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Levi flat hypersurfaces in C^2 with prescribed boundary : stability

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Levi Flat Hypersurfaces in C^2 with Prescribed Boundary: Stability (*).

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1. – Introduction.

Let $(z, w) = (x + iy, u + iv)$ be coordinates on $\boldsymbol{C}^2$, and let $D \subset\subset \boldsymbol{C} \times \boldsymbol{R} = \{v = 0\}$ be a domain such that $D \times i\boldsymbol{R}$ is strongly pseudoconvex. Given a function φ on ∂D, we set

$$\Gamma(\varphi) = \{(z, w) \in \partial D \times i\boldsymbol{R} \colon v = \varphi(z, u)\}\,, \tag{1}$$

and given Φ on $\bar{D}$, we set

$$\tilde{\Gamma}(\Phi) = \{(z, w) \in \bar{D} \times i\boldsymbol{R} \colon v = \Phi(z, u)\}\,. \tag{2}$$

We are interested in the following problem:

(3) Given a compact 2-manifold $\Gamma \subset \boldsymbol{C}^2$, find a Levi-flat hypersurface $\hat{\Gamma} \subset \boldsymbol{C}^2$ such that $\partial\hat{\Gamma} = \Gamma$.

For technical reasons we consider the following nonparametric form of (3):

(4) Given $\varphi \in C^2(\partial D)$, find a function $\Phi \in C(\bar{D})$ such that $\Phi|_{\partial D} = \varphi$ and $\tilde{\Gamma}(\Phi)$ is Levi flat.

A surface of class C^2 is said to be *Levi flat* if its Levi form vanishes identically. As is well known, a Levi flat surface in $\boldsymbol{C}^2$ of class C^2 may be foliated by complex manifolds. We will use this characterization of Levi-

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flatness, and we will construct the surface $\Gamma(\Phi)$ in certain cases by finding complex analytic disks with boundaries in $\Gamma(\varphi)$. Because of global difficulties involving the disk construction, we cannot show that the surface constructed is everywhere of class C^2. Thus by « Levi flat » in (4) we will mean that $\Gamma(\Phi)$ is pseudoconvex from both sides.

This problem may also be considered as a complex Plateau problem. This is because the Levi form of a hypersurface S in $\boldsymbol{C}^2$, suitably normalized (cf. [5]), is given at $p \in S$ by the curvature form of S, averaged on the unique complex line in TS_p. Since this does not involve the full mean curvature, the resulting expression, equation (16), is degenerate elliptic and does not seem to fall within the scope of standard P.D.E. techniques. (See the discussion of this problem by Debiard and Gaveau [8].)

Let $\Delta = \{\zeta \in \boldsymbol{C}\colon |\zeta| < 1\}$ denote the unit disk in $\boldsymbol{C}$. Our main result is the following

THEOREM 1. *Suppose that $\Phi_0 \in C^2(\bar{D}_0)$ solves (4), and let us set $\varphi_0 = \Phi_0|_{\partial D_0}$. If $\Gamma(\varphi_0)$ contains no parabolic points and if all the complex manifolds in $\tilde{\Gamma}(\Phi_0)$ are disks and closed, then (4) is solvable for all φ and ∂D sufficiently close in C^2 to φ_0 and ∂D_0. The solution Φ has the following properties*:

(i) $\Phi \in \operatorname{Lip}(\bar{D})$;

(ii) *for each $(z_0, u_0) \in D$, there exists a unique holomorphic mapping $F = (z, w)\colon \Delta \to D \times i\boldsymbol{R}$ which is continuous on $\bar{\Delta}$ and such that*

$$F(\partial\Delta) \subset \Gamma(\varphi)\,,$$

$$F(0) = \big(z_0, u_0, \Phi(z_0, u_0)\big)\,, \qquad z'(0) > 0\,,$$

and

$$F(\Delta) \subset \tilde{\Gamma}(\Phi)\,;$$

(iii) *if V is an open subset of a complex variety lying in $\tilde{\Gamma}(\Phi)$, then V must lie in one of the disks of* (ii);

(iv) *if $\varphi \in C^\infty(\partial D)$, then Φ is C^∞ in a neighborhood of each point $(z_1, u_1) \in \bar{D}$ such that: (z_1, u_1) is not an elliptic point of ∂D, and if $(z_1, u_1) \in F(\bar{\Delta})$ for one of the functions F in* (ii), *then $F(\partial\Delta)$ does not contain a hyperbolic point.*

We remark that if ∂D is a 2-sphere, then by Theorem 3.1 of [4], the complex manifolds contained in $\tilde{\Gamma}(\Phi_0)$ are disks and are closed, and so this hypothesis of Theorem 1 is automatically satisfied.

Of course, we would like to solve (4) for all $\varphi \in C^2(\partial D)$, in which case it would suffice to show that the set of φ for which (4) can be solved is both open and closed. By Section 4, this set is closed. The difficulty for us, then, is to show that this set is also open, and Theorem 1 may be seen as a step in that direction.

One motivation for solving (4) is that (4) is related to the construction of certain envelopes of holomorphy, i.e., the « tin cans » with tops cut, C_ϱ in (4) in [4]. The surface $\tilde{\Gamma}(\Phi)$ sometimes also gives a polynomial hull.

THEOREM 2. *Let $D \subset\subset C \times R$ be smoothly bounded, let $D \times iR$ be strongly pseudoconvex, and let ∂D be homeomorphic to S^2. If $\Phi_0 \in \mathrm{Lip}\,(\bar{D}_0)$, $\varphi_0 = \Phi_0|_{\partial D_0} \in C^2(\partial D_0)$, is a stable solution of* (4) *with property* (ii), (i.e., *if* (4) *is solvable for small C^2-perturbations of φ_0 and ∂D_0, and the solution Φ satisfies* (ii)), *then $\tilde{\Gamma}(\Phi)$ is the polynomial hull of $\Gamma(\varphi)$.*

Since the solution given in Theorem 1 is stable in this sense, it satisfies the hypotheses of Theorem 2. Thus Theorems 1 and 2 give a family of examples of polynomial hulls, obtained by solving a generalized Dirichlet problem. They also give the hull regularity and complex structure.

The condition that a 2-manifold be of the form (1) is rather restrictive, but if this condition is dropped, several things can go wrong (cf. Section 2 of [4]). The imbedding of the torus $T^2 = \partial\Delta \times \partial\Delta \subset C^2$ is in some sense « opposite » to the 2-manifolds satisfying (1). Yet if Γ is close to this « standard » T^2, a global disk construction is still possible (see [1] and [2]).

The points where a 2-manifold $\Gamma \subset C$ is not totally real are generically « elliptic », « parabolic », or « hyperbolic » (see Section 3). Bishop [7] gave a method of constructing disks near an elliptic point such that the boundaries of the disks lie in Γ. In [4], the hypothesis that $\Gamma(\varphi)$ have no hyperbolic points was adopted. In this case $\Gamma(\varphi)$ is necessarily a 2-sphere with two elliptic points, and it was shown in [4] that a disk construction can be pushed all the way from one elliptic point to the other.

For more general $\Gamma(\varphi)$, the method of [4] permits us to continue the construction of a 1-parameter family of complex disks until the boundary of one of the disks arrives at a complex tangency. Figure 1 shows the kind of degeneracy that we hope to have in the more general case: An elliptic point E corresponds to a « vanishing » disk, and a hyperbolic point H corresponds to a « bifurcation ». The present paper, therefore, studies the bifurcation at a hyperbolic point in some detail.

In order to illustrate the technical point that we will be working with, let us consider the case where $\Gamma(\varphi)$ is a 2-sphere with three elliptic points and one hyperbolic point. Let Φ be a solution of (4) with boundaries of disks filling $\Gamma(\varphi)$ as in Figure 1. To discuss the stability of the problem (4), we consider perturbations $\Gamma(\varphi')$ of $\Gamma(\varphi)$. For instance, suppose that φ' is any smooth function which coincides with φ in a neighborhood of γ and such that the only singularities of $\Gamma(\varphi')$ away from H are elliptic. Then following

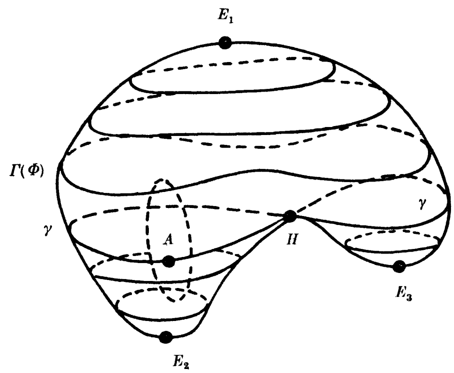

Fig. 1

the recipe of [4] we may start with the curves in a neighborhood of γ and construct disks to fill $\Gamma(\varphi')$, thus ending up at the perturbed elliptic points E_1', E_2', and E_3'.

The possibility which is not covered in [4] is that we could have a perturbation φ' of φ which differs from φ in a neighborhood of a point $A \in \gamma$. In this case, we may construct disks starting at the elliptic points. The disks will be the same as in Figure 1 until they touch the region of the perturbation (the shaded region containing A in Figure 2). Once they touch this region, it is *a priori* possible that they degenerate in the manner depicted in Figure 2. In this case we would not be able to construct disks in the region between the curves γ_1', γ_2', and γ_3'.

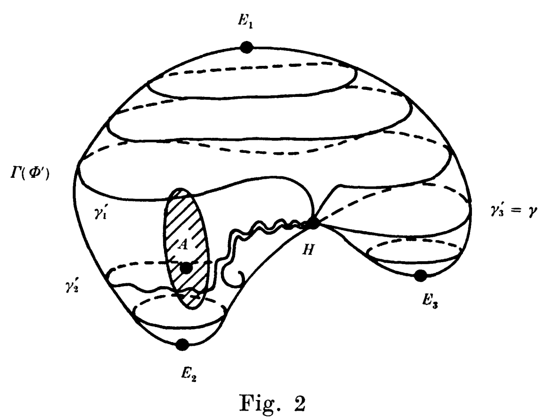

Fig. 2

In Figure 2 the disks below $\gamma = \gamma_3$ are unchanged. This illustrates the « shadow » effect and gives a possible explanation why the solution is in general not smooth at a hyperbolic point: The disks in Figure 1 that are below the curve γ_3 do not « see » the perturbation at A, whereas the disks above γ_3 are affected by the perturbation.

We will show that under the hypotheses of Theorem 1 the curves γ_2' and γ_3' fit together nicely, and as a consequence γ_1' coincides with $\gamma_2' \cup \gamma_3'$, so we are essentially back to Figure 1. The crux of the matter is to study a free boundary problem which is equivalent to showing that the curve γ in Figure 1 is piecewise smooth. This problem is different from those considered, for instance, in [13] and [15] because the nondegeneracy condition fails at the hyperbolic point.

This paper is organized as follows. Section 2 is devoted to the problem of the stability of a single disk with boundary in a totally real 2-manifold in C^2. Stability is determined by a « topological » index. In Section 3, the elementary properties of hyperbolic points are presented. In Section 4, some basic results from the use of barrier functions are assembled. These are used with more precision in Section 5 to control how a disk may asymptotically approach a hyperbolic point. With this asymptotic control, we are able in Section 6 to adapt the reflection argument of Lewy [15] for plane free boundary problems to our degenerate case. This gives the needed regularity of γ_2 and γ_3. Section 7 discusses two forms of « almost » holomorphic flattening $\Gamma(\varphi)$ globally, along the curve $\gamma_2 \cup \gamma_3$. Section 8 is mainly technical and combines the results of the previous sections to prove Theorem 1. Theorem 2 is proved in Section 9.

An appendix discusses the relation between problem (3) and a question of holomorphic flattening. An example is included to show that the « true » regularity of Φ lies somewhere between Lip $(\bar{D})$ and $C^3(\bar{D})$. In Sections 3 and 6 it was shown that the angle $\theta = \alpha\pi$ of the opening of γ at H is determined by the second derivatives of $\Gamma(\varphi)$ at H. In the appendix it is shown that if α is irrational then γ is *never* piecewise real analytic at H, except in the trivial case.

2. – Stability of disks.

Given a complex disk with boundary in a totally real, orientable manifold Γ in C^2, the question arises whether, for any small perturbation Γ' of Γ, there is again a complex disk near the original one with its boundary in Γ'. We show in this section that stability in this sense is determined by a geometric index of Γ about the disk.

Let $\Gamma \subset C^2$ be a smooth, totally real 2-manifold, and let $f\colon \Delta \to C^2$, $f \in C^1(\bar{\Delta})$, $f' \neq 0$ on $\bar{\Delta}$, be a holomorphic mapping with $f(\partial\Delta) \subset \Gamma$. We note (cf. [12] or Theorem 4.5 of [4]) that it follows automatically that f is as smooth as Γ. Let us choose a smooth holomorphic mapping $g = (g_1, g_2)\colon \Delta \to C^2$

such that

$$f_1' g_2 - f_2' g_1 = 1 .$$

Then $\tilde{F} = f + wg$ gives an imbedding of $\bar{\Delta}\times(-\varepsilon, \varepsilon)$ into $\boldsymbol{C}^2$. For an arbitrary smooth extension $\tilde{F}$ of F to a neighborhood of $\bar{\Delta}\times(-\varepsilon, \varepsilon)$, we consider $\tilde{\Gamma} = F^{-1}(\Gamma)$.

We will identify the complex vector $(a_1 + ib_1, a_2 + ib_2) \in \boldsymbol{C}^2$ with the real vector $(a_1, b_1, a_2, b_2) \in \boldsymbol{R}^4$. Let $X = \partial/\partial\theta = (ie^{i\theta}, 0)$ be the tangent to $\partial\Delta\times\{0\}$. Then there is a vector field $Y = (\alpha, \beta)$ such that X and Y span the tangent space to $\tilde{\Gamma}$ along $\partial\Delta\times\{0\}$. Since $\tilde{\Gamma}$ is totally real, $\beta \neq 0$, so we find a real function $\lambda > 0$ and an invertible holomorphic function $h(z)$ such that $\beta(z) = \lambda(z)z^p h(z)$. As in [4], we will say that p is the *index of Γ about the disk $f(\Delta)$*. If we make the biholomorphic change of coordinates $z^* = z$, $w^* = h(z)w$, then we may assume that $\beta(z) = \lambda(z)z^p$. Thus $X = \partial/\partial\theta$ and $Y = \operatorname{Re}\left(\alpha(\partial/\partial z) + \lambda z^p(\partial/\partial w)\right)$ span the tangent space to $\tilde{\Gamma}$ at $\partial\Delta\times\{0\}$. It follows that we may find a real $\mu \neq 0$ such that

$$V_1 = \operatorname{Im} z^p \frac{\partial}{\partial w} , \qquad V_2 = \frac{\partial}{\partial r} + \mu \operatorname{Re} z^p \frac{\partial}{\partial w}$$

are orthogonal to $\tilde{\Gamma}$ along $\partial\Delta\times\{0\}$.

