

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

V. I. OLIKER

The boundary value Minkowski problem. The parametric case

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 9, n° 3
(1982), p. 463-490

<http://www.numdam.org/item?id=ASNSP_1982_4_9_3_463_0>

© Scuola Normale Superiore, Pisa, 1982, tous droits réservés.

L'accès aux archives de la revue « *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* » (<http://www.sns.it/it/edizioni/riviste/annaliscienze/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

The Boundary Value Minkowski Problem. The Parametric Case (*).

V. I. OLIKER

0. – Introduction.

0.1. The celebrated Minkowski problem consists of finding a closed convex hypersurface in \mathbf{R}^{n+1} whose Gaussian curvature at the point with a unit normal ξ is prescribed in advance as a positive continuous function $K(\xi)$. The latter is given on a unit hypersphere Σ and should satisfy the necessary condition

$$(0.1) \quad \int_{\Sigma} \frac{\xi}{K(\xi)} d\mu = 0,$$

where $d\mu$ is the n -volume element of Σ .

Minkowski himself solved first the problem for convex polyhedral surfaces; in this case instead of the Gaussian curvature one prescribes the areas of the faces and the normals to them of the polyhedron which is to be found. Passing to a limit from polyhedrons to general convex surfaces one finds the generalized (weak) solution of the problem in terms of certain associated measures. The very difficult problem of existence of a regular solution under various assumptions on K has been solved for $n = 2$ by H. Lewy [7] in the analytic case, Pogorelov [12] and Nirenberg [9] in the class of smooth functions, and for arbitrary n by Pogorelov [14] (see also Cheng and Yau [4]).

(*) Partially supported by NSF Grant MCS80-02779. Part of research for this paper was done while the author was a guest of the Sonderforschungsbereich Theoretische Mathematik of the Universität Bonn in the summer of 1980. The final version was written when the author was visiting the Heidelberg University in July of 1981. The author would like especially to thank Prof. W. Jäger of Heidelberg University for his hospitality.

Pervenuto alla Redazione l'11 Novembre 1981 ed in forma definitiva il 5 Marzo 1982.

0.2. The original problem admits various generalizations. Gluck [6] studied the situation when the Gaussian curvature is prescribed on Σ but the hypersurface in question does not have to be an imbedding via the inverse of the Gauss map, as is the case in the classical formulation. A generalization more directly related to the original problem can be stated as follows.

Let ω be a domain on the unit hypersphere Σ and K a positive continuous function in ω . Find a convex hypersurface with boundary whose spherical image is ω and at the point with the unit normal ξ its Gaussian curvature is $K(\xi)$. It is natural to impose some boundary conditions. It turns out that prescribing the boundary of the hypersurface leads to an over-determined problem; this follows from our results in [11].

The analytic formulation of the problem suggests naturally to prescribe only the normal component of the position vector along the boundary of ω . In such a form (actually, slightly more restricted) the problem was proposed by Aleksandrov [1], p. 319, and also by Pogorelov [13], p. 657. When ω is convex and lies strictly inside a hemisphere, the problem has been investigated extensively [1, 2, 5, 12, 15]. However, very little is known in the case when ω is not contained in a hemisphere. Geometrically the latter case means that we are dealing with hypersurfaces in parametric form, and, since the closedness is not assumed, serious difficulties arise. Already the question of uniqueness is quite nontrivial. For example, the maximum principle, which is applicable in the case when ω is inside a hemisphere does not hold anymore for the corresponding linearized equation, and, in fact, the linearized problem may have geometrically nontrivial solutions [10, 11]. Apparently, for arbitrary given data the problem may not be solvable at all.

0.3. The purpose of this paper is an investigation of the solvability of the boundary value Minkowski problem in the following setting.

Fix a domain ω on the unit hypersphere Σ such that its complement $\bar{\Omega} = \Sigma \setminus \omega$ is a convex set lying strictly inside a hemisphere, and let $\varphi(\xi)$ be a positive continuous function in $\bar{\omega}$. We want to find a convex hypersurface S which is a relatively compact subset of some closed convex hypersurface \tilde{S} , its spherical image is $\bar{\omega}$, the Gaussian curvature at the point with the unit normal ξ is $1/\varphi(\xi)$, $\xi \in \bar{\omega}$, and the normal component h of the position vector of S on the boundary ∂S coincides with a linear function in \mathbf{R}^{n+1} . The last condition has a simple geometric meaning. Namely, if we consider the envelope of the set of hyperplanes tangent to S along its boundary, then the boundary condition on h means that this envelope has a vertex. In other words, the hypersurface S «sits» on a cone.

Since no assumptions on smoothness of φ or $\partial\omega$ are made so far, all the terms and conditions above should be understood in a generalized sense usually given to those concepts when dealing with general convex hypersurfaces. (A brief review of the necessary information is given in section 1 of this paper.)

Because of the fact that S must be a portion of a closed convex hypersurface, the function $\varphi(\xi)$ should satisfy some necessary condition similar to (0.1) but expressed in terms of ω . Analytically, such a condition can be given in the form

$$(0.2) \quad \int_{\omega} \xi \varphi(\xi) d\mu = -\lambda \bar{\xi},$$

where $\lambda > 0$ and $\bar{\xi} \in \Omega$, and geometrically it means that the n -volume of the projection of S (taken with the sign) on any hyperplane with the normal corresponding to the «missing» part of S is nonnegative.

It turns out that the requirement $\lambda > 0$ in (0.2) along with the other conditions stated before are sufficient to produce a generalized solution of the boundary value problem posed above. This solution is unique.

0.4. The proof consists essentially of two parts. At first, in section 2, the problem is formulated and solved in the class of convex polyhedrons, and then a generalized solution is obtained by a passage to a limit. This step is described in section 3. The reader familiar with the existence proof in the case of closed hypersurfaces will recognize here the method which goes back to Minkowski and which has been used successively by A. D. Aleksandrov, Pogorelov, and other mathematicians. However, its concrete realization for the boundary value problem is far from being a standard repetition of the arguments used in the case of closed hypersurfaces. In particular, the information that the problem in the latter case is solvable does not seem to be helpful, and the condition (0.2) is used in a much more subtle way than (0.1) in the classical situation.

0.5. In the last section of the paper we prove interior regularity of the generalized solution. The results here are based essentially on the a priori estimates for Monge-Ampère equations obtained by Pogorelov [14] and Calabi [3].

1. – Generalized solutions.

1.1. In the $(n + 1)$ -dimensional Euclidean space we fix a unit hypersphere Σ with the center \mathcal{O} which coincides with the origin of a system of

Cartesian coordinates. On Σ we introduce smooth local coordinates u^1, \dots, u^n . The standard metric on Σ will be denoted by (g_{ij}) , $i \leq i, j \leq n$; (g^{ij}) stands for the inverse matrix. Let S be a piece of a C^2 hypersurface in \mathbf{R}^{n+1} whose Gauss map γ is a diffeomorphism on a domain $\omega \subset \Sigma$. Obviously, S can be considered as an immersion $r: \omega \rightarrow \mathbf{R}^{n+1}$; moreover, we can identify the position vector r with γ^{-1} . Under such circumstances the correspondence between S and ω is established via correspondence between points with parallel unit normals. The expression for r , given as γ^{-1} , can be found in terms of the support function h of S , which is defined as

$$(1.1) \quad h(\xi) = \langle r(\xi), \xi \rangle,$$

where \langle, \rangle is the inner product in \mathbf{R}^{n+1} , and ξ is a point in ω . Here, and in the rest of the paper we make the obvious identification of the vector with its endpoint on Σ .

