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Kelvin's Solution and Nuclei of Strain in a Solid Mixture.

PIERO VILLAGGIO

1. – Introduction.

In the relatively recent times the balance laws and constitutive equations of a mixture of two linearly-elastic solids have been formulated in full agreement with the principles of thermodynamics and other restrictions necessary for the related boundary value problems to be well-posed. The thermodynamical restrictions on the constitutive equations have been examined by Green and Steel [1966] and, subsequently, by Green and Naghdi [1978]. The uniqueness of solutions of the boundary value problems has been proved by Atkin, Chadwick, Steel [1967] and by Knops and Steel [1969]; the existence of weak solutions for bounded regions has been investigated by Aron [1974].

However, even if the question of existence and uniqueness of solutions can be answered by the natural generalization of techniques successfully applied in classical elasticity, the task of finding explicit solutions to some particular boundary value problems is extremely hard, even in relatively simple situations.

Such considerations induced me to consider the simplest conceivable problem in the elasticity of a mixture of two homogeneous isotropic solids: the problem of the transmission of a force operative at a point of a solid of infinite extent. When the body consists of a single component material, the corresponding solution was given by Kelvin in 1848. It is rather surprising that the method employed by Kelvin can be extended to elastic mixtures, and, accordingly, an exact solution can be obtained. In a mixture, however, the way of prescribing the force is not unique, and there are many

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solutions corresponding to the same concentrated force. Another (expected) property of the solution is that the stress tensor is non-symmetric and its eigenvalues are not all real, except under particular circumstances.

Construction of the explicit solution to Kelvin's problem for a mixture of two elastic solids was done by Tiersten and Jahanmir [1977] in their theory of interacting continua, but their equations (which are more complete since they contain also linear terms in the displacements) are not immediately integrable by Kelvin's method.

2. – Equations of equilibrium in terms of displacements.

Let us consider a three-dimensional medium of infinite extent, whose particles are referred to a cartesian x, y, z-system of coordinates. The medium is a mixture of two elastic constituents, and each point (x, y, z) may be regarded as occupied simultaneously by two different particles, one for each constituent according to Fick and Stefan's hypothesis (1). There are two different displacements at each point:

$$u_i' = u_i'(x, y, z) \qquad (i = x, y, z)\,, \tag{2.1}$$

for the first constituent;

$$u_i'' = u_i''(x, y, z) \qquad (i = x, y, z)\,, \tag{2.2}$$

for the second.

The corresponding strains are

$$\varepsilon_{ij}' = \tfrac{1}{2}(u_{i,j}' + u_{j,i}')\,, \qquad \varepsilon_{ij}'' = \tfrac{1}{2}(u_{i,j}'' + u_{j,i}'') \qquad (i, j = x, y, z)\,, \tag{2.3}$$

and the relative rotation between the two constituents is

$$h_{ij} = \tfrac{1}{2}(u_{j,i}' - u_{i,j}' + u_{i,j}'' - u_{j,i}'') \qquad (i, j = x, y, z)\,. \tag{2.4}$$

The cubical dilatations $\varepsilon_{ii}', \varepsilon_{ii}''$ (i summed) are denoted by Δ', Δ''.

(1) Truesdell and Toupin [1960, § 158] ascribe to Fick and Stefan this basic assumption for the subsequent theories of mixtures.

If the constituents are both homogeneous and isotropic, the linear constitutive equations (suggested by Green and Steel [1966]) are

$$\sigma'_{ij} = (-\alpha_2 + \lambda_1 \Delta' + \lambda_3 \Delta'')\delta_{ij} + 2\mu_1 \varepsilon'_{ij} + 2\mu_3 \varepsilon''_{ij} - 2\lambda_5 h_{ij}\,, \tag{2.5}$$

$$\sigma''_{ij} = (\alpha_2 + \lambda_4 \Delta' + \lambda_2 \Delta'')\delta_{ij} + 2\mu_3 \varepsilon'_{ij} + 2\mu_2 \varepsilon''_{ij} + 2\lambda_5 h_{ij}\,, \tag{2.6}$$

$$\pi_i = \frac{\varrho_2}{\varrho}\alpha_2 \Delta'_{,i} + \frac{\varrho_1}{\varrho}\alpha_2 \Delta''_{,i}\,, \tag{2.7}$$

where σ'_{ij}, σ''_{ij} denote the stress in each constituent; π_i the mutual forces between the constituents; ϱ_1, ϱ_2 the partial densities and $\varrho = \varrho_1 + \varrho_2$ denotes the total density. In (2.5), (2.6), (2.7) the coefficients $\alpha_2, \mu_1, \mu_2, \ldots, \lambda_5$ are the elastic moduli of the two substances. These moduli must obey the inequalities (2)

$$\mu_1 > 0\,, \qquad \mu_2 > 0\,, \qquad \mu_1\mu_2 > \mu_2^3\,, \qquad \lambda_5 < 0\,, \tag{2.8}$$

and the requirement that the matrix

$$\left\| \begin{matrix} \lambda_1 + \frac{2}{3}\mu_1 - \frac{\varrho_2}{\varrho}\alpha_2 & \lambda_3 + \frac{2}{3}\mu_3 - \frac{\varrho_1}{\varrho}\alpha_2 \\ \lambda_3 + \frac{2}{3}\mu_3 - \frac{\varrho_1}{\varrho}\alpha_2 & \lambda_2 + \frac{2}{3}\mu_2 + \frac{\varrho_1}{\varrho}\alpha_2 \end{matrix} \right\| \tag{2.9}$$

be positive definite.