We will consider perturbations of f of the form

$$\mathcal{F}(a, b) = f + \zeta(a + iTa)f' + i\zeta^p(b + iTb)g , \tag{5}$$

where $(a, b) \in C^{m,\alpha}(\partial\Delta)^2$, and $T\colon C^{m,\alpha}(\partial\Delta) \to C^{m,\alpha}(\partial\Delta)$ denotes the harmonic conjugate operator on $\partial\Delta$. (That is, if $a \in C^{m,\alpha}(\partial\Delta)$, then there is a holomorphic function $A + iA^*$ on Δ with boundary values $a + iTa$, and $A^*(0) = 0$.) We note that along $\partial\Delta\times\{0\}$, $\tilde{F}_*(X) = f_\theta(e^{i\theta}) = i\zeta f'$ is tangential to Γ, and $\tilde{F}_*(\partial/\partial r) = \zeta f'$. Thus a gives the deformation normal to Γ and in the direction of the disk. The term involving g will account for the other direction orthogonal to Γ. Using equation (5), we may identify our space of perturbations of f with the space

$$\mathfrak{D} = \{(a, b) \in C^{m,\alpha}(\partial\Delta)^2\} .$$

We will write our perturbations of Γ similarly. We may choose defining functions ϱ_1, ϱ_2 for $\tilde{\Gamma}$ in a neighborhood of $\partial\Delta\times\{0\}$ such that

$$\nabla\varrho_1 = \operatorname{Im}\left(z^p \frac{\partial}{\partial w}\right) , \qquad \nabla\varrho_2 = \frac{\partial}{\partial r} + \mu \operatorname{Re}\left(z^p \frac{\partial}{\partial w}\right)$$

holds on $\partial\Delta\times\{0\}$. Mapping forward via $\tilde{F}$ we have defining functions $r_j = \varrho(\tilde{F}^{-1})$, $j = 1, 2$, for Γ.

The space of deformations of Γ may now be written as

$$\mathcal{M} = \{\sigma = (\sigma_1, \sigma_2) \in C_0^{m+2}(\mathcal{U})^2 \colon \|\sigma_1\|_{C^{m+2}(\mathcal{U})} + \|\sigma_2\|_{C^{m+2}(\mathcal{U})} < \delta\}$$

where $\delta > 0$ will be chosen to be small, and $\mathcal{U}$ is a small neighborhood of Γ. A function $\sigma \in \mathcal{M}$ is identified with the surface

$$\Gamma(\sigma) = \{(z, w) \in \mathcal{U} \colon (r_j + \sigma_j)(z, w) = 0,\ j = 1, 2\}\ .$$

For δ sufficiently small, $\Gamma(\sigma)$ is a small, smooth perturbation of $\Gamma = \Gamma(0)$ in a neighborhood of $f(\partial\Delta)$.

Now we consider the mapping

$$S \colon \mathcal{M} \times \mathfrak{D} \to C^{m,\alpha}(\partial\Delta)^2$$

given by

$$S((a, b), \sigma) = \big((r_1 + \sigma_1)(\mathcal{F}(a, b)),\ (r_2 + \sigma_2)(\mathcal{F}(a, b))\big)\ .$$

This mapping is of class C^1 (cf. Lemma 5.1 of [11]). We want to find a continuous mapping $\sigma \to (a(\sigma), b(\sigma))$ taking $\mathcal{M}$ into $\mathfrak{D}$, with the property that

$$S\big(a(\sigma), b(\sigma), \sigma\big) = 0\ . \tag{7}$$

For this we consider the differential dS of $S \colon \{\sigma\} \times \mathfrak{D} \to C^{m,\alpha}(\partial\Delta)^2$, i.e., the differential of S in terms of a and b, with σ fixed. By (5), we see that the differential at $(a, b) = 0$, $\sigma = 0$ is

$$\begin{aligned} (8) \qquad & dS_j\big(0, 0; (\delta a, \delta b)\big) = \\ & = \langle dr_j, \zeta f'\rangle\, \delta a + \langle dr_j, i\zeta f'\rangle\, T(\delta a) + \langle dr_j, i\zeta^p g\rangle\, \delta b - \langle dr_j, \zeta^p g\rangle\, T(\delta b)\ , \end{aligned}$$

where $\langle\, ,\rangle$ indicates the pairing between 1-forms and vectors. Since $\tilde F^*(\partial/\partial\theta) = i\zeta f'$, we have, for instance,

$$\langle dr_j, \zeta f'\rangle = \langle d\varrho_j, \tilde F_*^{-1}(\zeta f')\rangle = \left\langle d\varrho_j, \frac{\partial}{\partial r}\right\rangle\ .$$

Thus, $\langle dr_1, \zeta f'\rangle = \langle dr_1, i\zeta f'\rangle = \langle dr_2, i\zeta f'\rangle = 0$ and $\langle dr_2, \zeta f'\rangle = 1$ on $\partial\Delta \times \{0\}$ by (6).

Similarly, we have

$$\begin{aligned} \langle dr_j, i\zeta^p g\rangle = \langle d\varrho_j, \tilde F_*^{-1}(i\zeta^p g)\rangle = \left\langle d\varrho_j, \tilde F_*^{-1}\left(\zeta^p \tilde F_*\left(\frac{\partial}{\partial v}\right)\right)\right\rangle = \\ = \left\langle d\varrho_j, \tilde F_*^{-1}\left(\tilde F_* \operatorname{Im}\left(z^p \frac{\partial}{\partial w}\right)\right)\right\rangle = \langle d\varrho_j, \nabla\varrho_1\rangle\ . \end{aligned}$$

Thus

$$\langle dr_2, i\zeta^p g\rangle = 0 \quad \text{and} \quad \langle dr_1, i\zeta^p g\rangle = |\nabla\varrho_1|^2 = \tfrac{1}{4}\,.$$

Repeating this argument, we have

$$\langle dr_j, \zeta^p g\rangle = \left\langle d\varrho_j, \operatorname{Re}\left(z^p\frac{\partial}{\partial w}\right)\right\rangle = \begin{cases} 0\,, & j=1 \\ \mu/4\,, & j=2\,. \end{cases}$$

By these computations, (8) may be written in matrix form as

$$dS(0,0)\begin{pmatrix}\delta a\\ \delta b\end{pmatrix} = \begin{bmatrix}1 & -(\mu/4)T\\ 0 & \frac{1}{4}\end{bmatrix}\begin{pmatrix}\delta a\\ \delta b\end{pmatrix}.$$

Since μ is smooth of class C^{m+1}, it follows that $dS\colon \big(C^{m,\alpha}(\partial\Delta)\big)^2 \to \big(C^{m,\alpha}(\partial\Delta)\big)^2$ is invertible.

Thus, we may apply the usual Implicit Function Theorem to the mapping S to deduce that if $\delta > 0$ is chosen sufficiently small, then for each $\sigma \in \mathcal{M}$ there exists a pair $(a, b) \in \mathfrak{D}$ solving (7). We now summarize our discussion by stating a theorem.

THEOREM 2.1. *Let $\Gamma \subset \boldsymbol{C}^2$ be a totally real, orientable, 2-dimensional manifold which is smooth of class C^{m+2}. If $f\colon \Delta \to \boldsymbol{C}^2$ is a holomorphic mapping with $f \in C^{m,\alpha}(\bar\Delta)$, $0<\alpha<1$, $f' \neq 0$ on $\bar\Delta$, $f(\partial\Delta)\subset\Gamma$ and Γ has nonnegative index about $f(\Delta)$, then f is stable*; i.e., *if $\hat\Gamma$ is a small C^{m+2} perturbation of Γ, then there exists $\hat f$ close to f in $C^{m,\alpha}(\bar\Delta)\cap\mathcal{O}(\Delta)$ with $\hat f(\partial\Delta)\subset\hat\Gamma$.*

EXAMPLE. Let us show that if the index is negative, then disks are not stable. We let $\Gamma(-p)$ be given as the image of $\partial\Delta\times(-\varepsilon,\varepsilon)$ under the mapping $(z, w)\to(z, \bar z^p w)$. With this representation it is clear that the index of $\Gamma(-p)$ about the disk $\Delta\times\{0\}$ is $-p$. We may also write $\Gamma(-p) = \{\sigma=\tau=0\}$ with

$$\sigma = z\bar z - 1\,, \qquad \tau = \operatorname{Im} z^p w\,.$$

Now let us replace τ by $\tau+\delta$, and let us suppose that there exists a disk $F_\delta(\zeta) = (\zeta, 0) + g_\delta(\zeta)$, where $g_\delta(\zeta) = \big(g_1(\zeta), g_2(\zeta)\big)$ is uniformly small, and $F_\delta(\partial\Delta)\subset\Gamma_\delta(-p) = \{\tau+\delta = 0 = \sigma\}$. If this is the case, then

$$\operatorname{Im}\Big(\big(\zeta+g_1(\zeta)\big)^p g_2(\zeta)\Big) = -\delta$$

holds for $|\zeta|=1$, which implies that

$$\big(\zeta+g_1(\zeta)\big)^p g_2(\zeta) = C - i\delta \neq 0$$

for all $\zeta \in \Delta$. In particular, $\zeta + g_1(\zeta) \neq 0$ on Δ. But if $|g_1| < 1$, then this contradicts Rouché's Theorem, so we conclude that the disk $\Delta \times \{0\}$ is not stable in $\Gamma(-p)$.

3. – Hyperbolic points.

If $M \subset C^2$ is a smooth real submanifold of dimension 2, then M has a complex tangent at $p \in M$ if and only if there is a linear holomorphic change of coordinates in a neighborhood of p such that $p = 0$, and M is given by

$$w = \sigma(z)$$

in a neighborhood of p, where

$$|\sigma(z)| \leqslant C|z|^2 .$$

Now we consider a complex tangency of a 2-manifold $\Gamma(\varphi)$ given by (1). We take a point $p \in \Gamma(\varphi)$, and we set $p = 0$. Further, we assume that we may write ∂D near 0 in the form

$$u = \operatorname{Re} \alpha_1 z + z\bar{z} + \operatorname{Re} \alpha_2 z^2 + o(|z|^2) , \tag{9}$$

i.e., $D = \{r(z, u) < 0\}$, where

$$r(z, u) = -u + \operatorname{Re} \alpha_1 z + z\bar{z} + \operatorname{Re} \alpha_2 z^2 + o(|z|^2) . \tag{10}$$

Further, at 0 we may write $\varphi \in C^2(\partial D)$ as

$$\varphi(z) = \operatorname{Re} \beta_1 z + bz\bar{z} + \operatorname{Re} \beta_2 z^2 + o(|z|^2) \tag{11}$$

with $b \in \boldsymbol{R}$. It is easily seen that the 2-plane

$$u = \operatorname{Re} \alpha_1 z , \qquad v = \operatorname{Re} \beta_1 z$$

is totally real, i.e., not a complex line, if and only if

$$\beta_1 \neq -i\alpha_1 .$$

Thus $|i\alpha_1 + \beta_1| = |\alpha_1 - i\beta_1|$ measures the « distance » of the tangent space of $\Gamma(\varphi)$ at $p = 0$ to $\boldsymbol{CP}^1$ inside $Gr(2, 4)$.

Now we assume that $\beta_1 = -i\alpha_1$, i.e., $\Gamma(\varphi)$ has a complex tangent at (0, 0).

We note that if ∂D cannot be written in the form (9), i.e., if the tangent to ∂D at $(0, 0)$ is $\{(x, y, u) \in \boldsymbol{C} \times \boldsymbol{R}\colon \operatorname{Re} \alpha z = 0\}$, then Γ is necessarily totally real at $(0, 0)$. Thus a complex tangency of $\Gamma(\varphi)$ always has the form (9) and (11). For the rest of this section it will be convenient to drop the $o(|z|^2)$ terms. Now $\Gamma(\varphi)$ has the form

$$(12) \qquad \begin{aligned} w &= \alpha_1 z + (1 + ib)z\bar{z} + \operatorname{Re} \alpha_2 z^2 + i \operatorname{Re} \beta_2 z^2 \\ &= \alpha_1 z + (1 + ib)z\bar{z} + \tfrac{1}{2}[(\alpha_2 + i\beta_2)z^2 + (\bar{\alpha}_2 + i\bar{\beta}_2)\bar{z}^2] . \end{aligned}$$

If we make the change of coordinates

$$w^* = (1 + ib)^{-1}\big(w - \bar{\alpha}_1 z - \tfrac{1}{2}(\alpha_2 + i\beta_2)z^2\big) ,$$

then $\Gamma(\varphi)$ has the form

$$w^* = z\bar{z} + \frac{1}{2}\frac{\bar{\alpha}_2 + i\bar{\beta}_2}{1 + ib}\bar{z}^2 .$$

Now we choose τ real with $|\tau| < \pi$ such that

$$(13) \qquad \tau = \arg \frac{\bar{\alpha}_2 + i\bar{\beta}_2}{1 + ib} .$$

With coordinates $z^* = e^{-i\tau/2}z$, we have

$$(14) \qquad w^* = z^*\bar{z}^* + \tfrac{1}{2}\lambda(\bar{z}^*)^2$$

where

$$(14') \qquad \lambda = \left| \frac{\bar{\alpha}_2 + i\bar{\beta}_2}{1 + ib} \right| .$$

It is sometimes convenient to replace w^* by $w^{**} = w^* + \frac{1}{2}\lambda(z^*)^2$ so that $\Gamma(\varphi)$ has the form

$$(15) \qquad w^{**} = z^*\bar{z}^* + \lambda \operatorname{Re}(z^*)^2 .$$

We use the terminology of Bishop [7] and say that p is *hyperbolic* if $\lambda > 1$, p is *elliptic* if $\lambda < 1$, and p is *parabolic* if $\lambda = 1$. We now suppose that $\lambda > 1$ and consider families of disks in the (z^*, w^{**})-coordinates, whose boundaries lie in $\Gamma(\varphi)$; an obvious family of these is given in terms of a real parameter σ as $\{w^{**} = \sigma\}$. The disk $\{w^{**} = 0\}$, which passes through the hyperbolic point, intersects $\Gamma(\varphi)$ in the two lines

$$\{\arg z^* = \theta_1\} \quad \text{and} \quad \{\arg z^* = \theta_2\} ,$$

where θ_1 and θ_2 are the two solutions of

$$\cos 2\theta = \frac{-1}{\lambda} \tag{15'}$$

with $\pi/4 < \theta < 3\pi/4$. If $\lambda = 1$, these two solutions coincide, but we note that for $\lambda > 1$ but λ very close to 1, the solutions are given by

$$\theta \sim \frac{\pi}{2} \pm \sqrt{\frac{\lambda - 1}{2\lambda}}\,.$$

Now we consider smooth solutions of (4). If $\tilde{\Gamma}(\Phi)$ is Levi flat, then the Levi determinant vanishes (see equation (10) of [8]), i.e.,

$$\begin{aligned} L(\Phi) = (\Phi_{xx} + \Phi_{yy})(1 + \Phi_u^2) + \\ + \Phi_{uu}(\Phi_x^2 + \Phi_y^2) - 2\Phi_{ux}(\Phi_u\Phi_x - \Phi_y) - 2\Phi_{uy}(\Phi_u\Phi_y + \Phi_x) = 0\,. \end{aligned} \tag{16}$$

The condition of being Levi flat is holomorphically invariant, so we can apply it to a solution in the coordinates (15). If we suppress the ** from the coordinates, we see that D is given by

$$r(z, u) = -u + z\bar{z} + \lambda \operatorname{Re} z^2\,;$$

the surface $\Gamma(\varphi)$ is given by $\varphi = 0$. The solution Φ is *not unique* near $(z, u) = (0, 0)$ if $\lambda > 1$, so we cannot conclude that $\Phi \equiv 0$.