Since $r(\xi) = \gamma^{-1}(\xi)$, we find by differentiating $h(\xi)$

$$\frac{\partial h(\xi)}{\partial u^i} = \left\langle r(\xi), \frac{\partial \xi}{\partial u^i} \right\rangle, \quad i = 1, \dots, n.$$

The vectors $\partial \xi / \partial u^1, \dots, \partial \xi / \partial u^n, \xi$ form a basis in \mathbf{R}^{n+1} and we can express $r(\xi)$ in terms of h and $\text{grad } h$. The result is

$$(1.2) \quad r = \text{grad } h + h\xi,$$

where $\text{grad } h = g^{ij}(\partial h / \partial u^i)(\partial \xi / \partial u_j)$. (The summation convention is in effect here and in the sequel.)

The second fundamental form $\langle dr, d\xi \rangle = b_{ij} du^i du^j$ can also be expressed in terms of h . Namely, if we differentiate (1.2) we get

$$\frac{\partial^2 h}{\partial u^i \partial u^j} = \left\langle \frac{\partial r}{\partial u^j}, \frac{\partial \xi}{\partial u^i} \right\rangle + \left\langle r, \frac{\partial^2 \xi}{\partial u^i \partial u^j} \right\rangle.$$

By the Gauss formulas for Σ we have

$$\frac{\partial^2 \xi}{\partial u^i \partial u^j} = \Gamma_{ij}^k \frac{\partial \xi}{\partial u^k} - g_{ij} \xi,$$

where Γ_{ij}^k are the Christoffel symbols of the metric g_{ij} . Substituting it into the last expression and taking into account (1.1) and (1.2) we obtain

$$(1.3) \quad b_{ij} = \nabla_{ij} h + g_{ij} h,$$

where $\nabla_{ij} h = \partial^2 h / (\partial u^i \partial u^j) - \Gamma_{ij}^k (\partial h / \partial u^k)$.

The principal radii of curvature of S are the eigenvalues of the differential of γ (by the Rodrigues formula), and therefore are the roots R_1, \dots, R_n of the equation $\det(b_{ij} - Rg_{ij}) = 0$.

Since γ is a diffeomorphism, $R_i \neq 0, i = 1, \dots, n$, and we can consider the Gaussian curvature $K(\xi) = \left(\prod_{i=1}^n (R_i(\xi))^{-1}\right)$. In view of (1.3) we have

$$(1.4) \quad K^{-1}(\xi) = \frac{\det(\nabla_{ij}h(\xi) + g_{ij}h(\xi))}{\det(g_{ij})}.$$

Let $d\nu$ denote the n -dimensional volume element of S and $d\mu$ the n -volume element of Σ . Then the following relationship holds:

$$(1.5) \quad d\nu = K^{-1}(\xi) d\mu.$$

For an arbitrary Borel set G on Σ such that $G \subset \omega$ the set $\gamma^{-1}(G)$ is also a Borel set and its n -volume is given by

$$(1.6) \quad \nu(G) \equiv n\text{-volume of } \gamma^{-1}(G) = \int_{\gamma^{-1}(G)} d\nu = \int_G \frac{\det(\nabla_{ij}h(\xi) + g_{ij}h(\xi))}{\det(g_{ij})} d\mu = \int_G K^{-1}(\xi) d\mu.$$

1.2. Assume now that \tilde{S} is a closed convex hypersurface of class C^2 with positive Gaussian curvature. Let ω be a simply connected domain on Σ , different from Σ , and S a portion of \tilde{S} corresponding to $\bar{\omega} = \omega + \partial\omega$. The previous discussion shows that in order to solve the problem posed in section 0.3 of the introduction we have to consider the differential equation

$$(1.7) \quad \frac{\det(\nabla_{ij}h + g_{ij}h)}{\det(g_{ij})} = \varphi \quad \text{in } \omega,$$

subject to the boundary condition

$$h|_{\partial\omega} = \langle a, \xi \rangle,$$

where a is the position-vector of the vertex of the enveloping cone along ∂S . Making a parallel translation of \mathbf{R}^{n+1} we can assume that this vertex is located at the origin \mathcal{O} .

Note that $\langle a, \xi \rangle$ is simply a restriction to Σ of a linear function in \mathbf{R}^{n+1} . Also, a direct computation shows that $\nabla_{i,j} \xi + g_{i,j} \xi = 0$. This actually becomes obvious if we notice that $\langle a, \xi \rangle$ is the support function of the constant vector a . That also means that the equation (1.7) remains invariant under parallel translations of the origin. In order not to introduce new symbols, we leave the notation h for the support function of \tilde{S} with respect to the new position of the origin in \mathbf{R}^{n+1} , relative to which the boundary condition assumes the form

$$(1.8) \quad h|_{\partial\omega} = 0.$$

In view of (1.6) and (1.7) we have

$$(1.9) \quad \nu(G) = \int_G \varphi \, d\mu,$$

for any Borel subset of Σ contained in ω . The last formula, however, makes sense if we assume only that S is a portion of a closed convex hypersurface not necessarily of class C^2 and φ is a positive continuous function in $\bar{\omega}$ (and, obviously, even under more general assumptions).

The appropriate generalization is made in the following way. Let \tilde{S} be a closed convex hypersurface which is understood to be the boundary of an arbitrary finite convex body in \mathbf{R}^{n+1} . Then the convex hypersurface S is a simply connected relatively compact subset of \tilde{S} . If x is an interior point of S then for any supporting hyperplane at x with the exterior unit normal ξ we put in correspondence a point on Σ which is the endpoint of a unit vector parallel to ξ and starting at the center \mathcal{O} of Σ . The mapping defined in such a way is obviously a generalization of the standard Gauss map, and, in general, it is a set valued function. We map all interior points of S into Σ , and then take the relative closure of the image; the latter is called the spherical image of S .

Let G be a subset of Σ . Define on \tilde{S} a set G' as the set of points whose spherical images are in G . If G is a Borel subset of Σ then G' is also a Borel set on \tilde{S} ; we denote by $\nu(G)$ the n -volume of the set G' . The function ν is called the *area function*. It is a nonnegative completely additive measure on the σ -algebra of Borel subsets of Σ . Since \tilde{S} is finite, ν is also finite. If the Radon-Nikodym derivative of ν with respect to the measure μ on Σ exists at a point ξ and it is different from zero then its reciprocal is called the Gaussian curvature. When \tilde{S} is smooth then in view of (1.5) this derivative is equal to $K^{-1}(\xi)$ (see [2], section 8).

For a convex hypersurface $S \subset \tilde{S}$, where \tilde{S} is not assumed to be smooth, by its area function we mean the area function of the closed hypersurface \tilde{S} .

A generalized solution of (1.7) in $\omega \subset \Sigma$ is now defined as a convex hypersurface S contained in some closed convex hypersurface \tilde{S} and such that its spherical image is $\bar{\omega}$, and for any Borel subset $G \subset \Sigma$, contained in ω ,

$$v(G) = \int_G \varphi \, d\mu .$$

The support function $h(\xi)$, $\xi \in \bar{\omega}$, of S is a function of a unit vector ξ giving the oriented distance from the origin \mathcal{O} to a supporting hyperplane to \tilde{S} with the exterior unit normal ξ . It is known (see [2], section 6) that $h(\xi)$ is a continuous function on the entire \tilde{S} , and, in particular, $h(\xi)$ is defined on $\partial\omega$. Thus, (1.8) is also defined.

2. - Solution of the problem in the class of polyhedrons.