Stresses and mutual forces must satisfy the equilibrium equations. If F'_i, F''_i are the body forces *explicitly* prescribed in each constituent, these equations are

$$\sigma'_{ij,i} - \pi_j + \varrho_1 F'_j = 0\,, \tag{2.10}$$

$$\sigma''_{ij,i} + \pi_j + \varrho_2 F''_j = 0\,. \tag{2.11}$$

If we substitute the expression of the constitutive equations written in terms of displacements we obtain six equations of the type

$$(\mu_1 - \mu_5)\nabla^2 u'_j + (\mu_3 + \lambda_5)\nabla\, u''_j + \left(\mu_1 + \lambda_1 + \lambda_5 - \frac{\varrho_2}{\varrho}\alpha_2\right)\Delta'_{,j} + \left(\mu_3 + \lambda_3 - \lambda_5 - \frac{\varrho_1}{\varrho}\alpha_2\right)\Delta''_{,j} + \varrho_1 F'_j = 0\,, \tag{2.12}$$

(2) These solutions imply the uniqueness of classical solutions (cf. Atkin, Chadwick, Steel [1967]), and the existence of weak solutions (cf. Aron [1974]) for boundary value problems.

$$(\mu_3+\lambda_5)\nabla^2 u'_j+(\mu_2-\lambda_5)\nabla^2 u''_j+\left(\mu_3+\lambda_4-\lambda_5+\frac{\varrho_2}{\varrho}\alpha_2\right)\Delta'_{,j}+\left(\mu_2+\lambda_2+\lambda_5+\frac{\varrho_1}{\varrho}\alpha_2\right)\Delta''_{,j}+\varrho_2 F''_j=0\,, \tag{2.13}$$

where ∇^2 is the Laplace operator in three dimensions.

We shall consider certain particular solutions of (2.12), (2.13) which tend to become infinite in the neighbourhood of the origin, at which concentrated forces act on the body.

3. – Forces operative at a point.

Let us assume that the body forces F'_j, F''_j are different from zero within a ball B of radius R, surrounding the origin, and vanish outside B. Suppose, in addition, that the body forces are vector functions of the type

$$\boldsymbol{F}'\equiv(X',\,Y',\,Z')\,,\qquad \boldsymbol{F}''\equiv(X'',\,Y'',\,Z'')\,. \tag{3.1}$$

If $\boldsymbol{F}, \boldsymbol{F}''$ are continuous in $\bar{B}$ and continuously differentiable in B, by a Helmholtz's theorem, there are two scalar functions Φ', Φ'' and two vector functions (L', M', N'), (L'', M'', N'') such that

$$X'=\frac{\partial\Phi'}{\partial x}+\frac{\partial N'}{\partial y}-\frac{\partial M'}{\partial z}\,,\qquad X''=\frac{\partial\Phi''}{\partial x}+\frac{\partial N''}{\partial y}-\frac{\partial M''}{\partial z}\,,\ \dots\,, \tag{3.2}$$

where

$$\frac{\partial L'}{\partial x}+\frac{\partial M'}{\partial y}+\frac{\partial N'}{\partial z}=\frac{\partial L''}{\partial x}+\frac{\partial M''}{\partial y}+\frac{\partial N''}{\partial z}=0\,.$$

The displacements can be expressed in like manner by means of two scalar potentials φ', φ'' and two vector potentials (F', G', H'), (F'', G'', H''):

$$u'_x=\frac{\partial\varphi'}{\partial x}+\frac{\partial H'}{\partial y}-\frac{\partial G'}{\partial z}\,,\qquad u''_x=\frac{\partial\varphi''}{\partial x}+\frac{\partial H''}{\partial y}-\frac{\partial G''}{\partial z}\,,\ \dots, \tag{3.3}$$

in which F', G', H' and F'', G'', H'' satisfy the equations

$$\frac{\partial F'}{\partial x}+\frac{\partial G'}{\partial y}+\frac{\partial H'}{\partial z}=\frac{\partial F''}{\partial x}+\frac{\partial G''}{\partial y}+\frac{\partial H''}{\partial z}=0\,.$$

This resolution of body forces and displacements can be effected in many different ways (see, for instance, Love [1927, § 15]).

