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Alan Howard
Andrew J. Sommese

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On the Theorem of de Franchis.

ALAN HOWARD - ANDREW J. SOMMESE

1. – Introduction.

Let X be a compact Riemann surface of genus $g\geqslant 2$, and let $\mathrm{Hol}(X)$ denote the set of all surjective holomorphic mappings whose domain is X and whose range has genus $\geqslant 2$. The de Franchis theorem is the assertion that $\mathrm{Hol}(X)$ is finite [F; S. p. 75]. In this note we show that in fact there is a bound on the size of $\mathrm{Hol}(X)$ which depends only on the genus of X. An explicit bound will be constructed in the course of the proof, but it seems far from being sharp. As a corollary we show in section 4 that if M is a projective manifold with ample canonical bundle then there is a bound, depending only on the Chern numbers of M, on the number of surjective holomorphic maps whose domain is M and whose range is a compact Riemann surface of genus $\geqslant 2$.

Several related results should be mentioned. First of all, J. Harris (*) has proved the existence of a bound on $\mathrm{Hol}(X)$ in terms of the genus of X, but without, as far as we can see, giving an explicit estimate. His proof uses the Hilbert scheme. Previously, Henrik Martens showed that there is a bound (explicity computable but probably not sharp), depending only on the genus of X, for the set of all surjective holomorphic mappings from X to a fixed Riemann surface Y of genus $\geqslant 2$. A similar result valid in all dimensions (with « genus » replaced by « Chern numbers ») for X and Y projective manifolds with ample canonical bundle was proved by T. M. Bandman [B]. At the risk of belaboring the obvious we point out the previous two results are valid for a fixed target space, whereas ours (and Harris') bounds the number of target spaces as well.

We would like to thank Bun Wong for suggesting this problem.

(*) Private communication.

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2. – The correspondence associated to a morphism.

In this section we follow, with one small but important modification, the proof of the de Franchis theorem given in [S]. We let $\mathrm{Hol}'(X)$ denote the set of $f\colon X \to Y$ for which $X \neq Y$, i.e. those $f \in \mathrm{Hol}(X)$ which are not automorphisms of X. If $f \in \mathrm{Hol}'(X)$ we consider the correspondence (i.e. subvariety of $X \times X$) defined by

$$T_f = \{(x_1, x_2) \in X \times X\colon f(x_1) = f(x_2)\}$$

and write $T_f = \Delta + S_f$ where Δ is the diagonal. Letting F_i denote the fiber over a generic point of the i-th factor, we have the obvious intersection numbers:

$$S_f \cdot F_1 = S_f \cdot F_2 = d - 1 \quad \text{where } d = \text{degree of } f\,. \tag{2.1}$$

Given two correspondences A and B in $X \times X$, one defines a product as follows:

$$A \circ B = p_{13}((A \times X) \cdot (X \times B))$$

where $p_{13}\colon X \times X \times X \to X \times X$ is the projection onto the product of the first and third factors. The following identities are easily derived [S, p. 70]:

$$T_f \circ T_f = dT_f \tag{2.2}$$

$$S_f \circ S_f = (d-1)\Delta + (d-2)S_f\,. \tag{2.3}$$

Moreover, since the intersection number satisfies [S, p. 70]

$$A \cdot B = (A \circ B) \cdot \Delta\,, \tag{2.4}$$

one obtains the following.

LEMMA 1. *If $f \in \mathrm{Hol}'(X)$ then $0 > S_f \cdot S_f \geqslant (d-1)(2-2g)$.*

PROOF. Combining (2.3) and (2.4) yields

$$S_f \cdot S_f = (d-1)\Delta \cdot \Delta + (d-2)S_f \cdot \Delta\,.$$

To compute $\Delta \cdot \Delta$ we note that the normal bundle of Δ in $X \times X$ is the tangent bundle of Δ, and therefore $\Delta \cdot \Delta = 2 - 2g$.

Next we observe that $S_f \cdot \Delta$ is equal to the total branching index b of f. To see this, first note that $(p, p) \in S_f \cap \Delta$ if and only if p is a branch point of f. Given such a p, choose local coordinates t and u around p and $f(p)$ respectively so that f is given locally $u = t^m$ where $(m-1)$ is the branching index at p. In terms of the local coordinate t, S_f is given in a neighborhood of (p, p) as $\{(t_1, t_2): t_1^{m-1} + t_1^{m-2}t_2 + \ldots + t_2^{m-1} = 0\}$, so that the intersection multiplicity of S_f and Δ at (p, p) is $(m-1)$.