2.1. Let M be a closed convex polyhedron in \mathbf{R}^{n+1} . Take an arbitrary « true » vertex (i.e., different from an interior point of any $n - k$ dimensional face, $0 \leq k \leq n - 1$) of M and suppose a parallel translation is made so that this vertex coincides with origin \mathcal{O} of a Cartesian coordinate system in \mathbf{R}^{n+1} . We distinguish two types of faces and exterior unit normals of M . Namely, denote by g_1, \dots, g_m the faces for which $\mathcal{O} \cap g_i = \emptyset$, $i = 1, \dots, m$. Their corresponding exterior unit normals we denote by ξ_1, \dots, ξ_m . Here and in the sequel, unless otherwise stated, we understand n -dimensional faces. By q_1, \dots, q_s we denote the faces for which \mathcal{O} is a boundary point; η_1, \dots, η_s are their corresponding exterior unit normals. Let $\alpha_1, \dots, \alpha_s$ be the hyperplanes containing faces q_1, \dots, q_s . The intersection of the half-spaces determined by $\alpha_1, \dots, \alpha_s$, and containing M , is an infinite convex body whose boundary is an infinite convex polyhedron. Let h_1, \dots, h_m be the support numbers of the faces g_1, \dots, g_m , that is, the oriented distances from \mathcal{O} to the hyperplanes containing the corresponding faces. Obviously, the support numbers of q_1, \dots, q_s are equal to zero.

Now consider a convex polyhedron M' which also has a true vertex at \mathcal{O} . Suppose it has the same number of faces as M and (1) g_i and g'_i are parallel for all $i = 1, \dots, m$; the latter means that they have the same exterior unit normals; (2) the hyperplanes $\alpha'_1, \dots, \alpha'_s$ containing the faces q'_1, \dots, q'_s coincide with $\alpha_1, \dots, \alpha_s$, and for every q'_j , $j = 1, \dots, s$, \mathcal{O} is a boundary point. Geometrically, (2) means that the infinite polyhedral cones formed by the hyperplanes of the faces of M and M' adjacent to \mathcal{O} are the same.

2.2. PROPOSITION. *Let M and M' be as above. Assume, in addition, that the n -volumes F_i and F'_i of g_i and g'_i , $i = 1, \dots, m$, are correspondingly equal. Then M coincides with M' .*

PROOF. We will use a method of Minkowski based on his theory of mixed volumes. Details concerning mixed volumes theory can be found in [2, section 6].

The mixed volume of the polyhedrons M and M' is defined as

$$V(M, M') = \frac{1}{n} \left\{ \sum_{i=1}^m h_i F'_i + \sum_{j=1}^s h_j F'_j \right\},$$

where h_j are the support numbers of the faces q_j , $j = 1, \dots, s$, and F'_j are the n -volumes of q'_j . Since q_j pass through \mathcal{O} , $h_j = 0$, $j = 1, \dots, s$, and the second sum in $V(M, M')$ vanishes. Replacing in $V(M, M')$ the polyhedron M' by M we get the usual $(n+1)$ -volume of the convex body bounded by M . Similarly, $V(M', M') = (n+1)$ -volume of the convex body bounded by M' .

According to the Minkowski inequality (see [2], section 7),

$$V^n(M, M') \geq V(M, M) V^{n-1}(M', M'),$$

and the equality holds if and only if M and M' are homothetic. Thus,

$$\left(\sum_{i=1}^m h_i F'_i \right)^n \geq \left(\sum_{i=1}^m h_i F_i \right) \left(\sum_{i=1}^m h'_i F'_i \right)^{n-1}.$$

Since $F'_i = F_i$, $i = 1, \dots, m$, we have

$$\sum_{i=1}^m h_i F_i \geq \sum_{i=1}^m h'_i F'_i.$$

Changing the roles of M and M' , we get

$$\sum_{i=1}^m h'_i F'_i \geq \sum_{i=1}^m h_i F_i.$$

Thus, M and M' must be homothetic. Since the n -volumes of g_i and g_i are equal, M must coincide with M' . The proposition is proved.

2.3. LEMMA. *Let $\bar{\Omega}$ be a closed convex set on the unit hypersphere Σ lying strictly inside a hemisphere, and P a convex cone with vertex at \mathcal{O} at the center*

of Σ and with spherical image $\bar{\Omega}$. (By definition the spherical image of P consists of all exterior unit normals to hyperplanes supporting to P .)

Also let R be the infinite convex body whose boundary is P , and $\tau = R \cap \Sigma$. Denote by $\bar{\Omega}' \subset \Sigma$ the set symmetric to $\bar{\Omega}$ with respect to \mathcal{O} and put $\mathcal{K} = \tau \cap \bar{\Omega}'$. Then $\text{int } \mathcal{K} \neq \emptyset$. Moreover, for any point $\eta \in \text{int } \mathcal{K}$ there exists a positive number ε (depending on η) such that

$$(*) \quad \min \left\{ \min_{\xi \in \bar{\Omega}'} \langle \xi, \eta \rangle, \min_{\xi \in \tau} \langle \xi, \eta \rangle \right\} \geq \varepsilon.$$

2.3.1. PROOF. Denote by Σ_0^* the hypersphere containing $\bar{\Omega}$ strictly inside and by σ_0 the equatorial hyperplane determining Σ_0^* . Let ξ_0 be the pole of Σ_0 . We consider two possibilities: a) $\xi_0 \notin \text{int } \Omega$, b) $\xi_0 \in \text{int } \Omega$.

We start with the case a). Consider the infinite convex cone K formed by the rays originating at \mathcal{O} and of direction ξ , where $\xi \in \bar{\Omega}$. Since $\xi_0 \notin \text{int } \Omega$, there exists a hyperplane σ_1 containing ξ_0 and supporting to K . The latter means that K lies in one of the dihedral angles formed by σ_0 and σ_1 . Note also that $(\sigma_0 \cap \sigma_1) \cap \bar{\Omega} = \emptyset$, since $\sigma_0 \cap \bar{\Omega} = \emptyset$.

Let α be the dihedral angle containing K and Σ_α the portion of Σ_0 which is contained in α . Consider a point $\xi_1 \in \Sigma_\alpha$ such that the vector $\mathcal{O}\xi_1$ is orthogonal to the subspace $\sigma_0 \cap \sigma_1$. Again it may happen that $\xi_1 \notin \text{int } \Omega$. In this case we repeat the above construction by taking a hyperplane σ_2 containing ξ_1 and identifying the trihedral angle containing $\bar{\Omega}$. This process is continued until either after k steps, $k \leq n$, we construct $\xi_k \in \text{int } \Omega$, or after n steps we construct a hyperoctant containing the cone K . In the first case the hyperplane σ_k with the normal $-\xi_k$ will be clearly strictly supporting to K , that is, the hemisphere Σ_k determined by σ_k will contain $\bar{\Omega}$ strictly inside. Since ξ_k is an interior point of Ω , the normals to supporting hyperplanes of K form with ξ_k angles greater than $\pi/2$; that is, the hyperplane σ_k is also strictly supporting to the cone R . (Note that τ is the spherical image of K .) On the other hand the $\text{int } \mathcal{K}$ consists of points corresponding to hyperplanes strictly supporting to both R and K . Thus, $\text{int } \mathcal{K}$ is not empty, and the inequality (*) follows immediately.

Now consider the situation when after n steps we obtained a hyperoctant T containing K . Let $\bar{\xi} \in \text{int } \Omega$. Since T contains K , the hyperplane β with the unit normal $\bar{\xi}$ is strictly supporting to both K and T . Obviously, the same hyperplane will be strictly supporting to R , since τ is the spherical image of K . Thus, $-\bar{\xi} \in \text{int } \tau$, and therefore $\text{int } \mathcal{K} \neq \emptyset$. As before, the inequality (*) follows from the openness of $\text{int } \mathcal{K}$. The lemma is proved.

2.4. LEMMA. Let $\bar{\Omega}$ be a closed convex set on the hypersphere Σ with polyhedral boundary lying strictly inside a hemisphere, and P a convex cone

with vertex \mathcal{O} and spherical image $\bar{\Omega}$. Denote by $\alpha_1, \dots, \alpha_s$ the hyperplanes containing the infinite n -dimensional faces of P . Finally, let $\omega = \Sigma \setminus \bar{\Omega}$, and ξ_1, \dots, ξ_m be unit vectors with origins at \mathcal{O} and endpoints in ω .