On substituting (3.2), (3.3) into (2.12), (2.13), these equations can be written in such form as,

$$(3.4)\quad (\mu_1-\lambda_5)\left(\frac{\partial}{\partial x}\nabla^2\varphi'+\frac{\partial}{\partial y}\nabla^2 H'-\frac{\partial}{\partial z}\nabla^2 G'\right)+\left(\mu_1+\lambda_1+\lambda_5-\frac{\varrho_2}{\varrho}\alpha_2\right)\frac{\partial}{\partial x}\nabla^2\varphi'$$
$$+(\mu_3+\lambda_5)\left(\frac{\partial}{\partial x}\nabla^2\varphi''+\frac{\partial}{\partial y}\nabla^2 H''-\frac{\partial}{\partial z}\nabla^2 G''\right)+\left(\mu_3+\lambda_3-\lambda_5-\frac{\varrho_1}{\varrho}\alpha_2\right)\frac{\partial}{\partial x}\nabla^2\varphi''$$
$$+\varrho_1\left(\frac{\partial}{\partial x}\Phi'+\frac{\partial N'}{\partial y}-\frac{\partial M'}{\partial z}\right)=0\,,$$

$$(3.5)\quad (\mu_3+\lambda_5)\left(\frac{\partial}{\partial x}\nabla(\varphi'+\frac{\partial}{\partial y}\nabla^2 H'-\frac{\partial}{\partial z}\nabla^2 G'\right)+\left(\mu_3+\lambda_4-\lambda_5+\frac{\varrho_2}{\varrho}\alpha_2\right)\frac{\partial}{\partial x}\nabla^2\varphi'$$
$$+(\mu_2-\lambda_5)\left(\frac{\partial}{\partial x}\nabla^2\varphi''+\frac{\partial}{\partial y}\nabla^2 H''-\frac{\partial}{\partial z}\nabla^2 G''\right)+\left(\mu_2+\lambda_2+\lambda_5+\frac{\varrho_1}{\varrho}\alpha_2\right)\frac{\partial}{\partial x}\nabla^2\varphi''$$
$$+\varrho_2\left(\frac{\partial\Phi''}{\partial x}+\frac{\partial N''}{\partial y}-\frac{\partial M''}{\partial z}\right)=0,\ldots,$$

and particular solutions of these equations can be obtained from the solutions of the equations

$$(3.6)\quad \begin{cases}\left(2\mu_1+\lambda_1-\frac{\varrho_1}{\varrho}\alpha_2\right)\nabla^2\varphi'+\left(2\mu_3+\lambda_3-\frac{\varrho_1}{\varrho}\alpha_2\right)\nabla^2\varphi''+\varrho_1\Phi'=0\,.\\ (\mu_1-\lambda_5)\nabla^2 H'+(\mu_3+\lambda_5)\nabla^2 H''+\varrho_1 N'=0\,,\\ (\mu_1-\lambda_5)\nabla^2 G'+(\mu_3+\lambda_5)\nabla^2 G''+\varrho_1 M'=0\,,\\ (\mu_1-\lambda_5)\nabla^2 F'+(\mu_3+\lambda_5)\nabla^2 F''+\varrho_1 L'=0\,,\end{cases}$$

and of the equations

$$(3.7)\quad \begin{cases}\left(2\mu_3+\lambda_3+\frac{\varrho_2}{\varrho}\alpha_2\right)\nabla^2\varphi'+\left(2\mu_2+\lambda_2+\frac{\varrho_1}{\varrho}\alpha_2\right)\nabla^2\varphi''+\varrho_2\Phi''=0\,,\\ (\mu_3+\lambda_5)\nabla^2 H'+(\mu_2-\lambda_5)\nabla^2 H''+\varrho_2 N''=0\,,\\ (\mu_3+\lambda_5)\nabla^2 G'+(\mu_2-\lambda_5)\nabla^2 G''+\varrho_2 M''=0\,,\\ (\mu_3+\lambda_5)\nabla^2 F'+(\mu_2-\lambda_5)\nabla^2 F''+\varrho_2 L''=0\,.\end{cases}$$

It is possible to express (3.6), (3.7) in terms of $\nabla^2\varphi'$, $\nabla^2\varphi''$, $\nabla^2 H'$.... If we set

$$\text{(3.8)}\qquad \begin{cases} 2\mu_2+\lambda_2+\dfrac{\varrho_1}{\varrho}\alpha_2=a'\,, & 2\mu_3+\lambda_4+\dfrac{\varrho_2}{\varrho}\alpha_2=a''\,,\\ 2\mu_3+\lambda_3-\dfrac{\varrho_1}{\varrho}\alpha_2=b'\,, & 2\mu_1+\lambda_1-\dfrac{\varrho_2}{\varrho}\alpha_2=b''\,,\\ \mu_2-\lambda_5=c' \,, & \mu_1-\lambda_5=c''\,,\\ \mu_3+\lambda_5=d'=d'' \,, & \end{cases}$$

we derive

$$\text{(3.9)}\qquad \begin{cases} (a'b''-a''b')\nabla^2\varphi' = -\varrho_1 a'\Phi'+\varrho_2 b'\Phi'',\\ (a'b''-a''b')\nabla^2\varphi'' = \varrho_1 a''\Phi'-\varrho_2 b''\Phi'',\\ (c'c''-d'd'')\nabla^2 H' = -\varrho_1 c' N'+\varrho_2 d' N'',\\ (c'c''-d'd'')\nabla^2 H'' = \varrho_1 d'' N'-\varrho_2 c'' N'',\\ (c'c''-d'd'')\nabla^2 G' = -\varrho_1 c' M'+\varrho_2 d' M'',\\ (c'c''-d'd'')\nabla^2 G'' = \varrho_1 d'' M'-\varrho_2 c'' M'',\\ (c'c''-d'd'')\nabla^2 F' = -\varrho_1 c' L'+\varrho_2 d' L'',\\ (c'c''-d'd'')\nabla^2 F'' = \varrho_1 d'' L'-\varrho_2 c'' L''. \end{cases}$$

