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The $\oplus_c$-topology is not completable


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1. Introduction.

G. D'Este [5] introduced and studied an interesting and difficult functorial topology defined on the category of abelian groups: Let $\oplus_c$ be the class of all direct sums of cyclic $p$-groups. For each group $A$ let $\mathcal{U}_A = \{U < A : A/U \in \oplus_c\}$. Then $\mathcal{U}_A$ is a neighborhood basis at 0, a «local basis» for short, for some topology on $A$ which makes $A$ into a topological group. We write $A[\mathcal{U}_A] = A[\oplus_c]$ for this topological group. Every homomorphism $f : A \to B$ is then a continuous map $f : A[\oplus_c] \to B[\oplus_c]$. In the terminology of Boyer-Mader [2], $\oplus_c$ is a discrete class and $A \to A[\oplus_c]$, $f \to f$ is the corresponding minimal functorial topology. This minimal functorial topology as well as the associated topology on an individual group is called the $\oplus_c$-topology. Every group $A[\oplus_c]$ has a (Hausdorff) completion $\hat{A}$ and if the completion topology of $\hat{A}$ is the $\oplus_c$-topology then $A$ is called completable; if every $A$ is completable then the $\oplus_c$-topology is completable. A crucial result in [5], Theorem 1.4, states that the $\oplus_c$-topology is indeed completable. In this note we disprove this claim. This is achieved by noting that separable $p^{\omega+1}$-projective $p$-groups are either $\oplus_c$-complete or not completable. We then construct such groups which are $\oplus_c$-incomplete as well as some which are $\oplus_c$-complete. Unfortunately, the error invalidates most of D'Este's results, and as it stands very little is known about the $\oplus_c$-topology.

In Section 2 we summarize what is known about the $\oplus_c$-topology. Section 3 contains our examples.

All groups in this paper are abelian. The notation is standard and follows Fuchs [6]. The background on linear functorial topologies can be found in Mader [9]. Unless indicated otherwise a topological group carries the $\oplus_c$-topology. $\hat{A}$ denotes the $\oplus_c$-completion of $A$, and $\hat{A}$ the $p$-adic
completion. The explicit construction of the completion of a group with linear topology can be found in [6; Vol. I, pp. 68/69] as well as the definition of the appropriate topology which is called the completion topology. Suppose $T$ is a functorial topology so that, for every abelian group $A$, we obtain the topological group $TA$ with $A$ as the underlying group. Every subgroup of $A$ then has two topologies: its own functorial topology and the topology induced by the topology of $TA$. The subgroup is called $T$-concordant if these two topologies coincide. Maps are written on the right.

I owe thanks to Ray Mines with whom I began studying the paper of G. D’Este and who first noted the likely errors.

3. - Properties of the $\oplus_c$-topology.

Most of the results in this section are due to D’Este [5]. We indicate how the results follow from the general considerations of Mader [9].

The first observation follows from the fact that the class $\oplus_c$ is closed under arbitrary direct sums ([9; 3.21 and 4.1c]).

(2.1) $\bigoplus_c A_i \sim = \bigoplus_c \tilde{A}_i$. In particular, ([5; 2.1]) any direct sum of $\oplus_c$-complete groups is $\oplus_c$-complete. □

The following fact is true for any minimal functorial topology and follows from [9; 3.21 and 4.1c].

(2.2) ([5; Lemma 1.3]). A direct summand of a $\oplus_c$-complete group is $\oplus_c$-complete. □

The next result essentially follows from the fact that an extension of a direct sum of cyclic groups by a bounded group is a direct sum of cyclic groups ([6; 18.3]).