If we write

$$\Phi(z, u) = \varphi(z) + \varphi_1(z)\,r(z, u) + \varphi_2(z)\,r^2(z, u) + \ldots,$$

where $\varphi \equiv 0$, then we may evaluate condition (16). It is easy to check that (16) yields $\Phi_{xx} + \Phi_{yy} = 0$ at $(z, u) = (0, 0)$, and thus $\varphi_1(0) = 0$. We conclude, then, that the gradient of a solution to (4) is determined at a hyperbolic point of $\Gamma(\varphi)$. Returning to the original coordinates (15), we see that the condition $dv^{**}/du^{**} = 0$ yields

$$\Phi(z, u) = bu - \operatorname{Re} i\alpha_1 z + \ldots\,. \tag{17}$$

4. – Regularity of the solution.

In this section we assume that the solution Φ of (4) is continuous, and we derive an *a priori* estimate on the modulus of continuity of Φ.

With this we may see that the set of $\varphi \in C(\partial D)$ for which (4) is solvable is closed.

Given a function f on a set K, we define the modulus of continuity as

$$\omega(f, \delta) = \Big(\sup_{\substack{p_1, p_2 \in K \\ |p_1 - p_2| \leqslant \delta}} |f(p_1) - f(p_2)| \Big)^{\wedge},$$

where $h^{\wedge}$ means the smallest concave function $\geq h$; i.e., we assume our modulus of continuity is concave. Given functions $\Psi_1, \Psi_2 \in C(\bar{D})$, we write

$$\mathfrak{R}(\Psi_1, \Psi_2) = \{(z, u + iv) \in D \times i\boldsymbol{R}: \Psi_1(z, u) < v < \Psi_2(z, u)\} .$$

Now we recall an observation from [4]. We suppose that $\varphi \in C^1(\partial D)$ has barriers Ψ^+, Ψ^-, i.e., $\Psi^+ = \Psi^- = \varphi$ on ∂D, and $\mathfrak{R}(\Psi^-, \Psi^+)$ is pseudoconvex. Now let $M \subset \mathfrak{R}(\Psi^-, \Psi^+)$ be a complex manifold with smooth boundary $\partial M \subset \Gamma(\varphi)$. If at each point $(z_0, w_0) \in M$, we may write M locally as a graph $w = w(z)$, then it follows that

$$\left| \frac{dw}{dz} \right| \leq \|\Psi^-\|_{\mathrm{Lip}(\bar{D})} + \|\Psi^+\|_{\mathrm{Lip}(\bar{D})} . \tag{18}$$

(This is seen because the analytic function dw/dz takes its maximum modulus at ∂M. There it is bounded by $\nabla(\Psi^+ - \Psi^-)$ because $M \subset \mathfrak{R}(\Psi^-, \Psi^+)$.)

We will consider solutions Φ of (4) such that

(19) for each $(z_0, w_0) \in \tilde{\Gamma}(\Phi)$ there is a nonconstant holomorphic map $f: \Delta \to \boldsymbol{C}^2$ such that $(z_0, w_0) \in f(\Delta) \subset \tilde{\Gamma}(\Phi)$.

It follows that if $\omega \subset \tilde{\Gamma}(\Phi)$ is any open subset and if F is holomorphic in a neighborhood of $\bar{\omega}$, then

$$\sup_{\omega} |F| \subseteq \sup_{\partial\omega} |F| .$$

(Otherwise, for some $\delta > 0$, $|F| + \delta|z|^2$ takes an interior maximum at some point $(z_0, w_0) \in \omega$, which contradicts the maximum principle on $f(\Delta)$.)

The assumption (19) also gives the following maximum principle:

(20) If Φ_1, Φ_2 solve (4) and (19) on D, and if $\Phi_1 \geq \Phi_2$ on ∂D, then $\Phi_1 \geq \Phi_2$ on D.

(Otherwise, for some $\delta > 0$, $\Phi_2 + \delta|z|^2 - \Phi_1$ has an interior maximum at some (z_0, u_0). But this contradicts the maximum principle on a disk $f_1(\Delta) \subset \tilde{\Gamma}(\Phi_1)$, where Φ_1 is harmonic.)

LEMMA 4.1. *Let $\Psi_1, \Psi_2 \in C(\bar{D})$ be given with $\Psi_1|_{\partial D} = \Psi_2|_{\partial D} = \varphi$, and assume that $\mathcal{R}(\Psi_1, \Psi_2)$ is pseudoconvex. If $\Phi \in C(\bar{D})$ is a solution of* (4) *satisfying* (19), *then*

$$\omega(\Phi, \delta) \leq \omega(\Psi_1, \delta) + \omega(\Psi_2, \delta). \tag{21}$$

PROOF. First we note that by (19) we have

$$\Psi_1 \leq \Phi \leq \Psi_2 . \tag{22}$$

This is because each point of $\tilde{\Gamma}(\Phi)$ lies in a variety $V \subset D \times i\boldsymbol{R}$ with $\partial V \subset \Gamma(\varphi)$. Since $\{(z, w) \in D \times i\boldsymbol{R} \colon v < \Psi_2\}$ is pseudoconvex, V must lie in this set, and so $\Phi \leq \Psi_2$. Similarly, $\Psi_1 \leq \Phi$.

Let us fix $\alpha \in \boldsymbol{C} \times \boldsymbol{R}$. It follows that

$$\Phi(p + \alpha) - \omega(\Psi_2, |\alpha|) \leq \varphi(p)$$

for all $p \in \partial D$ such that $p + \alpha \in D$. Thus for $p \in \bar{D}$

$$\tilde{\Psi}(p) = \begin{cases} \max\big(\Psi_1(p), \Phi(p + \alpha) - \omega(\Psi_1, |\alpha|) - \omega(\Psi_2, |\alpha|)\big) & p + \alpha \in \bar{D} \\ \Psi_1(p) & p + \alpha \notin \bar{D} \end{cases}$$

is continuous on $\bar{D}$, and the set $\{(z, w) \in D \times i\boldsymbol{R} \colon v > \tilde{\Psi}\}$ is pseudoconvex. It follows as above that $\tilde{\Psi} \leq \Phi$. Thus for $p, p + \alpha \in D$

$$\Phi(p + \alpha) - \Phi(p) \leq \omega(\Psi_1, |\alpha|) + \omega(\Psi_2, |\alpha|) .$$

Replacing α by $-\alpha$, we have (21).

LEMMA 4.2. *Let $\varphi \in C^2(\partial D)$ be given, and let $\Phi \in C(\bar{D})$ be a solution of* (4) *satisfying* (19). *Then there is a constant C (depending only on D) so that*

$$\|\Phi\|_{\mathrm{Lip}^1(\bar{D})} \leq C \|\varphi\|_{C^2(\partial D)} . \tag{23}$$

PROOF. Since $D \times i\boldsymbol{R}$ is strongly pseudoconvex, there exists a function $r = r(z, u) \in C^2(\bar{D})$ which is strongly plurisubharmonic. Since $\varphi \in C^2(\partial D)$, there exists $\tilde{\varphi} \in C^2(\bar{D})$ with $\tilde{\varphi}|_{\partial D} = \varphi$. We may choose $C > 0$ so that $Cr \pm \tilde{\varphi}$ is plurisubharmonic on D. Thus $\Phi \in \mathcal{R}(Cr + \tilde{\varphi}, -Cr + \tilde{\varphi})$, and so (23) follows from (21).

LEMMA 4.3. *Let $\Phi \in C(\bar{D})$ be a solution of* (4) *satisfying* (19), *and let $\varphi = \Phi|_{\partial D}$. If D is convex with nonvanishing principal curvatures, then there is a constant C, depending only on D, such that $\omega(\Phi, \delta) \leq C\omega(\varphi, C\sqrt{\delta})$.*

PROOF. By Lemma 4.1, it suffices to show that we can find barriers Ψ_1, Ψ_2 above and below such that

$$\omega(\Psi_j, \delta) \leq C\omega(\varphi_j, C\sqrt{\delta}), \quad j = 1, 2 .$$

It is easily seen that it will suffice to show that there is a lower barrier Ψ_1^p for the function $\varphi(z) = -\omega(|z-p|)$ on ∂D, for any fixed $p \in \partial D$. For general φ, we take a supremum of functions Ψ_1^p.

By hypothesis, $-\omega(|z-p|)$ is convex, so we will take Ψ_1^p to be the convex minorant of $-\omega(|z-p|)$. Now by the special convexity assumption on ∂D, there exists a linear supporting function $L(z, u)$ for ∂D at p such that

$$|z-p|^2 \geq -L(z, u) \geq 0$$

holds for $(z, u) \in \partial D$, and $|\nabla L| > k > 0$ where k depends only on ∂D. It follows that

$$\Psi_1^p \geq -\omega\left((-L(z, u))^{\frac{1}{2}}\right)$$

since the right-hand side is a convex function.

REMARK. An estimate of this sort was given for solutions of the complex Monge-Ampère equation by Gaveau [10]. Using the argument above, one can modify the proof of Theorem 6.2 of [6] to yield a nonprobabilistic proof of Gaveau's result.

THEOREM 4.4. *Let D be convex with strictly positive normal curvature. Suppose that $\Phi_j \in C(\bar{D})$ solve* (4) *and satisfy* (19). *Suppose further that $\Phi_j|_{\partial D} = \varphi_j$ converges uniformly to $\varphi \in C(D)$, and there is a common modulus of continuity $\omega(\delta)$ for the family $\{\varphi, \varphi_1, \varphi_2, \dots\}$. Then the limit $\Phi = \lim_{j\to\infty} \Phi_j$ is taken uniformly, Φ has modulus of continuity $C\omega(C\sqrt{\delta})$, and Φ solves* (4).

PROOF. By Lemma 4.3, the functions Φ_j all have a common modulus of continuity $C\omega(C\sqrt{\delta})$. Further, since $\{\varphi_j\}$ converges uniformly, we may assume that it is monotone increasing. By (20), the Φ_j are also monotone increasing. Since there is a common modulus of continuity, the convergence is uniform. It is now clear that the uniform limit Φ solves (4).

THEOREM 4.5. *Suppose that $\Phi_j \in C^2(\bar{D})$ solves* (4), *and let us set $\varphi_j = \Phi_j|_{\partial D} \in C^2(\partial D)$. If*

$$\sup_{1 \leqslant j \leqslant \infty} \|\varphi_j\|_{C^2(\partial\omega)} < \infty ,$$

then there is a subsequence $\{\Phi_{j'}\} \subset \{\Phi_j\}$ *such that* $\Phi = \lim_{j\to\infty} \Phi_{j'}$ *exists and belongs to* $\mathrm{Lip}^1(\bar{D})$. *Further,* Φ *solves* (4) *and satisfies* (19) *and has property* (iii) *of Theorem* 1.

PROOF. By Lemma 4.2 and the proof of Theorem 4.4, $\Phi \in \mathrm{Lip}^1(\bar{D})$ exists and solves (4). Now let us consider a complex manifold lying in the graph $\tilde{\Gamma}(\Phi_j)$. If we write it locally as $w = w(z)$, then by (18) we have the estimate

$$\left|\frac{dw}{dz}\right| \leq k\|\varphi_j\|_{C^2(\partial\omega)},$$

where k depends only on D. Let us fix a point $(z_0, w_0) \in \tilde{\Gamma}(\Phi_j)$. Then there is a disk $\{|z - z_0| < \varepsilon\} = \Delta_\varepsilon$ such that the manifold M_j passing through (z_0, w_0) may be written as $w = f_j(z)$, where $f_j(z)$ is single-valued and analytic on Δ_ε. Further, $\varepsilon > 0$ is independent of j. Since $|f_j'|$ is uniformly bounded, and $f(z_0) = w_0$, it follows that $\{f_j\}$ converges uniformly to $f(z) \in \mathcal{O}(\Delta_\varepsilon)$. Thus $\{(z, f(z)) \colon z \in \Delta_\varepsilon\}$ is a complex disk in $\tilde{\Gamma}(\Phi)$, containing (z_0, w_0), and so $\tilde{\Gamma}(\Phi)$ satisfies (19).

The property (iii) of Theorem 1 is a consequence of the following lemma, since we may move to a regular point of V.

LEMMA 4.6. *Let* Φ *be an arbitrary function. If* V_1, V_2 *are germs of varieties in* C^2 *at* (z_0, w_0) *with* V_1 *nonsingular and* $V_1 \cup V_2 \subset \tilde{\Gamma}(\Phi)$, *then* $V_1 = V_2$.

PROOF. We let σ be the intersection of the (z, u)-projections of V_1 and V_2. Let us write the projection of V_1 as $\{u = h(z)\}$. Now $h - u|_{V_2}$ is a harmonic function vanishing at (z_0, w_0). Thus $\sigma = \{h_{V_2} = u\}$ is a 1-dimensional real analytic curve. Since $V_1 \cup V_2 \subset \tilde{\Gamma}(\Phi)$, it follows that

$$\sigma_\Phi = \{v = \Phi(z, u) \colon (z, u) \in \sigma\}$$

lies in both V_1 and V_2. Since $V_1 \cup V_2$ contains a 1-dimensional set, V_1 and V_2 must have a component in common. Thus $V_1 = V_2$.

5. – Barrier estimate.

Now we look at the estimate (22) in a neighborhood of a hyperbolic point. Let us assume that for a fixed solution Φ_0 we have upper and lower barriers $\Psi^+, \Psi^- \in C^1(\bar{D})$. That is, $\Psi^+|_{\partial D} = \Psi^-|_{\partial D} = \varphi_0$, and the domain $\mathcal{R}(\Psi^-, \Psi^+)$ is pseudoconvex. We note that the quantity

$$\varkappa = \max_{\partial D} |\nabla(\Psi^+ - \Psi^-)|$$

will be taken to be small.