The material inequalities (2.8), (2.9) ensure that the quantities

$$\text{(3.10)}\qquad \mathcal{A}=a'b''-a''b'\,,\qquad \mathcal{B}=c'c''-d'd''\,,$$

are strictly positive. In fact $\mathcal{A}>0$ is an immediate consequence of the positive-definiteness of (2.9), and $\mathcal{B}>0$ follows from (2.8) on writing

$$\mathcal{B}=(\mu_2-\lambda_5)(\mu_1-\lambda_5)-(\mu_3+\lambda_5)^2=\mu_1\mu_2-\mu_3^2-\lambda_5(\mu_1+\mu_2+2\mu_3)$$
$$>\mu_1\mu_2-\mu_3^2-\lambda_5(\mu_1+\mu_2-2\sqrt{\mu_1\mu_2})\geqslant\mu_1\mu_2-\mu_3^2>0\,.$$

Thus equations (3.9) yield unique solutions for $\nabla^2\varphi'$, $\nabla^2\varphi''$, ...

We now assume that the body forces act exclusively in the direction of the x-axis. This means that Y', Y'', Z', Z'' vanish and X', X'' can be expressed in forms of the type (3.2) by putting

$$\Phi'=-\frac{1}{4\pi}\iiint_B X'(x',y',z')\,\frac{\partial r^{-1}}{\partial x}\,dx'\,dy'\,dz'\,,$$

$$\Phi'' = -\frac{1}{4\pi}\iiint_B X''(x', y', z')\frac{\partial r^{-1}}{\partial x}\,dx'\,dy'\,dz' ,$$

$$L' = L'' = 0 ,$$

$$M' = \frac{1}{4\pi}\iiint_B X'(x', y', z')\frac{\partial r^{-1}}{\partial z}\,dx'\,dy'\,dz' ,$$

$$M'' = \frac{1}{4\pi}\iiint_B X''(x', y', z')\frac{\partial r^{-1}}{\partial z}\,dx'\,dy'\,dz' ,$$

$$N' = -\frac{1}{4\pi}\iiint_B X'(x', y', z')\frac{\partial r^{-1}}{\partial y}\,dx'\,dy'\,dz' ,$$

$$N'' = -\frac{1}{4\pi}\iiint_B X''(x', y', z')\frac{\partial r^{-1}}{\partial y}\,dx'\,dy'\,dz' ,$$

where (x', y', z') is any point within B and r is the distance of this point from (x, y, z).

If we pass to the limit by diminishing R, the radius of B, and suppose that $\iiint_B X'(x', y', z')\,dx'dy'dz'$, $\iiint_B X''(x', y', z')\,dx'dy'dz'$ have finite limits as $R \to 0$, we can put

$$\lim_{R\to 0} \varrho_1 \iiint_B X'(x', y', z')\,dx'\,dy'\,dz' = X'_0 ,$$

$$\lim_{R\to 0} \varrho_2 \iiint_B X''(x', y', z')\,dx'\,dy'\,dz' = X''_0 ,$$

defining in this way two forces X'_0, X''_0 acting at the origin in the direction on the x-axis.

We then have

$$(3.11)\qquad \begin{cases} \Phi' = -\dfrac{1}{4\pi\varrho_1} X'_0 \dfrac{\partial r^{-1}}{\partial x} , & L' = 0 , \\[2ex] M' = \dfrac{1}{4\pi\varrho_1} X'_0 \dfrac{\partial r^{-1}}{\partial z} , & N' = -\dfrac{1}{4\pi\varrho_1} X'_0 \dfrac{\partial r^{-1}}{\partial y} , \\[2ex] \Phi'' = -\dfrac{1}{4\pi\varrho_2} X''_0 \dfrac{\partial r^{-1}}{\partial x} , & L'' = 0 , \\[2ex] M'' = \dfrac{1}{4\pi\varrho_2} X''_0 \dfrac{\partial r^{-1}}{\partial z} , & N'' = -\dfrac{1}{4\pi\varrho_2} X''_0 \dfrac{\partial r^{-1}}{\partial y} . \end{cases}$$

On recalling that $\nabla^2(\partial r/\partial x)=2(\partial r^{-1}/\partial x)$ we may therefore obtain from (3.9)

$$
(3.12)\quad
\begin{cases}
\varphi' = \dfrac{1}{8\pi\mathcal{A}}(a'X_0'-b'X_0'')\dfrac{\partial r}{\partial x}, & \varphi''=\dfrac{1}{8\pi\mathcal{A}}(-a''X_0'+b''X_0'')\dfrac{\partial r}{\partial x},\\[2mm]
H'=\dfrac{1}{8\pi\mathcal{B}}(c'X_0'-d'X_0'')\dfrac{\partial r}{\partial y}, & H''=-\dfrac{1}{8\pi\mathcal{B}}(d''X_0'-c''X_0'')\dfrac{\partial r}{\partial y},\\[2mm]
G'=-\dfrac{1}{8\pi\mathcal{B}}(c'X_0'-d'X_0'')\dfrac{\partial r}{\partial z}, & G''=\dfrac{1}{8\pi\mathcal{B}}(d''X_0'-c''X_0'')\dfrac{\partial r}{\partial z},\\[2mm]
F'=F''=0\,.
\end{cases}
$$

