

ANNALI DELLA SCUOLA NORMALE SUPERIORE DI PISA *Classe di Scienze*

F. COLOMBINI

E. JANNELLI

S. SPAGNOLO

Well-posedness in the Gevrey classes of the Cauchy problem for a non-strictly hyperbolic equation with coefficients depending on time

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 10, n° 2 (1983), p. 291-312

http://www.numdam.org/item?id=ASNSP_1983_4_10_2_291_0

© Scuola Normale Superiore, Pisa, 1983, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (<http://www.sns.it/it/edizioni/riviste/annaliscienze/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Well-Posedness in the Gevrey Classes of the Cauchy Problem for a Non-Strictly Hyperbolic Equation with Coefficients Depending on Time.

F. COLOMBINI - E. JANNELLI - S. SPAGNOLO (*)

1. – Introduction.

We shall consider here the Cauchy problem

$$(1) \quad \begin{cases} u_{tt} - \sum_{i,j}^{1,n} a_{ij}(t) u_{x_i x_j} = 0 \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) \end{cases}$$

on $\mathbb{R}^n \times [0, T]$, under the *non-strict hyperbolicity* condition

$$(2) \quad \sum a_{ij}(t) \xi_i \xi_j \geq 0, \quad \forall \xi \in \mathbb{R}^n.$$

It is known (see [1]) that (1) is *well-posed* ⁽¹⁾ in the space $\mathcal{A}$ of analytic functions on $\mathbb{R}^n$, whenever the coefficients belong to $L^1([0, T])$. On the other side (1) can fail to be well posed in the class $\mathcal{E}$ of the C^∞ functions, even if the coefficients are C^∞ (see [2]).

The aim of this paper is to prove the well-posedness of (1) in some Gevrey class $\mathcal{E}^s$, assuming only the minimum of regularity on the coefficients.

Going into detail, we shall prove (see th. 1 and Remark 2 below) that:

If the coefficients $a_{ij}(t)$ belong to $C^{k,\alpha}([0, T])$, with k integer ≥ 0 and $0 \leq \alpha \leq 1$, then problem (1) is well posed in the Gevrey class $\mathcal{E}^s$ provided that

$$(3) \quad 1 \leq s < 1 + \frac{k + \alpha}{2}.$$

If the coefficients are analytic on $[0, T]$, then (1) is well posed in $\mathcal{E}$.

(*) The A.A. are members of G.N.A.F.A. (C.N.R.).

⁽¹⁾ We shall say that problem (1) is well-posed in some space $\mathcal{F}$ of functions on functionals on $\mathbb{R}_x^n$ if for any φ, ψ in $\mathcal{F}$ it admits one and only one solution u in $C^1([0, T], \mathcal{F})$.

Pervenuto alla Redazione il 24 Ottobre 1982.

Such a result is *optimal*, in the sense that *there exist* $a_{ij}(t)$ of class $C^{k,\alpha}$ and $\varphi(x)$, $\psi(x)$, belonging to $\mathcal{E}^s$ for every $s > 1 + (k + \alpha)/2$, for which problem (1) is not solvable in the space of distributions (see § 4 below).

It can be expected that similar results also hold for the more general hyperbolic equation

$$u_{tt} - \sum (a_{ij}(x, t) u_{x_i x_j}) = 0.$$

For instance, we can conjecture that the Cauchy problem for such an equation is well-posed in $\mathcal{E}^s$ when the coefficients $a_{ij}(x, t)$ belong to $C^{k,\alpha}([0, T], \mathcal{E}^s)$ while k, α, s satisfy (3) (see [6] for the case $k = \alpha = 0, s = 1$), and that it is well-posed in $\mathcal{E}$ when the coefficients are analytic in t and C^∞ in x (cf. OLEINIK [8] and NISITANI [7]).

A consequence of th. 1 is that (1) is well posed in every Gevrey class when the coefficients $a_{ij}(t)$ are C^∞ . In this connexion we can observe that such a conclusion can become false if we replace the equation in (1) by a non homogeneous equation as

$$(4) \quad u_{tt} - \sum a_{ij}(t) u_{x_i x_j} + \sum b_i(t) u_{x_i} = 0.$$

(For instance, if we consider the equation $u_{tt} - u_x = 0$ the corresponding Cauchy problem is well-posed in $\mathcal{E}^s$ only if $1 \leq s < 2$).

Here (Remark 2 below) we get also some result for an equation like (4). For instance we prove that the Cauchy problem for (4), with $a_{ij}(t)$ in $C^{k,\alpha}([0, T])$ and $b_i(t)$ in $L^1([0, T])$, is well-posed in $\mathcal{E}^s$ for

$$1 \leq s < 1 + \text{Min} \left\{ 1, \frac{k + \alpha}{2} \right\}.$$

As a special case we have the well-posedness in every $\mathcal{E}^s$ with $1 \leq s < 2$ as soon as the a_{ij} have first derivatives Lipschitz-continuous and the b_i are integrable on $[0, T]$.

An extensive study of the necessary Levi conditions for the well-posedness in the Gevrey classes has been made by Ivrii and Petkov in [5].

Finally we remark that the present paper can be considered an extension of [1], where problem (1) was extensively studied under the *strict hyperbolicity* condition

$$(5) \quad \sum a_{ij}(t) \xi_i \xi_j \geq \lambda_0 |\xi|^2 \quad (\lambda_0 > 0).$$

In this case, to get the well-posedness in $\mathcal{E}$ of the Cauchy problem (1) it is sufficient that the coefficients $a_{ij}(t)$ are Lipschitz-continuous, while a

very little regularity on the a_{ij} insures the well-posedness in some Gevrey class. More precisely (see [1]) *if the $a_{ij}(t)$ belong to $C^{0,\alpha}([0, T])$, the Cauchy problem $\{(1), (5)\}$ is well posed in $\mathcal{E}^s$ for*

$$1 \leq s < 1 + \frac{\alpha}{1 - \alpha}.$$

The techniques used in the present paper are fundamentally the same of [1], namely the Fourier-Laplace transform and the *approximate energy* estimate. Besides this, we shall use the following result of real analysis (Lemma 1 below): *if $f(t)$ is a function ≥ 0 of class $C^{k,\alpha}$ on $[0, T]$, then $f^{1/(k+\alpha)}$ is absolutely continuous on $[0, T]$* . We have not been able to find this result in the literature, but for the case $k = 2, \alpha = 0$ (Gleaser [4], see also Dieudonné [3]). For this reason we shall exhibit a proof (see § 2) of it. Such a proof has been essentially suggested to us by F. Conti, whom we thank warmly.

NOTATIONS:

$\mathcal{K}$ is the topological vector space of entire functions on $\mathbb{R}^n$.

$\mathcal{A}$ is the t.v.s. of analytic functions on $\mathbb{R}^n$.

$\mathcal{E}^s$, for s real ≥ 1 , is the t.v.s. of Gevrey functions on $\mathbb{R}^n$, i.e. the C^∞ functions φ verifying

$$|D^r \varphi(x)| \leq C_K A_K^{|r|} |r|^{s|r|}, \quad \forall x \in K, \forall r,$$

for any compact subset $K \subset \mathbb{R}^n$.