(2.3) ([5; Lemma 2.3]). If $B$ is a subgroup of $A$ such that $A/B$ is a bounded $p$-group then $B$ is a $\oplus_c$-concordant open and hence closed subgroup of $A$. Thus $A$ is $\oplus_c$-complete if and only if $B$ is $\oplus_c$-complete. □

It is helpful to compare the $\oplus_c$-topology with better understood topologies. If $A/U$ is a bounded $p$-group then $A/U \in \oplus_c$. Hence the $p$-adic topology is weaker than the $\oplus_c$-topology. On the other hand it is easy to see that each subgroup $U$ with $A/U \in \oplus_c$ is closed in the $p$-adic topology. Hence [9; 4.11] applies:
The natural map $\tilde{A} \to \hat{A}$, where $\hat{A}$ denotes the $p$-adic completion of $A$, is injective. □

If $A$ is a $p$-group and $U$ is a large subgroup of $A$ then $A/U \in \bigoplus_c$ by [6, 67.4]. Hence the large subgroup or inductive topology is weaker than the $\bigoplus_c$-topology. It is well-known ([3; 3.9] or [4; 2.8]) that the completion of a $p$-group $A$ in the large subgroup topology is its torsion-completion $\tilde{A}$, i.e. the maximal torsion subgroup of $\hat{A}$. Also the $p$-adic topology is weaker than the large subgroup topology. Hence there are natural maps $\tilde{A} \to \hat{A} \to \tilde{A}$. The following fact now follows from (2.4).

(2.5) ([5; Lemma 1.2]). For a $p$-group $A$ the $\bigoplus_c$-completion $\tilde{A}$ is naturally imbedded in the torsion-completion $\hat{A}$. In particular, $\tilde{A}$ is again $p$-primary. □

If $\tilde{A}$ is purely imbedded in $\hat{A}$ then $\tilde{A}$ is also the torsion completion of $\tilde{A}$ and by (2.5) we have $(\tilde{A})^\vee$ imbedded in $\tilde{A}$. Thus, if $\tilde{A}$ were always purely imbedded in $\hat{A}$ then $(\tilde{A})^\vee$ and all transfinitely iterated $\bigoplus_c$-completions would be contained in $\tilde{A}$ and hence the chain of iterated completions would have to become stationary. It will be shown below that for minimal functorial topologies in general, the chain of iterated completions of a group $A$ becomes stationary if and only if it is constant, i.e. $A$ is completable.

This is D'Este's idea. It fails because $\tilde{A}$ need not be pure in $\hat{A}$, and the error is made in the middle of page 244 by equating two distinct imbeddings in $\tilde{A}$.

(2.6) ITERATED COMPLETIONS. Let $T$ be a minimal functorial topology on the category of abelian groups. Let $L^0A = A$, $L^1A = LA$, $\varepsilon_{01}: L^0A \to L^1A$: $\varepsilon_{01} = \varepsilon_A$ and let $\varepsilon_{ii}: L^iA \to L^{i-1}A$: $\varepsilon_{ii} = 1$. Suppose $L^\alpha A$ and maps $\varepsilon_{\alpha\beta}: L^\alpha A \to L^\beta A$ have been defined for $\alpha < \beta < \lambda$ satisfying $\varepsilon_{\alpha\beta} \varepsilon_{\beta\gamma} = \varepsilon_{\alpha\gamma}$ for $\alpha < \beta < \gamma < \lambda$. If $\lambda - 1$ exists let $L\lambda A = L(L^{\lambda-1}A)$ and $\varepsilon_{\lambda\alpha} = \varepsilon_{\alpha\lambda-1} \varepsilon_{\lambda-1\alpha}$; if $\lambda$ is a limit ordinal let $L\lambda A = \lim L^\alpha A$; $\alpha < \lambda$ and $\varepsilon_{\alpha\lambda} = \lim \varepsilon_{\alpha\beta}$; $\alpha < \beta < \lambda$. In any case let $\varepsilon_{\lambda\lambda} = 1$. Then, clearly, each $\varepsilon_{\alpha\beta}$ is injective and $\varepsilon_{\alpha\beta} \varepsilon_{\beta\gamma} = \varepsilon_{\alpha\gamma}$ for $\alpha < \beta < \gamma$. Furthermore, if some $\varepsilon_{\alpha\beta}$ with $\alpha < \beta$ is bijective then $\tilde{A}$ is completable, and if so the whole chain of iterated completions is constant.