The corresponding expressions for the displacements follow from (3.3). Setting

$$
(3.13)\quad
\begin{cases}
A'=-\dfrac{a'}{8\pi\mathcal{A}}+\dfrac{c'}{8\pi\mathcal{B}}, & B'=\dfrac{b'}{8\pi\mathcal{A}}-\dfrac{d'}{8\pi\mathcal{B}},\\[2mm]
\gamma'A'=\dfrac{a'}{8\pi\mathcal{A}}+\dfrac{c'}{8\pi\mathcal{B}}, & \delta'B'=-\dfrac{b'}{8\pi\mathcal{A}}-\dfrac{d'}{8\pi\mathcal{B}}
\end{cases}
$$

we obtain the final expressions for u_i':

$$
(3.14)\quad
\begin{cases}
u_x'=(A'\gamma'X_0'+B'\delta'X_0'')\dfrac{1}{r}+(A'X_0'+B'X_0'')\dfrac{x^2}{r^3},\\[2mm]
u_y'=(A'X_0'+B'X_0'')\dfrac{xy}{r^3},\\[2mm]
u_z'=(A'X_0'+B'X_0'')\dfrac{xy}{r^3}\,.
\end{cases}
$$

Similarly, by setting

$$
(3.15)\quad
\begin{cases}
A''=\dfrac{a''}{8\pi\mathcal{A}}-\dfrac{d''}{8\pi\mathcal{B}}, & B''=-\dfrac{b''}{8\pi\mathcal{A}}+\dfrac{c''}{8\pi\mathcal{B}},\\[2mm]
\gamma''\mathcal{A}''=-\dfrac{a''}{8\pi\mathcal{A}}-\dfrac{d''}{8\pi\mathcal{B}}, & \delta''B''=\dfrac{b''}{8\pi\mathcal{A}}+\dfrac{c''}{8\pi\mathcal{B}},
\end{cases}
$$

we derive the field of displacements u_i'':

$$
(3.16)\quad
\begin{cases}
u_x''=(A''\gamma''X_0'+B''\delta''X_0'')\dfrac{1}{r}+(A''X_0'+B''X_0'')\dfrac{x^2}{r^2},\\[2mm]
u_y''=(A''X_0'+B''X_0'')\dfrac{xy}{r^3},\\[2mm]
u_z''=(A''X_0'+B''X_0'')\dfrac{xz}{r^3}\,.
\end{cases}
$$

The dilatation corresponding to the displacements (3.14), (3.16) are given by

$$
(3.17) \qquad \begin{cases} \Delta' = [A'(1-\gamma')X'_0 + B'(1-\delta')X''_0]\dfrac{x}{r^3}, \\ \Delta'' = [A''(1-\gamma'')X'_0 + B''(1-\delta'')X''_0]\dfrac{x}{r^3}. \end{cases}
$$

If X'_0, X''_0 act separately on each constituent (fig. 3.1) formulae (3.13), (3.14) represent the solutions for a condition of loading which is called « separate device ».

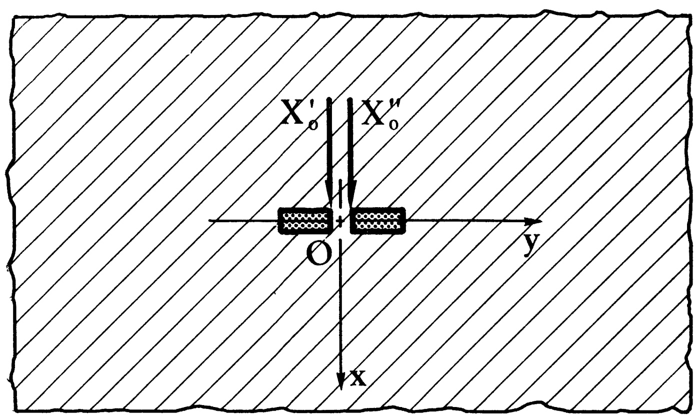

Figure 3.1

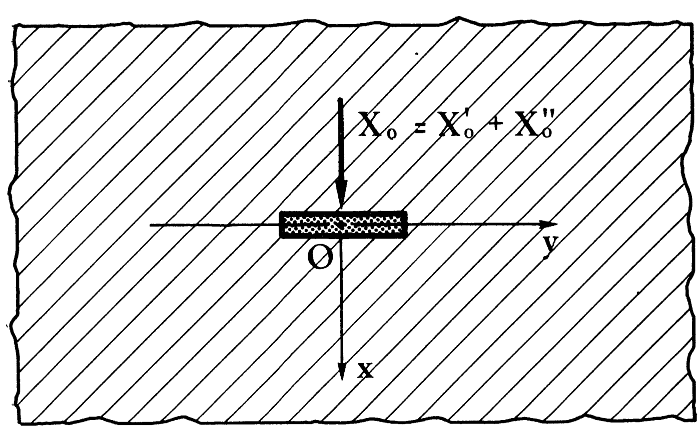

Figure 3.2

One can instead prescribe the total load $X_0 = X_0' + X_0''$ acting at the origin in the x-direction (fig. 3.2). In this case, which is called « common device », X_0' and X_0'' are indeterminate and can be evaluated by imposing that