Let $U = \{\zeta \in \boldsymbol{C} \colon \operatorname{Im} \zeta > 0\}$, and let us consider a holomorphic mapping $f\colon U \to \boldsymbol{C}^2$ such that

$$f \in C^1(\overline{U} \setminus f^{-1}(0)) \cap \mathcal{O}(U) \cap C(\overline{U}) , \tag{24a}$$

$$f(U) \subset \mathfrak{R} , \tag{24b}$$

$$f(\boldsymbol{R}) \subset \Gamma(\varphi) \quad \text{and} \quad f(0) = 0 , \tag{24c}$$

$$f(\boldsymbol{R}) \text{ has no self-intersections in } \Gamma(\varphi) \setminus \{0\} . \tag{24d}$$

Let us write $f(\boldsymbol{R}) = \gamma$ and $\gamma^{\pm} = f(\boldsymbol{R}^{\pm})$. We write $z = s + it$ and set $f(\zeta) = (z(\zeta), w(\zeta))$. If Γ has the form in (12), then for z sufficiently small,

$$w(s) = \alpha_1 z + (1 + ib) z\bar{z} + \operatorname{Re} \alpha_2 z^2 + i \operatorname{Re} \beta_2 z^2 + o(|z|^2) .$$

It follows that, taking the partial derivative with respect to s, we have

$$w_s = \alpha_1 z_s + 2(1 + ib) \operatorname{Re} \bar{z} z_s + 2 \operatorname{Re} \alpha_2 z z_s + 2i \operatorname{Re} \beta_2 z z_s + o(z) z_s . \tag{25}$$

Near ∂D, (17) gives us

$$\Phi_0 = \varphi_0(z) - br + O(r^2 + r|z|) . \tag{26}$$

Thus the barrier estimate (22) is equivalent to

$$|v - \varphi(z) + br(z, u)| \leq \varkappa C |r| \tag{27}$$

for (z, u) sufficiently close to O. If we differentiate (27) with respect to t for $\zeta \in \boldsymbol{R}$, we obtain

$$|v_t - \varphi_t + br_t| \leq \varkappa C |r_t| . \tag{28}$$

By the Cauchy-Riemann equations, $iz_s = z_t$ and $iw_s = w_t$. Thus

$$\begin{aligned} r_t &= 2 \operatorname{Re} r_z z_t + r_u u_t \\ &= \operatorname{Re} i(\alpha_1 + 2\bar{z} + 2\alpha_2 z) z_s - u_t + o(z) z_s \\ &= 2 \operatorname{Re} [(i + b)\bar{z} + (i\alpha_2 + \beta_2) z] z_s + o(z) z_s \end{aligned}$$

where the last line is obtained by substituting v_s from (25). Similarly, we see from (25) that

$$\begin{aligned} v_t - \varphi_t &= u_s - \varphi_t \\ &= \operatorname{Re}(\alpha_1 z_s + 2\bar{z} z_s + 2\alpha_2 z\bar{z}_s) - \operatorname{Re}(\beta_1 z_t + 2b\bar{z}z_t + 2\beta_2 z z_t) + o(z)z_s \\ &= \operatorname{Re}(\alpha_1 - i\beta_1)z_s + 2\operatorname{Re}\big((1-ib)\bar{z} + [\alpha_2 - i\beta_2)z\big)z_s + o(z)z_s \\ &= 2\operatorname{Re}\big((1-ib)\bar{z} + (\alpha_2 - i\beta_2)z\big)z_s + o(z)z_s\,. \end{aligned}$$

Let us consider the vector field

$$\hat{X}(z) = 2[(-i+b)z + (-i\bar{\alpha}_2 + \bar{\beta}_2)\bar{z}]\,,$$

which is defined for $z \in C$ and has a singular point at $z=0$. Since

$$|\alpha_2 + i\beta_2| > |1+ib|,$$

i.e., O is a hyperbolic point, $\hat{X}$ has index -1 at $z=0$. Thus we have

$$r_t = \operatorname{Re}\hat{X}(z)\bar{z}_s = \langle \hat{X}, z_s\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual Euclidean inner product in the plane Similarly,

$$v_t - \varphi_t = \langle i\hat{X}, z_s\rangle\,.$$

Thus the estimate (28) becomes

$$\left|\frac{\langle i\hat{X}, z_s\rangle}{\langle \hat{X}, z_s\rangle} + b\right| \le \varkappa C\,. \tag{29}$$

Geometrically, this means that

$$|\text{angle}(\hat{X}, z_s) + \arctan b| \le \varkappa C'\,. \tag{30}$$

Now let us choose $-\pi/2 < k < \pi/2$ such that $\tan k = -b$, and let us set

$$X = e^{ik}\hat{X}\,.$$

Then (30) is equivalent to

$$|\text{angle}(X, z_s)| \le \varepsilon\,, \tag{31}$$

where $\varepsilon = \varepsilon(\varkappa)$ is a small number depending on $\varkappa$. We depict (31) geometrically in Figure 3. The vector field X is depicted on the unit circle $|z| = 1$, and the angle about X corresponding to (31) is indicated.

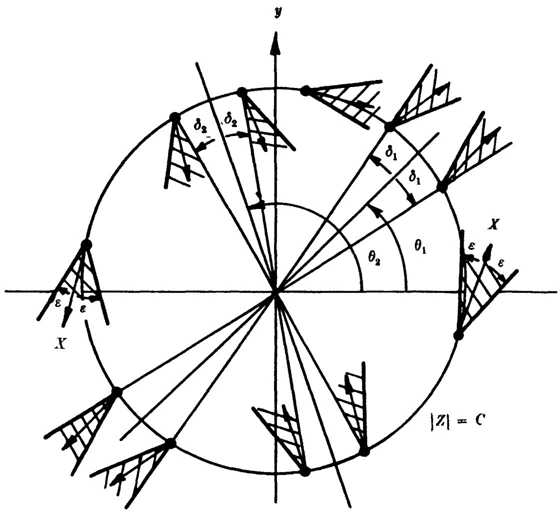

Fig. 3

Let us note that we may write

$$X(z) = -2ie^{ik}(1+ib)\left(z + \left(\frac{\bar{\alpha}_2 + i\bar{\beta}_2}{1+ib}\right)\bar{z}\right) = -2i|1+ib|^{-1}(z + e^{i\tau}\lambda\bar{z})$$

since $\tan\mu = b$. Now the point z_j with argument θ_j satisfies

$$(-1)^{j+1}X(z_j)\bar{z}_j > 0\,, \quad j = 1, 2\,.$$

Thus

$$\mathrm{Im}\left(i(z_j\bar{z}_j + e^{i\tau}\lambda z_j^2)\right) = 0\,,$$

which yields

$$\cos\,(2\theta_j + \tau) = -\frac{1}{\lambda}\,.$$

Thus, as expected, the solutions obtained here agree with the solutions obtained in Section 3 (for the *-coordinates). In particular, we conclude that

$$0 < |\theta_2 - \theta_1| < \frac{\pi}{2}\,.$$

Let us use the notation

$$A(\phi_1, \phi_2) = \{z \in C \colon \phi_1 < \arg z < \phi_2\}\,.$$

Let $\delta_1, \delta_2 > 0$ be the angles indicated in Figure 3. We may choose $\varkappa > 0$ (and thus $\varepsilon(\varkappa) > 0$) sufficiently small that

$$A(\theta_2 - \delta_2, \theta_2 + \delta_2) \cap A(\theta_1 - \delta_1, \theta_1 + \delta_1) = \emptyset\,.$$

Now we consider the possibility of a simple curve approaching $z = 0$ while under the constraint of Figure 3. It is clear that a simple curve $\gamma = f(\boldsymbol{R})$ can reach the origin only if it remains within $A(\theta_2 - \delta_2, \theta_2 + \delta_2)$ or $A(\theta_2 + \pi - \delta_2, \theta_2 + \pi + \delta_2)$ when $|f|$ is very small. Similarly, it can only exit through $A(\theta_1 - \delta_1, \theta_1 + \delta_1)$ or $A(\theta_1 + \pi - \delta_1, \theta_1 + \pi + \delta_1)$.

In the proof of Theorem 1, the disk satisfying (24) will arise as a limit of a 1-parameter family of disks, all of which are subject to (31). For $\eta > 0$ let us write

$$U_\eta = U \cap \{|\zeta| < \eta\}\,, \qquad \bar{\Omega}_\eta = z\text{-projection}\,(f(U_\eta))\,.$$

By the Lipschitz estimate (18), the disk $f(U_\eta)$ is given as a graph over Ω_η.

LEMMA 5.1. *For $\eta > 0$ sufficiently small, the disk $f(U_\eta)$ is given, near the hyperbolic point $(0, 0)$, as a graph $\{(z, w(z)) \colon z \in \Omega_\eta\}$ where the function w is holomorphic and w' is continuous on $\bar{\Omega}_\eta$.*

PROOF. The fact that $f(U_\eta)$ is a graph over Ω_η follows from the discussion above. Further, $w' \in L^\infty(\Omega_\eta)$ by (18). By the arguments of [4], w is smooth at all points of $\partial\Omega_\eta \setminus \{0\}$. For z near 0, we use (25) to see that for $z \in \partial\Omega_\eta$

$$\frac{dw}{ds} = \big(\alpha_1 + O(z)\big)\frac{dz}{ds}\,.$$

Thus it follows that

$$\lim_{\substack{z \to 0\\ z \in \partial\Omega_\eta}} w'(z) = \alpha_1\,,$$

and thus

$$w' \in C(\bar{\Omega}_\eta)\,.$$

Now we make our essential observation concerning Figure 3.

LEMMA 5.2. *If γ^- approaches 0 inside the angle $A(\theta_2 - \delta_2, \theta_2 + \delta_2)$, then γ^+ exits through the angle $A(\theta_1 - \delta_1, \theta_1 + \delta_1)$, and the region Ω_η lies on the side of γ containing the angle $A(\theta_1 + \delta_1, \theta_2 - \delta_2)$.*

PROOF. By the preceding discussion, it is sufficient to show that there is a contradiction if we assume γ exits through the angle about $\theta_1 + \pi$. Since f preserves orientation, the region Ω_η contains the angle running from $\theta_2 - \delta_2$ to $\theta_1 + \pi + \delta_1$ and containing the origin. Since we have noted that $0 < \theta_2 - \theta_1 < \pi/2$, it follows that this angle is greater than π, and thus we can find points $\alpha \in \Omega_\eta$ such that

$$\{|z-\alpha|^2 < |\alpha|^2\} \subset \Omega_\eta . \tag{32}$$

Since $\operatorname{Re} f(\zeta) = u(\zeta) \in D$ for $\zeta \in U$, it follows that

$$u \geq \operatorname{Re} \alpha_1 z + \alpha_2 z^2 + z\bar{z} .$$

Thus it follows that

$$u \geq \operatorname{Re} (\alpha_1 z + \alpha_2 z^2) + h(z)$$

for z in the circle (32), and where $h(z)$ is the harmonic function with boundary values $z\bar{z}$ on the boundary of the circle (32). Now h reaches its minimum at $z = 0$, so $h_z \neq 0$ there. Thus by Lemma 5.1, we may compute

$$u_z(0) \neq \tfrac{1}{2} \alpha_1 .$$

On the other hand, we have

$$u_z = \tfrac{1}{2} \alpha_1$$

since $u + iv = \alpha_1 z + O(|z|^2)$ holds on γ.

This proves the first assertion; the second statement follows because f preserves orientation.

We conclude that the region Ω_η is either located as in Figure 4 or is the mirror image of it below the x-axis.

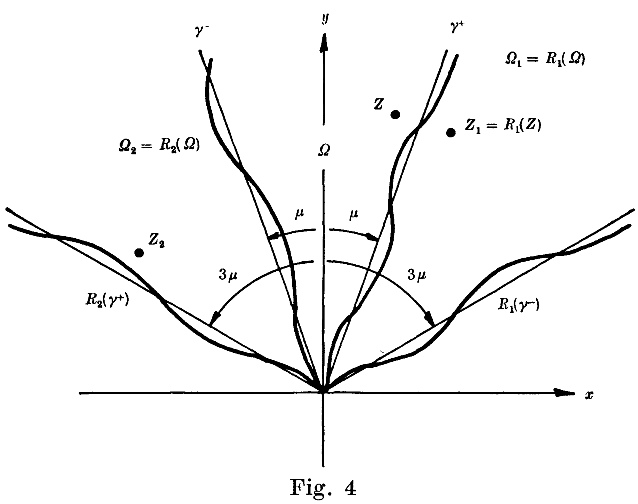

Fig. 4

6. – Reflection principle.

As in the previous Section, we consider a complex disk $f\colon U \to \boldsymbol{C}^2$ satisfying (24). Let us assume that $\Gamma(\varphi)$ has exactly the form (12) (without any $o(|z|^2)$ terms). Let Ω_η be the region as in Lemma 5.1, and let us simply write $\Omega = \Omega_\eta$. Let us also write $\gamma^\pm = \gamma^\pm_\eta$ and $U = U_\eta$. Thus we may assume that γ^- approaches $z = 0$ through the angle $A(\theta_2 - \delta_2, \theta_2 + \delta_2)$ and γ^+ leaves $z = 0$ through the angle $A(\theta_1 - \delta_1, \theta_1 + \delta_1)$. Finally, let us choose $\varkappa > 0$ sufficiently small that

$$\theta_2 - \theta_1 + \delta_1 + \delta_2 = \pi/2 - \delta \tag{33}$$

for some $\delta > 0$. This is possible since $0 < \theta_2 - \theta_1 < \pi/2$.

Now let us use the coordinates of (15). Suppressing the *'s, we have the surface given by

$$w = z\bar{z} + \lambda \operatorname{Re} z^2 .$$

By Lemma 5.1, $f(U)$ is given as a graph of a function $g\colon \Omega \to \boldsymbol{C}$, and

$$g(z) = z\bar{z} + \lambda \operatorname{Re} z^2 \quad \text{for } z \in \partial\Omega . \tag{34}$$

Thus

$$g(z(\zeta))\colon U \to \boldsymbol{C}$$

is analytic, and $g(z(\zeta))$ is real for $\zeta \in \boldsymbol{R}$. By the Schwarz reflection principle, $g(z(\zeta))$ is analytic in a neighborhood of $\zeta = 0$, and

$$g(z(\zeta)) = a_1\zeta + \dots \tag{35}$$

since $g(0) = 0$.

We will use the following result on Lipschitz regularity for conformal mappings.

LEMMA 6.1. *Let $\omega_1, \omega_2 \subset \boldsymbol{C}$ be star-shaped regions of the form*

$$\omega_j = \{r < h_j(\theta) + k_j(\theta)\}$$

where

$$1 < h_j + k_j < \infty ,$$

k_j is smooth, and

$$\|h_j\|_{\mathrm{Lip}^1} < \varepsilon .$$

If $F: \omega_1 \to \omega_2$ is a conformal equivalence, then $F \in \mathrm{Lip}^\eta(\bar\omega_1)$, where

$$\eta = \frac{\pi - 2\arctan\varepsilon}{\pi + 2\arctan\varepsilon}.$$

PROOF. At each point $z_0 \in \partial\omega_j$ we may construct comparison domains. After translating and rotating coordinates, we may assume that $z_0 = 0$ and there exists $\bar\eta > 0$ (independent of z_0) such that

$$A(\varepsilon, \pi-\varepsilon) \cap \{|z| < \bar\eta\} \subset \omega_j \cap \{|z| < \bar\eta\} \subset A(-\varepsilon, \pi+\varepsilon).$$

If $H_j(z)$ is a function on ω_j with $H_j < 0$, $H_j = 0$ on $\partial\omega_j$ and $\Delta H_j \geqq 0$, $\Delta H_j \in C_0^\infty(\omega_j)$, then using the comparison domains above, we have

$$-c_j'|y|^{1+(2\arctan\varepsilon)/\pi} \geq H_j(iy) \geq -c_j''|y|^{1-(2\arctan\varepsilon)/\pi} \tag{36}$$

for $0 < y < \bar\eta$. By a similar comparison, we can bound the growth of the Carathéodory metric:

$$c_j''\frac{|\xi|}{y} \leq C_j(iy;\xi) \leq c_j'\frac{|\xi|}{y}.$$

In this estimate, y is approximately proportional to $\mathrm{dist}(y, \partial\omega_j)$.

Since $C_1(z;\xi) = C_2(f(z); f'(\xi))$, it follows that

$$\frac{|\xi|}{\mathrm{dist}(z, \partial\omega_1)} \simeq \frac{|f'(z)|\,|\xi|}{\mathrm{dist}(f(z), \partial\omega_2)}.$$

Thus we have

$$|f'(z)| \leq \mathrm{Const}\,\frac{\mathrm{dist}(f(z), \partial\omega_2)}{\mathrm{dist}(z, \partial\omega_1)}.$$

If H_j is a function as above, then by (36), $H_j(z)$ is estimable by a certain power of the distance to the boundary. Since $H_2(f)$ is again a function of the form H_1, we have

$$|f'(z)| \leq \mathrm{Const}\,[\mathrm{dist}(z, \partial\omega_1)]^{\eta-1}$$

where $\eta = (1 - 2\arctan\varepsilon)/(1 + 2\arctan\varepsilon)$. It follows now that $f \in \mathrm{Lip}^\eta(\bar\omega_1)$, which is the desired result.

