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On the Stability of Strongly Continuous Semigroups of Positive Operators on $L^2(\mu)$.

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One of the fundamental problems in infinite-dimensional linear stability theory is to decide whether

$$(*) \qquad s(A) = \omega_0$$

for the generator A of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space (see [9]). Here,

$$s(A) := \sup \{\operatorname{Re} \lambda\colon \lambda \in \sigma(A)\}$$

denotes the *spectral bound* of A while

$$\omega_0 := \inf \{\omega\colon \|T(t)\| \leqslant M_\omega \cdot \exp [\omega t] \text{ for every } t \geqslant 0\}$$

is the *growth bound* of $\{T(t)\}_{t\geq 0}$. The coincidence of the spectral and growth bounds means that stability of the semigroup depends on the location of the spectrum of the generator. More precisely, suppose that $(*)$ is true. Then $s(A) < 0$ implies $\lim_{t\to\infty} \|T(t)\| = 0$.

We refer to [7] for a complete discussion of $(*)$, but recall that « $s(A) = \omega_0$ » does not hold in general, neither for semigroups on Hilbert spaces (see [10] or [2], th. 2.17) nor for positive semigroups on Banach lattices (see [4]). In this note we combine the Hilbert space and the

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order structure and show that « $s(A) = \omega_0$ » for every strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ on $L^2(\mu)$ consisting of positive operators, i.e. such that $T(t)f$ is a positive function whenever $f \in L^2(\mu)$ is positive and $t \geqslant 0$.

The proof of this result was inspired by a recent paper of L. Monauni [6]. In fact, using implicitely Lemma 1 below, he characterizes the growth bound ω_0 for semigroups on Hilbert spaces by the boundedness of the resolvent along imaginary axes $\lambda + i\mathbb{R}$, $\lambda > \omega_0$. For positive semigroups on L^2-spaces we obtain this property from the integral representation stated in Lemma 2. For the basic concepts on one-parameter semigroups we refer to [1] or [2].

LEMMA 2. *Let H be a Hilbert space. Then the vector-valued Fourier transform $\mathcal{F}_H\colon L^2(\mathbb{R}, H) \to L^2(\mathbb{R}, H)$ defined by $F \mapsto \hat{F}$,*

$$\hat{F}(\lambda) := (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} \exp[-i\lambda s]F(s)\, ds \qquad \text{for } \lambda \in \mathbb{R}$$

and suitable $F \in L^2(\mathbb{R}, H)$, is an isometry.

PROOF. Every Hilbert space H is isomorphic to $L^2(\mu)$ for some measure space (X, Σ, μ). Given a Banach space G and an operator $T \in \mathfrak{L}(G)$, then $F(\cdot) \mapsto T(F(\cdot))$ defines an operator $Id \otimes T$ on $L^2(\mu; G)$ satisfying $\|Id \otimes T\| \leqslant \|T\|$. In particular, $Id \otimes T$ is an isometric isomorphism if the same is true for T. Now, observe that the Hilbert space $L^2(\mathbb{R}; H) = L^2(\mathbb{R}; L^2(\mu))$ is canonically isomorphic to $L^2(\mu; L^2(\mathbb{R}))$. Moreover, this isomorphism transforms $\mathcal{F}_H$ into $Id \otimes \mathcal{F}$, where $\mathcal{F}$ denotes the scalar-valued Fourier transform on $L^2(\mathbb{R})$. Since $\mathcal{F}$ is unitary the assertion follows.

LEMMA 2 ([3], 3.2 or [4], 3.3). *Let $\{T(t)\}_{t\geqslant 0}$ be a strongly continuous semigroup of positive operators on a Banach lattice E and denote by $(A, D(A))$ its generator. Then the resolvent integral*

$$\int_0^{\infty} \exp[-\lambda s]T(s)f\, ds\,, \qquad f \in E\,,$$

exists for every $\lambda \in \mathbb{C}$ satisfying $\operatorname{Re}\lambda > s(A)$. In particular, the resolvent $R(\lambda) := (\lambda - A)^{-1}$, $\lambda \in \mathbb{C}\setminus\sigma(A)$, satisfies $\|R(\lambda)\| \leqslant \|R(\operatorname{Re}\lambda)\|$ for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > s(A)$.

