic module with the same p-rank. Such a group is $\oplus_c$-completable.* □

More interesting examples are provided by the theory of Howard [7]. If a group A is of second category in its p-adic topology then every reduced p-primary epimorphic image of A is bounded hence the p-adic topology and the $\oplus_c$-topology on A coincide. Examples of second category groups are the p-adically complete groups, but ([7; 4.3]) there are others as well, a situation very much reminiscent of thick groups. In [8; 4.6] it was shown that every reduced p-primary epimorphic image of a group K is bounded if and only if K is not the union of an ascending sequence of p-adically nowhere dense subgroups. Thus such groups are $\oplus_c$-completable and their completions are just the p-adic completions.

3. – Groups which are not completable.

It appears to be rather difficult to determine the $\oplus_c$-completions in general. As far as completability is concerned $p^{\omega+1}$-projective p-groups are particularly simple since they are either complete or not completable as we will show first. Recall that a $p^{\omega+1}$-projective group is an extension of an elementary p-group by a direct sum of cyclic p-groups (*). Thus a $p^{\omega+1}$-projective group contains an open subgroup which is elementary. This fact is exploited in the first lemma.

(3.1) LEMMA. *Let A be a separable p-group having a subsocle T with $A/T \in \oplus_c$. Then $\check{A}/A \cong T^{\sharp}/T$ where $T^{\sharp}$ is both the topological closure of T in $\check{A}$ and the completion of T when T has the topology induced by the $\oplus_c$-topology of A. Hence $\check{A}/A$ is p-bounded and A is completable if and only if A is complete.*

PROOF. We have ([9; 4.5]) the following commutative diagram with exact rows:

$$\begin{array}{rccccccccc} E\colon & 0 & \to & T & \to & A & \to & A/T & \to & 0 \\ & & & \Big\downarrow{\scriptstyle\eta} & & \Big\downarrow & & \Big\| & & \\ \eta E\colon & 0 & \to & T^{\sharp} & \to & \check{A} & \to & A/T & \to & 0 \end{array}$$

A diagram chase yields $\check{A}/A \cong T^{\sharp}/T$. Thus $\check{A}/A$ is p-elementary. By the Completability Criterion [9; 5.10 (6)] A is completable if and only if $\check{A}/A$ is p-divisible, i.e. if and only if $\check{A} = A$. □

(*) R. NUNKE, *Purity and subfunctors of the identity*, *Topics in Abelian Groups*, Scott, Foresman & Co, Chicago, 1963, pp. 121-171.

The problem is now reduced to deciding whether or not T is complete with the induced topology. In general it is neither clear what this induced topology might look like nor what the completion is. Fortunately, a method due to Benabdallah-Irwin [1] permits to construct a group A such that the induced topology on T is the topology induced by the p-adic topology on A, and this case can be handled.

We first need a special case of a theorem by Benabdallah-Irwin [1; Theorem 2.2].

(3.2) LEMMA. *If G is a p-group and K a pure subgroup of G such that $G/K[p]\in\bigoplus_c$ then K is a direct summand of G.*

Starting with any p-group G the method of Benabdallah-Irwin [1; pp. 326-327] yields a $p^{\omega+1}$-projective group A whose properties are related to those of G.

(3.3) CONSTRUCTION. *Let G be a given p-gruop. Let $\tilde{G}=\bigoplus\{\langle\tilde{g}\rangle : g\in G\}$ where $\langle\tilde{g}\rangle\cong\langle g\rangle$, and let $\varepsilon:\tilde{G}\to G:\tilde{g}\varepsilon=g$. It is well-known that $K=\mathrm{Ker}\,\varepsilon$ is pure in $\tilde{G}$. Put $A=\tilde{G}/K[p]$. Then $T=\tilde{G}[p]/K[p]$ is a subsocle of A with $A/T\cong\tilde{G}/\tilde{G}[p]\cong p\tilde{G}\in\bigoplus_c$. Hence A is $p^{\omega+1}$projective. Furthermore by 3.2, $A\in\bigoplus_c$ if and only if $G\in\bigoplus_c$.*

In the following we always refer to this situation placing stronger and stronger conditions on G.