LEMMA 6.2. *Let $z(\zeta)$ be the conformal mapping satisfying* (24). *Then if $\varkappa > 0$ is small enough, we have*

$$c_1|\zeta|^{\nu_1} \leq |z(\zeta)| \leq c_2|\zeta|^{\nu_2}$$

for some $\nu_1 < \frac{1}{2}$.

PROOF. Let us consider the conformal mapping

$$H\colon \boldsymbol{C}\setminus[0, -i\infty) \to \boldsymbol{C}$$

given by $H(z) = (e^{-i\theta_1}z)^{\pi/(\theta_2-\theta_1)}$. It follows that $H(\gamma^+)$ lies in the sector $\{|\arg z| < 2\delta_1\pi/(\theta_2-\theta_1)\}$. Furthermore, since H preserves angles, Figure 3 gives

$$|\arg H(z)_s(\zeta)| < \frac{2(\delta_1+\varepsilon)\pi}{\theta_2-\theta_1}$$

for $0<\zeta<\eta$. We have a similar estimate for $-\eta<\zeta<0$. Since $z(\zeta)$ is smooth on ∂U for $\zeta\neq 0$, we may shrink U so that the domain $\omega = H(\Omega)$ satisfies the hypotheses of Lemma 6.1. Thus we have $H(z(\zeta))\in \mathrm{Lip}^{1-\varepsilon'}(\overline{U})$ where we may take ε' arbitrarily small by choosing $\varkappa>0$ small. It follows, then, that

$$|z(\zeta)|^{\pi/(\theta_2-\theta_1)} = |H(z(\zeta))-0| \le C|\zeta|^{1-\varepsilon'}.$$

Thus we have

$$|z(\zeta)| \le C_2|\zeta|^{\nu_2}$$

where $\nu_2 = (1/\pi)(\theta_2-\theta_1)(1-\varepsilon') < \frac{1}{2}$ by (33). Now we have the other inequality by applying Lemma 6.1 to the inverse mapping $f(z) = H^{-1}(z(\zeta))$. This gives the reverse inequality with $\nu_1 = (1/\pi)(\theta_2-\theta_1)(1-\varepsilon')^{-1}$, which completes the proof.

Let $\chi(z)=\chi(\theta)$ be the function homogeneous of degree zero such that

$$z\bar{z} + \lambda \operatorname{Re} z^2 = |z|^2\chi(\theta).$$

In the *-coordinates of (15), which we are using this Section, the lines $\{\arg z = \theta_j\}$, $j=1, 2$, in Figure 3 correspond to the lines $\{\arg z = \pi/2 \pm \mu\}$, as in Figure 4. These lines are also the set where $\chi = 0$.

It follows from (35) and Lemma 6.2 that for $-\eta<\zeta<\eta$

$$|g(z(\zeta))| \simeq |a_1\zeta| \le C|z(\zeta)|^{1/\nu_1}.$$

Thus for some $\delta>0$ we have

$$|z|^2|\chi(z)| \le C|z|^{1/\nu_1} = C|z|^{2+\delta}, \tag{37}$$

and thus

$$|\chi(z)| \le C|z|^{\delta}. \tag{38}$$

Since $\chi'(\theta) \neq 0$ for $\theta = \pi/2 \pm \mu$, it follows that the curve γ^- is asymptotic to $\{\arg z = \theta_2\}$ (in the old coordinates) and

$$|\arg z(s) - \theta_2| \leq C|z(s)|^\delta \tag{39}$$

for $-\eta < s < 0$. The corresponding statement also holds for γ^+.

LEMMA 6.3. *We have the estimate* $|g(z)| \leq c|z|^{2+\delta}$ *for all* $z \in \Omega$ *and some* $\delta > 0$.

PROOF. Since $g(z)$ is bounded on Ω, it is given by the Poisson integral formula of its boundary values $\chi(z)|z|^2$ which vanish at $z = 0$ as in (37). By using comparison domains, as in the proof of Lemma 6.1, we may show that the harmonic function $h(z)$ on Ω with boundary values $|z|^{2+\delta}$ is estimated by $0 < h(z) < c|z|^{2+\delta'}$ for some $\delta' > 0$. This gives the desired estimate. Now we use a reflection argument motivated by arguments given in [13] and [15]. Since

$$g(z) = z\bar{z} + \lambda \operatorname{Re} z^2$$

holds for $z \in \gamma$, we consider the complexified equation:

$$g(z) = z\tilde{z} + \frac{\lambda}{2}(z^2 + \tilde{z}^2) . \tag{40}$$

Solving (40), we have two solutions

$$\tilde{z}_\pm = \frac{z}{\lambda}\left(-1 \pm i\sqrt{\lambda^2 - 1 - \frac{2\lambda g(z)}{z^2}}\right) \tag{41}$$

and we define

$$R_1(z) = (\bar{\tilde{z}}_-) , \qquad R_2(z) = (\bar{\tilde{z}}_+) .$$

Clearly R_j is an anti-conformal mapping. Geometrically, R_j reflects Ω about $\gamma^\pm$ as indicated in Figure 4. If $z \in \gamma^+$, then $R_1(z) = z$, and if $z \in \gamma^-$, $R_2(z) = z$. In fact

$$R_1(z) = -\frac{\bar{z}}{\lambda}\left(1 + i\sqrt{\lambda^2 - 1} + O(|z|)\right) \tag{42}$$

and thus $R_1(\gamma^-)$ as asymptotic to the ray $\{\arg z = \pi/2 - 3\mu\}$. Similarly $R_2(\gamma^+)$ is asymptotic to the ray $\{\arg z = \pi/2 + 3\mu\}$.

LEMMA 6.4. *For some* $\delta > 0$,

$$|g''(z)| \le C|z|^\delta$$

for $z \in \Omega$. *In particular* $g \in C^{2+\delta}(\bar{\Omega})$.

PROOF. Let us choose α_1, α_2 with $\pi/2 - 3\mu < \alpha_1 < \pi/2 - \mu < \pi/2 + \mu < \alpha_2 < \pi/2 + 3\mu$. Then we may define $\tilde{g}$ on

$$\omega = A(\alpha_1, \alpha_2) \cap \{|z| < \varepsilon\}$$

by setting

$$\tilde{g}(z) = \begin{cases} g(z) & \text{if } z \in \omega \cap \bar{\Omega} \\ \overline{g(\tilde{z})} & \text{if } z \in \Omega_j \cap \omega \text{ and } R_j(\tilde{z}) = z\,. \end{cases}$$

Since $R_1(\gamma^-)$ is asymptotic to the ray $\{\arg z = \pi/2 - 3\mu\}$, it follows that the winding number of $R_1(\partial\Omega)$ about $z_1 \in \omega \cap \Omega_1$ is -1. Thus by the argument principle, applied to the anticonformal R_1, there is a unique $z \in \Omega$ with $R_1(z) = z_1$. Since the corresponding statement for R_2 is also true, it follows that $\tilde{g}$ is well-defined.

It follows now that $\tilde{g}$ is an analytic function on ω, and by (42), we see that

$$|\tilde{g}(z)| \le c|z|^{2+\delta}$$

for $z \in \omega$. We may choose $\alpha > 0$ such that for each $z_0 \in \Omega$, $|z_0| < \varepsilon/2$, $\{|z - z_0| < \alpha|z_0|\} \subset \omega$. By the familiar Cauchy estimate, we have, for $z \in \Omega$,

$$|\tilde{g}''(z_0)| \le \frac{1}{\alpha^2|z_0|^2} \sup\{|\tilde{g}(z)|: |z - z_0| < \alpha|z_0|\} \le c'|z_0|^\delta$$

which gives the desired estimate for g since $g = \tilde{g}$ on $\Omega \cap \omega$.

A similar argument gives

$$|g'''(z)| \le c|z|^{\delta-1}$$

for $z \in \Omega$, and thus we have $g'' \in C^\delta(\bar{\Omega})$.

7. – Almost holomorphic flattenings.

In this Section we discuss « almost » flattenings from two points of view. First we show (Lemma 7.4) that flattening is possible with a mapping that is almost holomorphic. This allows us in Lemma 7.5 to construct barriers $\Psi^\pm$

so that we may apply the arguments of Sections 5 and 6. Then it is shown that a chain may be flattened with a holomorphic mapping if we first allow a small C^2 perturbation.

Throughout this Section we assume that φ_0, Φ_0, *and* D *satisfy the hypotheses of Theorem* 1, *and* Φ_0 *is* C^2 *and solves* (4) *in a neighborhood of* $\bar{D}$.

Let $H_1(\varphi), \ldots, H_m(\varphi)$ denote the hyperbolic points of $\Gamma(\varphi)$. Then $H_j(\varphi)$ varies continuously under small C^2-perturbations of φ and ∂D. Let $\mathfrak{L}(\tilde{\Gamma}(\Phi_0))$ denote the Levi foliation, i.e. the foliation of $\tilde{\Gamma}(\Phi_0)$ by complex manifolds. Thus $\mathfrak{L}(\tilde{\Gamma}(\Phi_0))$ induces a singular foliation $\mathcal{F}(\Gamma(\varphi_0))$ of $\Gamma(\varphi_0)$. (See [4] for a more detailed discussion of the relation between $\mathfrak{L}$ and $\mathcal{F}$.) We note that we may apply the arguments of Section 5 to $\Gamma(\varphi_0)$ with $\varkappa = 0$ because $\Phi_0 \in C^2$. Thus a complex disk satisfying (24) may approach a hyperbolic point only if it is asymptotic to either of the approach regions $A(\theta_1, \theta_2)$ or $A(\theta_1 + \pi, \theta_2 + \pi)$.

Let us define a *chain* $\mathcal{C}$ to be a union of simple, closed curves $\gamma_1, \ldots, \gamma_q \subset \subset \Gamma(\varphi)$ with the following properties;

(a) for $1 \leq j \leq q$ there is a mapping f_j: $\Delta \to D \times i\boldsymbol{R}$, $f_j \in \mathcal{O}(\Delta) \cap C(\bar{\Delta})$ such that $f_j(\partial\Delta) = \gamma_j$:

(b) γ_j is smooth at the totally real points of $\Gamma(\varphi)$;

(c) $\mathcal{C}$ is connected;

(d) if $H \in \mathcal{C}$ is a hyperbolic point, then $\mathcal{C}$ approaches H through each of the asymptotic approach regions exactly once.

Given a chain $\mathcal{C}$ we will denote by $\tilde{\mathcal{C}}$ the set $\{f_1(\Delta), \ldots, f_q(\Delta)\}$ of complex disks whose boundaries are given by $\mathcal{C}$.

LEMMA 7.1. *If* M *is a leaf of* $\mathfrak{L}(\tilde{\Gamma}(\Phi_0))$, *then* M *can approach a hyperbolic point* $H \in \Gamma(\varphi_0)$ *only once.*

PROOF. Since the Levi foliation $\mathfrak{L}(\tilde{\Gamma}(\Phi_0))$ extends to be a foliation of a neighborhood of $\bar{D}$, a leaf can approach H through each asymptotic approach region only once. We need to show that the leaf M cannot fill both approach regions to H.

If γ is the boundary of M, and if γ approaches H through both approach regions, then $M \cup \{H\}$ is not simply connected. Let M_ε be the leaf of $\mathfrak{L}(\tilde{\Gamma}(\Phi_0))$ passing through the point H^ε obtained by taking the inward normal to $\partial\tilde{\Gamma}(\Phi_0)$ a distance $\varepsilon > 0$ from H. Then the saddle point at H connects these two regions in M_ε. Thus M_ε is not simply connected, which contradicts the hypotheses of Theorem 1.

REMARK. Without the assumption that Φ_0 is a C^2 solution in a neighborhood of $\bar{D}$ we are not able to rule out the possibility that more than one leaf approaches H from the same angle.

From Lemma 7.1 it is clear that we can piece leaves of $\mathfrak{L}(\tilde{\Gamma}(\Phi_0))$ together to form chains.

LEMMA 7.2. *Every hyperbolic point $H \in \Gamma(\varphi_0)$ is contained in a chain.*

We recall that a Riemann domain $(\mathfrak{D}, \pi)$ over $\boldsymbol{C}$ is a complex manifold $\mathfrak{D}$ with a locally biholomorphic mapping $\pi\colon \mathfrak{D} \to \boldsymbol{C}$. With the following Lemma, we may assume that the complex manifolds of $\tilde{\Gamma}(\Phi_0)$ are globally graphs (cf. Lemma 5.1).

LEMMA 7.3. *If $\mathcal{C}$ is a chain, then there is a Riemann domain $(\mathfrak{D}, \pi)$ and a subdomain $\Omega \subset\subset \mathfrak{D}$ with $g \in C^1(\bar{\Omega}) \cap \mathcal{O}(\Omega)$ such that $G\colon \bar{\Omega} \to \bar{\tilde{\mathcal{C}}}$ given by $G(q) = (\pi(q), g(q))$ is a homeomorphism.*

PROOF. Clearly, no components of $\tilde{\mathcal{C}}$ can intersect, and two curves of $\mathcal{C}$ can intersect only at a hyperbolic point. Let $H \in \mathcal{C}$ by a hyperbolic point. Then by Lemmas 7.1 and 5.1, a neighborhood of H in $\bar{\tilde{\mathcal{C}}}$ can be written as a graph over a domain in $\boldsymbol{C}$. Similarly for any point $(z_0, w_0) \in \tilde{\mathcal{C}}$ we may find $\varepsilon > 0$ such that if we write

$$\Omega(z_0, w_0, \varepsilon) = z\text{-projection } \left(\tilde{\mathcal{C}} \cap \{|(z, w) - (z_0, w_0)| < \varepsilon\}\right)$$

then there is a continuous function $g\colon \bar{\Omega}(z_0, w_0, \varepsilon) \to \boldsymbol{C}$ such that

$$\{w = g(z)\colon z \in \bar{\Omega}(z_0, w_0, \varepsilon)\} = \bar{\tilde{\mathcal{C}}} \cap \{|(z, w) - (z_0, w_0)| \le \varepsilon\}\,.$$

Now we may cover $\bar{\tilde{\mathcal{C}}}$ with a finite number of these ε-neighborhoods. We let $\mathfrak{D}$ be the disjoint union of disks $\Delta(z_0, w_0, \varepsilon) = \{|z - z_0| < \varepsilon\} \times \{w_0\}$ with the appropriate identification; that is, $\Delta(z', w', \varepsilon) = \Delta(z'', w'', \varepsilon)$ if $\Omega(z', w', \varepsilon) \cap \Omega(z'', w'', \varepsilon) \neq \emptyset$ and the functions g agree on the overlap. The mapping π is just the z-projection.

By Lemma 7.3, we may canonically identify a chain $\tilde{\mathcal{C}}$ with a graph over a Riemann domain $\mathfrak{D}$, i.e. $\tilde{\mathcal{C}} = \{w = g(q)\colon q \in \Omega\}$. We say that $\tilde{\mathcal{C}}$ is *regular* if $g \in C^2(\bar{\Omega})$. The next result is that $\Gamma(\varphi_0)$ is almost flat in a neighborhood of a regular chain.