Then there exists a convex polyhedron M with a true vertex at \mathcal{O} and such that it has s faces q_1, \dots, q_s adjacent to \mathcal{O} , each of which lies on the corresponding hyperplane α_j , $j = 1, \dots, s$. M also has m other faces g_1, \dots, g_m with exterior normals ξ_1, \dots, ξ_m . Moreover, there exists a neighborhood of \mathcal{O} on P free of points of the faces g_1, \dots, g_m . In addition, if the normals ξ_1, \dots, ξ_m are all directed in the hemisphere containing $\bar{\Omega}$ then M is an infinite polyhedron; otherwise, it is a closed convex polyhedron.

2.5. PROOF. As in Lemma 2.3 we denote by R the infinite convex body whose boundary is P , and $\tau = R \cap \Sigma$. Correspondingly, $\bar{\Omega}'$ denotes the set symmetric to $\bar{\Omega}$ with respect to \mathcal{O} and $\mathcal{K} = \tau \cap \bar{\Omega}'$. Also, we denote by K the infinite convex cone formed by the rays originating at \mathcal{O} and passing through ξ , where $\xi \in \bar{\Omega}$. Finally, put $C = \partial K$.

By Lemma 2.3, $\text{int } \mathcal{K} \neq \emptyset$, and we choose $\bar{\eta} \in \text{int } \mathcal{K}$. Let l be a straight line passing through \mathcal{O} and containing $\bar{\eta}$, and H a hyperplane passing through \mathcal{O} and perpendicular to l . Denote by \mathcal{O}' an arbitrary point on l different from \mathcal{O} and taken in the direction $\bar{\eta}$. Let C' be the cone obtained from C by parallel translation so that in the new position its vertex coincides with \mathcal{O}' . The intersection of the infinite convex body bounded by C' with the infinite convex body R gives a finite convex body whose boundary is a closed convex polyhedron. Let T denote this polyhedron.

We note that every point of the intersection $C' \cap P$ is a positive distance away from \mathcal{O} . In fact, let $\bar{x} \in C' \cap P$ and be such that $s_0 \equiv \text{dist}(\mathcal{O}, \bar{x}) = \min_{x \in C' \cap P} \text{dist}(\mathcal{O}, x)$. Also let $\gamma \equiv \max_{\xi \in \partial \bar{\Omega}} \text{arc cos } \langle \bar{\eta}, \xi \rangle$. Since H is strictly supporting, $\pi/2 < \gamma < \pi$, and a calculation shows that $\text{dist}(\mathcal{O}, x) \geq s_0 = \text{dist}(\mathcal{O}, \mathcal{O}') \cdot \sin \gamma > 0$ for any $x \in C' \cap P$.

Next we consider the hyperplanes $\alpha_1, \dots, \alpha_m$ supporting to T whose exterior unit normals are ξ_1, \dots, ξ_m . None of those hyperplanes passes through \mathcal{O} , since $\xi_i \in \omega$, $i = 1, \dots, m$. Among $\alpha_1, \dots, \alpha_m$ we take an arbitrary hyperplane, say α_i , and move it parallel to itself by a small distance in the direction $-\xi_i$. If this distance is sufficiently small then clearly none of the faces of T will disappear. The intersection of T with a halfspace determined by α_i in the direction $-\xi_i$ gives a new convex polyhedron with additional face whose normal is ξ_i . However, it could happen that α_i already contained an n -dimensional face of T before we moved it. For convenience, we assume in this case that α_i was «moved» but by a distance equal to zero. Now we move each α_i in the fashion described above, and then consider the intersection of the halfspaces determined by $\alpha_1, \dots, \alpha_m$ in

the directions $-\xi_1, \dots, -\xi_m$ and the infinite body R . This intersection is a convex body, whose boundary is a closed convex polyhedron with all properties stated in the Proposition 2.4, except for the last one which is yet to be verified.

Let M be the convex polyhedron constructed above. Suppose ξ_1, \dots, ξ_m are all directed in the hemisphere Σ^+ containing $\bar{\Omega}$. Let $\bar{\xi} \in \Sigma^+$ be the unit vector perpendicular to the equatorial hyperplane bounding Σ^+ . Then are $\cos \langle \bar{\xi}, \xi_i \rangle \leq \pi/2, i = 1, \dots, m$, and $\arccos \langle \bar{\xi}, \xi \rangle < \pi/2$ for $\xi \in \bar{\Omega}$. Obviously, the ray with the origin at O and of direction $-\bar{\xi}$ intersects neither $\alpha_1, \dots, \alpha_m$ nor P which means that M is infinite. The remaining case is treated similarly. The lemma is proved.

2.6. REMARK. *In the following the cone P in Lemma 2.4 will be referred to as the « boundary » cone of M .*

2.7. We will use the mapping lemma of Aleksandrov to prove the existence of a polyhedron with prescribed boundary cone, normals, and n -volumes of the corresponding faces. For convenience of the reader we quote it here. More details can be found in [1], p. 90.

Let A and B be two m -dimensional manifolds without boundaries (not necessarily connected) and τ a mapping of A into B satisfying the conditions:

- 1) each connected component of B contains images of points in A ;
- 2) τ is a one-to-one correspondence;
- 3) τ is continuous;
- 4) if a sequence $b_k \in B, k = 1, 2, \dots$, is such that there exists a sequence $a_k \in A$ for which $\tau(a_k) = b_k, k = 1, 2, \dots$, and b_k converges to $b \in B$, then there exists $a \in A$, for which $\tau(a) = b$, and one can select a subsequence of a_k converging to a .

Under the conditions 1)-4), $\tau(A) = B$.

2.8. THEOREM. *Let $\bar{\Omega}$ be as in Lemma 2.4, i.e., a closed convex set on the unit hypersphere Σ with polyhedral boundary lying strictly inside a hemisphere, P a convex cone with vertex O at the center of Σ and spherical image $\bar{\Omega}$, $\omega = \Sigma \setminus \bar{\Omega}$, and ξ_1, \dots, ξ_m points in ω . Further, let F_1, \dots, F_m be positive numbers such that*

$$(2.1) \quad \sum_{i=1}^m F_i \xi_i = -\lambda \bar{\xi},$$

where $\bar{\xi} \in \Omega$ and $\lambda > 0$.

Then there exists a closed convex polyhedron M with a true vertex at \mathcal{O} for which P is a boundary cone, and which, in addition to faces generated by P , has m other faces g_1, \dots, g_m with exterior unit normals ξ_1, \dots, ξ_m and n -volumes F_1, \dots, F_m .

2.9. In order to prove the theorem we need some preparatory statements.

Consider the set Π of closed convex polyhedrons which have \mathcal{O} as their true vertex and P as the boundary cone. Assume also that besides the faces lying on P they have only m other faces with exterior unit normals ξ_1, \dots, ξ_m . By «definition» of the boundary cone the faces with the normals ξ_1, \dots, ξ_m cannot be adjacent to the vertex \mathcal{O} . It follows from Lemma 2.4 that $\Pi \neq \emptyset$.

Each polyhedron $M \in \Pi$ is defined by its support numbers h_1, \dots, h_m corresponding to the normals ξ_1, \dots, ξ_m . The support numbers of the faces lying on P are equal to zero. Clearly, $h_i > 0$, $i = 1, \dots, m$, and therefore, with each $M \in \Pi$ we can associate a point (h_1, \dots, h_m) in the positive coordinate angle of the m -dimensional Euclidean space E^m . Denote by \mathcal{A} the set of points in E^m corresponding to polyhedrons from Π .

The set \mathcal{A} is open. Indeed, for any $M \in \Pi$ if we make a parallel translation by a sufficiently small distance of a hyperplane α_i containing a face with the normal ξ_i , none of the other faces will disappear and P will still remain a boundary cone. Therefore, the new polyhedron obtained from M will also lie in Π . Of course, we can move all α_i simultaneously and still remain in Π .