When $s = 1$, we have the coincidence $\mathcal{E}^1 = \mathcal{A}$.

$\mathcal{E}$ is the t.v.s. of C^∞ functions on $\mathbb{R}^n$.

$\mathcal{D}$ is the t.v.s. of C^∞ functions with compact support in $\mathbb{R}^n$

$\mathcal{D}^s = \mathcal{E}^s \cap \mathcal{D}$.

$\mathcal{K}', \mathcal{A}', \mathcal{D}', (\mathcal{D}^s)'$ are the dual spaces of $\mathcal{K}, \mathcal{A}, \mathcal{D}, \mathcal{D}^s$.

All these spaces are endowed by the usual topologies.

$C^k([0, T], \mathcal{F})$, with $\mathcal{F}$ equal to one of the t.v.s. introduced above. is the t.v.s. of functions $u: [0, T] \rightarrow \mathcal{F}$ having k continuous derivatives on $[0, T]$. The elements u of $C^k([0, T], \mathcal{F})$ shall be treated, as usual, as functions or functionals on $\mathbb{R}^n \times]0, T[$. In this sense we shall write $u(x, t)$, $\partial u / \partial x_j$, $\partial u / \partial t$.

$C^{k,\alpha}([0, T])$, with k integer ≥ 0 and $0 \leq \alpha \leq 1$, is the Banach space of the functions having k derivatives continuous on $[0, T]$, and the k -th derivative Hölder-continuous with exponent α when $\alpha > 0$.

The norm in this space is

$$\|u\|_{C^{k,\alpha}} = \sum_{h=0}^k \sup_{[0, T]} |u^{(h)}| + \sup_{t \neq s} |u^{(k)}(t) - u^{(k)}(s)| |t - s|^{-\alpha}.$$

2. - A lemma of real analysis.

LEMMA 1. Let $f(t)$ be a real function of class $C^{k,\alpha}$ on some compact interval $I \subset \mathbb{R}$, with k integer ≥ 1 and $0 \leq \alpha \leq 1$, and assume that

$$f(t) \geq 0 \quad \text{on } I.$$

Then the function $f^{1/(k+\alpha)}$ is absolutely continuous on I . Moreover

$$(6) \quad \|(f^{1/(k+\alpha)})'\|_{L^1(I)}^{k+\alpha} \leq C(k, \alpha, I) \|f\|_{C^{k,\alpha}(I)}.$$

PROOF. The conclusion of the Lemma is obvious when $k = 1, \alpha = 0$. Moreover the case $k = \nu \geq 2, \alpha = 0$, can be reduced to the case $k = \nu - 1, \alpha = 1$. Thus we shall consider only the case $\alpha > 0$.

Let us firstly assume that $f(t) > 0$ on I . In such a case the function $f^{1/(k+\alpha)}$ is C^1 as well as f , and we must only prove that

$$(7) \quad \left(\int_I f^{(1/(k+\alpha)-1)} |f'| \, dt \right)^{k+\alpha} \leq C(k, \alpha, I) \|f\|_{C^{k,\alpha}(I)}.$$

In order to treat the general case ($f(t) \geq 0$) we must only approximate $f(t)$ by $f(t) + \varepsilon$, $\varepsilon \rightarrow 0$. Since $(f + \varepsilon)^{1/(k+\alpha)-1} |f'|$ is increasing for ε decreasing to zero, then, by Beppo Levi's theorem and inequality (7) for $f + \varepsilon$, we get that the functions $(f + \varepsilon)^{1/(k+\alpha)-1} |f'|$ are equi-integrable on I . This gives the conclusion of Lemma 1.

Hence we assume that $f(t) > 0$ on I and we are aiming at inequality (7). We shall also assume, without a real loss of generality, that f is C^∞ on I .

Now let $\mathcal{I} = \{x_0, x_1, \dots, x_N\}$, with $a = x_0 < x_1 < \dots < x_N = b$, be a *partition* of $I \equiv [a, b]$. We define, for every real function g on I ,

$$(8) \quad V_s(\mathcal{I}, g) = \sum_{j=0}^{N-1} |g(x_{j+1}) - g(x_j)|^{1/s}, \quad s > 0,$$

and

$$V_s^*(g) = \sup_{\mathcal{F} \in P(g)} V_s(\mathcal{F}, g),$$

where $P(g)$ is the class of partitions $\mathcal{F}$ of I such that

$$(9) \quad g'(x_j) = 0 \quad \text{for } j = 1, \dots, N-1.$$

We claim that the following inequalities hold:

$$(10) \quad \text{Var}(g) \leq V_1^*(g),$$

$$(11) \quad V_1^*(|g|^{1/s}) \leq V_s^*(g), \quad \text{for } s \geq 1,$$

$$(12) \quad V_s^*(g) \leq \|g\|_{C^{0,s}(I)} \cdot |I|, \quad \text{for } 0 < s \leq 1,$$

$$(13) \quad V_s^*(g) \leq [V_{s-1}^*(g')^{(s-1)/s} + \|g'\|_{C^0(I)}^{1/s}] |I|^{1/s}, \quad \text{for } s > 1,$$

where $|I|$ denotes the length of I and $\text{Var}(g)$ the variation on I of g , i.e. the supremum of $V_1(\mathcal{F}, g)$ as $\mathcal{F}$ runs in the class of *all* partitions of I .

From these inequalities it is easy to derive (7), i.e. the conclusion of the Lemma.

Indeed, by applying successively (13) with $g = f$ and $s = k + \alpha$; $g = f'$ and $s = k + \alpha - 1$; ...; $g = f^{(k-1)}$ and $s = \alpha + 1$; and finally using (12) with $g = f^{(k)}$ and $s = \alpha$, we get

$$(14) \quad V_{k+\alpha}^*(f) \leq C_0(k, \alpha, |I|) \|f\|_{C^{k,\alpha}(I)}^{1/(k+\alpha)} \quad (k \geq 1; 0 < \alpha \leq 1).$$

Now from (10), (11) and (14) it follows

$$\begin{aligned} \text{Var}(f^{1/(k+\alpha)}) &\leq V_1^*(f^{1/(k+\alpha)}) \leq V_{k+\alpha}^*(f) \\ &\leq C_0(k, \alpha, |I|) \|f\|_{C^{k,\alpha}(I)}^{1/(k+\alpha)} \end{aligned}$$

and hence (7).

Let us then prove (10), (11), (12) and (13).

In order to prove (10) we show that for every partition $\mathcal{F}$ on I , there exists another partition $\tilde{\mathcal{F}}$ verifying (9) and such that

$$(15) \quad V_1(\mathcal{F}, g) \leq V_1(\tilde{\mathcal{F}}, g).$$

To this end, if $\mathcal{F} = \{x_0, \dots, x_N\}$ we consider these values of j such that g' has some zero on $[x_j, x_{j+1}]$ and correspondingly we denote by y_j and z_j respectively the first and the last of these zeros. Then the partition $\tilde{\mathcal{F}}$ whose endpoints are a, b, y_j, z_j belongs to $P(g)$ and verifies (15).

Inequalities (11) and (12) are obvious.