LEMMA 7.4. *Let $\mathcal{C}$ be a regular chain in $\Gamma(\varphi_0)$, and let $i\colon \tilde{\mathcal{C}} \to \mathfrak{D}$ be the associated Riemann domain. Then there is a neighborhood $\bar{\tilde{\mathcal{C}}} \subset \mathfrak{U} \subset \boldsymbol{C}^2$ and* *a C^2 mapping $F: \mathcal{U} \to \mathfrak{D} \times \boldsymbol{C}$ such that*

(a) $F(q) = i(q)$ *for* $q \in \tilde{\mathcal{C}}$;

(b) $F(\mathcal{U} \cap \tilde{\Gamma}(\Phi_0)) \subset \mathfrak{D} \times \boldsymbol{R}$;

(c) $|\bar{\partial} F(q)| = o(\operatorname{dist}(q, \tilde{\mathcal{C}}))$.

PROOF. Using the mapping i, we may identify $\mathcal{U}$ with a neighborhood $\hat{\mathcal{U}}$ of $\bar{\Omega} \times \{0\} = i(\mathcal{C}) \times \{0\} \subset \mathfrak{D} \times \boldsymbol{C}$. Without loss of generality, we may assume that $g = 0$, so $\tilde{\mathcal{C}} = \bar{\Omega} \subset \mathfrak{D}$. Also $\tilde{\Gamma}(\Phi_0) \cap \mathcal{U}$ is naturally identified with a surface, which we again call $\tilde{\Gamma}(\Phi_0)$, in $\hat{\mathcal{U}}$.

In a neighborhood of $\tilde{\mathcal{C}}$, we may choose a smooth parameter t for the leaves of $\mathcal{L}(\tilde{\Gamma}(\Phi_0))$. Locally, we may choose a parameter, and the orientation may be determined globally by requiring $\langle \partial/\partial t, \partial/\partial u \rangle > 0$. By Lemma 7.1, and since the leaves are closed disks, we may do this globally. The mapping

$$F: \mathcal{U} \cap \tilde{\Gamma}(\Phi_0) \to \mathfrak{D} \times \boldsymbol{C}$$

may be defined by

$$\hat{F}(q, w) = (q, t)$$

where (q, w) is in the leaf with parameter t in $\tilde{\Gamma}(\Phi_0)$. Defined this way, $\hat{F}$ satisfies (a) and (b). Further, $\hat{F}$ is easily seen to be a C^2, CR map. Thus we may extend $\hat{F}$ to a neighborhood of $\mathcal{U} \cap \tilde{\Gamma}(\Phi_0)$ to satisfy (c).

REMARK. By the Appendix it is not possible to find a mapping F which is actually holomorphic. For our purposes, however, it will suffice to flatten a small C^2-perturbation of $\Gamma(\varphi_0)$, which we do in Lemma 7.6.

Now we construct barriers.

LEMMA 7.5. *Let $\mathcal{C}$ be a chain in $\Gamma(\varphi_0)$. For $\varkappa > 0$, there is an open set $\tilde{\mathcal{C}} \subset \mathcal{U} \subset \boldsymbol{C}^2$ and $\delta > 0$ with the following property: If $\|\varphi - \varphi_0\|_{C^2} < \delta$, then there exist $\Psi^+, \Psi^- \in C^2(\mathcal{U} \cap \bar{D})$ such that $\{\pm \Psi^\pm > v\}$ is strongly pseudoconvex, $\Psi^+ = \Psi^- = \varphi$ on $\mathcal{U} \cap \Gamma(\varphi)$ and*

$$\max_{\mathcal{U} \cap \partial D} |\nabla(\Psi^+ - \Psi^-)| \leq \varkappa .$$

PROOF. We consider the map F given by Lemma 7.4, and we let $\hat{\Gamma} = F(\Gamma(\varphi_0))$ denote the image inside $\mathfrak{D} \times \boldsymbol{R}$. Since the only complex tangencies near $\Omega \times \{0\}$ are hyperbolic points, and since ∂D is strongly pseudoconvex, we may find a defining function r for $F(\mathcal{U} \cap \Gamma(\Phi_0))$ in a neighborhood of $\bar{\Omega} \times \{0\}$ such that r is C^2 and $r_{z\bar{z}} > 0$. Now by (16) we see that for $\varkappa > 0$ sufficiently small, $\hat{\Psi}^\pm = \mp \varkappa r$ form upper and lower barriers for $\hat{\Gamma}$ in a neighborhood of $\bar{\Omega} \times \{0\}$.

It follows, then, that we may take

$$\Psi^\pm = \hat{\Psi}^\pm(F)$$

for $\varkappa > 0$ small, which completes the proof.

Let us continue to develop the idea of the proof of Lemma 7.4. We consider a regular chain $\mathfrak{C}$ in $\Gamma(\varphi)$, and we let $\mathfrak{D}$, Ω, and g give $\tilde{\mathfrak{C}}$ as in Lemma 7.3. Let Ω_ε denote an ε-neighborhood of Ω in $\mathfrak{D}$. We may choose $\hat{g} \in \mathcal{O}(\bar{\Omega}_\varepsilon)$ such that $\|\hat{g} - g\|_{C^2(\bar{\Omega})}$ is arbitrarily small.

Thus we may make a small C^2 perturbation $\hat{\Gamma}$ of $\Gamma(\varphi)$ such that $\hat{\Gamma}$ is C^∞, and the chain $\hat{\mathfrak{C}} = \{\hat{g}(\partial\Omega)\}$ is contained in $\hat{\Gamma}$. We will use the notation $\hat{\Gamma} = \Gamma(\hat{\varphi})$, although this is a little imprecise; $\hat{\Gamma}$ is in fact a graph over a small perturbation $\partial\hat{D}$ of ∂D.

Via the mapping $w^* = w + g(q)$, $q^* = q$, we may identify a neighborhood of $\bar{\hat{\mathfrak{C}}}$ with a neighborhood of $\Omega \times \{0\}$ in $\mathfrak{D} \times \boldsymbol{C}$. We will now find a neighborhood $\mathfrak{U}$ of $\bar{\Omega} \times \{0\}$ in $\mathfrak{D} \times \boldsymbol{C}$ and a holomorphic mapping

$$F\colon \mathfrak{U} \to \mathfrak{D} \times \boldsymbol{C}$$

such that

$$|\operatorname{Im} w| \leq o\big(|\operatorname{Re} w|^2\big)$$

holds for $(q, w) \in \hat{\Gamma} \cup \mathfrak{U}$.

Now we return to the notation of Section 2. We let X be a vector field tangent to $\partial\Omega$ such that $X \neq 0$ at the regular points of $\partial\Omega$. We may choose a vector field Y such that X and Y together span the tangent space $T\hat{\Gamma}$ at all regular points of $\partial\bar{\Omega}$. Let us write

$$Y = \operatorname{Re}\left(\alpha \frac{\partial}{\partial q} + \beta \frac{\partial}{\partial w}\right).$$

Since it is not clear a priori how to define Y at the complex tangencies of $\hat{\Gamma}$, we recall that there are no parabolic points. In this case, we can choose Y to vary continuously. In fact, if we work in the local coordinates of (15), then we have $\alpha = 0$, $\beta \in \boldsymbol{R}$, $\beta \neq 0$ at the complex tangencies. Further, we may make an arbitrarily small C^2 perturbation of $\hat{\Gamma}$ and have $\alpha = 0$ and β constant in a small neighborhood of the complex tangency, i.e. we may in fact remove the $o\big(|z|^2\big)$ terms that were removed for convenience in (15).

We claim that $\arg\beta$ is well defined on $\partial\Omega$, i.e. that the indexes of $\hat{\Gamma}$ about the disks in Ω are all zero. This follows because the index, being an integer, is constant under small C^2 perturbations, and $\hat{\Gamma}$ is a small C^2 perturbation of $\Gamma(\varphi_0)$. In the case of $\Gamma(\varphi_0)$, the index is seen to be zero because of the existence of the smooth surface $\tilde{\Gamma}(\Phi_0)$.

Let $b \in C(\bar{\Omega})$ be the harmonic extension of $\arg\beta$ from $\partial\Omega$ to Ω. Since the asymptotic behavior of $\partial\Omega$ at hyperbolic points (i.e. it is as in Figure 4), it follows that $b \in C^{2+\delta}(\bar{\Omega})$ for some $\delta > 0$. We wish to obtain the harmonic conjugate by simply integrating the holomorphic function db/dz on $\bar{\Omega}$. If $\bar{\Omega}$ is simply connected, we can do this, and we obtain $b + ib^* \in C^{2+\delta}(\bar{\Omega})$. It follows now that after the change of coordinates

$$w^* = w \exp(b + ib^*)\,, \qquad q^* = q$$

we have

$$\operatorname{Im} w^* = a_2(q)(\operatorname{Re} w^*)^2 + o\big((\operatorname{Re} w^*)^2\big)$$

for $(q^*, w^*) \in \hat{\Gamma}$. Now we let A_2 denote the harmonic extension of a_2 from $\partial\Omega$ to Ω and we let $\tilde{A}_2$ be the harmonic conjugate. Then $A_2 + i\tilde{A}_2 \in \in C^{2+\delta}(\bar{\Omega}) \cap \mathcal{O}(\Omega)$, and in the new coordinate system

$$w^{**} = w^* - (iA_2 - \tilde{A}_2)(w^*)^2\,, \qquad q^{**} = q^*$$

we have

$$|\operatorname{Im} w^{**}| = o\big(|\operatorname{Re} w^{**}|^2\big)$$

for $(q^{**}, w^{**}) \in \hat{\Gamma}$.

We conclude from this that we may make an arbitrarily small C^2 perturbation of $\hat{\Gamma}$, which we call $\hat{\Gamma}$, such that

$$\operatorname{Im} w^{**} = 0$$

holds in a small neighborhood of $\hat{\mathcal{C}}$ in $\hat{\Gamma}$.

Thus we have proved the following.

LEMMA 7.6. *Let $\mathcal{C} = \{\gamma_1, \dots, \gamma_q\}$ be a regular chain in $\Gamma(\varphi)$. Assume that the barriers Ψ^+, Ψ^- as in Section 6 exist. Then there exist an arbitrarily small neighborhood $\hat{\mathcal{U}}$ of $\tilde{\mathcal{C}} \cup \mathcal{C}$ and arbitrarily small C^2 perturbations $\Gamma(\hat{\varphi})$ of $\Gamma(\varphi)$ with the following properties*:

(i) $(\boldsymbol{C}^2 \setminus \hat{\mathcal{U}}) \cap \Gamma(\hat{\varphi}) = (\boldsymbol{C}^2 \setminus \hat{\mathcal{U}}) \cap \Gamma(\varphi)$,

(ii) *there is a chain* $\hat{\mathcal{C}} = \{\gamma_1, \dots, \gamma_q\} \subset \hat{\mathcal{U}} \cap \Gamma(\hat{\varphi})$ *with* $\tilde{\hat{\mathcal{C}}} \subset \hat{\mathcal{U}}$,

(iii) *there is a nonsingular holomorphic function w^{**} on $\hat{\mathcal{U}}$ such that*

$$\tilde{\hat{\mathcal{C}}} \subseteq \{w^{**} = 0\}\,,$$

(iv) $\hat{\mathcal{U}}_0 \cap \Gamma(\varphi) \subset \{\operatorname{Im} w^{**} = 0\}$ *for some open set* $\hat{\mathcal{U}}_0$ *containing* $\hat{\mathcal{C}}$.

8. – Proof of Theorem 1.

We wish to show that there exists $\delta > 0$ such that the conclusion of Theorem 1 holds for φ and ∂D which differ from φ_0 and ∂D_0 by at most δ in the C^2 norm. This number $\delta > 0$ will be determined by Lemma 7.5 and the restrictions on $\varkappa = \varkappa(\delta) > 0$, such as (33), depending on 2nd order derivatives, at each hyperbolic point. However, during the proof, it will be convenient to assume that φ_0 and ∂D_0 have certain generic « good » properties. Thus we will replace φ_0 and ∂D_0 by C^2 perturbations of size $\varepsilon > 0$, where $\varepsilon \ll \varkappa$.

A small C^2 perturbation of ∂D or ∂D_0 will be again denoted as ∂D or ∂D_0.

In particular, if we bump D_0 inward a small amount, then we may assume that $\bar{\Phi}_0$ is a C^2 solution of (4) in a neighborhood of $\bar{D}_0$.

By (14′) the case $\lambda = \infty$ cannot occur for a surface of the form (1) if D is strongly pseudoconvex. And by hypothesis there are no parabolic points, so all complex tangencies of $\Gamma(\varphi)$ are either elliptic or parabolic.

Let us use the terminology that a curve $\sigma \subset \Gamma(\varphi)$ is a *complex disk* if it is in fact the boundary of a complex disk $\tilde{\sigma} \subset D \times i\boldsymbol{R}$.

We recall that by (18), for every disk $\sigma \subset \Gamma(\varphi)$, $\tilde{\sigma}$ is locally a graph over the z-axis. In particular, if we consider the projection $\pi(\tilde{\sigma})$ of $\tilde{\sigma}$ from $D \times i\boldsymbol{R}$ to D, we see that the u-axis is always transversal to $\pi(\tilde{\sigma})$. Thus we may speak of « up » and « down » locally at a point of $\tilde{\sigma}$.

In particular, we consider a chain $\mathcal{C}$, and we may consider leaves of $\mathcal{L}(\tilde{\Gamma}(\Phi_0))$ which are just « above » and just « below » $\hat{\mathcal{C}}$, in the sense of the u-coordinate of the projection. Since the leaves are closed, this serves to construct a family of disks which effectively separate $\mathcal{C}$ from the rest of $\Gamma(\varphi_0)$.

Let us summarize this as a Lemma.

LEMMA 8.1. *Let Φ_0 be C^2 in a neighborhood of $\bar{D}_0$, and let $\mathcal{C} = (\gamma_1, \dots, \gamma_t\}$ be a chain in $\Gamma(\varphi_0)$. Then for any neighborhood $\mathcal{U}$ of the closure of $\hat{\mathcal{C}}$, there are families of complex disks $\Sigma^* = \{\sigma_1^*, \dots, \sigma_r^*\}$ and $\Sigma^{**} = \{\sigma_1^{**}, \dots, \sigma_s^{**}\}$ which are just below and above $\mathcal{C}$, respectively, and which have the following properties*:

(i) $\tilde{\Sigma}^* \cup \tilde{\Sigma}^{**} \subset \mathcal{U}$;

(ii) *the connected component Γ_Σ of $\Gamma(\varphi_0) \setminus (\Sigma^* \cup \Sigma^{**})$ containing* $\mathcal{C}$ *also lies in* $\mathcal{U}$;

(iii) *every hyperbolic point of Γ_Σ lies in* $\mathcal{C}$.

The case of one hyperbolic point.