Consider now an arbitrary polyhedron $M \in \Pi$. Since P is a boundary cone for M , there exists a neighborhood V of the vertex \mathcal{O} on P free of points of the faces g_i with normals ξ_i , $i = 1, \dots, m$. Let $\tilde{M} = \bigcup_{i=1}^m g_i$. The orthogonal projection of \tilde{M} on the hyperplane H , constructed in the proof of Lemma 2.4, obviously covers the projection of V on H . Therefore, there exists $\varepsilon > 0$ such that

$$(2.2) \quad \sum_{i=1}^m f_i \langle \xi_i, \bar{\eta} \rangle > \varepsilon,$$

where f_i is the n -volume of the face g_i . The left hand side in (2.2) is the n -volume of the projection of \tilde{M} on H .

In general, the number ε in (2.2) depends on the particular polyhedron. We define the set $\Pi_\varepsilon \subset \Pi$ as the set of polyhedrons in Π which satisfy (2.2). It is clear that for any $\varepsilon > 0$ the set Π_ε is not empty. This follows from Lemma 2.4 if we take the polyhedron constructed there and make a homothetic transformation with the center at \mathcal{O} and appropriate coefficient. Correspondingly to the set Π_ε we have the open set \mathcal{A}_ε .

2.10. Next we consider the set \mathcal{B}_δ of positive numbers f_1, \dots, f_m which satisfy the inequality

$$(2.3) \quad \sum_{i=1}^m f_i \langle \xi_i, \bar{\eta} \rangle > \delta$$

for some fixed positive δ . In the m -dimensional Euclidean space with coordinates f_1, \dots, f_m (2.3) defines an open convex subset of the positive coordinate angle. This subset lies above the hyperplane $\sum_{i=1}^m f_i \langle \xi_i, \bar{\eta} \rangle = \delta$.

We choose δ so that \mathcal{B}_δ contains the point (F_1, \dots, F_m) given in Theorem 2.8. Let us show that it can be done. It follows from the construction of vector $\bar{\eta}$ that $\alpha \equiv \max_{\xi \in \partial\Omega} \langle \xi, \bar{\eta} \rangle < 0$, since the hyperplane H is strictly supporting to the cone C made up of rays which originate at \mathcal{O} and go into points of $\partial\Omega$. On the other hand, because of (2.1) and since $\bar{\xi} \in \Omega$, we have

$$\sum_{i=1}^m F_i \langle \xi_i, \bar{\eta} \rangle = -\lambda \langle \bar{\xi}, \bar{\eta} \rangle > -\lambda \alpha > 0.$$

Thus, if we take $\delta = -\lambda \alpha$ then \mathcal{B}_δ will contain the point (F_1, \dots, F_m) . We fix this δ , and also in (2.2) take $\varepsilon = \delta$.

2.11. The proof of Theorem 2.8 consists of verification of the conditions 1)-4) of the Mapping Lemma with $A = \mathcal{A}_\delta$, $B = \mathcal{B}_\delta$. As for the mapping τ one takes it as

$$\tau(M) \equiv \tau(h_1, \dots, h_m) = (f_1, \dots, f_m).$$

Since \mathcal{B}_δ is an open convex set, it has only one component. It contains images of points in \mathcal{A}_δ because by Lemma 2.4 and the discussion in 2.9, \mathcal{A}_δ is not empty. Thus, condition 1) is verified.

Condition 2) holds because of Proposition 2.2.

To verify 3) one should just note that under a small change of support numbers h_1, \dots, h_m the corresponding n -volumes f_1, \dots, f_m change continuously.

2.12. Finally, we establish 4). Let $f^k = (f_1^k, \dots, f_m^k)$, $k = 1, 2, \dots$, be a sequence of points from \mathcal{B}_δ converging to $f = (f_1, \dots, f_m)$, and $\{M^k\}$ is the corresponding sequence of polyhedrons from Π_δ with support numbers $h^k = (h_1^k, \dots, h_m^k)$ of the faces g_i^k , $i = 1, \dots, m$. We will prove now that the sequence $\{h^k\}$ is bounded.

Fix an arbitrary polyhedron M^k from the sequence $\{M^k\}$. Let Q be a point on M^k such that the $\text{dist}(\mathcal{O}, Q)$ is maximal. In order to prove that

$\{h^k\}$ is bounded it suffices to show that $\text{dist}(\mathcal{O}, Q)$ is bounded by a constant independent on M^k .

Let H be the hyperplane with the unit normal $\bar{\eta}$ constructed in the proof of Lemma 2.4. In view of the condition (2.2) and because of the way II_δ was constructed there exists a neighborhood U of \mathcal{O} on P free of points of the faces with normals ξ_1, \dots, ξ_m of any polyhedron in II_δ . Therefore, we can move the hyperplane H parallel to itself in the direction $\bar{\eta}$ so that in the new position H' will still have common points only with U . Denote by H' the translated hyperplane H . The intersection of H' with the convex body bounded by P is a closed convex body Φ whose n -volume is bounded away from zero, say by a constant $\gamma > 0$. Consider a pyramid with the base Φ and vertex Q . It lies entirely within the convex body \tilde{M}^k bounded by M^k and therefore its $(n + 1)$ -volume V does not exceed the $(n + 1)$ -volume of \tilde{M}^k . Thus, we have

$$(2.4) \quad V \leq \frac{1}{n} (n\text{-volume of } \Phi) \cdot s \leq (n + 1)\text{-volume of } \tilde{M}^k,$$

where s is the distance from Q to H' . By the isoperimetric inequality the right hand side of (2.4) is bounded provided the n -volume of M^k is bounded.

The total n -volume of M^k is equal to $\sum_{i=1}^m f_i^k + \sum_{i=1}^m e_j^k$, where e_j^k denote the n -volumes of the faces of M^k adjacent to \mathcal{O} . Let us show that the n -volumes e_j^k are bounded. Since the polyhedron M^k is closed,

$$\sum_{i=1}^m f_i^k \xi_i + \sum_{j=1}^s e_j^k \eta_j = 0,$$

where η_j are the unit normals to the faces of the cone P . Then

$$\sum_{j=1}^s e_j^k \langle \eta_j, -\bar{\eta} \rangle = \sum_{i=1}^m f_i^k \langle \xi_i, \bar{\eta} \rangle \leq \sum_{i=1}^m f_i^k.$$

On the other hand $-\bar{\eta}$ is an interior point of Ω , $\eta_j \in \partial\Omega$, and $e_j^k > 0$ (in fact $e_j^k > 0$, since none of the faces of M^k adjacent to \mathcal{O} degenerate). In the second paragraph of 2.10 it was shown that $\bar{\alpha} \equiv \min_{\xi \in \partial\Omega} \langle \xi, -\bar{\eta} \rangle > 0$. Therefore,

$$(2.5) \quad \sum_{j=1}^s e_j^k \leq \frac{\sum_{i=1}^m f_i^k}{\bar{\alpha}}.$$

Thus, the n -volume of M^k is bounded by a quantity depending only on $\sum_{i=1}^m f_i^k$ and the domain $\bar{\Omega}$. Finally, (2.4) and the isoperimetric inequality show that the distance s is bounded. Since the distance between H and H' is taken to be bounded and does not depend on the particular polyhedron from Π_δ , the distance from Q to H is bounded. On the other hand H is strictly supporting to the cone R . Therefore, any ray originating at \mathcal{O} and going in R forms with $\bar{\eta}$ an angle less than $\pi/2$. Then, obviously the boundedness of the distance from Q to H implies boundedness of the distance from Q to \mathcal{O} .

Since $\{f^k\}$ converges to f , the sums $\sum_{i=1}^m f_i^k$ can be uniformly bounded in terms of $f_i, i = 1, \dots, m$, for all sufficiently large k .