In order to get inequality (13) it is sufficient to prove that for any partition $\mathcal{F}$ belonging to $P(g)$, i.e. verifying (9), there exists a partition $\tilde{\mathcal{F}} \in P(g')$ in such a way that

$$(16) \quad V_s(\mathcal{F}, g) \leq (V_{s-1}(\tilde{\mathcal{F}}, g')^{(s-1)/s} + \|g'\|_{C^s(I)}^{1/s}) |I|^{1/s}$$

for $s > 1$.

To this end, if $\mathcal{F} = \{x_0, x_1, \dots, x_N\}$, we denote by y_j the first point of maximum of $|g'|$ on the interval $[x_j, x_{j+1}]$, for $j = 0, 1, \dots, N-1$. Afterwards we denote by z_j the first point of minimum (resp. of maximum) of g' on the interval $[y_j, y_{j+1}]$ if $g'(y_j) \geq 0$ (resp. $g'(y_j) \leq 0$), for $j = 0, 1, \dots, N-2$. In particular, taking into account that $g'(x_{j+1}) = 0$ and x_{j+1} belongs to $[y_j, y_{j+1}]$, we have

$$(17) \quad g'(y_j) \cdot g'(z_j) \leq 0.$$

Now let $\tilde{\mathcal{F}}$ be the partition having as endpoints a, b and y_j, z_j . We shall verify that $\tilde{\mathcal{F}}$ belongs to $P(g')$, i.e. g'' vanishes at every endpoint different from a and b , and that (16) holds.

Let y_j be different from a and b . Two cases are then possible: either y_j lies at the interior of $[x_j, x_{j+1}]$, or it coincides with x_j or with x_{j+1} . In the first case we get immediately that $g''(y_j) = 0$; in the second case we know that $g'(y_j) = 0$ since $\mathcal{F}$ verifies (9), and by consequence g' must be identically zero on $[x_j, x_{j+1}]$. In both cases we have $g''(y_j) = 0$.

Let now z_j be different from a and b . Since $z_j \in [y_j, y_{j+1}]$, if z_j is equal to y_j or to y_{j+1} we have just seen that $g''(z_j) = 0$, while if z_j is internal to $[y_j, y_{j+1}]$ we get obviously $g''(z_j) = 0$.

Thus $\tilde{\mathcal{F}}$ belongs to $P(g')$.

It remains only to verify (16). Now, remembering the definition of y_j and using (17) and the Hölder inequality, we get, for $s > 1$,

$$\begin{aligned} \sum_{j=0}^{N-1} |g(x_{j+1}) - g(x_j)|^{1/s} &\leq \sum_{j=0}^{N-1} |g'(y_j)|^{1/s} |x_{j+1} - x_j|^{1/s} \\ &\leq \sum_{j=0}^{N-2} |g'(y_j) - g'(z_j)|^{1/s} |x_{j+1} - x_j|^{1/s} + |g'(y_{N-1})|^{1/s} |x_N - x_{N-1}|^{1/s} \\ &\leq \left[\sum_{j=0}^{N-2} |g'(y_j) - g'(z_j)|^{1/(s-1)} \right]^{(s-1)/s} |I|^{1/s} + \|g'\|_{C^s(I)}^{1/s} |I|^{1/s}, \end{aligned}$$

whence (16) follows.

This completes the proof of Lemma 1. //

3. – The existence theorem.

THEOREM 1. *Let us consider the problem*

$$(18) \quad \begin{cases} u_{tt} - \sum_{i,j}^{1,n} a_{ij}(t) u_{x_i x_j} = 0 \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) \end{cases}$$

on $\mathbb{R}^n \times [0, T]$, assuming that

$$(19) \quad \sum a_{ij}(t) \xi_i \xi_j \geq 0, \quad \forall \xi \in \mathbb{R}^n,$$

and

$$(20) \quad a_{ij} \in C^{k,\alpha}([0, T]), \quad k \text{ integer} > 0, \quad 0 \leq \alpha \leq 1.$$

Then for every φ and ψ in $\mathcal{E}^s$, the problem admits one and only one solution $u \in C^2([0, T], \mathcal{E}^s)$, provided that

$$(21) \quad 1 \leq s < 1 + \frac{k + \alpha}{2}.$$

REMARK 1. When $k = \alpha = 0$, (21) does not make sense. However in [1], § 8, has been proved that problem (18) is well posed in $\mathcal{E}^1 (= \mathcal{A})$ whenever the coefficients a_{ij} belong to $C^0([0, T])$, or even to $L^1([0, T])$.

PROOF OF TH. 1. We can devote ourselves to the case $s > 1$ (see Remark 1 here above).

The coefficients $a_{ij}(t)$ are taken continuous on $[0, T]$, thus we can assume that, for some $\Lambda > 0$,

$$(22) \quad \sum a_{ij}(t) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \forall t.$$

By Holmgren's theorem we know that every solution $u(x, t)$ of (18), whose initial data are identically zero on some ball $\{|x - x_0| \leq r\}$, is zero on the cone $\{|x - x_0| \leq r - (1 + \Lambda)t\}$ (more precisely, $u \equiv 0$ on the cone $\{|x - x_0| \leq r - \sqrt{\Lambda}t$; cf. [1], formula (90)).

This fact gives the uniqueness of solutions to (18), and moreover allows us to reduce ourselves to the case of initial data having a compact support in $\mathbb{R}^n$.

Hence we assume, from now on, that $\varphi(x)$ and $\psi(x)$ belong to $\mathcal{D}^s$.

Now $\mathcal{D}^s$ is a subspace of the space $\mathcal{H}'$ of holomorphic functionals on $\mathbb{C}^n$ and the Ovcianikov theorem ensures the well-posedness of (18) in $\mathcal{H}'$ (even without the hyperbolicity assumption (19)). Hence (18) admits a solution $u \in C^2([0, T], \mathcal{H}')$: our task is to prove that u belongs to $C^2([0, T], \mathcal{D}^s)$ when (19) and (21) are satisfied. To this purpose, denoting by

$$v(\xi, t) = \langle u(x, t), \exp[-i(\xi, x)] \rangle, \quad \xi \in \mathbb{R}^n,$$

the Fourier transform of u with respect to x , it will be sufficient to prove that

$$(23) \quad |v(\xi, t)| \leq M \exp[-\delta|\xi|^{1/s}]$$

for every $\xi \in \mathbb{R}^n$ and $t \in [0, T]$, and some $M, \delta > 0$.

Indeed from (23) it follows, in virtue of Paley-Wiener theorem, that $u(\cdot, t)$ belongs to $\mathcal{D}^s$ or rather that $\{u(\cdot, t) | t \in [0, T]\}$ is bounded in $\mathcal{D}^s$. Thus, taking into account that u is a solution of (18), (23) gives that $u \in C^2([0, T], \mathcal{D}^s)$.

Let us hence prove inequality (23), assuming that $\hat{\phi}(\xi)$ and $\hat{\psi}(\xi)$, i.e. the Fourier transforms of the initial data, verify an analogous inequality and that $1 < s < 1 + (k + \alpha)/2$.