$$\lim_{x\to 0} \frac{u_x'(x, 0, 0)}{u_x''(x, 0, 0)} = 1 .$$

This condition implies that

$$A'(1+\gamma')X_0' + B'(1+\delta')X_0'' = A''(1+\gamma'')X_0' + B''(1+\delta'')X_0'' ,$$

whence, by (3.13), (3.15),

$$X_0'(\mu_2 + \mu_3) = X_0''(\mu_1 + \mu_3) .$$

Since $X_0' + X_0'' = X_0$, we find

$$X_0' = \frac{\mu_1 + \mu_3}{\mu_1 + \mu_2 + 2\mu_3} X_0 , \qquad X_0'' = \frac{\mu_2 + \mu_3}{\mu_1 + \mu_2 + 2\mu_3} X_0 . \tag{3.18}$$

It is interesting to remark that the moduli $\lambda_1, \ldots, \lambda_5$ do not affect the values of X_0', X_0''.

4. – Properties of Kelvin's solution.

In order to shorten subsequent developments we only analyse the properties of solution (3.14), representing the displacements for the first costituent. The analogous formulae for the second constituent are obtainable in an obvious manner.

If we introduce spherical coordinates r, θ, φ, such that

$$x = r \cos\theta ,$$

$$y = r \sin\theta \cos\varphi ,$$

$$z = r \sin\theta \sin\varphi ,$$

the displacements (3.14) become

$$(4.1)\quad \begin{cases} u'_x = (A'\gamma' X'_0 + B'\delta' X''_0)\dfrac{1}{r} + (A'X'_0 + B'X''_0)\dfrac{\cos^2\theta}{r}, \\ u'_y = (A'X'_0 + B'X''_0)\dfrac{1}{r}\cos\theta \sin\theta \cos\varphi, \\ u'_z = (A'X'_0 + B'X''_0)\dfrac{1}{r}\cos\theta \sin\theta \sin\varphi. \end{cases}$$

The first of these components, which is directed along the x-axis, vanishes on the (possibly imaginary) surface of a cone given by the equation

$$\cos^2\theta = -\frac{a'\gamma' X'_0 + B'\delta' X''_0}{A'X'_0 + B'X''_0},$$

provided that $A'X'_0 + B'X''_0 \neq 0$. If instead $A'X'_0 + B'X''_0 = 0$ the other components of displacement u'_y, u'_z vanish everywhere.

In the polar coordinates r, φ, θ the displacement becomes

$$(4.2)\quad \begin{cases} u'_r = u'_x \cos\theta + u'_y \sin\theta\cos\varphi + u'_z \sin\theta \sin\varphi \\ \qquad = [A'(1+\gamma')X'_0 + B'(1+\delta')X''_0]\dfrac{\cos\theta}{r}, \\ u'_\theta = -u'_x \sin\theta + u'_y \cos\theta\cos\varphi + u'_z\cos\theta\sin\varphi \\ \qquad = -(A'\gamma' X'_0 + B'\delta' X''_0)\dfrac{\sin\theta}{r}, \\ u'_\varphi = -u'_y \sin\varphi + u'_z \cos\varphi = 0. \end{cases}$$

The radial displacement u'_r vanishes for $\theta = \pi/2$, on the y, z-plane, but if

$$A'(1+\gamma')X'_0 + B'(1+\delta')X''_0 = 0,$$

that is

$$\frac{X'_0}{X''_0} - = \frac{B'(1+\delta')}{A'(1+\gamma')} = \frac{\mu_3 + \lambda_5}{\mu_2 - \lambda_5},$$

u'_r vanishes everywhere. Conversely, the other component u'_θ vanishes if

$$A'\gamma' X'_0 + B'\delta' X''_0 = 0.$$

From (4.2) it is easy to obtain the strain by using the strain-displacement relations in polar coordinates (see, for instance, Love [1927, § 22]):

$$(4.3)\quad \begin{cases} \varepsilon'_{rr} = -[A'(1+\gamma')X'_0 + B'(1+\delta')X''_0]\dfrac{\cos\theta}{r^2}\,, \\ \varepsilon'_{\theta\theta} = \varepsilon'_{\varphi\varphi} = (A'X'_0 + B'X''_0)\dfrac{\cos\theta}{r^2}\,, \qquad \varepsilon'_{\theta\varphi} = \varepsilon'_{\theta\varphi} = 0\,, \\ \varepsilon'_{r\theta} = \dfrac{1}{2}[A'(\gamma'-1)X'_0 + B'(\delta'-1)X''_0]\dfrac{\sin\theta}{r^2}\,. \end{cases}$$

The cubical dilatation thus becomes

$$(4.4)\qquad \Delta' = -[A'(\gamma'-1)X'_0 + B'(\delta'-1)X''_0]\frac{\cos\theta}{r^2}\,.$$