Therefore, the sequence $\{h^k\}$ is bounded, and we can select from it a converging subsequence. Correspondingly, we will have a subsequence of convex polyhedrons converging to some convex polyhedron M . The volumes of the faces of polyhedrons of that subsequence converge to the numbers f_1, \dots, f_m . Thus, the condition 4) of the Mapping Lemma also holds. From this we conclude that to any $f \in \mathcal{B}_\delta$ there corresponds a polyhedron from \mathcal{A}_δ . On the other hand, it has been shown in 2.10 that the point $F = (F_1, \dots, F_m)$ given in Theorem 2.8 belongs to \mathcal{B}_δ . This completes the proof of Theorem 2.8.

3. – Generalized solution of the boundary value Minkowski problem.

3.1. THEOREM. *Let ω be a domain on a unit hypersphere Σ such that $\bar{\Omega} = \Sigma \setminus \omega$ is a convex domain lying strictly inside a hemisphere. Let $\varphi(\xi)$ be a continuous function defined in $\bar{\omega}$ and such that*

$$1) \quad 0 < \text{const} = c_0 \leq \varphi(\xi) \leq c_1;$$

$$2) \quad \int_{\bar{\omega}} \xi \varphi(\xi) \, d\mu = -\lambda \bar{\xi},$$

where $\lambda > 0$ and $\bar{\xi} \in \Omega$.

Then there exists a convex hypersurface S , contained in a closed convex hypersurface \bar{S} , such that the spherical image of S is $\bar{\omega}$, the Gaussian curvature (that is, the Radon-Nikodym derivative of the area function) at an interior point ξ of ω is $1/\varphi(\xi)$, and the support function vanishes at the boundary $\partial\omega$. Moreover, the hypersurface \bar{S} can be constructed in such a way that the support function of \bar{S} vanishes on the entire $\bar{\Omega}$.

In order to prove the theorem we construct a converging sequence of convex polyhedrons with prescribed set of unit normals, n -volumes of the

corresponding faces, and certain boundary cones. Existence of such polyhedrons is assured by Theorem 2.8. The sequence is constructed in such a way that the area functions of the polyhedrons converge weakly to the area function generated by the function $\varphi(\xi)$. Also the boundary cones of these polyhedrons converge to a cone which will be the boundary cone of the hypersurface in question.

3.1.1. Consider a convex polyhedral domain Ω_k , whose boundary $\partial\Omega_k$ consists of k $(n-1)$ -dimensional faces, and is circumscribed about $\partial\Omega$. The $(n-1)$ -dimensional faces are pieces of $(n-1)$ -dimensional spheres formed by intersections of Σ with n -dimensional hyperplanes passing through the center \mathcal{O} of Σ . Suppose k is large and the diameters of the faces of $\partial\Omega_k$ are small enough so that $\partial\Omega_k$ lie strictly inside the hemisphere containing Ω . If $k \rightarrow \infty$ and diameters of the faces of $\partial\Omega_k$ converge uniformly to zero, then $\partial\Omega_k \rightarrow \partial\Omega$. Convergence is understood in the standard sense given to convergence of sets in Euclidean space, which is applicable here, since we are dealing with convex sets lying inside a hemisphere.

To each Ω_k there corresponds a closed domain $\bar{\omega}_k = \Sigma \setminus \Omega_k$, which is contained in the domain ω . We partition $\bar{\omega}_k$ into small domains β_k^l and define positive numbers F_k^l and unit vectors ξ_k^l by the condition

$$(3.1) \quad F_k^l \xi_k^l = \int_{\beta_k^l} \xi \varphi(\xi) d\mu, \quad l = 1, \dots, N.$$

A certain care is needed in order to assure that vectors ξ_k^l lie inside β_k^l . This condition will be satisfied if one takes, for example, a triangulation of ω_k into convex sets. The latter can be done in several quite obvious ways, though describing the actual constructions is rather tedious. For that reason we omit it here.

It follows from conditions 1), 2) in the theorem and (3.1) that for sufficiently large k there exist $\lambda_k > 0$ and unit vectors $\bar{\xi}_k \in \Omega$ such that

$$(3.2) \quad \sum_l F_k^l \xi_k^l = -\lambda_k \bar{\xi}_k.$$

Moreover, since the diameters of β_k^l tend to zero when $k \rightarrow \infty$, and $\omega_k \rightarrow \omega$, we have $\lambda_k \xi_k \rightarrow \lambda \xi$.

Denote by P_k the convex cone with vertex \mathcal{O} and spherical image $\bar{\Omega}_k$. Then in view of (3.2) it follows from Theorem 2.8 that there exists a convex polyhedron M_k with a true vertex at \mathcal{O} for which P_k is a boundary cone and which, besides the faces adjacent to \mathcal{O} , has N other faces with exterior normals ξ_k^l and n -volumes F_k^l , $l = 1, \dots, N$.

3.2. Let us show that the polyhedrons P_k all lie within a certain hypersphere with the center at \mathcal{O} . In order to do that we need to construct at first a unit vector $\bar{\eta}$ and a hyperplane H similar to those constructed in the proof of Lemma 2.5.

3.2.1. Consider the cone R with vertex at \mathcal{O} and spherical image $\bar{\Omega}$. Put $\tau = R \cap \Sigma$, $\bar{\Omega}' = \{\xi \in \Sigma | -\xi \in \bar{\Omega}\}$, $\mathcal{K} = \bar{\Omega}' \cap \tau$. Correspondingly we have the cones R_k , whose boundaries are P_k , and sets τ_k , $\bar{\Omega}'_k$, and \mathcal{K}_k . Since $\bar{\Omega}_k \supset \bar{\Omega}_{k+1} \supset \dots \supset \bar{\Omega}$, we have the inclusions $\tau_k \supset \tau_{k+1} \supset \dots \supset \tau$. Suppose K is a positive number large enough so that $\bar{\Omega}_k$ lies strictly inside the hemisphere containing $\bar{\Omega}$ for all $k \geq K$. Then τ_k is a convex set with interior points. By Lemma 2.3 the $\text{int } \mathcal{K} \neq \emptyset$. Let $\bar{\eta} \in \text{int } \mathcal{K}$ and be the center of a ball of largest radius inscribed in \mathcal{K} . The hyperplane H passing through \mathcal{O} and perpendicular to $\mathcal{O}\bar{\eta}$ is strictly supporting to both cones P and C , where, as before, C is the cone formed by the rays of direction $\mathcal{O}\xi$, $\xi \in \partial\bar{\Omega}$, and with vertex \mathcal{O} . Since the sets $\Omega_k \xrightarrow{k \rightarrow \infty} \Omega$, we can assume that for $k \geq K$ the hyperplane H is strictly supporting to all cones C_k corresponding to $\partial\Omega_k$. Obviously, it is also strictly supporting to all cones P_k . Therefore,

$$(3.3) \quad \bar{\alpha}_k \equiv \min_{\xi \in \bar{\Omega}_k} \langle \xi, -\bar{\eta} \rangle \geq \min_{\xi \in \bar{\Omega}} \langle \xi, -\bar{\eta} \rangle \equiv \bar{\alpha} > 0 .$$

It has been shown in 2.12 that the total n -volume of faces q_j , $j = 1, \dots, s$, of M_k adjacent to \mathcal{O} satisfies the inequality

$$(3.4) \quad \sum_j e_j \leq \frac{\sum_i F_k^i}{\bar{\alpha}_k} ,$$

where e_j is the n -volume of q_j , $\bar{\alpha}_k = \min_{\xi \in \partial\Omega_k} \langle \xi, -\bar{\eta}_k \rangle$, and $\bar{\eta}_k \in \text{int } \mathcal{K}_k$.