Let us assume that $\Gamma(\varphi_0)$ is a 2-sphere with one hyperbolic point, as in Figure 1. We will prove that Theorem 1 holds for small perturbations of $\Gamma(\varphi_0)$. Let $\mathcal{C} = \{\gamma_2, \gamma_3\}$ be the unique chain in $\Gamma(\varphi_0)$, given by Lemmas 7.1 and 7.2. By Lemma 7.5 there is a neighborhood $\mathcal{U}$ of the closure of $\tilde{\mathcal{C}}$ and strongly pseudoconvex barrier functions $\Psi^{\pm}$ on $\mathcal{U}$ for small perturbations of $\Gamma(\varphi_0)$.

Now we choose complex disks Σ^* and Σ^{**} as in Lemma 8.1, and open sets $\mathcal{U}'$, $\mathcal{U}''$ with $\mathcal{C} \subset \mathcal{U}'' \subset\subset \mathcal{U}' \subset\subset \Gamma_\Sigma \subset \mathcal{U}$. Let us also choose a partition of unity $\{\chi_1, \chi_2\}$ for a neighborhood of ∂D_0 such that $\{\chi_1 = 1\} \supset \mathcal{U}'' \supset \mathcal{C}$ and supp $\chi_1 \subset \mathcal{U}'$. Every perturbation φ of φ_0 may be written as $\varphi = \varphi_0 + \varphi_1$. If φ_1 is small in C^2, then so is $\chi_j\varphi_1$, $j = 1, 2$. In particular, when φ_1 is small we still have barriers for $\Gamma(\varphi_0 + \chi_1\varphi_1)$, and the hyperbolic point $H(\varphi_0 + \chi_1\varphi_1)$ stays within $\mathcal{U}''$.

Without loss of generality, we may assume that $\Sigma^* = \{\sigma_2^*, \sigma_3^*\}$ contains two disks. Following the construction of 1-parameter families of disks given in [4], we may construct families $\{\sigma_2(t)\}$ and $\{\sigma_3(t)\}$ in $\Gamma(\varphi_0 + \chi_1\varphi_1)$, starting with σ_2^* and σ_3^*. These may proceed « upward » until they reach complex disks $\bar{\sigma}_2$ and $\bar{\sigma}_3$, whose closure contain the hyperbolic point. It follows from the regularity theorems of [4] that these disks may be given as in (24). For topological reasons, $\bar{\sigma}_2$ and $\bar{\sigma}_3$ cannot approach $H(\varphi_0 + \chi_1\varphi_1)$ through the same angle. Thus by Section 6, $\mathcal{C}_1 = \{\bar{\sigma}_2, \bar{\sigma}_3\}$ forms a regular chain.

Let $\hat{\mathcal{U}}$ be a neighborhood of the closure of $\tilde{\mathcal{C}}_1$ such that $\Gamma(\varphi_0 + \chi_1\varphi_1) \cap \cap \hat{\mathcal{U}} \subset \mathcal{U}''$. By Lemma 7.6, with a shrinking of $\hat{\mathcal{U}}$ if necessary, there is a small perturbation $\Gamma(\hat{\varphi})$ of $\Gamma(\varphi_0 + \chi_1\varphi_1)$ and a holomorphic function w^* on $\hat{\mathcal{U}}$ such that

$$\hat{\mathcal{U}} \cap \Gamma(\hat{\varphi}) \subset \{\operatorname{Im} w^* = 0\} .$$

We consider

$$\Sigma(t) = \{(z, w) \colon w^* = t,\ (z, u) \in \hat{D}\} .$$

Then $\partial\Sigma(0)$ is a chain containing the hyperbolic point $H(\hat{\varphi})$, where we set $\partial\Sigma(t) = \overline{\Sigma(t)} \setminus \Sigma(t)$.

For $|t| < t_0$ small, $\partial\Sigma(t) \subset \Gamma(\hat{\varphi})$. Further, the connected components of $\Sigma(t)$ are complex disks. Thus, after possibly shrinking t_0 and interchanging $\pm t_0$, we may take $\Sigma^* = \Sigma(-t_0)$ and $\Sigma^{**} = \Sigma(t_0)$ to have the properties (i), (ii), and (iii) in Lemma 8.1.

At this stage, Γ_Σ is holomorphically flat, and is spanned by a 1-parameter family of complex disks (i.e. $\Sigma(t)$, $-t_0 < t < t_0$). It remains to fill the rest of $\Gamma(\varphi)$.

Since supp $\chi_2 \cap \hat{\mathcal{U}}'' = \emptyset$, Σ^* and Σ^{**} also have properties (i), (ii), and (iii) for the surface $\Gamma(\hat{\varphi} + \chi_2\varphi_1)$. If φ_1 is small in C^2, then so is $\chi_2\varphi_1$, and thus the only complex tangencies of $\Gamma(\hat{\varphi} + \chi_2\varphi_1)$ over supp χ_2 will be small perturbations E'_j of the elliptic points E_j, $j = 1, 2, 3$. It follows again from the method of [4], that we may build 1-parameter families of disks « up » from the disks of Σ^{**} and « down » from the disks of Σ^* until they exit through the perturbed elliptic points E'_1, E'_2 and E'_3, as in Figure 1.

In this way we construct a smooth family of disks whose boundaries fill $\Gamma(\hat{\varphi} + \chi_2\varphi_1)$. We note that $\hat{\varphi} + \chi_2\varphi_1$ is an arbitrarily small C^2 perturbation of φ and that the surface obtained, $\tilde{\Gamma}(\hat{\Phi})$, satisfies conditions (i)-(iv) of Theorem 1.

In order to make the preceding argument work in the case of several hyperbolic points, we need to know that we may bump $\Gamma(\varphi_0)$ slightly to make the chains smaller.

LEMMA 8.2. *Let $\mathcal{C}$ be a chain of $\Gamma(\varphi_0)$, and let $H \in \mathcal{C}$ be a hyperbolic point. Then for each $\varepsilon > 0$, we may make an arbitrarily small C^2 perturbation Γ' to have the properties:*

(i) $\Gamma(\varphi_0) \cap \{|(z, w) - H| > \varepsilon\} = \Gamma' \cap \{|(z, w) - H| > \varepsilon\}$;

(ii) *there is a chain $\mathcal{C}' \subset \Gamma'$ such that*

$$\mathcal{C}' \cap \{|(z, w) - H| > \varepsilon\} = \mathcal{C} \cap \{|(z, w) - H| > \varepsilon\};$$

(iii) *$\mathcal{C}'$ contains no hyperbolic point in $\{|(z, w) - H| < \varepsilon\}$.*

PROOF. The perturbation only involves a small neighborhood of H, so we may use local coordinates as in (15), so that $\Gamma(\varphi_0)$ has the form

$$w = z\bar{z} + \lambda \operatorname{Re} z^2 + \sigma(z)$$

where $\sigma = o(|z|^2)$.

Now we let $\mathcal{C}$ be given near $H = (0, 0)$ as $\{w = g(z) : z \in \Omega\}$, where Ω is the asymptotic approach region one half of which is pictured in Figure 4. By Lemmas 6.3 and 6.4 we have $g \in \mathcal{O}(\Omega) \cap C^{2+\delta}(\bar{\Omega})$, and $|g(z)| = O(|z|^{2+\delta})$.

Let $\chi \in C_0^\infty(\boldsymbol{C})^+$ be such that $\chi = 1$ on $\{|z| < 1\}$ and $\chi = 0$ on $\{|z| > 2\}$. We consider the new 2-manifold, Γ', given near $z = 0$ as

$$w = z\bar{z} + \lambda \operatorname{Re} z^2 + \varepsilon^3 \chi\left(\frac{3z}{\varepsilon}\right) + i\chi\left(\frac{z}{\varepsilon}\right) \operatorname{Im} g + \left(1 - \chi\left(\frac{z}{\varepsilon}\right)\right)\sigma(z) .$$

For $\varepsilon > 0$ small, Γ' is a small C^2 perturbation of $\Gamma(\varphi_0)$, and (i) is satisfied (with a different ε).

Next we show the existence of the chain $\mathcal{C}'$. We note that for $\eta > 0$

$$\gamma_0 = \left\{z \in \overline{A\left(\frac{\pi}{2} - 3\mu, \frac{\pi}{2} + 3\mu\right)} : |z| < \eta, z\bar{z} + \lambda \operatorname{Re} z^2 + \varepsilon^3 \chi\left(\frac{3z}{\varepsilon}\right) = 0\right\}$$

is relatively compact inside $A(\pi/2 - 3\mu, \pi/2 + 3\mu)$. Similarly, the set

$$\Omega' = \left\{z \in A\left(\frac{\pi}{2} - 3\mu, \frac{\pi}{2} + 3\mu\right) : |z| < \eta, \operatorname{Re} g(z) > z\bar{z} + \lambda \operatorname{Re} z^2 + \right.$$
$$\left. + \varepsilon^3 \chi\left(\frac{3z}{\varepsilon}\right) + \operatorname{Re} \sigma(z)\right\}$$

is a set whose boundary is a small perturbation of γ_0 if $\varepsilon > 0$ is small. (The $\eta > 0$ now comes from Section 5.)

Now we see that the chain $\mathcal{C}'$ is given by the chain $\mathcal{C}$ away from $\{|z| < \eta\}$, and over $\{|z| < \eta\}$, $\mathcal{C}'$ is given as

$$\mathcal{C}' = \{w = \tilde{g}(z) : z \in \partial\Omega', |z| < \eta\},$$

where $\tilde{g}$ is the extension of g given by reflection.

Finally, we observe that the point $(0, \varepsilon^3)$ is a hyperbolic point of the surface Γ'. Since $(0, \varepsilon^3) \notin \mathcal{C}'$, and since no new hyperbolic points are introduced under a small perturbation of $\Gamma(\varphi_0)$ we see that (iii) holds.

The case of several hyperbolic points.

Now we show that Theorem 1 holds for small C^2 perturbations of φ and ∂D in the case of an arbitrary number of hyperbolic points. We will show here that chains of arbitrary length may be perturbed by Lemma 8.2 and then flattened as in Lemma 7.6.

Once this is done, we may take families of disks Σ^* and Σ^{**} (satisfying (i), (ii), and (iii) of Lemma 8.1) about each flattened chain $\mathcal{C}$. Following [4], we may build 1-parameter families of complex disks $\{\sigma_j^*(t)\}$ and $\{\sigma_j^{**}(t)\}$, starting at σ_j^* and σ_j^{**}. By [4] the disks constructed in this manner fill out an open and closed subset of $\Gamma(\varphi)$. Thus we fill $\Gamma(\varphi)$ with a smooth family of disks satisfying the conclusions of Theorem 1.

To show that chains may be flattened, we proceed by induction. We have already shown that a chain with two disks and one hyperbolic point can be suitably flattened. Let us suppose that we have completed our argument for chains of length $\leq p$, and let $\mathcal{C} \subset \Gamma(\varphi_0)$ be a chain containing $p + 1$ disks.

Let Σ^* and Σ^{**} be the sets of disks given as in Lemma 8.1, and let $H \in \mathcal{C}$ be a hyperbolic point. Let Γ' be the small perturbation of $\Gamma(\varphi_0)$

near H given by Lemma 8.2; let $\mathfrak{C}'$ be the resulting chain with p hyperbolic points, and let H' be the new hyperbolic point.

By Lemma 7.5, there are barrier functions defined in a neighborhood of the closure of $\tilde{\mathfrak{C}}$, and thus the closure of $\tilde{\mathfrak{C}}'$. The induction hypothesis allows us to flatten a small neighborhood of $\tilde{\mathfrak{C}}'$, after a preliminary, small C^2 perturbation in the interior of Γ_Σ. Now let Σ_1^*, $\Sigma_1^{**} \subset \Gamma_\Sigma$ be disks above and below $\mathfrak{C}'$ as in Lemma 8.1. Without loss of generality, we may assume that H' lies between Σ_1^* and Σ^*.

Now we start with the disks of Σ_1^* and build 1-parameter families of disks with boundaries in Γ' and moving down. One of these families must have a limit disk $\bar{\sigma}_1$ which contains H' in its closure. Similarly, we build 1-parameter families starting at the disks of Σ^* and proceeding upward. And likewise, there must be a limit disk $\bar{\sigma}$ touching H' from below. However, since none of these families of disks can intersect, one of these limiting disks, say $\bar{\sigma} = \sigma(t_0)$, must approach the asymptotic approach region from the « outside ». Let us suppose that $\bar{\sigma}$ fills the asymptotic approach region containing the negative y-axis (not pictured in Figure 4). Then the disks $\sigma(t)$ approaching $\bar{\sigma}$ from the outside must project to the upper approach region, too (cf. Figure 3). Thus the limiting set of $\sigma(t)$ as $t \to t_0$ contains a pair of disks $\bar{\sigma}'$ and $\bar{\sigma}''$, each corresponding to an asymptotic approach region.

These disks are trapped within appropriate barriers, so by Section 6, they fit together in a C^2 manner. Since H' is the only hyperbolic point in the region between Σ^* and Σ_1^*, $\mathfrak{C}_1 = \{\bar{\sigma}', \bar{\sigma}''\}$ is a regular chain with one hyperbolic point. Now by our previous arguments, it can be flattened. This completes the induction step.

Conclusion of the Proof.

As was noted in the Introduction, Theorem 1 is more easily proved in the absence of hyperbolic points. For instance, if $\Gamma(\varphi_0)$ is a totally real 2-torus, then we may use Theorem 2.1 to obtain the solution of [4]. By the Bishop Index Theorem in [7], the only other case without hyperbolic points is when $\Gamma(\varphi_0)$ is a 2-sphere with 2-elliptic points, in which case we use Theorem 1 of [4].

In case there are one or more hyperbolic points, we have shown that there exist φ_j and D_j for which Theorem 1 holds and such that φ_j, D_j converge to φ, D in C^2. It follows by Theorem 4.5 that Φ_j converges uniformly to a solution Φ of (4) satisfying (19), and $\Phi \in \mathrm{Lip}^1(\bar{D})$. The regularity statement (iv) follows from Theorem 5.1 of [4]. The uniqueness part of (ii) and (iii) comes from Lemma 4.6. The existence of a complex disk $f\colon \Delta \to \tilde{\Gamma}(\Phi)$ in (ii) follows from the arguments of [4], since we may consider the corresponding disk $f_j\colon \Delta \to \tilde{\Gamma}(\Phi_j)$ with $f_j(0) = (z_0, w_0)$, $\pi_z f_j'(0) > 0$, and use the a priori estimate of [4] to conclude that the sequence f_j converges uniformly on $\bar{\Delta}$.

9. – Polynomial hull.

In this Section we give a proof of Theorem 2. The main step is the following.

LEMMA 9.1. *Let $D \subset\subset \boldsymbol{C}\times\boldsymbol{R}$ be smoothly bounded, and let $D\times i\boldsymbol{R}$ be strongly pseudoconvex. If ∂D is a 2-sphere, then $\bar{D}$ is polynomially convex.*

PROOF. We will consider a sequence of domains $D_1 \supset D_2 \supset \dots$ with $\cap \bar{D}_j = \bar{D}$, and it will suffice to show that $\bar{D}_j$ is polynomially convex. We obtain D_j by a small perturbation of D, so we may assume that ∂D is a 2-sphere, and all the complex tangencies of ∂D are either elliptic or hyperbolic. Further, if for $c \in \boldsymbol{R}$ we denote the slices by

$$\bar{D}(c) = \bar{D} \cap \{u = c\}\,,$$

we may assume that there is only one complex tangency of ∂D on each $\bar{D}(c)$.