By Fourier transform, (18) becomes

$$(24) \quad v'' + (a(t)\xi, \xi)v = 0, \quad t \in [0, T],$$

where we have put

$$(a(t)\xi, \xi) = \sum a_{ij}(t)\xi_i\xi_j.$$

Now we approximate $a(t)$, in a suitable way, by a family $\{a_\varepsilon(t)\}_{\varepsilon>0}$ of C^1 strictly positive quadratic forms, and we introduce, for any $\varepsilon > 0$, the ε -approximate energy of u

$$(25) \quad E_\varepsilon(\xi, t) = (a_\varepsilon(t)\xi, \xi)|v|^2 + |v'|^2.$$

Our goal will be to get a good estimate of the growth of E_ε as $|\xi| \rightarrow \infty$. By differentiating in t , we have

$$E'_\varepsilon(\xi, t) = (a'_\varepsilon\xi, \xi)|v|^2 + 2(a_\varepsilon\xi, \xi) \operatorname{Re}(v\bar{v}') + 2 \operatorname{Re}(\bar{v}'v'),$$

whence, taking (24) into account,

$$E'_\varepsilon \leq |(a'_\varepsilon\xi, \xi)||v|^2 + 2|(a_\varepsilon - a)\xi, \xi)||v||v'|$$

i.e.

$$E'_\varepsilon \leq \frac{|(a'_\varepsilon \xi, \xi)|}{(a_\varepsilon \xi, \xi)} E_\varepsilon + \frac{|((a_\varepsilon - a)\xi, \xi)|}{(a_\varepsilon \xi, \xi)^{\frac{1}{2}}} E_\varepsilon.$$

By Gronwall lemma we then derive, $\forall t \in [0, T]$,

$$(26) \quad E_\varepsilon(\xi, t) \leq E_\varepsilon(\xi, 0) \exp \left[\int_0^t \frac{|(a'_\varepsilon \xi, \xi)|}{(a_\varepsilon \xi, \xi)} ds + \int_0^t \frac{|((a_\varepsilon - a)\xi, \xi)|}{(a_\varepsilon \xi, \xi)^{\frac{1}{2}}} ds \right].$$

Let us now define the approximating coefficients $a_\varepsilon(t)$, by considering separately the case in which $a(t)$ belongs to $C^{k,\alpha}$ with $k \geq 1$ and the case in which $a(t)$ belongs to $C^{0,\alpha}$.

In the first case we take

$$a_\varepsilon(t) = a(t) + \varepsilon I,$$

where I denotes the identity matrix.

We have then obviously

$$(27) \quad (a_\varepsilon \xi, \xi) \geq (a_\varepsilon \xi, \xi)^{1-1/(k+\alpha)} (\varepsilon |\xi|^2)^{1/(k+\alpha)}$$

and

$$(28) \quad \frac{|((a_\varepsilon - a)\xi, \xi)|}{(a_\varepsilon \xi, \xi)^{\frac{1}{2}}} \leq \sqrt{\varepsilon} |\xi|.$$

On the other hand, using Lemma 1 with $f(t) = (a(t)\xi, \xi)$ and remarking that $\text{Var}_{[0,T]}(a_\varepsilon \xi, \xi) = \text{Var}_{[0,T]}(a \xi, \xi)$, we get

$$(29) \quad \int_0^T \frac{(a'_\varepsilon \xi, \xi)}{(a_\varepsilon \xi, \xi)^{1-1/(k+\alpha)}} ds \leq C(k, \alpha, T) \|a\|_{C^{k,\alpha}}^{1/(k+\alpha)} |\xi|^{2/(k+\alpha)}.$$

Introducing (27), (28) and (29) in (26), we obtain then the estimate

$$(30) \quad E_\varepsilon(\xi, t) \leq E_\varepsilon(\xi, 0) \exp [C_1(\varepsilon^{-1/(k+\alpha)} + \sqrt{\varepsilon} |\xi|)],$$

where $C_1, \dots, C_i, \dots$ denote constants depending only on $\|a\|_{C^{k,\alpha}([0,T])}$:

Now let us compare the ε -energy E_ε with the functional E defined as

$$E(\xi, t) = |\xi|^2 |v(\xi, t)|^2 + |v'(\xi, t)|^2.$$

We see immediately that

$$\varepsilon E(\xi, t) \leq E_\varepsilon(\xi, t) \leq (1 + \Lambda) E(\xi, t)$$

for $\varepsilon < 1$, Λ being defined by (22).

By consequence, (30) with $\varepsilon = (1 + |\xi|)^{-2(k+\alpha)/(2+k+\alpha)}$ gives

$$(31) \quad E(\xi, t) \leq C_2(1 + |\xi|)^{2(k+\alpha)/(2+k+\alpha)} E(\xi, 0) \exp[C_3|\xi|^{2/(2+k+\alpha)}].$$

But the initial data φ, ψ of (18) belong to $\mathcal{D}^s$, thus their transforms $\hat{\varphi}(\xi), \hat{\psi}(\xi)$, and consequently $E(\xi, 0)$, can be estimated by $M_0 \cdot \exp(-\delta_0|\xi|^{1/s})$ for some $M_0, \delta_0 > 0$.

Therefore by (31) we get

$$E(\xi, t) \leq M_0 C_4 \exp\left(-\frac{\delta_0}{2}|\xi|^{1/s} + C_3|\xi|\right)$$

and hence (23), as $1/s > 2/(2+k+\alpha)$.

Let us now pass to examine the case $k=0$, in which $a(t)$ belongs to $C^{0,\alpha}([0, T])$. In this case we must not only make $a(t)$ strictly positive but also regularise it.

We then take

$$a_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^{+\infty} \tilde{a}(t+s) \varrho\left(\frac{s}{\varepsilon}\right) ds + \varepsilon^\alpha I,$$

where $\tilde{a}(t)$ is the continuation of $a(t)$ on $[0, +\infty[$ such that $\tilde{a} \equiv a(T)$ on $[T, +\infty[$, and $\varrho(t)$ is a non negative C^∞ function such that $\varrho \equiv 0$ on $]-\infty, 0]$ and on $[1, +\infty[$, and $\int_{-\infty}^{+\infty} \varrho ds = 1$.

The α -Hölder continuity of $a(t)$ gives

$$\int_0^T |(a'_\varepsilon \xi, \xi)| ds \leq C_5 \varepsilon^{\alpha-1} |\xi|^2$$

and

$$\int_0^T |((a_\varepsilon - a)\xi, \xi)| ds \leq C_6 \varepsilon^\alpha |\xi|^2,$$

while by definition

$$(a_\varepsilon \xi, \xi) \geq \varepsilon^\alpha |\xi|^2.$$

Introducing these estimates in (26) we get

$$E_\varepsilon(\xi, t) \leq E_\varepsilon(\xi, 0) \exp [C_7(\varepsilon^{-1} + \varepsilon^{\alpha/2}|\xi|)].$$

From now on, we proceed in the same manner that in the case $k \geq 1$. The only difference is the choice of ε , now taken equal to $(1 + |\xi|)^{-2/(2+\alpha)}$.

In both cases, (23) is obtained and the theorem is proved. //

REMARK 2. As a corollary of th. 1, we have that problem (18) is well posed in $\mathcal{E}^s$ for every $s \geq 1$, when the coefficients $a_{ij}(t)$ are C^∞ on $[0, T]$.

Concerning the well-posedness in $\mathcal{E}$, we must assume further regularity on the a_{ij} (see the example of [2]).