Since $\bar{\eta} \in \text{int } \mathcal{K}_k$ for all $k \geq K$, we can take $\bar{\eta}_k = \bar{\eta}$. Then (3.4) and (3.3) give

$$(3.5) \quad \sum_j e_k^j \leq \frac{\sum_i F_k^i}{\bar{\alpha}} .$$

On the other hand, we have

$$(3.6) \quad \sum_i F_k^i = \sum_i \int_{\beta_k^i} \langle \xi, \xi_k^i \rangle \varphi(\xi) \, d\mu \leq \int_{\cup \beta_k^i} \varphi(\xi) \, d\mu \leq \int_{\omega} \varphi(\xi) \, d\mu \leq c_1 \cdot (n\text{-volume of } \omega) ,$$

and we conclude that the total n -volume of faces adjacent to \mathcal{O} is bounded by a constant independent on k (when it is $\geq K$).

3.2.2. Consider now a point $Q \in M_k$ at a maximal distance from \mathcal{O} . Put $L = \text{dist}(\mathcal{O}, Q)$ and consider the hyperplane σ perpendicular to the vector $\zeta = \mathcal{O}Q$ and at a distance $L/2$ away from \mathcal{O} . Consider the projection of M_k on σ , and denote by S_ζ^k the n -volume of the projection. We need to estimate S_ζ^k from below.

We have, in view of (3.1),

$$S_\zeta^k = \frac{1}{2} \left\{ \sum_i F_k^i |\langle \xi_k^i, \zeta \rangle| + \sum_j e_k^j |\langle \xi_k, \zeta \rangle| \right\} \geq \frac{1}{2} \sum_i F_k^i |\langle \xi_k^i, \zeta \rangle| = \frac{1}{2} \sum_i \left| \int_{\beta_k^i} \langle \xi, \zeta \rangle \varphi(\xi) \, d\mu \right|.$$

The function $\langle \xi, \zeta \rangle$, $\xi \in \Sigma$, vanishes only on $(n-1)$ -dimensional sphere which is the intersection of Σ and the hyperplane passing through \mathcal{O} perpendicular to ζ . On the other hand by our hypothesis $\varphi(\xi) \geq c_0 > 0$ in $\bar{\omega}$ and it is continuous. It follows from this and the fact that diameters of β_k^i tend to zero when $k \rightarrow \infty$ that there exists $\alpha > 0$ such that

$$\frac{1}{2} \sum_i \left| \int_{\beta_k^i} \langle \xi, \zeta \rangle \varphi(\xi) \, d\mu \right| \geq \frac{1}{2} \int_I \langle \xi, \zeta \rangle \varphi(\xi) \, d\mu - \varepsilon,$$

where $I = \{\xi \in \omega \mid |\langle \xi, \zeta \rangle| > \alpha\}$, $\varepsilon \xrightarrow{k \rightarrow \infty} 0$, and α is the same for all sufficiently large k . Since

$$\frac{1}{2} \int_I \langle \xi, \zeta \rangle \varphi(\xi) \, d\mu > \frac{1}{2} c_0 \alpha \int_I d\mu,$$

the constant α can be assumed small enough so that $\int_I d\mu > (n\text{-volume of } \omega)/2$. Finally we conclude that

$$S_\zeta^k \geq \left[\frac{c_0}{4} \alpha (n\text{-volume of } \omega) - \varepsilon \right] > 0.$$

In order to estimate the distance L we symmetrize the polyhedron M_k relative to the hyperplane σ . Denote by \tilde{M}_k the resulting convex polyhedron. Under this symmetrization $\sigma \cap M_k \equiv \sigma \cap \tilde{M}_k$, and $(n+1)$ -volume of $M_k = (n+1)$ -volume of \tilde{M}_k . Inscribe in \tilde{M}_k two pyramids with the common base $\sigma \cap \tilde{M}_k$ and vertices at \mathcal{O} and Q . The total $(n+1)$ -volume of the two pyramids does not exceed the $(n+1)$ -volume V_k of \tilde{M}_k . Therefore,

$$(3.7) \quad \frac{1}{n} L \frac{c_0}{4} \alpha (n\text{-volume of } \omega) - \varepsilon \leq \frac{1}{n} L S_\zeta^k \leq V_k.$$

It follows from (3.5) and (3.6) that the total n -volume of the hypersurface M_k is uniformly bounded. Under such circumstances the isoperimetric inequality implies that the $(n + 1)$ -volume V_k is uniformly bounded. Therefore, we can conclude from (3.7) that the distance L is uniformly bounded for all sufficiently large k . The latter means that the polyhedrons M_k are contained in a hypersphere of some fixed radius independent on k .

3.3. Since the polyhedrons M_k are uniformly bounded, we can apply Blaschke's theorem asserting that the sequence M_k contains a subsequence M_{k_i} converging to a convex hypersurface. Denote the limiting hypersurface by \tilde{S} . Since the boundary cones P_k converge uniformly to the cone P , and the support functions of M_{k_i} converge uniformly to the support function of \tilde{S} (see [2], p. 43), the latter must vanish at the boundary of ω . In fact, it is clear that the support function of \tilde{S} vanishes everywhere in $\bar{\Omega}$.

3.4. It is known (see [2], section 8) that the area functions of the subsequence M_{k_i} converge weakly to the area function of \tilde{S} . Let us show that for any Borel subset G of Σ contained in ω the area function $\nu(G)$ of \tilde{S} is equal to $\int_G \varphi(\xi) d\mu$.

Let $\nu_{k_i}(G)$ be the values of area functions of polyhedrons M_{k_i} on the set G . By definition of the area function

$$\nu_{k_i}(G) = \sum_l F_{k_i}^l,$$

where the sum is taken over the faces of M_{k_i} for which the normals lie in G . We have

$$\left| \sum_l F_{k_i}^l - \int_G \varphi(\xi) d\mu \right| \leq \left| \sum_l F_{k_i}^l - \sum_l \int_{\beta_{k_i}^l} \langle \xi, \xi_{k_i}^l \rangle \varphi(\xi) d\mu \right| + \left| \sum_l \int_{\beta_{k_i}^l} \langle \xi, \xi_{k_i}^l \rangle \varphi(\xi) d\mu - \int_G \varphi(\xi) d\mu \right|.$$

Because of (3.1) the first term on the right hand side vanishes. For the second term we note that when the diameters of $\beta_{k_i}^l$ are uniformly small, say less than $\varepsilon_{k_i} > 0$, then $\langle \xi, \xi_{k_i}^l \rangle \leq 1 - \varepsilon_{k_i}$ for any $\xi \in \beta_{k_i}^l$. Thus,

$$\left| \sum_l \int_{\beta_{k_i}^l} \langle \xi, \xi_{k_i}^l \rangle \varphi(\xi) d\mu - \int_G \varphi(\xi) d\mu \right| \leq \left| (1 - \varepsilon_{k_i}) \int_{(\cup \beta_{k_i}^l) \cap G} \varphi(\xi) d\mu - \int_G \varphi(\xi) d\mu \right| \leq \left| \int_{G \setminus (\cup \beta_{k_i}^l)} \varphi(\xi) d\mu \right| + \varepsilon_{k_i} \left| \int_G \varphi(\xi) d\mu \right|.$$

Since $\varphi(\xi)$ is a positive bounded function on $\bar{\omega}$, the terms on the right hand side tend to zero when $k_i \rightarrow \infty$. Thus,

$$\nu_{k_i}(G) \rightarrow \int_G \varphi(\xi) d\mu$$

for any Borel subset $G \subset \omega$.

It follows from our assumptions on $\varphi(\xi)$ that at any interior point of ω there exists the Radon-Nikodym derivative of $\nu(G)$ equal to $\varphi(\xi)$. That is, the limiting hypersurface \tilde{S} at each interior point of ω has the Gaussian curvature equal to $1/\varphi(\xi)$.