First, it is clear that the polynomial hull of $\bar{D}$ lies inside $\{v = 0\}$, for we may consider a sequence of polynomials approximating $\exp(\pm iw)$.

Next, if we show that $\bar{D}(c)$ is a simply connected subset of $\boldsymbol{C}$, then $\bar{D}$ is polynomially convex. In this case, for each $z_0 \in \boldsymbol{C}\setminus\bar{D}$, we may choose a polynomial $p(z)$ such that $|p(z_0)| > \sup\limits_{\bar{D}(c)} |p|$. Now by considering $p(z)\cdot \exp(-k(w-c)^2)$ for large k, we see that (z_0, c) is not in the polynomial hull of $\bar{D}$.

Let us review the possibilities for $\bar{D}(c)$. The intersection of ∂D and $\{w = c\}$ is transverse unless c is one of the finite number of critical values, i.e. $\bar{D}(c)$ contains a complex tangency. At a complex tangency $(z_0, c) \in \partial D$ we may write

$$u = c + a(z-z_0)(\bar{z}-\bar{z}_0) + \operatorname{Re} b(z-z_0)^2 + o(|t-z_0|^2)\,.$$

Since (z_0, c) is either elliptic or hyperbolic, we have $|a| \neq |b|$. In other words, the coordinate function u is a Morse function on ∂D.

If c is a regular value for u, then $\partial\bar{D}(c)$ consists of a finite number of smooth curves, and $\bar{D}(c_1)$ is diffeomorphic to $\bar{D}(c_2)$ if $[c_1, c_2]$ consists of regular values (see [15]). If $p \in \bar{D}(c_0)$ is an elliptic point, and if p is isolated, then for $c = c_0 + \varepsilon$ or $c = c_0 - \varepsilon$, $\bar{D}(c)$ contains a small disk which is shrinking away to p. A hyperbolic point $q \in \bar{D}(c_0)$ is a place where two curves of $\partial D(c_0)$ cross (analogous to γ_2 and γ_3 in Figure 1).

Now let us suppose that $\bar{D}(c^*)$ is not simply connected for some fixed value c^*. Let $\omega_1(c^*)$ denote a bounded component of $\boldsymbol{C}\setminus\bar{D}(c^*)$. This choice of $\omega_1(c^*)$ may be used to define a connected component $\omega(c)$ of $\boldsymbol{C}\setminus\bar{D}(c)$ in the following manner. If (c', c'') is an open interval of regular points containing c^*, we define $\omega_1(c)$ by the natural diffeomorphism between $\bar{D}(c^*)$ and $\bar{D}(c)$, which is induced by the Morse function u. If we have defined $\omega_1(c)$ for $c \in (c', c'')$ and c'' is a critical value, we will define $\omega_1(c'')$ by continuity.

First we note that by the pseudoconvexity of $D \times i\boldsymbol{R}$, $\omega_1(c)$ cannot vanish to a point as $c \nearrow c''$, for this could occur only at a pseudoconcave point. Let us suppose that $p'' = (z'', c'') \in \bar{D}(c'')$ is critical. If p'' is elliptic, it cannot be isolated, so either a component of $\boldsymbol{C}\setminus\overline{D(c)}$ is vanishing to $\{z''\}$ as $c \nearrow c''$ or a small «hole» in $\boldsymbol{C}\setminus\overline{D(c'')}$ is being created for $c = c'' + \varepsilon$. But both possibilities contradict the pseudoconvexity of $D \times i\boldsymbol{R}$.

Thus p'' is a hyperbolic point. Now it is possible that p'' «splits» $\omega_1(c'')$ as $c \nearrow c''$. (An example would be the curves γ in Figure 1 *decreasing* to γ_1.) In this case we arbitrarily choose one component $\omega_1(c'')$ and use that to define $\omega_1(c)$ for $c = c'' + \varepsilon$. The other possibility is that p'' serves to adjoin a new component, such as γ increasing to γ_2 in Figure 1. Again, the definition of $\omega_1(c)$ is clear.

Now we proceed in this manner for all $c \in \boldsymbol{R}$. For $|c|$ very large, we have $\omega_1(c) = \boldsymbol{C}$ since $\bar{D}$ is compact. It follows now that D is not simply connected, since we may define a continuous curve $t \to (z(t), t), -\infty < t < \infty$ where $z(t) \in \omega_1(t)$. Now we see that ∂D is not a 2-sphere, so by this contradiction we conclude that $D(c)$ is simply connected, which completes the proof.

PROOF OF THEOREM 2. Let ϱ be a strongly plurisubharmonic defining function for D_0, and let $D_\varepsilon = \{\varrho < \varepsilon\}$. Thus D_ε is strongly pseudoconvex for $0 \leq \varepsilon \leq \varepsilon_0$. Let $\tilde{\varphi}$ denote a C^2 extension of φ_0 to a neighborhood of ∂D_0: By Lemma 4.2, there is a constant K such that

$$\|\Phi_\varepsilon\|_{\mathrm{Lip}^1(\bar{D}_\varepsilon)} \leq K \tag{43}$$

where Φ_ε is the solution to (4) on the domain D_ε with boundary data $\tilde{\varphi}|_{\partial D_\varepsilon}$, $0 \leq \varepsilon \leq \varepsilon_0$.

It follows, then, that the domains

$$\Omega_\varepsilon = \{(z, w) : (z, u) \in D_\varepsilon,\ |v - \Phi_\varepsilon(z, u)| < 2K\varepsilon\}$$

are pseudoconvex if $0 \le \varepsilon \le \varepsilon_0$. Further, we claim that $\bar{\Omega}_{\varepsilon_1} \subset \Omega_{\varepsilon_2}$ if $0 \le \varepsilon_1 < < \varepsilon_2 \le \varepsilon_0$. This follows because by (43), we see that

$$\Phi_{\varepsilon_2} + 2K\varepsilon > \varphi_{\varepsilon_1}$$

on $\partial D_{\varepsilon_1}$. Thus by (18), we have the same inequality on all of D_{ε_1}.

We may construct another increasing family of pseudoconvex domains by setting

$$\Omega'_t = \{(z, w): (z, u) \in D_{\varepsilon_0}, \max(-\|\bar{\Phi}\|_{L^\infty}, \bar{\Phi} - t) < v < \min(\bar{\Phi} + t, \|\bar{\Phi}\|_{L^\infty})\},$$

where $\bar{\Phi} = \Phi_{\varepsilon_0}$. The family $\{\Omega'_t\}$ is an increasing 1-parameter family of domains of holomorphy, and $\Omega'_0 = \Omega_{\varepsilon_0}$.

By Lemma 8.1, it follows that $\bar{D}_\varepsilon \times [-ic, ic]$ is polynomially convex. Thus it follows from the theorem of Docquier and Grauert [9] that $\tilde{\Gamma}(\Phi) = = \cap \Omega_\varepsilon$ is holomorphically convex in $D_\varepsilon \times (-ic, ic)$, and thus it is polynomially convex.

We note that $\tilde{\Gamma}(\Phi)$ satisfies property (19), and so by (20) the polynomial hull of $\Gamma(\varphi)$ contains $\tilde{\Gamma}(\Phi)$, which completes the proof.

Appendix: Holomorphic flattening.

The local construction of the solution of (4) may be shown to be related to the problem of « local flattening » of $\Gamma(\varphi)$. We will use this to show that the question of regularity of the solution of (4) is delicate. If $S \subset \mathbf{C}^2$ is a real analytic Levi-flat hypersurface, then it is a classical result that there is locally a holomorphic change of coordinates (z^*, w^*) such that (locally) $S = \{\operatorname{Im} w^* = 0\}$. In particular, if the solution of (4) is real analytic in a neighborhood of $\bar{D}$, then $\tilde{\Gamma}(\Phi)$ can be flattened over the points ∂D. Thus, locally, we also have $\Gamma(\varphi) \subset \{\operatorname{Im} w^*\}$.

EXAMPLE. *There are real analytic ∂D and φ such that there is no local flattening of $\Gamma(\varphi)$ at a hyperbolic point.*

Let us write ∂D near $(0, 0)$ as

$$\partial D = \{u = z\bar{z} + z^2 + \bar{z}^2 + O(|z|^4)\}$$

and let us set

$$\varphi(z) = 2 \operatorname{Im} z\bar{z}^2 + O(|z|^4).$$

Clearly we can find global D and φ to arrange this. Let us suppose that there is polynomial change of coordinates in a neighborhood of $(0, 0)$ which flattens the surface $S = \Gamma(\varphi)$ to 3rd order. That is, there is a nonsingular holomorphic change of coordinates (z^*, w^*) such that

$$S \subset \left\{\operatorname{Im} w^* + o\big(|z^*|^3 + |\operatorname{Re} w^*|^3\big) = 0\right\}.$$

Any possible coordinate change must have the form

$$w^* = aw + \dots$$

$$z^* = bz + cw + \dots$$

where $a \in \boldsymbol{R}$, and $ab \neq 0$. Without loss of generality we may assume that $a = b = 1$. Thus in the new coordinates $\Gamma(\varphi)$ becomes

$$w^* = z^* \bar{z}^* + 2 \operatorname{Re} (z^*)^2 + O_3\big(|z^*|\big)$$

since $w = O\big(|z|^2\big)$. Thus we may assign « weight one » to z^* and « weight two » to w^*. The only terms in w^* which will influence $\operatorname{Im} z^2 \bar{z}$ will be of weight 3, so we may consider coordinates

$$w = w^* - \alpha_1 z^* w^* - \alpha_3 (z^*)^3,$$

leaving $z^* = z^*(z, w)$ unspecified. Thus we see that for $(z^*, w^*) \in \Gamma(\varphi)$, we have

$$w^* - \alpha_1 z^* w^* - \alpha_3 (z^*)^3 = \text{real terms} + 2i \operatorname{Im} (z^*)^2 \bar{z}^*.$$

Now to have $\operatorname{Im} w^* = 0$ on $\Gamma(\varphi)$, we must choose α_1 and α_3 such that

$$\alpha_1 z^* \big(z^* \bar{z}^* + 2 \operatorname{Re} (z^*)^2\big) + \alpha_3 (z^*)^3 + (z^*)^2 \bar{z}^* - (\bar{z}^*)^2 z^*$$

is real. Comparing the coefficients of $(z^*)^2 \bar{z}^*$ and $z^* (\bar{z}^*)^2$ we must have

$$\alpha_1 + 1 = \overline{(\alpha_1 - 1)},$$

which is clearly impossible. Thus $\Gamma(\varphi)$ cannot be flattened to third order.

Now we use this Example to show that:

> *there exist real analytic ∂D and φ such that the solution Φ of* (4) *is not smooth of class $C^3(\bar{D})$.*

This follows from the Example and the Proposition below. For if Φ is C^3 at $(0, 0)$, then $\tilde{\Gamma}(\Phi)$ can be flattened at $(z, u) = (0, 0)$ to order 3. Thus $\Gamma(\varphi)$ can also be flattened, which is a contradiction.

Since the holomorphic flattening of S is a bit subtle when S is not C^∞ (cf. [3]), we will indicate the proof of the following.

PROPOSITION. *Let Φ, ∂D, and φ be as above. If $\Phi \in C^k(\bar{D})$, then there is a holomorphic change of coordinates (z^*, w^*) at $(0, 0)$ such that*

$$\operatorname{Im} w^* = o\big(|\operatorname{Re} w^*| + |z^*|\big)^k$$

holds for $(z^*, w^*) \in \tilde{\Gamma}(\Phi)$.

PROOF. We may assume that $d\Phi/dz = 0$ at $(0, 0)$. Thus we may make a change of coordinates of the form $w^* = w + a_2 w^2 + \ldots$, i.e. transforming the w-axis within itself to obtain $\Phi(u, 0) = o(u^k)$, $u \geq 0$. Since the complex manifolds in $\tilde{\Gamma}(\Phi)$ are given by $\partial(v - \Phi)$, they form a C^k-foliation of $\tilde{\Gamma}(\Phi)$. Working modulo $o\big(|z|^k + u^k\big)$, we see that the leaf passing through the point $(z_0, w_0) = \big(0, t + i\, o(t^k)\big)$ is given by

$$w = t + a_1(t)\, z + a_2(t)\, z^2 + \ldots + a_k(t)\, z^k$$

where $a_j(t)$ is a polynomial of degree $k - j$. Our coordinate change is now:

$$w^*(z, w) = w - a_1(w) z - \ldots - a_k(w) z^k\,, \qquad z^* = z\,,$$

which completes the proof.

Now we may consider the regularity of the curves γ at H. We suppose that the hyperbolic point has the form

$$w = z\bar{z} + \lambda \operatorname{Re} z^2 + \psi(z) \tag{44}$$

where $\psi(z)$ is real, real analytic and vanishing to order 3 at 0. By (15′) and Section 6 the angle of opening of γ at H is

$$\alpha\pi = \theta_2 - \theta_1\,.$$

EXAMPLE. *If the hyperbolic point H has the form* (44) *and if $1 < \lambda < \infty$ is chosen so that α is irrational, then γ is not piecewise real analytic at H unless it is trivial* (i.e. *the curve γ lies in the z-axis and is given by setting $w = 0$ in* (44).)

As in Sections 5 and 6, we let Ω denote the z-projection of the hyperbolic disk near H. If α is irrational and if γ is piecewise analytic, then by Lehman [14], the Riemann mapping function $f\colon U \to \Omega$ has the asymptotic expansion

$$f(\zeta) \sim A\zeta^{\alpha} + \sum_{\substack{+\alpha k > \alpha \\ j,k \geq 0}} A_{jk}\zeta^{j+\alpha k}.$$

On the other hand, γ is the boundary of a complex analytic disk, and so it follows that $w(f(\zeta))$ is the boundary value of an analytic function in ζ. Since by (44) w is real, we apply the Schwarz reflection principle to obtain

$$w(f(\zeta)) = C_1\zeta + C_2\zeta^2 + \dots . \tag{45}$$

If γ is not trivial then there must be a nonzero C_m in (45). Now we note that we cannot have $A_{j0} = 0$ for $j = 1, 2, \dots$. This is because α is irrational, and in this case all the nonzero terms of $w(f(\zeta))$ would have irrational exponents. But since there is a nonzero C_M in (45), we must be able to write $f(\zeta)$ in the form

$$f(\zeta) \sim A\zeta^{\alpha} + \sum_{j \geqslant m} B_j\zeta^j + \zeta^{\alpha} \sum_{\substack{j,k \geqslant 0 \\ j+k \geqslant 1}} C_{jk}\zeta^{j+\alpha k}$$

where A and B_m are both nonzero. Using this expansion, we see that the term of order $m + \alpha$ in $w(f(\zeta))$ is given by

$$2\,\mathrm{Re}\,[A\zeta^{\alpha}\bar{B}_m\bar{\zeta}^m + \lambda A\zeta^{\alpha}B_m\zeta^m]\,.$$

Since this must vanish by (45) and since ζ^m is real for $\zeta \in R$, we have

$$\mathrm{Re}\, A(\bar{B}_m + \lambda B_m)\zeta^{\alpha} = 0\,.$$

But if this holds for $\zeta \in \boldsymbol{R}^+$ and $\zeta \in \boldsymbol{R}^-$, then we conclude that

$$\bar{B}_m + \lambda B_m = 0\,.$$

Thus $B_m = 0$. From this contradiction we conclude that γ is trivial.

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