We thus obtain

$$S_f \cdot S_f = (d-1)(2-2g) + (d-2)b \geqslant (d-1)(2-2g) .$$

To obtain the upper bound we use the Riemann-Hurwitz formula to get $b < (2g-2)$, so that

$$S_f \cdot S_f < 2 - 2g < 0 .$$

For any correspondence A let $[A]$ denote the homology class determined by A, so that $[A] \in H_{1,1}(X \times X, \mathbb{Z})$.

LEMMA 2. *The map $f \in \mathrm{Hol}'(X) \to [S_f] \in H_{1,1}(X \times X, \mathbb{Z})$ is at most $g^2(g-1)(\sqrt{2})^{g(g-1)}$ to one.*

PROOF. Fix an $f \in \mathrm{Hol}'(X)$ and consider all $h \in \mathrm{Hol}'(X)$ for which $[S_f] = [S_h]$ but $S_f \neq S_h$. For any such h we have $S_f \cdot S_h = S_f \cdot S_f < 0$, so that S_f and S_h must have a common component. We may write $S_f = D + \hat{S}_f$ and $S_h = D + \hat{S}_h$ where $\hat{S}_f$ and $\hat{S}_h$ have no common component. We then have

$$\begin{aligned} D \circ D + D \circ \hat{S}_f + \hat{S}_f \circ D + \hat{S}_f \circ \hat{S}_f &= S_f \circ S_f \\ &= (d-1)\Delta + (d-2)D + (d-2)\hat{S}_f \end{aligned}$$

(from (2.3)), so that as divisors

$$D \circ D \leqslant (d-1)\Delta + (d-2)D + (d-2)\hat{S}_f .$$

Applying the same argument to S_h we also obtain

$$D \circ D \leqslant (d-1)\Delta + (d-2)D + (d-2)\hat{S}_h ,$$

and since $\hat{S}_f$ and $\hat{S}_h$ have no common component, it follows that

$$D \circ D \leqslant (d-1)\Delta + (d-2)D . \tag{2.5}$$

Now consider the correspondence $G = \Delta + D$. It follows from (2.5) that set-theoretically we have

$$G \circ G \subset G \,. \tag{2.6}$$

Moreover G is easily seen to be a symmetric correspondence, which together with (2.6) shows that G is the correspondence associated to a morphism $\pi\colon X \to X'$ for some Riemann surface X'. This can be seen by letting (p, p_i), $i = 1, \ldots, k$, be the points of G lying above a point $p \in X$ (where $k = G \cdot F_1$), and mapping p into into the point of the k-fold symmetric product of X determined by the k-tuple $(p_1, \ldots, p_k)$. Letting X' be the non-singular model of the image of X under this map, we obtain a map $\pi\colon X \to X'$, and one checks easily that $G = T_\pi$.

We observe that $G \cdot F_1 = \Delta \cdot F_1 + D \cdot F_1 \geqslant 2$, so that π has degree at least 2 and thus the genus g' of X' satisfies $g' < g$. Furthermore, one checks easily that there are morphisms f' and h' making the following diagrams commute:

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y_f & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "h"'] & & X' \arrow[dl, "h'"] \\ & Y_h & \end{tikzcd} \tag{2.7}$$

where Y_f and Y_h are the target spaces of f and h. From the morphisms f' and h' we obtain correspondences T'_f and T'_h in $X' \times X'$, and one sees easily that

$$T'_f = (\pi \times \pi)(T_f) = \Delta' + S'_f$$

and

$$T'_h = (\pi \times \pi)(T_h) = \Delta' + S'_h$$

where Δ' is the diagonal in $X' \times X'$, $S'_f = (\pi \times \pi)(\hat{S}_f)$, and similarly for S'_h.

We note that, having fixed f, the number of possible maps π and target spaces X', and hence the number of possible S'_f, are bounded by the number of possible $D \leqslant S_f$ which can occur as common component with some S_h. This number cannot be greater than the total number of possible divisors $D \leqslant S_f$, which is bounded by 2^{d-1} since from (2.1) it follows that S_f can have no more than $(d-1)$ irreducible components (counted with multiplicity). Since by Riemann-Hurwitz, $d \leqslant g$, we can bound the number of possible S'_f that occur by 2^{g-1}.

We further note that the genus g' of X' is $\geqslant 2$, as follows from diagram (2.7), and that $X' \neq Y_f$ since $F_1 \cdot G < F_1 \cdot T_f$. It follows that S'_f satisfies lemma 1. Moreover S'_f and S'_h are homologous since S_f and S_h are. Therefore either $S'_h = S'_f$ or they are unequal but have a common component.