PROOF. Suppose $\varepsilon_{\alpha\beta}$ is bijective for $\lambda < \beta$. Then $\varepsilon_{\lambda\lambda+1}: L^\lambda A \to L(L^\lambda A)$ is bijective, i.e. $TL^\lambda A$ is complete. We identify all $L^\alpha A$, $\alpha < \lambda$, with their images in $L^\lambda A$. By [9; 5.7] $L^\alpha A = LA \oplus K_\alpha$. Suppose that $K_\alpha$, $\alpha < \mu < \lambda$, has been found such that $L^\alpha A = LA \oplus K_\alpha$ and $K_\alpha < K_\beta$ for $\alpha < \beta < \mu$. If
If $\mu - 1$ exists then $L^\mu A = L(L^{\mu - 1}A) = L^\mu A \oplus LK_\alpha = LA \oplus (K_1 \oplus LK_\alpha)$ and we let $K_\mu = K_1 \oplus LK_\alpha$. If $\mu$ is a limit ordinal then $L^\mu A = \bigcup_{\alpha < \mu} L^\alpha A = LA \oplus \bigcup_{\alpha < \mu} K_\alpha$ and we let $K_\mu = \bigcup_{\alpha < \mu} K_\alpha$. Hence, by induction, $L^\alpha A = LA \oplus K_\alpha$ and $T^\alpha LA$ is complete as a direct summand of a complete group. 

Megibben [10] called a $p$-group $A$ thick if $A/U \in \oplus_c$ implies that $U$ contains a large subgroup of $A$.

(2.7) ([5; 1.1]). A $p$-group $A$ is thick if and only if the $\oplus_c$-topology on $A$ coincides with the large subgroup topology. The completion of a thick group $A$ is its torsion-completion $\bar{A}$ and every thick group is $\oplus_c$-completable.

**Proof.** It has been mentioned earlier that $\bar{A}$ is the completion of $A$ in the large subgroup topology. Since $\bar{A}/A$ is divisible, it is $\oplus_c$-indiscrete and it follows from the completability criterion of Mines-Oxford (see [9; 5.10 (6)]) that $A$ is completable. 

The next result follows immediately from (2.7) and (2.1).

(2.8) ([5; 2.2]). Direct sums of torsion-complete $p$-groups are $\oplus_c$-complete. 

A little more can be asserted.

(2.9) If $A$ is the direct sum of thick groups $A_i$ then $\bar{A} = \oplus_i \bar{A}_i$ and $A$ is completable although usually not thick.

**Proof.** $\bar{A} = \oplus_i \bar{A}_i$ and $\bar{A}_i = \bar{A}_i$ since $A_i$ is thick. Completablity follows since $\bar{A}/A \cong \oplus \bar{A}_i/A_i$ is divisible. If $A = \oplus_i A_i$ and if $A$ is thick then $\bar{A} = \oplus_i \bar{A}_i$. By [6; 71.3] there is $m$ such that $p^m \bar{A}_i = 0$ for almost all $i$. Hence $A$ is usually not thick. 

(2.10) **Remark.** We just showed: If $A = \oplus_i A_i$ is thick then, for some positive integer $m$, $p^m A_i = 0$ for almost all $i$. 

There is also a large class of groups for which the $\oplus_c$-topology coincides with the $p$-adic topology. This is trivially the case for torsion-free groups of finite $p$-rank ($= \dim A/pA$).

(2.11) The $\oplus_c$-completion of a direct sum of torsion-free groups of finite rank is the free $p$-adic module with the same $p$-rank. Such a group is $\oplus_c$-completable. 

More interesting examples are provided by the theory of Howard [7]. If a group $A$ is of second category in its $p$-adic topology then every reduced $p$-primary epimorphic image of $A$ is bounded hence the $p$-adic topology and the $\oplus_c$-topology on $A$ coincide. Examples of second category groups are the $p$-adically complete groups, but ([7; 4.3]) there are others as well, a situation very much reminiscent of thick groups. In [8; 4.6] it was shown that every reduced $p$-primary epimorphic image of a group $K$ is bounded if and only if $K$ is not the union of an ascending sequence of $p$-adically nowhere dense subgroups. Thus such groups are $\oplus_c$-completable and their completions are just the $p$-adic completions.

3. – Groups which are not completable.