For instance, when the $a_{ij}(t)$ are analytic on $[0, T]$ it is easy to prove that (18) is well posed in $\mathcal{E}$. Indeed, in virtue of the analyticity, one can prove that $(a'(t)\xi, \xi)$ admits at most N isolated zeros for every $\xi \in \mathbb{R}^n$, with N independent on ξ . Therefore

$$\int_0^T \frac{(a'(t)\xi, \xi)}{(a(t)\xi, \xi) + \varepsilon|\xi|^2} dt \leq (N+1) \log \frac{\Lambda + \varepsilon}{\varepsilon},$$

where Λ is defined by (22). Thus, going back to the proof of th. 1, we see that (26) becomes

$$E_\varepsilon(\xi, t) \leq E_\varepsilon(\xi, 0) \exp \left((N+1) \log \frac{\Lambda + \varepsilon}{\varepsilon} + \sqrt{\varepsilon} |\xi| T \right).$$

Hence, taking $\varepsilon = |\xi|^{-2}$, we obtain that $E(\xi, t)/E(\xi, 0)$ has a polynomial growth for $|\xi| \rightarrow \infty$, so that (18) is well posed in $\mathcal{E}$.

REMARK 3. Let us consider the more general equation

$$(32) \quad u_{tt} - \sum a_{ij}(t) u_{x_i x_j} + \sum b_i(t) u_{x_i} + c(t) u + d(t) u_t = 0$$

where the a_{ij} are in $C^{k,\alpha}([0, T])$, k integer > 0 and $0 < \alpha \leq 1$, and satisfy (2), while b_i , c and d belong to $L^1([0, T])$.

Moreover let us assume the following sort of Levi's condition:

$$|\sum b_i(t) \xi_i| \leq \lambda(t, \xi) (\sum a_{ij}(t) \xi_i \xi_j)^\beta$$

for some $\beta \in [0, \frac{1}{2}]$ and some λ such that

$$\sup_{|\xi|=1} \int_0^T \lambda(t, \xi) dt < +\infty.$$

Therefore, using the same technique of th. 1, we can prove that the Cauchy problem for the equation (32) is well posed in $\mathcal{E}^s$ for every s verifying

$$1 \leq s < 1 + \text{Min} \left\{ \frac{k + \alpha}{2}, \frac{1}{1 - 2\beta} \right\}.$$

For $\beta = \frac{1}{2}$ we get in particular the same conclusion as in the homogeneous equation (th. 1).

Finally let us observe that if $\beta = 0$, i.e. if no condition is imposed on the coefficients $b_i(t)$, we cannot have in general the well-posedness in $\mathcal{E}^s$ for $s \geq 2$.

REMARK 4. Under the same assumptions of th. 1, we can prove, in a similar way, that problem (18) is well posed in $(\mathcal{D}^s)'$, space of the Gevrey ultradistributions with order $s < 1 + (k + \alpha)/2$.

4. - Counter-examples.

In this section we put ourselves in the case $n = 1$, considering the problem

$$(33) \quad u_{tt} - a(t)u_{xx} = 0$$

$$(34) \quad u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x)$$

for $x \in \mathbb{R}$, $t \geq 0$, with $a(t) \geq 0$.

Our purpose is to show that th. 1 cannot be improved, by constructing for any (k, α) a coefficient $a(t)$ of class $C^{k, \alpha}$ in such a way that $\{(33), (34)\}$ is not well-posed in $\mathcal{E}^s$ if $s > 1 + (k + \alpha)/2$.

More precisely we shall prove the following result.

THEOREM 2. *For every $T_* > 0$ and every (k, α) (k integer ≥ 0 , $0 \leq \alpha < 1$) it is possible to construct a function $a(t)$, C^∞ and strictly positive on $[0, T_*[$, identically zero on $[T_*, +\infty[$, and a solution $u(x, t)$ of (33) in such a way that*

$$(35) \quad a(t) \text{ belongs to } C^{k, \alpha}([0, +\infty[)$$

and

$$(36) \quad u \text{ belongs to } C^1([0, T_*[, \mathcal{E}^s), \quad \forall s > 1 + \frac{k + \alpha}{2},$$

whereas

$$(37) \quad \{u(\cdot, t)\} \text{ is not bounded in } \mathcal{D}', \text{ as } t \rightarrow T_*^-.$$

REMARK 5. From (36) it follows in particular that $u(\cdot, 0)$ and $u_t(\cdot, 0)$ belong to $\mathcal{E}^s$, $\forall s > 1 + (k + \alpha)/2$. Hence $u(x, t)$ is a solution (in fact the *unique* solution) of problem $\{(33), (34)\}$ with $\varphi(x) = u(x, 0)$ and $\psi(x) = u_t(x, 0)$.

Thus, th. 2 says that this problem is not well-posed in the Gevrey space $\mathcal{E}^s$ if $s > 1 + (k + \alpha)/2$.

PROOF OF TH. 2. The construction of $a(t)$ and $u(x, t)$ will be very similar to the one made in [2], where it was given an example of $a(t)$ of class C^∞ such that the Cauchy problem $\{(33), (34)\}$ is not well-posed in C^∞ (the example of [2] can be in some sense considered as the limit case of th. 2 as $k + \alpha \rightarrow \infty$).

However we shall give here for sake of completeness a self-consistent exposition, referring to [2] for some technical step.

Fixed $T_* > 0$, let us introduce the following *parameters*, whose values will be chosen at the end of the proof:

a sequence $\{\varrho_i\}$ of positive numbers, decreasing to zero and verifying

$$(38) \quad \sum_{i=1}^{\infty} \varrho_i = T_*;$$

a sequence $\{\delta_i\}$ of positive numbers, decreasing to zero;

a sequence $\{\nu_i\}$ of integers ≥ 1 , increasing to ∞ .

Correspondingly let us consider the points of $[0, T_*[$

$$t_j = \varrho_1 + \dots + \varrho_{j-1} + \frac{\varrho_j}{2},$$

and the intervals

$$J_j = \left[t_j - \frac{\varrho_j}{2}, t_j + \frac{\varrho_j}{2} \right].$$

We have then

$$[0, T_*[= \bigcup_{j=1}^{\infty} J_j.$$

Finally let us consider, inside J_j , the points

$$t'_j = \left(t_j - \frac{\varrho_j}{2} \right) + \frac{\varrho_j}{8\nu_j}, \quad t''_j = \left(t_j + \frac{\varrho_j}{2} \right) - \frac{\varrho_j}{8\nu_j},$$

and denote by

$$\tilde{I}_j = \left[t_j - \frac{\varrho_j}{2}, t'_j \right] \quad \text{and} \quad I_j = \left[t'_j, t_j + \frac{\varrho_j}{2} \right]$$

the intervals into which J_j is divided by t'_j .

The definition of $a(t)$ will be given piece by piece on each J_j and it will be based on two auxiliary functions, $\alpha(\tau)$ and $\beta(\tau)$.

As $\beta(\tau)$ we take any C^∞ function on $\mathbb{R}$, strictly increasing on $[0, 1]$, equal to zero on $]-\infty, 0]$ and equal to 1 on $[1, +\infty[$.