In the first case we observe that the number of possible h for which $S'_h = S'_f$ is bounded by $(\deg \pi)^2$ times the number of components of S_f, and is therefore, again using Riemann-Hurwitz, no larger than $g^2 \cdot (g-1)$. If, on the other hand, $S'_h \neq S'_f$ we can decompose into $S'_f = D' + \hat{S}'_f$ and $S'_h = D' + \hat{S}'_h$ and obtain, as before, a map $\pi'\colon X' \to X''$. Repeating this process as often as necessary, but not more than $(g-2)$ times, we arrive at commutative diagrams

$$\begin{tikzcd} X \arrow[rr, "\tilde{\pi}"] \arrow[dr, "f"'] & & \tilde{X} \arrow[dl, "\tilde{f}"] \\ & Y_f & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "\tilde{\pi}"] \arrow[dr, "h"'] & & \tilde{X} \arrow[dl, "\tilde{h}"] \\ & Y_h & \end{tikzcd}$$

and correspondences $\tilde{T}_f = \tilde{\Delta} + \tilde{S}_f = \tilde{\Delta} + \tilde{S}_h = \tilde{T}_h$. The number of possible such diagrams is bounded by the number of possible sequences $D, D', \ldots$, and thus by the number $2^{g-1} \cdot 2^{g-2} \ldots 2 = (\sqrt{2})^{g(g-1)}$. For a given diagram the number of possible h with $\tilde{S}_h = \tilde{S}_f$ is, as before, bounded by $g^2(g-1)$, and by combining these estimates we arrive at the lemma.

3. – A bound for $\mathrm{Hol}(X)$.

We now prove the main result.

THEOREM 1. *Given a Riemann surface X of genus $g \geqslant 2$, the number of surjective holomorphic maps from X onto a Riemann surface of genus $\geqslant 2$ is no larger than*

$$\left(2\sqrt{6}(g-1)+1\right)^{2+2g^2} g^2(g-1)(\sqrt{2})^{g(g-1)} + 84(g-1)\,.$$

PROOF. Since $\mathrm{Hol}(X)$ is the disjoint union of $\mathrm{Hol}'(X)$ and $\mathrm{Aut}(X)$, and since by Hurwitz's theorem $\mathrm{Aut}(X)$ has at most $84(g-1)$ elements, we need only find a bound for $\mathrm{Hol}'(X)$. Moreover by lemma 2 it suffices to bound the number of homology classes in $H_{1,1}(X \times X, \mathbb{Z})$ satisfying the intersection properties given by lemma 1 and (2.1).

To this end let $A = F_1 + F_2$. Then $A \cdot A = 2 > 0$, and by the Hodge index theorem [GH, p. 472] we must have $C \cdot C < 0$ for every real non-zero homology class of type $(1,1)$ which satisfies $A \cdot C = 0$. This allows us to define a norm as follows on $H_{1,1}(X \times X, \mathbb{R})$. Given $\eta \in H_{1,1}(X \times X, \mathbb{R})$ there is a unique decomposition $\eta = \eta_1 + \eta_2$, where $\eta_1 = \left((\eta \cdot A)/(A \cdot A)\right)A$ and

$A\cdot\eta_2=0$. The norm $\|\eta\|$ is defined by

$$\begin{aligned}\|\eta\|^2&=\eta_1\cdot\eta_1-\eta_2\cdot\eta_2\\&=\frac{(\eta\cdot A)^2}{A\cdot A}-\left(\eta-\left(\frac{\eta\cdot A}{A\cdot A}\right)A\right)^2\\&=2\frac{(\eta\cdot A)^2}{(A\cdot A)}-\eta\cdot\eta\\&=(\eta\cdot A)^2-\eta\cdot\eta\,.\end{aligned}$$

It is a norm since $\eta_1\cdot\eta_1\geqslant 0$ and $-(\eta_2\cdot\eta_2)\geqslant 0$, with both zero implying $\eta_1=0$ (since $A\cdot A>0$) and $\eta_2=0$ (by the Hodge index theorem).

If η is integral then so is $\|\eta\|^2$, and thus

$$\|\eta\|\geqslant 1\,. \tag{3.1}$$

If S_f is the correspondence introduced in section 2, then

$$\begin{aligned}\|S_f\|^2&=(A\cdot S_f)^2-(S_f\cdot S_f)\\&\leqslant(2d-2)^2+(2d-2)(g-1)\\&\leqslant 6(g-1)^2\end{aligned} \tag{3.2}$$

using (2.1), lemma 1 and the bound $d\leqslant g$.

