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[On the Stability of Strongly Continuous Semigroups] of Positive Operators on $L^2(\mu)$.

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One of the fundamental problems in infinite-dimensional linear stability theory is to decide whether

$$(*) \quad s(A) = \omega_0$$

for the generator A of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ of bounded linear operators on a Banach space (see [9]). Here,

$$s(A) := \sup \{ \operatorname{Re} \lambda : \lambda \in \sigma(A) \}$$

denotes the *spectral bound* of A while

$$\omega_0 := \inf \{ \omega : \|T(t)\| \leq M_\omega \cdot \exp[\omega t] \text{ for every } t \geq 0 \}$$

is the *growth bound* of $\{T(t)\}_{t \geq 0}$. The coincidence of the spectral and growth bounds means that stability of the semigroup depends on the location of the spectrum of the generator. More precisely, suppose that $(*)$ is true. Then $s(A) < 0$ implies $\lim_{t \rightarrow \infty} \|T(t)\| = 0$.

We refer to [7] for a complete discussion of $(*)$, but recall that « $s(A) = \omega_0$ » does not hold in general, neither for semigroups on Hilbert spaces (see [10] or [2], th. 2.17) nor for positive semigroups on Banach lattices (see [4]). In this note we combine the Hilbert space and the

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order structure and show that « $s(A) = \omega_0$ » for every strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on $L^2(\mu)$ consisting of positive operators, i.e. such that $T(t)f$ is a positive function whenever $f \in L^2(\mu)$ is positive and $t \geq 0$.

¶ The proof of this result was inspired by a recent paper of L. Monna [6]. In fact, using implicitly Lemma 1 below, he characterizes the growth bound ω_0 for semigroups on Hilbert spaces by the boundedness of the resolvent along imaginary axes $\lambda + i\mathbb{R}$, $\lambda > \omega_0$. For positive semigroups on L^2 -spaces we obtain this property from the integral representation stated in Lemma 2. For the basic concepts on one-parameter semigroups we refer to [1] or [2].

LEMMA 2. *Let H be a Hilbert space. Then the vector-valued Fourier transform $\mathcal{F}_H: L^2(\mathbb{R}, H) \rightarrow L^2(\mathbb{R}, H)$ defined by $F \mapsto \hat{F}$,*

$$\hat{F}(\lambda) := (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp[-i\lambda s] F(s) ds \quad \text{for } \lambda \in \mathbb{R}$$

and suitable $F \in L^2(\mathbb{R}, H)$, is an isometry.

PROOF. Every Hilbert space H is isomorphic to $L^2(\mu)$ for some measure space (X, Σ, μ) . Given a Banach space G and an operator $T \in \mathcal{L}(G)$, then $F(\cdot) \mapsto T(F(\cdot))$ defines an operator $Id \otimes T$ on $L^2(\mu; G)$ satisfying $\|Id \otimes T\| \leq \|T\|$. In particular, $Id \otimes T$ is an isometric isomorphism if the same is true for T . Now, observe that the Hilbert space $L^2(\mathbb{R}; H) = L^2(\mathbb{R}; L^2(\mu))$ is canonically isomorphic to $L^2(\mu; L^2(\mathbb{R}))$. Moreover, this isomorphism transforms $\mathcal{F}_H$ into $Id \otimes \mathcal{F}$, where $\mathcal{F}$ denotes the scalar-valued Fourier transform on $L^2(\mathbb{R})$. Since $\mathcal{F}$ is unitary the assertion follows.

LEMMA 2 ([3], 3.2 or [4], 3.3). *Let $\{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup of positive operators on a Banach lattice E and denote by $(A, D(A))$ its generator. Then the resolvent integral*

$$\int_0^{\infty} \exp[-\lambda s] T(s) f ds, \quad f \in E,$$

exists for every $\lambda \in \mathbb{C}$ satisfying $\operatorname{Re} \lambda > s(A)$. In particular, the resolvent $R(\lambda) := (\lambda - A)^{-1}$, $\lambda \in \mathbb{C} \setminus \sigma(A)$, satisfies $\|R(\lambda)\| \leq \|R(\operatorname{Re} \lambda)\|$ for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > s(A)$.