As $\alpha(\tau)$ we take the function

$$(39) \quad \alpha(\tau) = 1 - \frac{4}{10} \sin 2\tau - \frac{1}{100} (1 - \cos 2\tau)^2.$$

Observe that $\alpha(\tau)$ is π -periodic and valued in $[\frac{1}{2}, 2]$.

Now let us define $a(t)$ by taking

$$(40) \quad \begin{cases} a = a_j b_j + a_{j-1}(1 - b_j) & \text{on } J_j \ (j \geq 1), \\ a \equiv 0 & \text{on } [T_*, +\infty[, \end{cases}$$

where a_j, b_j are defined by

$$(41) \quad \begin{cases} a_j(t) = \delta_j \cdot \alpha\left(2\nu_j \pi \frac{t - t_j}{\varrho_j}\right), & j \geq 1, \\ b_j(t) = \beta\left(8\nu_j \frac{t - (t_j - \varrho_j/2)}{\varrho_j}\right), & j \geq 1, \\ a_0(t) = 2\delta_1. \end{cases}$$

Observe that $a(t) \equiv a_j(t)$ on I_j and that $a(t)$ is C^∞ on $[0, T_*[$.

Now let us define the solution $u(x, t)$ as

$$(42) \quad u(x, t) = \sum_{j=1}^{\infty} v_j(t) \sin(h_j x),$$

with

$$(43) \quad h_j = 2\pi \frac{\nu_j}{\varrho_j} \frac{1}{\sqrt{\delta_j}},$$

and $v_j(t)$ equal to the solution of

$$(44) \quad \begin{cases} v'' + h_j^2 \cdot a(t) v = 0, & t \geq 0, \\ v(t_j) = 0, & v'(t_j) = 1. \end{cases}$$

Clearly (42) defines a solution, in some weak sense, of equation (33). Hence the problem is to find ϱ_j , δ_j , ν_j in such a way that (35), (36) and (37) are satisfied.

To get (35), let us differentiate k -times (40). We then obtain

$$a^{(k)}|_{J_j} = \sum_{r=0}^k \binom{k}{r} b_j^{(k-r)} \cdot (a_j^{(r)} - a_{j-1}^{(r)}) + a_{j-1}^{(k)},$$

whence, using the monotonicity of $\{\delta_j\}$ and $\{\varrho_j/\nu_j\}$, we derive the estimate

$$(45) \quad \|a\|_{C^{k,\alpha}(J_j)} \leq C(k, \alpha) \delta_{j-1} \left(\frac{\nu_j}{\varrho_j}\right)^{k+\alpha}.$$

Hence a sufficient condition for the C^k -regularity of $a(t)$ on $[0, +\infty]$ is that

$$(46) \quad \delta_{j-1} \left(\frac{\nu_j}{\varrho_j}\right)^k \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

As the C^α -regularity of $a^{(k)}(t)$ on $[0, +\infty[$, we can see that a sufficient condition is

$$(47) \quad \delta_{j-1} \left(\frac{\nu_j}{\varrho_j}\right)^{k+\alpha} \leq M, \quad \forall j.$$

Indeed from (47) we derive, using (45) with $\alpha = 0$,

$$\|a\|_{C^k(J_j)} \leq C(k, 0) M \frac{\varrho_j^\alpha}{\nu_j^\alpha} \leq C(k, 0) M \varrho_j^\alpha,$$

and this inequality, together with (45), enables us to get

$$|a^{(k)}(t'') - a^{(k)}(t')| \leq 2M(C(k, \alpha) + C(k, 0))|t'' - t'|^\alpha.$$

Let us now look for a sufficient condition on the parameters which ensures (36). To this end we must estimate the growth for $j \rightarrow \infty$ of the coefficients $v_j(t)$ of Fourier expansion (42).

Since $a(t) \equiv \delta_j \cdot \alpha(2\pi(\nu_j/\varrho_j)(t - t_j))$ on I_j , we can calculate $v_j(t)$ on I_j . In fact we have

$$(48) \quad v_j(t) = \frac{\varrho_j}{2\pi\nu_j} w\left(2\pi \frac{\nu_j}{\varrho_j}(t - t_j)\right), \quad \text{on } I_j,$$

having denoted by $w(\tau)$ the solution of

$$(49) \quad \begin{cases} w'' + \alpha(\tau)w = 0, & \text{on } \mathbb{R}, \\ w(0) = 0, & w'(0) = 1. \end{cases}$$

But we defined $\alpha(\tau)$ in such a way that (49) admits a solution of the form $w(\tau) = p(\tau) \cdot \exp(\gamma\tau)$, with $p(\tau)$ periodic and $\gamma > 0$. More precisely the solution of (49) is

$$(50) \quad w(\tau) = \sin \tau \cdot \exp \left[\frac{1}{10} \left(\tau - \frac{1}{2} \sin 2\tau \right) \right].$$

Thus (48) and (50) give an explicit expression of $v_j(t)$ on I_j , and in particular

$$(51) \quad \begin{cases} |v_j(t'_j)| = \tilde{c}_1 \frac{\varrho_j}{\nu_j} \exp \left(-\frac{\pi}{10} \nu_j \right) \\ |v'_j(t'_j)| = \tilde{c}_2 \exp \left(-\frac{\pi}{10} \nu_j \right) \end{cases}$$

and

$$(52) \quad \begin{cases} |v_j(t''_j)| = \tilde{c}_3 \frac{\varrho_j}{\nu_j} \exp \left(\frac{\pi}{10} \nu_j \right) \\ |v'_j(t''_j)| = \tilde{c}_4 \exp \left(\frac{\pi}{10} \nu_j \right) \end{cases}$$

with $\tilde{c}_j > 0$.

If we introduce the *energy* of $v_j(t)$ as

$$(53) \quad E_j(t) = \hbar_j^2 a(t) v_j^2 + v_j'^2,$$

we get by (51)

$$(54) \quad E_j(t'_j) = C_0 \exp \left(-\frac{\pi}{5} \nu_j \right).$$

Now, starting from (54), we estimate $E_j(t)$ for $t < t'_j$.

To this end we use the energy estimate

$$(55) \quad E_j(t) \leq E_j(s) \exp \left[\int_t^s \frac{|a'(\xi)|}{a(\xi)} d\xi \right], \quad t < s,$$

which can be easily derived from equation (44).

We use (55) with $s = t'_j$ and $t < t'_j$, thus we must estimate the integral $\int_t^{t'} |a'| a^{-1} d\xi$. For this purpose we take into account the behaviour of $a(t)$

on the interval

$$[0, t'_j] \equiv \tilde{I}_1 \cup I_1 \cup \dots \cup \tilde{I}_{j-1} \cup I_{j-1} \cup \tilde{J}_j,$$

and, more precisely, the following facts:

- $a(t)$ is decreasing near the points $t = 0$, $t = t'_i$, and $a(0) = 2\delta_1$, $a(t'_i) = c_1 \cdot \delta_1$ ($c_1 = \alpha(\pi/4)$);
- $a(t)$ has exactly $2\nu_h$ points of minimum and $2\nu_h$ points of maximum on I_h , where $\delta_h/2 \leq a(t) \leq 2\delta_h$;
- $a(t)$ is decreasing in a neighborhood of $\tilde{I}_h$.