The entire deformation of the mixture is not only defined by the strains (4.3), but also through the relative rotations (2.4). In the r, φ, θ-coordinates they are

$$(4.5)\quad \begin{cases} h_{r\varphi} = \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial r}(ru'_\varphi \sin\theta) - \dfrac{\partial u'_r}{\partial\varphi} + \dfrac{\partial u''_r}{\partial\varphi} - \dfrac{\partial}{\partial r}(ru''_\varphi\sin\theta)\right], \\ h_{\varphi\theta} = \dfrac{1}{r^2\sin\theta}\left[\dfrac{\partial}{\partial\varphi}(ru'_\theta) - \dfrac{\partial}{\partial\theta}(ru'_\varphi\sin\theta) + \dfrac{\partial}{\partial\theta}(ru''_\varphi\sin\theta) - \dfrac{\partial}{\partial\varphi}(ru''_\theta)\right], \\ h_{r\theta} = \dfrac{1}{r}\left[\dfrac{\partial u'_r}{\partial\theta} - \dfrac{\partial}{\partial r}(ru'_\theta) + \dfrac{\partial}{\partial r}(ru''_\theta) - \dfrac{\partial u''_r}{\partial\theta}\right], \end{cases}$$

or, from (3.14), (3.16),

$$(4.6)\quad \begin{cases} h_{r\varphi} = h_{\varphi\theta} = 0\,, \\ h_{\theta r} = -h_{r\theta} = \{[A'(1+\gamma')X'_0 + B'(1+\delta')X''_0] \\ \qquad\qquad -[A''(1+\gamma'')X'_0 + B''(1+\delta'')X''_0]\}\dfrac{\sin\theta}{r^2}\,. \end{cases}$$

The rotation $h_{r\theta}$ vanishes when

$$[A'(1+\gamma') - A''(1+\gamma'')]X'_0 = [B''(1+\delta'') - B'(1+\delta')]X''_0\,,$$

that is, when

$$(4.7)\qquad \frac{X'_0}{X''_0} = \frac{\mu_1+\mu_3}{\mu_2+\mu_3}\,,$$

which is also the condition of the common device.

From the strain, the stress components can be calculated readily in the forms

$$
(4.8)\quad \left\{\begin{aligned}
\sigma_{rr} = \{&-\alpha_2 - \lambda_1[A'(\gamma'-1)X_0' + B'(\delta'-1)X_0'']\\
&- \lambda_3[A''(\gamma''-1)X_0' + B''(\delta''-1)X_0'']\\
&- 2\mu_1[A'(1+\gamma')X_0' + B'(1+\delta')X_0'']\\
&- 2\mu_3[A''(1+\gamma'')X_0' + B''(1+\delta'')X_0'']\}\frac{\cos\theta}{r^2}\,,\\
\sigma'_{\theta\theta} = \sigma'_{\varphi\varphi} = \{&-\alpha_2 - \lambda_1[A'(\gamma'-1)X_0' + B'(\delta'-1)X_0'']\\
&- \lambda_3[A''(\gamma''-1)X_0' + B''(\delta''-1)X_0''] + 2\mu_1(A'X_0' + B'X_0'')\\
&+ 2\mu_3(AX_0' + B''X_0'')\}\frac{\cos\theta}{r^2}\,,\\
\sigma'_{\theta\varphi} = \sigma'_{\varphi r} = 0\,,&\\
\sigma'_{r\theta} = \{&\mu_1[A'(\gamma'-1)X_0' + B'(\delta'-1)X_0'']\\
&+ \mu_3[A''(\gamma''-1)X_0' + B''(\delta''-1)X_0'']\\
&+ 2\lambda_5[A'(1+\gamma')X_0' + B'(1+\delta')X_0''\\
&- A''(1+\gamma'')X_0' - B''(1+\delta'')X_0'']\}\frac{\sin\theta}{r^2}\,.
\end{aligned}\right.
$$

The stress is in general non-symmetric and, consequently, the principal stresses are not necessarily real. This fact has no counterpart in Kelvin's solution for a single elastic body (cf. Love [1927, § 141]). However, in case of common device, the stress tensor in each constituent is symmetric and its eigenvalues are all real.

5. – Nuclei of strain.

Nuclei of strain are singular solutions derived from (3.14), (3.16). In particular we may suppose two points, at which forces act, to coalesce, and obtain a new solution by calculating the limit as the distance between the two points tends to zero. The analogous solutions for a single elastic material are due to Dougall (Love [1927, § 132]).

Only some typical examples are considered here.

(*a*) Let a force $h^{-1}P$ be applied at the origin in the direction of the x-axis, and let an equal and opposite force be applied at the point $(h, 0, 0)$. The force P is given by two forces X_0', X_0'', each acting on one constituent.

If we pass to the limit as h tends to zero while X_0', X_0'' remain constant, the displacement in the first constituent is

$$\left(\frac{\partial u_x'}{\partial x}, \frac{\partial u_y'}{\partial x}, \frac{\partial u_z'}{\partial x}\right)$$

where u_x', u_y', u_z' are given by (3.14).