It appears to be rather difficult to determine the $\oplus_c$-completions in general. As far as completability is concerned $p^{a+1}$-projective $p$-groups are particularly simple since they are either complete or not completable as we will show first. Recall that a $p^{a+1}$-projective group is an extension of an elementary $p$-group by a direct sum of cyclic $p$-groups (*). Thus a $p^{a+1}$-projective group contains an open subgroup which is elementary. This fact is exploited in the first lemma.

(3.1) Lemma. Let $A$ be a separable $p$-group having a subsocle $T$ with $A/T \in \oplus_c$. Then $\bar{A}/A \cong T'/T$ where $T'$ is both the topological closure of $T$ in $\bar{A}$ and the completion of $T$ when $T$ has the topology induced by the $\oplus_c$-topology of $A$. Hence $\bar{A}/A$ is $p$-bounded and $A$ is completable if and only if $A$ is complete.

Proof. We have ([9; 4.5]) the following commutative diagram with exact rows:

$$
\begin{array}{ccc}
E: 0 \rightarrow T \rightarrow A \rightarrow A/T \rightarrow 0 \\
\eta\downarrow \quad \downarrow \quad \quad \\
\eta E: 0 \rightarrow T' \rightarrow \bar{A} \rightarrow A/T \rightarrow 0
\end{array}
$$

A diagram chase yields $\bar{A}/A \cong T'/T$. Thus $\bar{A}/A$ is $p$-elementary. By the Completability Criterion [9; 5.10 (6)] $A$ is completable if and only if $\bar{A}/A$ is $p$-divisible, i.e. if and only if $\bar{A} = A$.

The problem is now reduced to deciding whether or not $T$ is complete with the induced topology. In general it is neither clear what this induced topology might look like nor what the completion is. Fortunately, a method due to Benabdallah-Irwin [1] permits to construct a group $A$ such that the induced topology on $T$ is the topology induced by the $p$-adic topology on $A$, and this case can be handled.

We first need a special case of a theorem by Benabdallah-Irwin [1; Theorem 2.2].

(3.2) **Lemma.** If $G$ is a $p$-group and $K$ a pure subgroup of $G$ such that $G/K[p] \in \bigoplus_e$ then $K$ is a direct summand of $G$.

Starting with any $p$-group $G$ the method of Benabdallah-Irwin [1; pp. 326-327] yields a $p^{\omega_1}$-projective group $A$ whose properties are related to those of $G$.

(3.3) **Construction.** Let $G$ be a given $p$-group. Let $\bar{G} = \bigoplus \langle \bar{g} \rangle : g \in G \rangle$ where $\langle \bar{g} \rangle \cong \langle g \rangle$, and let $\varepsilon : \bar{G} \to G : \bar{g} = g$. It is well-known that $K = \text{Ker } \varepsilon$ is pure in $\bar{G}$. Put $A = \bar{G}/K[p]$. Then $T = \bar{G}/[p]/K[p]$ is a subsocle of $A$ with $T \cong \bar{G}/[p]$ and $\omega_1$. Hence $A$ is $p^{\omega_1}$-projective. Furthermore by 3.2, $A \in \bigoplus_e$ if and only if $G \in \bigoplus_e$.

In the following we always refer to this situation placing stronger and stronger conditions on $G$.

(3.4) **Let $G$ be separable.** Then $A$ is separable.

**Proof.** $G$ separable implies that $K$ is $p$-adically closed in $\bar{G}$. So is $\bar{G}[p]$, and hence $K[p] = K \cap \bar{G}[p]$. Thus $A = \bar{G}/K[p]$ is separable. □

(3.5) **Let $G$ be pure-complete.** Then for any subsocle $S$ with $K[p] < S < \bar{G}[p]$ there exists a pure subgroup $L$ of $\bar{G}$ containing $K$ with $L[p] = S$.

**Proof.** Since $S \in \bigoplus_e G[p]$ and $G$ is pure-complete there is a pure subgroup $M$ of $G$ with $M[p] = S$. Let $L = M^{-1}$. Then $L$ is pure in $\bar{G}$ and contains $K$. It is easily checked that $L[p] = S$. □

(3.6) **Let $G$ be quasi-complete.** If $K[p] < S < \bar{G}[p]$ and $\bar{G}/S \in \bigoplus_e$ then $\bar{G} = L \oplus M$ with $L[p] = S$ and $M$ bounded.