The first two of these facts are direct consequences of definition itself of $a(t)$, whereas to have the third we must impose a supplementary assumption on the parameters, namely that

$$(56) \quad 2\delta_j \leq \frac{\delta_{j-1}}{2}, \quad \forall j.$$

Using the properties of $a(t)$ enumerated above, we derive from (55)

$$(57) \quad E_j(t) \leq E_j(t'_j) \exp \left[2(\nu_1 + \dots + \nu_{j-1}) \lg 4 + \lg \left(\frac{2}{c_1} \cdot \frac{\delta_1}{\delta_j} \right) \right]$$

for any $t \leq t'_j$.

Finally, observing that $(h_j^2 a(t))^{-1} \leq c_2$ for $t \leq t'_j$, we derive from (57), (54) and (53) that

$$(58) \quad \sup_{[0, t'_j]} |v_j| \leq c_3 \exp \left[-\frac{\pi}{10} \nu_j + (\nu_1 + \dots + \nu_{j-1}) \lg 4 + \lg \frac{\delta_1}{\delta_j} \right].$$

On the other side, Paley-Wiener theorem ensures that the series (42) is converging near some $u(x, t)$ in $C([0, T_* - \varepsilon], \mathcal{E}^s)$ for some $\varepsilon > 0$ and $s \geq 1$, if and only if

$$\sup_{[0, T_* - \varepsilon]} |v_j| \leq M_\varepsilon \cdot \exp(-\mu_\varepsilon h_j^{1/s})$$

with M_ε and $\mu_\varepsilon > 0$.

Thus, taking into account that $t'_j \rightarrow T_*$ as $j \rightarrow \infty$, we get from (58) the following sufficient condition for $u(x, t)$ belong to $C([0, T_*], \mathcal{E}^s)$:

$$(59) \quad -\frac{\pi}{10} \nu_j + (\nu_1 + \dots + \nu_{j-1}) \lg 4 + \lg \frac{\delta_1}{\delta_j} \leq -\mu h_j^{1/s} + \lg M$$

for some $M, \mu > 0$.

Remembering that $h_j = 2\pi\nu_j\varrho_j^{-1}\delta_j^{-1/2}$, we see that (59) is true in particular when

$$(60) \quad (\nu_1 + \dots + \nu_{i-1}) \lg 4 < \frac{\pi}{11} \nu_i$$

and

$$(61) \quad \sup_j \nu_j^{1-s} \varrho_j^{-1} \delta_j^{-1/2} < \infty.$$

Let us moreover observe that, if the series in (42) converges in $C([0, T_*[, \mathfrak{E}^s)$, then $u(x, t)$ is a weak solution of equation (33); so that, by the regularity of $a(t)$ on $[0, T_*[$, we also get that $u \in C^\infty([0, T_*[, \mathfrak{E}^s)$.

In conclusion, in order that (36) holds, it is sufficient that (60) and (61), with $s > 1 + (k + \alpha)/2$, are satisfied.

It remains to find conditions ensuring (37). To this purpose let us go back to (52) and observe that if (59) holds for some $s \geq 1$, then (52) gives

$$(62) \quad |v_j(t_j'')| \geq \frac{1}{c_4} \exp\left(\frac{\mu}{2} h_j\right),$$

where $\mu > 0$.

Inequality (62) gives the unboundedness of $\{u(\cdot, t_j'')\}$ in $\mathfrak{D}'$. Hence no further assumption on the parameters is needed, in order to have (37).

Summarizing, in order to have (35), (36) and (37), we must only exhibit a choice of the parameters $\varrho_i, \delta_i, \nu_i$ verifying conditions (38), (46), (47), (56), (60) and (61) for $s > 1 + (k + \alpha)/2$. Incidentally, let us observe that it is impossible to satisfy simultaneously (46) and (61) if $s < 1 + (k + \alpha)/2$.

A good choice is the following

$$(63) \quad \begin{cases} \varrho_j = 2^{-j} T_* \\ \nu_j = 2^{j^2} \\ \delta_j = 2^{-(k+\alpha)(j+1)(j+2)-2j} \end{cases}$$

which gives in particular

$$h_j = \frac{2\pi}{T_*} 2^{j^2+2j+(k+\alpha)(j+1)(j+2)/2}. \quad //$$

REMARK 6. In th. 2 we have constructed on $\mathbb{R} \times [0, T_*[$ a solution of (33), $u(x, t)$, which cannot be continued on the *closed* interval $[0, T_*]$ as an element of the space $C([0, T_*], \mathfrak{D}')$.

Moreover, as it is easily seen, such a solution cannot be continued as a distribution on $\mathbb{R} \times [0, T_* + \varepsilon]$, for any $\varepsilon > 0$.

On the other side we know that u can be continued to some $\tilde{u} \in C^1([0, +\infty[, (\mathcal{D}^s)')$, with $s < 1 + (k + \alpha)/2$. Indeed, (36) gives in particular that $u(x, 0)$ and $u_t(x, 0)$ belong to $(\mathcal{D}')'$ for every $r \geq 1$, and problem $\{(33), (34)\}$ is well-posed in $(\mathcal{D}^s)'$ for $s < 1 + (k + \alpha)/2$ (see Rem. 4).

Now one can ask if the ultradistributions $\tilde{u}(\cdot, T_*)$ and $\tilde{u}_t(\cdot, T_*)$ are belonging to $\mathcal{D}'$.

The answer to this question is that they cannot *both* belong to $\mathcal{D}'$.

To prove this fact, let us introduce the λ -energy of $v_j(t)$ as

$$E_{j,\lambda}(t) = \lambda h_j^2 |v_j(t)|^2 + |v'_j(t)|^2 \quad (\lambda > 0),$$

with $h_j, v_j(t)$ as in the proof of th. 2.

It is then easy to prove, in a similar way that (26), the following energy estimate:

$$(64) \quad E_{j,\lambda}(s) \leq E_{j,\lambda}(t) \cdot \exp \left(\frac{h_j}{\sqrt{\lambda}} \left| \int_s^t |a(\xi) - \lambda| d\xi \right| \right), \quad \forall s, t.$$

Let us take $\lambda = \delta_{j+1}$, $s = t_j''$, $t = T_*$, and observe that

$$\begin{aligned} \int_{t_j''}^{T_*} |a(\xi) - \delta_{j+1}| d\xi &= \int_{t_j''}^{t_{j+1}'} |a(\xi) - \delta_{j+1}| d\xi + \int_{t_{j+1}'}^{T_*} |a(\xi) - \delta_{j+1}| d\xi \\ &\leq (t_{j+1}' - t_j'') 2\delta_j + (T_* - t_{j+1}') \delta_{j+1} \leq C \left(\frac{\varrho_j}{\nu_j} \delta_j + \left(\sum_{j+1}^{\infty} \varrho_n \right) \delta_{j+1} \right) \end{aligned}$$

and that (see (52))

$$E_{j,\lambda}(t_j'') \geq \frac{1}{C} \exp \left(\frac{\pi}{5} \nu_j \right).$$

Then, in virtue of our choice of $\varrho_j, \delta_j, \nu_j$ (see (63)), we get by (64) the estimate from below

$$|v_j(T_*)| + |v'_j(T_*)| \geq \frac{1}{C'} \exp(\mu \nu_j),$$

for some C' and $\mu > 0$ and j large enough, which shows that $\{|v_j(T_*)| + |v'_j(T_*)|\}$ has an exponential growth with respect to h_j for $j \rightarrow \infty$ and hence that $u(\cdot, T_*)$ and $u_t(\cdot, T_*)$ cannot be both distributions. //

The solution $u(x, t) \equiv \sum v_j(t) \sin(h_j x)$ constructed in th. 2 has the property to be very regular for $t < T_*$ and to become irregular at $t = T_*$. In fact $|v_j(t)|$ decreases to zero as $\exp(-\mu_1 h_j^{1/s})$ for $t < T_*$, whereas $|v_j(T_*)| + |v'_j(T_*)|$ grows as $\exp(\mu_2 h_j^{1/s})$, with $\mu_j > 0$, $s > 1 + (k + a)/2$ and $j \rightarrow \infty$.