A simple differentiation with respect to x yields

$$(5.1)\quad \begin{cases} \dfrac{\partial u_x'}{\partial x} = -(A'\gamma' X_0' + B'\delta' X_0'')\dfrac{x}{r^3} + (A'X_0' + B'X_0'')\left(2\dfrac{x}{,^3} - 3\dfrac{x^3}{r^5}\right), \\ \dfrac{\partial u_y'}{\partial x} = \quad (A'X_0' + B'X_0'')\left(\dfrac{y}{r^3} - 3\dfrac{x^2 y}{r^5}\right), \\ \dfrac{\partial u_z'}{\partial x} = \quad (A'X_0' + B'X_0'')\left(\dfrac{z}{r^3} - 3\dfrac{x^2 z}{r^5}\right). \end{cases}$$

The displacement may be resolved into a component along the x-axis and another parallel to the y, z-plane. The former vanishes on the surface

$$-(A'\gamma' X_0' + B'\delta' X_0'') + (A'X_0' + B'X_0'')(2 - 3\cos^2\theta) = 0 .$$

If $A'X_0' + B'X_0' \neq 0$ the surface is the (possibly imaginary) cone

$$\cos^2\theta = \frac{2(A'X_0' + B'X_0'') - (\gamma' A'X_0' + B'\delta' X_0'')}{3(A'X_0' + B'X_0'')} .$$

Under the same assumption $A'X_0' + B'X_0'' \neq 0$, the other displacement

$$\frac{\partial u_y'}{\partial x}\cos\varphi + \frac{\partial u_z'}{\partial x}\sin\varphi = (A'X_0' + B'X_0'')\frac{\sin\theta}{r^2}(1 - 3\cos^2\theta) ,$$

vanishes for $\theta = 0, \pi$ and for

$$\cos^2\theta = \tfrac{1}{3} .$$

(*b*) Let us now assume that a system of loads $X_0' = P$, $X_0'' = 0$ acts at the point $(0, 0, 0)$ and a system $X_0' = 0$, $X_0'' = -P$ acts at $(h, 0, 0)$.

The solution as h tends to zero is a simple superposition of (3.14), where

$X_0' = P$, $X_0'' = 0$, with (3.16), in which $X_0' = 0$, $X_0'' = -P$:

$$(5.2)\qquad \begin{cases} u_x' = (A'\gamma' - B'\delta')\dfrac{P}{r} + (A' - B')P\dfrac{x^2}{r^3}, \\ u_y' = (A' - B')P\dfrac{xy}{r^3}, \\ u_z' = (A' - B')P\dfrac{xz}{r^3}. \end{cases}$$

If $A' \neq B'$ the displacement u_x' vanishes on the surface of the cone

$$\cos^2\theta = \frac{A'\gamma' - B'\delta'}{A' - B'}.$$

(*c*) We may suppose a force $h^{-1}P$ to act at the origin in the positive direction of the x-axis, and an equal and opposite force to act at the point $(0, h, 0)$. The force P is generated by a force X_0' acting on the first constituent and by a force X_0'' acting on the second. On passing to the limit as before the resultant displacement in the first constituent is

$$\left(\frac{\partial u_x'}{\partial y}, \frac{\partial u_y'}{\partial y}, \frac{\partial u_z'}{\partial y}\right)$$

or, by differentiation of (3.14) with respect to y,

$$(5.3)\qquad \begin{cases} \dfrac{\partial u_x'}{\partial y} = -(A'\gamma' X_0' + B'\delta' X_0'')\dfrac{y}{r^3} - 3(A'X_0' + B'X_0'')\dfrac{x^2 y}{r^5}, \\ \dfrac{\partial u_y'}{\partial y} = \quad (A'X_0' + B'X_0'')\left(\dfrac{x}{r^3} - 3\dfrac{xy^2}{r^5}\right), \\ \dfrac{\partial u_z'}{\partial y} = -3(A'X_0' + B'X_0'')\dfrac{xyz}{r^5}. \end{cases}$$

This singularity is described as a « double force with moment », since the forces applied at the origin are statically equivalent to a couple of moment P about the z-axis.

(*d*) The singularity generated by the forces $X_0' = P$, $X_0'' = 0$ at the origin and the forces $X_0' = 0$, $X_0'' = -P$ at $(0, h, 0)$ can be simply obtained by superposition of (3.14), where $X_0' = P$, $X_0'' = 0$, with (3.16), where $X_0' = 0$, $X_0'' = -P$, and its expression is again (5.2).

(e) It is possible to combine two double forces with moment, the moments being about the same axis and of the same sign, and the directions of the forces being at right angles to each other. We take the forces to be $h^{-1}P$ ($P = X_0' + X_0''$) and $-h^{-1}P$ ($P = Y_0' + Y_0''$) parallel to the x and y-axis at the origin, $-h^{-1}P$ ($P = X_0' + X_0''$) parallel to the x-axis at the point $(0, h, 0)$ and $h^{-1}P$ ($P = Y_0' + Y_0''$) parallel to the y-axis at the point $(h, 0, 0)$ and we pass to the limit as before. The displacement generated by the forces Y_0' and Y_0'' is deducible from (3.15) and (3.16) by interchanging x with y and is denoted by $[v_x', v_y', v_z']$. The resulting displacement is

$$\left\{\left(\frac{\partial u_x'}{\partial y} - \frac{\partial v_x'}{\partial x}\right), \left(\frac{\partial u_y'}{\partial y} - \frac{\partial v_y'}{\partial x}\right), \left(\frac{\partial u_z'}{\partial y} - \frac{\partial v_z'}{\partial x}\right)\right\},$$

and can be described as a « centre of rotation about the z-axis ».

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