THEOREM. *Let $\{T(t)\}_{t\geqslant 0}$ be a strongly continuous semigroup of positive operators on $L^2(\mu)$. Then the spectral bound $s(A)$ of the generator A coincides with the growth bound ω_0 of $\{T(t)\}_{t\geqslant 0}$.*

PROOF. We may assume that $\omega_0 = 0$ which implies that $s(A) \leqslant 0$. For every $\alpha \in \mathbf{C}$, with $\operatorname{Re}\alpha > s(A)$, and $f \in L^2(\mu)$, we define continuous, $L^2(\mu)$-valued functions

$$F_f^\alpha(s) := \exp[-\alpha|s|]T(|s|)f\,, \qquad s \in \mathbb{R}\,,$$
$$G_f^\alpha(\lambda) := (2\pi)^{-\frac{1}{2}}\big(R(\alpha + i\lambda)f + R(\alpha - i\lambda)f\big)\,, \qquad \lambda \in \mathbb{R}\,.$$

For $\operatorname{Re}\alpha > 0$ it follows from the definition of ω_0 that F_f^α is contained in $L^2(\mathbb{R}, L^2(\mu)) \cap L^1(\mathbb{R}, L^2(\mu))$. Using the integral representation of the resolvent we infer from [2], th. 2.8 that

$$\hat{F}_f^\alpha(\lambda) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp[-i\lambda s]\exp[-\alpha|s|]T(|s|)f\,ds = G_f^\alpha(\lambda)\,. \tag{1}$$

By Lemma 1 we conclude $G_f^\alpha \in L^2(\mathbb{R}, L^2(\mu))$ and obtain

$$\|G_f^\alpha\|_2^2 = \|F_f^\alpha\|_2^2 = \int_{-\infty}^{\infty}\exp[-2\operatorname{Re}\alpha|s|]\|T(|s|)f\|^2\,ds \leqslant c_\alpha \cdot \|f\|^2 \tag{2}$$

for some constant $c_\alpha \in \mathbb{R}$ independent of f.

In order to prove the assertion, we assume to the contrary that $s(A) < 0$. Then take $\alpha \in \mathbf{C}$, $s(A) < \operatorname{Re}\alpha \leqslant 0$, and $f \in D(A^2)$ and recall the following identity:

$$\begin{aligned}
(2\pi)^{\frac{1}{2}}G_f^\alpha(\lambda) &= R(\alpha + i\lambda)R(\alpha)(\alpha - A)f + R(\alpha - i\lambda)R(\alpha)(\alpha - A)f \\
&= -(i\lambda)^{-1}\big(R(\alpha + i\lambda) - R(\alpha)\big)(\alpha - A)f + (i\lambda)^{-1}\big(R(\alpha - i\lambda) - R(\alpha)\big)(\alpha - A)f \\
&= -(i\lambda)^{-1}\big(R(\alpha + i\lambda)R(\alpha) - R(\alpha - i\lambda)R(\alpha)\big)(\alpha - A)^2 f \\
&= -\lambda^{-2}\big[\big(R(\alpha + i\lambda) + R(\alpha - i\lambda)\big)(\alpha - A)^2 f - 2(\alpha - A)f\big]\,.
\end{aligned} \tag{3}$$

Since $G_f^\alpha(\cdot)$ is continuous and $\|R(\alpha \pm i\lambda)\|$ is dominated by $\|R(\operatorname{Re}\alpha)\|$ (use Lemma 2) we obtain $G_f^\alpha \in L^2(\mathbb{R}, L^2(\mu)) \cap L^1(\mathbb{R}, L^2(\mu))$.

The inverse Fourier transform applied to G_f^α gives a new function

$$H_f^\alpha(s) := \check{G}_f^\alpha(s) = (2\pi)^{-1}\int_{-\infty}^{\infty}\exp[i\lambda s]\big(R(\alpha + i\lambda)f + R(\alpha - i\lambda)f\big)\,d\lambda\,.$$

We already stated in (1) that $H_f^\alpha = F_f^\alpha$ for every α with $\operatorname{Re}\alpha > 0$.