**Proof.** By [6; 74.2] $G$ is pure-complete. Hence, by (3.5) and (3.2), there exist groups $L$ and $M$ such that $\bar{G} = L \oplus M$, $K \subset L$ and $L[p] = S$, and
Now \( G \cong (L/K) \oplus M \) and \( M \in \oplus_e \). If \( G \) is torsion-complete then so is \( M \) and hence \( M \) is bounded. If \( G \) is not torsion-complete then by [6; 74.6] either \( L/K \) or \( M \) is bounded. But \( M \in \oplus_e \) cannot be unbounded in view of [6; 74.6]. □

(3.7) Let \( G \) be quasi-complete. The topology induced on \( T = \bar{G}[p]/K[p] \) by the \( \oplus_e \)-topology of \( A = \bar{G}/K[p] \) has the local basis \( \{(p^n\bar{G}[p] + K[p])/K[p]: n \in \omega \} \). Thus the \( \oplus_e \)-topology and the \( p \)-adic topology on \( A \) induce the same topology on \( T \).

**Proof.** It is clear that each \( (p^n\bar{G}[p] + K[p])/K[p] \) is open in \( T \). Suppose \( U \) is an open subgroup of \( A[\oplus_e] \). Then so is \( U \cap T \) and hence there exists a subgroup \( S \) of \( \bar{G} \) such that \( S/K[p] < U \cap T \) and \( \bar{G}/S \in \oplus_e \). By (3.6) there exists \( n \) such that \( (p^n\bar{G}[p]) < S \) hence

\[
(p^n\bar{G}[p] + K[p])/K[p] < U \cap T. \quad □
\]

We now relate the topological group \( T \) to the socle of \( G \).

(3.8) For any group \( G \), the groups \( G[p] \) and \( T = \bar{G}[p]/K[p] \) are isomorphic as topological groups with topologies induced by the \( p \)-adic topologies on \( G \) and \( \bar{G} \) respectively.

**Proof.** Clearly \( \varepsilon: \bar{G} \to G \) induces an isomorphism \( \varepsilon: T \to G[p] \) with

\[
((p^n\bar{G}[p] + K[p])/K[p]) \varepsilon = p^nG[p]. \quad □
\]

(3.9) **Theorem.** Let \( G \) be quasi-complete, \( 0 \to K \to \bar{G} \to G \to 0 \) the standard pure-projective resolution of \( G \) and \( A = \bar{G}/K[p] \). Then \( A \) is a separable \( p^{\omega+1} \)-projective group and \( A[\oplus_e] \) is complete if and only if \( G \) is torsion-complete.

**Proof.** By the construction (3.3) we have that \( A \) is \( p^{\omega+1} \)-projective, and \( A \) is separable by (3.4). Since \( G \) is quasi-complete the \( p \)-adic and the \( \oplus_e \)-topologies on \( A \) induce the same topology on \( T = \bar{G}[p]/K[p] \) by (3.7). By (3.1), \( A \) is complete if and only if \( T \) is complete. But \( T \) and \( G[p] \) are isomorphic topological groups by (3.8) where \( \bar{G}[p] \) has the topology induced by the \( p \)-adic topology on \( G \). By [6; 70.6] \( G[p] \) is complete if and only if \( G \) is torsion-complete. Thus \( A \) is complete if and only if \( G \) is torsion-complete. □

(3.10) **Corollary.** The \( \oplus_e \)-topology is not completable.
PROOF. There exist quasi-complete groups which are not torsion-complete ([6], Vol. II, p. 48). Results (3.9) and (3.1) complete the proof. □

Thus a \( p^{\omega+1} \)-projective group may or may not be complete. The class of \( \oplus_c \)-complete group is smaller than it appeared in [5], and many of the theorems of [5] now became open questions, e.g. are \( \oplus_c \)-complete \( p \)-groups determined by their valued socles?

REFERENCES