THEOREM. *Let $\{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup of positive operators on $L^2(\mu)$. Then the spectral bound $s(A)$ of the generator A coincides with the growth bound ω_0 of $\{T(t)\}_{t \geq 0}$.*

PROOF. We may assume that $\omega_0 = 0$ which implies that $s(A) \leq 0$. For every $\alpha \in \mathbb{C}$, with $\operatorname{Re} \alpha > s(A)$, and $f \in L^2(\mu)$, we define continuous, $L^2(\mu)$ -valued functions

$$\begin{aligned} F_f^\alpha(s) &:= \exp[-\alpha|s|]T(|s|)f, \quad s \in \mathbb{R}, \\ G_f^\alpha(\lambda) &:= (2\pi)^{-1/2}(R(\alpha + i\lambda)f + R(\alpha - i\lambda)f), \quad \lambda \in \mathbb{R}. \end{aligned}$$

For $\operatorname{Re} \alpha > 0$ it follows from the definition of ω_0 that F_f^α is contained in $L^2(\mathbb{R}, L^2(\mu)) \cap L^1(\mathbb{R}, L^2(\mu))$. Using the integral representation of the resolvent we infer from [2], th. 2.8 that

$$(1) \quad \hat{F}_f^\alpha(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp[-i\lambda s] \exp[-\alpha|s|]T(|s|)f \, ds = G_f^\alpha(\lambda).$$

By Lemma 1 we conclude $G_f^\alpha \in L^2(\mathbb{R}, L^2(\mu))$ and obtain

$$(2) \quad \|G_f^\alpha\|_2^2 = \|F_f^\alpha\|_2^2 = \int_{-\infty}^{\infty} \exp[-2\operatorname{Re} \alpha|s|] \|T(|s|)f\|^2 \, ds \leq c_\alpha \cdot \|f\|^2$$

for some constant $c_\alpha \in \mathbb{R}$ independent of f .

In order to prove the assertion, we assume to the contrary that $s(A) < 0$. Then take $\alpha \in \mathbb{C}$, $s(A) < \operatorname{Re} \alpha \leq 0$, and $f \in D(A^2)$ and recall the following identity:

$$\begin{aligned} (3) \quad (2\pi)^{1/2} G_f^\alpha(\lambda) &= R(\alpha + i\lambda)R(\alpha)(\alpha - A)f + R(\alpha - i\lambda)R(\alpha)(\alpha - A)f \\ &= -(i\lambda)^{-1}(R(\alpha + i\lambda) - R(\alpha))(\alpha - A)f + (i\lambda)^{-1}(R(\alpha - i\lambda) - R(\alpha))(\alpha - A)f \\ &= -(i\lambda)^{-1}(R(\alpha + i\lambda)R(\alpha) - R(\alpha - i\lambda)R(\alpha))(\alpha - A)^2f \\ &= -\lambda^{-2}[(R(\alpha + i\lambda) + R(\alpha - i\lambda))(\alpha - A)^2f - 2(\alpha - A)f]. \end{aligned}$$

Since $G_f^\alpha(\cdot)$ is continuous and $\|R(\alpha \pm i\lambda)\|$ is dominated by $\|R(\operatorname{Re} \alpha)\|$ (use Lemma 2) we obtain $G_f^\alpha \in L^2(\mathbb{R}, L^2(\mu)) \cap L^1(\mathbb{R}, L^2(\mu))$.

The inverse Fourier transform applied to G_f^α gives a new function

$$H_f^\alpha(s) := \check{G}_f^\alpha(s) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp[i\lambda s](R(\alpha + i\lambda)f + R(\alpha - i\lambda)f) \, d\lambda.$$

We already stated in (1) that $H_f^\alpha = F_f^\alpha$ for every α with $\operatorname{Re} \alpha > 0$.