Now keep $s \in \mathbb{R}$ and $f \in D(A^2)$ fixed and observe that the map

$$\alpha \mapsto F_f^\alpha(s) = \exp[-\alpha|s|]T(|s|)f$$

is analytic on $\mathbb{C}$. On the other hand the function

$$\alpha \mapsto H_f^\alpha(s)$$

is analytic on $\{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha > s(A)\}$. This can be seen from the theorem of Morera since

$$\int_\Gamma \int_{-\infty}^{\infty} \exp[i\lambda s]\big(R(\alpha+i\lambda)f + R(\alpha-i\lambda)f\big)\,d\lambda\,d\alpha$$
$$=\int_{-\infty}^{\infty} \int_\Gamma \exp[i\lambda s]\big(R(\alpha+i\lambda)f + R(\alpha-i\lambda)f\big)\,d\alpha\,d\lambda = 0$$

for every closed curve Γ contained in the right semiplane determined by $s(A)$. Here the order of integration may be reversed since the norm of the integrand is dominated by $2\lambda^{-2}(\|R(\operatorname{Re}\alpha)\|\cdot\|(\alpha-A)^2 f\| + \|(\alpha-A)f\|)$ (use (3) and Lemma 2). From the uniqueness theorem for analytic functions, we conclude that

$$H_f^\alpha = F_f^\alpha$$

on the semiplane $\{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha > s(A)\}$. Applying Lemma 1 again we obtain

$$\|F_f^\alpha\|_2^2 = \|G_f^\alpha\|_2^2 \tag{4}$$

for $\operatorname{Re}\alpha > s(A)$ and $f \in D(A^2)$.

For $\operatorname{Re}\alpha > 0$ we could estimate $\|G_f^\alpha\|_2^2$ by a suitable multiple of $\|f\|^2$ (see (2)). In order to obtain an analogous estimate for $s(A) < \operatorname{Re}\alpha \leqslant 0$ we use the following relation between $R(\operatorname{Re}\alpha)$ and $R(-\operatorname{Re}\alpha)$. Choose $0 < -\alpha < -s(A)$ in such a way that $(1-2\alpha R(\alpha))$ becomes invertible. Using the resolvent equation in the form

$$R(\alpha \pm i\lambda) = \big(1 - 2\alpha R(\alpha \pm i\lambda)\big)R(-\alpha \pm i\lambda)$$

one deduces the identity

$$\big(1-2\alpha R(\alpha-i\lambda)\big)\big(1-2\alpha R(\alpha+i\lambda)\big)[R(-\alpha+i\lambda) + R(-\alpha-i\lambda)]$$
$$= \big(1-2\alpha R(\alpha-i\lambda)\big)R(\alpha+i\lambda) + \big(1-2\alpha R(\alpha+i\lambda)\big)R(\alpha-i\lambda)$$

$$= R(\alpha + i\lambda) + R(\alpha - i\lambda) - 4\alpha R(\alpha + i\lambda) R(\alpha - i\lambda)$$

$$= R(\alpha + i\lambda) + R(\alpha - i\lambda) + \frac{2\alpha}{i\lambda}\big(R(\alpha + i\lambda) - R(\alpha)\big) - \frac{2\alpha}{i\lambda}\big(R(\alpha - i\lambda) - R(\alpha)\big)$$

$$= \big(1 - 2\alpha R(\alpha)\big)\big(R(\alpha + i\lambda) + R(\alpha - i\lambda)\big),$$

which yields the estimate

$$\|G_f^\alpha\|_2^2 \leqslant d_\alpha \|G_f^{-\alpha}\|_2^2 \tag{5}$$

for every $f \in D(A^2)$ and some constant d_α independent of f.

Putting together (4), (5) and (2) we have finally

$$\|F_f^\alpha\|_2^2 \leqslant d_\alpha \cdot c_{-\alpha} \|f\|_2^2$$

for every $f \in D(A^2)$ and a constant $d_\alpha c_{-\alpha}$ still independent of f.

Since $D(A^2)$ is dense in $L^2(\mu)$ we may extend this estimate and obtain

$$\int_{-\infty}^{\infty} \exp[-2\alpha|s|] \|T(|s|)f\|^2 \, ds < \infty$$

for every $f \in L^2(\mu)$. By Datko's theorem (see [8], p. 121) this implies $\omega_0 < 0$ contradicting our assumption. Therefore the spectral bound $s(A)$ has to be 0.

FINAL REMARK. While « $s(A) = \omega_0$ » holds for positive semigroups on $L^1(\mu)$ (see [3], 3.3), on $L^2(\mu)$ and on $L^\infty(\mu)$ ([3], 3.3, also [5]) we do not know whether the statement still holds for $L^p(\mu)$, $1 < p < \infty$, $p \neq 2$.

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