A bound on the number of cohomology classes is obtained from (3.1) and (3.2) together with the following standard lemma with

$$m=\dim_{\mathbb{R}}H_{1,1}(X\times X,\mathbb{R})=2+2g^2\quad\text{and}\quad r=\sqrt{6}\,(g-1)\,.$$

LEMMA 3. *Let G be an additive subgroup of $\mathbb{R}^m$ with euclidean norm, and suppose that each $x\in G$ satisfies $\|x\|\geqslant 1$. Then the number of points of G lying in the closed ball of radius r about the origin is no greater than $(2r+1)^m$.*

PROOF. Since G is a subgroup and each element has norm $\geqslant 1$, it follows that any two elements of G have distance at least one from each other. If we surround each point of G with an open ball of radius $\frac{1}{2}$ then these balls are disjoint, and each ball lies in the open ball of radius $(r+\frac{1}{2})$ about the origin provided its center lies in the closed ball of radius r. Letting ω_m denote the area of the unit ball in $\mathbb{R}^m$, we obtain

$$(\text{number of points})\cdot\omega_m\cdot(\tfrac{1}{2})^m\leqslant\omega_m(r+\tfrac{1}{2})^m$$

from which the lemma follows.

Combining the results of this section with lemma 2 we arrive at the asserted upper bound.

4. – The case of higher dimensional X.

We extend the previous result as follows.

THEOREM 2. *Let X be a compact complex manifold with ample canonical bundle. The number of surjective holomorphic maps whose domain is X and whose image is a Riemann surface of genus $\geqslant 2$ is bounded by a number depending only on the Chern numbers of X.*

PROOF. The proof is by induction on the dimension n of X, the case $n = 1$ having been proved in theorem 1. So assume that the statement is true of dimension $(n-1)$ and consider an X of dimension n. By a theorem of Matsusaka [LM] there is an integer q depending only on the Chern numbers of X such that the pluricanonical bundle K^q is very ample. Let Z be a smooth divisor of the complete linear system $|K^q|$. Since Z is a hyperplane section, it cannot be a fibre of a map defined on X. Hence the number of target spaces of maps from X cannot be larger than those from Z, as we see by restriction. By the induction assumption the number of possible target spaces from Z has a bound depending only on the Chern numbers of Z. Moreover, since the normal bundle of Z is $K^q|_Z$ with Chern class $-qc_1(X)$, an easy application of the Whitney product formula shows that the Chern numbers of Z depend on those of X.

The proof is thus reduced to showing the following. If Y is a fixed Riemann surface of genus $\geqslant 2$ and we let $\operatorname{Hol}(X, Y)$ $\big($resp. $\operatorname{Hol}(Z, Y)\big)$ denote the set of surjective holomorphic mappings from X (resp. Z) to Y, then the restriction map $\operatorname{Hol}(X, Y) \to \operatorname{Hol}(Z, Y)$ is injective. To do that we consider the commutative diagram:

$$\begin{array}{ccccc}
Z & \xrightarrow{\ i\ } & X & \xrightarrow{\ f\ } & Y \\
\big\downarrow{\scriptstyle a_Z} & & \big\downarrow{\scriptstyle a_X} & & \big\downarrow{\scriptstyle a_Y} \\
A(Z) & \xrightarrow{\ i_*\ } & A(X) & \xrightarrow{\ f_*\ } & A(Y)
\end{array}$$

where $f \in \operatorname{Hol}(X, Y)$, i is the inclusion map, A and a denote the Albanese variety and map respectively, and i_* and f_* are the induced maps. Since Y has complex dimension one, a_Y is an imbedding, and it follows that f is determined by $f_* a_X$, and therefore, since a_X is canonically defined, by f_*.

On the other hand, the Lefschetz theorem assures us that i_* is surjective so that f_* is determined by $f_* i_*$, which is itself determined by fi. Thus f is determined by its restriction to Z, and the statement is proved.

Finally, we remark that theorem 2 can be extended in an obvious way to any polarized projective manifold.

REFERENCES

[B] T. M. BANDMAN, *Surjective holomorphic mappings of projective manifolds*, Siberian Math. J., **22** (1981), pp. 204-210.

[F] M. DE FRANCHIS, *Un teorema sulle involuzioni irrazionali*, Rend. Circ. Mat. Palermo, **36** (1913), pg. 368.

[GH] P. GRIFFITHS - J. HARRIS, *Principles of Algebraic Geometry*, John Wiley, New York, 1978.

[LM] D. LIEBERMAN - D. MUMFORD, *Matsusaka's big theorem*, Amer. Math. Soc. Proceedings of Symposia in Pure Mathematics, **29** (1975), pp. 513-530.

[S] P. SAMUEL, *Lectures on old and new results on algebraic curves*, Tata Institute of Fundamental Research, Bombay, 1966.

University of Notre Dame
Department of Mathematics
P.O. Box 398
Notre Dame, Indiana 46556