Now keep $s \in \mathbb{R}$ and $f \in D(A^2)$ fixed and observe that the map

$$\alpha \mapsto F_f^\alpha(s) = \exp[-\alpha|s|]T(|s|)f$$

is analytic on $\mathbb{C}$. On the other hand the function

$$\alpha \mapsto H_f^\alpha(s)$$

is analytic on $\{\alpha \in \mathbb{C}: \operatorname{Re} \alpha > s(A)\}$. This can be seen from the theorem of Morera since

$$\begin{aligned} & \int_{\Gamma} \int_{-\infty}^{\infty} \exp[i\lambda s] (R(\alpha + i\lambda)f + R(\alpha - i\lambda)f) d\lambda d\alpha \\ &= \int_{-\infty}^{\infty} \int_{\Gamma} \exp[i\lambda s] (R(\alpha + i\lambda)f + R(\alpha - i\lambda)f) d\alpha d\lambda = 0 \end{aligned}$$

for every closed curve Γ contained in the right semiplane determined by $s(A)$. Here the order of integration may be reversed since the norm of the integrand is dominated by $2\lambda^{-2}(\|R(\operatorname{Re} \alpha)\| \cdot \|(\alpha - A)^2 f\| + \|(\alpha - A)f\|)$ (use (3) and Lemma 2). From the uniqueness theorem for analytic functions, we conclude that

$$H_f^\alpha = F_f^\alpha$$

on the semiplane $\{\alpha \in \mathbb{C}: \operatorname{Re} \alpha > s(A)\}$. Applying Lemma 1 again we obtain

$$(4) \quad \|F_f^\alpha\|_2^2 = \|G_f^\alpha\|_2^2$$

for $\operatorname{Re} \alpha > s(A)$ and $f \in D(A^2)$.

For $\operatorname{Re} \alpha > 0$ we could estimate $\|G_f^\alpha\|_2^2$ by a suitable multiple of $\|f\|^2$ (see (2)). In order to obtain an analogous estimate for $s(A) < \operatorname{Re} \alpha \leq 0$ we use the following relation between $R(\operatorname{Re} \alpha)$ and $R(-\operatorname{Re} \alpha)$. Choose $0 < -\alpha < -s(A)$ in such a way that $(1 - 2\alpha R(\alpha))$ becomes invertible. Using the resolvent equation in the form

$$R(\alpha \pm i\lambda) = (1 - 2\alpha R(\alpha \pm i\lambda))R(-\alpha \pm i\lambda)$$

one deduces the identity

$$\begin{aligned} & (1 - 2\alpha R(\alpha - i\lambda))(1 - 2\alpha R(\alpha + i\lambda))[R(-\alpha + i\lambda) + R(-\alpha - i\lambda)] \\ &= (1 - 2\alpha R(\alpha - i\lambda))R(\alpha + i\lambda) + (1 - 2\alpha R(\alpha + i\lambda))R(\alpha - i\lambda) \end{aligned}$$

$$\begin{aligned}
&= R(\alpha + i\lambda) + R(\alpha - i\lambda) - 4\alpha R(\alpha + i\lambda)R(\alpha - i\lambda) \\
&= R(\alpha + i\lambda) + R(\alpha - i\lambda) + \frac{2\alpha}{i\lambda} (R(\alpha + i\lambda) - R(\alpha)) - \frac{2\alpha}{i\lambda} (R(\alpha - i\lambda) - R(\alpha)) \\
&= (1 - 2\alpha R(\alpha))(R(\alpha + i\lambda) + R(\alpha - i\lambda)),
\end{aligned}$$

which yields the estimate

$$(5) \quad \|G_f^\alpha\|_2^2 \leq d_\alpha \|G_f^{-\alpha}\|_2^2$$

for every $f \in D(A^2)$ and some constant d_α independent of f .

Putting together (4), (5) and (2) we have finally

$$\|F_f^\alpha\|_2^2 \leq d_\alpha \cdot c_{-\alpha} \|f\|_2^2$$

for every $f \in D(A^2)$ and a constant $d_\alpha c_{-\alpha}$ still independent of f .

Since $D(A^2)$ is dense in $L^2(\mu)$ we may extend this estimate and obtain

$$\int_{-\infty}^{\infty} \exp[-2\alpha|s|] \|T(|s|)f\|^2 ds < \infty$$

for every $f \in L^2(\mu)$. By Datko's theorem (see [8], p. 121) this implies $\omega_0 < 0$ contradicting our assumption. Therefore the spectral bound $s(A)$ has to be 0.

FINAL REMARK. While « $s(A) = \omega_0$ » holds for positive semigroups on $L^1(\mu)$ (see [3], 3.3), on $L^2(\mu)$ and on $L^\infty(\mu)$ ([3], 3.3, also [5]) we do not know whether the statement still holds for $L^p(\mu)$, $1 < p < \infty$, $p \neq 2$.

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