Now, in view of Th. 3 below, we shall indicate how to construct a solution $\tilde{u}(x, t)$ of an equation of type (33), say

$$(65) \quad \tilde{u}_{tt} - \tilde{a}(t)\tilde{u}_{xx} = 0, \quad t \geq 0,$$

which has just the opposite property that u . Namely we look for some solution $\tilde{u}$ of (65) which is very irregular if $t < T_*$ but becomes regular when $t = T_*$.

To construct $\tilde{u}(x, t)$, we proceed as in the proof of th. 2, using in addition the techniques of Rem. 6. The main difference is actually that, to define $\tilde{a}(t)$, we use this time the function

$$\tilde{\alpha}(\tau) = 1 + \frac{4}{10} \sin 2\tau - \frac{1}{100} (1 - \cos 2\tau)^2$$

in place of the function $\alpha(\tau)$ defined by (39).

The solution of

$$\begin{cases} \tilde{w}'' + \tilde{\alpha}(\tau)\tilde{w} = 0 \\ \tilde{w}(0) = 0, \quad \tilde{w}'(0) = 1 \end{cases}$$

is given by

$$\tilde{w}(\tau) = -\sin \tau \exp \left[-\frac{1}{10} \left(\tau - \frac{1}{2} \sin 2\tau \right) \right],$$

so that

$$|\tilde{w}(\tau)| \leq C \exp \left(-\frac{\tau}{10} \right).$$

By means of $\tilde{\alpha}(\tau)$ we then construct the coefficient $\tilde{a}(t)$ of equation (65) in the same manner that $a(t)$ in the proof of th. 2 (see (40), (41)).

Let us now construct the wished solution $\tilde{u}$, belonging to $C^1([0, +\infty[, (\mathcal{D}')')$ for some $s > 1$, by taking again

$$\tilde{u}(x, t) = \sum \tilde{v}_j(t) \sin(h_j x)$$

with $\tilde{v}_j(t)$ such that

$$\begin{cases} \tilde{v}_j'' + h_j \tilde{a}(t) \tilde{v}_j = 0 \\ \tilde{v}_j(t_j) = 0, \tilde{v}_j'(t_j) = 1. \end{cases}$$

We have then (cf. (51), (52))

$$(66) \quad |v_j(t'_j)| = \tilde{c}_1 \frac{\varrho_j}{\nu_j} \exp \left(\frac{\pi}{10} \nu_j \right); \quad |v'_j(t'_j)| = \tilde{c}_2 \exp \left(\frac{\pi}{10} \nu_j \right)$$

and

$$(67) \quad |v_j(t_j'')| = \tilde{c}_3 \frac{\rho_j}{\nu_j} \exp\left(-\frac{\pi}{10} \nu_j\right); \quad |v_j'(t_j'')| = \tilde{c}_4 \exp\left(-\frac{\pi}{10} \nu_j\right),$$

with $\tilde{c}_j > 0$.

Now, using the energy estimate (64) with $\lambda = \delta_{i+1}$, $s = T_*$ and $t = t_j''$, we derive from (67) that $|v_j(T_*)|$ and $|v_j'(T_*)|$ are less than $C \cdot \exp(-\mu h_j^{1/s})$ for some $\mu > 0$ and every $s > 1 + (k + \alpha)/2$. Thus $u(\cdot, T_*)$ and $u_t(\cdot, T_*)$ are belonging to $\mathcal{E}^s$ for $s > 1 + (k + \alpha)/2$.

Finally we derive from (66) that u and u_t are not two distributions on any strip $\mathbb{R} \times]T_* - \varepsilon, T_*[$ for $\varepsilon > 0$.

In conclusion, if we effect the change of variable $t \mapsto T_* - t$, we get the following result.

THEOREM 3. *For every k, α, k integer ≥ 0 and $0 \leq \alpha < 1$, it is possible to construct a function $a(t)$, vanishing at $t = 0$ and strictly positive for $t > 0$, and two initial data $\varphi(x), \psi(x)$ which belong to $\mathcal{E}^s$ for any $s > 1 + (k + \alpha)/2$, in such a way that:*

- i) $a(t)$ belongs to $C^{k, \alpha}([0, +\infty[)$;
- ii) the problem $\{(33), (34)\}$ does not admit solutions in the space $\mathcal{D}'(\mathbb{R} \times]0, \varepsilon[)$, $\forall \varepsilon > 0$.

ADDED IN PROOF. After the drawing up of the present paper, T. Nishitani sent us a manuscript containing the extension of th. 1, when $k + \alpha \leq 2$, to the more general case of an equation whose coefficients $a_{ij}(x, t)$ are $C^{k, \alpha}$ with respect to t and Gevrey functions of order s with respect to x , and (k, α, s) satisfies condition (3).

REFERENCES

- [1] F. COLOMBINI - E. DE GIORGI - S. SPAGNOLO, *Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps*, Ann. Scuola Norm. Sup. Pisa, **6** (1979), pp. 511-559.
- [2] F. COLOMBINI - S. SPAGNOLO, *An example of weakly hyperbolic Cauchy problem not well posed in C^∞* , Acta Math., **148** (1982), pp. 243-253.
- [3] J. DIEUDONNÉ, *Sur un théorème de Glaeser*, J. Analyse Math., **23** (1970), pp. 85-88.
- [4] G. GLAESER, *Racine carrée d'une fonction différentiable*, Ann. Inst. Fourier, **13** (1963), pp. 203-210.
- [5] V. YA. IVRII - V. M. PETKOV, *Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed*, Uspehi Mat. Nauk, **29** (1974), pp. 3-70, English Transl. in Russian Math. Surveys.

- [6] E. JANNELLI, *Weakly hyperbolic equations of second order with coefficients real analytic in space variables*, Comm. in Partial Diff. Equations, **7** (1982), pp. 537-558.
- [7] T. NISHITANI, *The Cauchy problem for weakly hyperbolic equations of second order*, Comm. in Partial Diff. Equations, **5** (1980), pp. 1273-1296.
- [8] O. A. OLEINIK, *On the Cauchy problem for weakly hyperbolic equations*, Comm. Pure Appl. Math., **23** (1970), pp. 569-586.

Istituto Matematico
Università
Piazzale Europa, 1
34127 Trieste

Scuola Normale Superiore
Piazza dei Cavalieri, 7
56100 Pisa

Istituto Matematico
Università
Via F. Buonarroti, 2
56100 Pisa