(3.4) *Let G be separable. Then A is separable.*

PROOF. G separable implies that K is p-adically closed in $\tilde{G}$. So is $\tilde{G}[p]$, and hence $K[p]=K\cap\tilde{G}[p]$. Thus $A=\tilde{G}/K[p]$ is separable. □

(3.5) *Let G be pure-complete. Then for any subsocle S with $K[p]\leqslant S\leqslant\tilde{G}[p]$ there exists a pure subgroup L of $\tilde{G}$ containing K with $L[p]=S$.*

PROOF. Since $S\varepsilon\leqslant G[p]$ and G is pure-complete there is a pure subgroup M of G with $M[p]=S\varepsilon$. Let $L=M\varepsilon^{-1}$. Then L is pure in $\tilde{G}$ and contains K. It is easily checked that $L[p]=S$. □

(3.6) *Let G be quasi-complete. If $K[p]\leqslant S\leqslant\tilde{G}[p]$ and $\tilde{G}/S\in\bigoplus_c$ then $\tilde{G}=L\oplus M$ with $L[p]=S$ and M bounded.*

PROOF. By [6; 74.2] G is pure-complete. Hence, by (3.5) and (3.2). there exist groups L and M such that $\tilde{G}=L\oplus M$, $K\leqslant L$ and $L[p]=S$, Now $G \cong (L/K) \oplus M$ and $M \in \oplus_c$. If G is torsion-complete then so is M and hence M is bounded. If G is not torsion-complete then by [6; 74.6] either L/K or M is bounded. But $M \in \oplus_c$ cannot be unbounded in view of [6; 74.6]. $\square$

(3.7) *Let G be quasi-complete. The topology induced on $T = \tilde{G}[p]/K[p]$ by the $\oplus_c$-topology of $A = \tilde{G}/K[p]$ has the local basis $\{(p^n\tilde{G}[p] + K[p])/K[p]: n \in \omega\}$. Thus the $\oplus_c$-topology and the p-adic topology on A induce the same topology on T.*

PROOF. It is clear that each $(p^n\tilde{G}[p] + K[p])/K[p]$ is open in T. Suppose U is an open subgroup of $A[\oplus_c]$. Then so is $U \cap T$ and hence there exists a subgroup S of $\tilde{G}$ such that $S/K[p] \leqslant U \cap T$ and $\tilde{G}/S \in \oplus_c$. By (3.6) there exists n such that $(p^n\tilde{G})[p] \subset S$ hence

$$(p^n\tilde{G}[p] + K[p])/K[p] \leqslant U \cap T. \quad \square$$

We now relate the topological group T to the socle of G.

(3.8) *For any group G, the groups $G[p]$ and $T = \tilde{G}[p]/K[p]$ are isomorphic as topological groups with topologies induced by the p-adic topologies on G and $\tilde{G}$ respectively.*

PROOF. Clearly $\varepsilon: \tilde{G} \to G$ induces an isomorphism $\varepsilon: T \to G[p]$ with

$$\big((p^n\tilde{G}[p] + K[p])/K[p]\big)\varepsilon = p^nG[p]. \quad \square$$

(3.9) THEOREM. *Let G be quasi-complete, $0 \to K \to \tilde{G} \to G \to 0$ the standard pure-projective resolution of G and $A = \tilde{G}/K[p]$. Then A is a separable $p^{\omega+1}$-projective group and $A[\oplus_c]$ is complete if and only if G is torsion-complete.*

PROOF. By the construction (3.3) we have that A is $p^{\omega+1}$-projective, and A is separable by (3.4). Since G is quasi-complete the p-adic and the $\oplus_c$-topologies on A induce the same topology on $T = \tilde{G}[p]/K[p]$ by (3.7). By (3.1), A is complete if and only if T is complete. But T and $G[p]$ are isomorphic topological groups by (3.8) where $G[p]$ has the topology induced by the p-adic topology on G. By [6; 70.6] $G[p]$ is complete if and only if G is torsion-complete. Thus A is complete if and only if G is torsion-complete. $\square$

(3.10) COROLLARY. *The $\oplus_c$-topology is not completable.*

PROOF. There exist quasi-complete groups which are not torsion-complete ([6], Vol. II, p. 48). Results (3.9) and (3.1) complete the proof. □

Thus a $p^{\omega+1}$-projective group may or may not be complete. The class of $\oplus_c$-complete group is smaller than it appeared in [5], and many of the theorems of [5] now became open questions, e.g. are $\oplus_c$-complete p-groups determined by their valuated socles?

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