3.5. The convex hypersurface S whose existence was asserted in Theorem 3.1 is now defined as the closure of the set of points on \tilde{S} for which the supporting hyperplanes have normals parallel to vectors going from \mathcal{O} in the interior of ω . It is easy to see that the spherical image of S coincides with $\bar{\omega}$.

3.6. It remains to establish uniqueness of the hypersurface S . Let S_1 and S_2 be two convex hypersurfaces with area functions ν_1 and ν_2 defined on Σ and such that for any Borel set $G \subset \omega$,

$$(3.8) \quad \nu_1(G) = \nu_2(G).$$

Assume that the support functions h_1 and h_2 of S_1 and S_2 are such that $h_1 = h_2 = 0$ on $\partial\omega$, and the domain $\bar{\Omega} = \Sigma \setminus \omega$ is a convex domain lying strictly inside a hemisphere. Consider a cone P with the vertex \mathcal{O} at the center of Σ whose spherical image is $\bar{\Omega}$. The hyperplanes which contain a generating ray of P will be supporting to the hypersurfaces S_1 and S_2 . We define two convex hypersurfaces \bar{S}_1 and \bar{S}_2 by the support functions

$$\bar{h}_1 = \begin{cases} h_1 & \text{when } \xi \in \omega, \\ 0 & \text{when } \xi \in \bar{\Omega}, \end{cases}$$

$$\bar{h}_2 = \begin{cases} h_2 & \text{when } \xi \in \omega, \\ 0 & \text{when } \xi \in \bar{\Omega}. \end{cases}$$

Both function \bar{h}_1 and \bar{h}_2 are continuous on Σ , and for the hypersurfaces \bar{S}_1 and \bar{S}_2 the relation (3.8) holds for Borel sets $G \subset \omega$. The area functions of \bar{S}_1 and \bar{S}_2 vanish identically for any Borel subset of Ω .

The mixed volume of the convex bodies T_1 and T_2 bounded by \bar{S}_1 and \bar{S}_2 is by definition given by the formula

$$V(T_1, T_2) = \frac{1}{n+1} \int_{\Sigma} \bar{h}_1 \nu_2(d\sigma),$$

where $\nu_2(d\sigma)$ denotes the value of the area function on the Borel set $d\sigma \subset \Sigma$. Correspondingly, the volumes of T_1 and T_2 are given by

$$V(T_1) = \frac{1}{n+1} \int_{\Sigma} \bar{h}_1 \nu_1(d\sigma)$$

and

$$V(T_2) = \frac{1}{n+1} \int_{\Sigma} \bar{h}_2 \nu_2(d\sigma).$$

By the Minkowski inequality ([2], p. 48)

$$(3.9) \quad V^n(T_1, T_2) \geq V(T_1, T_1) V^{n-1}(T_2, T_2),$$

and the equality holds if and only if T_1 and T_2 are homothetic (possibly after a parallel translation of one of them).

In order to apply (3.9) and make use of (3.8), we need to be able to « replace » $\nu_1(d\sigma)$ in $V(T_1)$ by $\nu_2(d\sigma)$. However, this is impossible, since for the sets containing portions of $\partial\omega$ the area functions may not coincide, and, in general, they are not absolutely continuous. To overcome this difficulty we consider a closed subset ω_ε of Ω which is a strip along the boundary $\partial\omega$ such that $\bar{h}_1 < \varepsilon$ and $\bar{h}_2 < \varepsilon$ in ω_ε . This is possible since \bar{h}_1 and \bar{h}_2 are both continuous and vanish on $\partial\omega$. Since both ν_1 and ν_2 are completely additive ([2], p. 60) we can write

$$V(T_1, T_2) = \frac{1}{n+1} \cdot \int_{\omega \setminus \omega_\varepsilon} \bar{h}_1 \nu_2(d\sigma) + \frac{1}{n+1} \int_{\Omega \cup \omega_\varepsilon} \bar{h}_1 \nu_2(d\sigma),$$

where the second term tends to zero when $\varepsilon \rightarrow 0$. (Note that ν_2 is a finite measure.) Similar expressions hold for $V(T_1)$ and $V(T_2)$. Now applying (3.9) and taking into account (3.8) we get

$$\left[\int_{\omega \setminus \omega_\varepsilon} \bar{h}_1 \nu_1(d\sigma) + \int_{\Omega \cup \omega_\varepsilon} \bar{h}_1 \nu_2(d\sigma) \right]^n \geq \left[\int_{\omega \setminus \omega_\varepsilon} \bar{h}_1 \nu_1(d\sigma) + \int_{\Omega \cup \omega_\varepsilon} \bar{h}_1 \nu_1(d\sigma) \right] \cdot \left[\int_{\omega \setminus \omega_\varepsilon} \bar{h}_2 \nu_2(d\sigma) + \int_{\Omega \cup \omega_\varepsilon} \bar{h}_2 \nu_2(d\sigma) \right]^{n-1}.$$

Letting $\varepsilon \rightarrow 0$ we obtain

$$V(T_1) \geq V(T_2).$$

Since $V(T_1, T_2) = V(T_2, T_1)$ (see [2], p. 51), we can interchange the roles of T_2 and T_1 . Then we obtain $V(T_2) \geq V(T_1)$. Therefore $V(T_1) = V(T_2)$, which means that we have equality in (3.9). Then T_1 and T_2 are homothetic. But their volumes are equal, and therefore they may differ at most by a parallel translation. On the other hand they have a common boundary cone P with the vertex at \mathcal{O} . The latter means that $\bar{h}_1 \equiv \bar{h}_2$.

This completes the proof of uniqueness and the theorem is now completely proved.

3.7. REMARK. It is easy to see that the hypersurface \tilde{S} containing S does not have to be unique. An obvious example is obtained by taking a hypersphere and a right circular cone tangential to this hypersphere.

4. – Interior regularity of the generalized solution.

4.1. In this section we intend to show that the generalized solution of the boundary value problem (1.7), (1.8) constructed in section 3 is regular inside the domain ω provided the prescribed data is sufficiently regular. Namely, we have the following result.

4.2. THEOREM. *Suppose that the conditions of Theorem 3.1 are satisfied and S the convex hypersurface constructed in section 3. Assume, in addition, that the function φ given in Theorem 3.1 is of class C^k , $k \geq 3$, in ω . Then the support function h of S is of class $C^{k+1, \alpha}$, $0 < \alpha < 1$, in ω . If φ is analytic in ω , then h is analytic.*

In order to prove the theorem we need some preparatory results.

4.3. We recall a few facts related to graphs of convex functions. Let $\bar{\omega}$ be the spherical image of a smooth convex hypersurface F with support function $u(\xi)$, $\xi \in \bar{\omega}$. Take an interior point $\eta \in \omega$ and let D be a subdomain of ω such that it also lies strictly inside a hemisphere Σ^+ . Assume that η is the pole of Σ^+ . Denote by T the hyperplane tangent to Σ^+ at η , and x_1, \dots, x_n the Cartesian coordinate system on T . By central projection of Σ^+ from \mathcal{O} onto T those coordinates are introduced in Σ^+ . Now we project orthogonally T onto the hyperplane T' parallel to T and passing through \mathcal{O} . Let D' denote the image of D on T' obtained by means of these two projections. We assume that in \mathbf{R}^{n+1} we have the Cartesian coordinate system x_1, \dots, x_n, z and $\eta = (0, \dots, 0, 1)$. The metric (g_{ij}) on Σ in coordinates

x_1, \dots, x_n assumes the form $g_{ij} = (1 + \varrho^2)^{-1}[\delta_{ij} - (1 + \varrho^2)^{-1}x_i x_j]$, where $\varrho^2 = \sum_i x_i^2$.

We associate with the support function $u(\xi)$ defined in \bar{D} the function $\tilde{u} = (1 + \varrho^2)^{\frac{1}{2}}u$. A computation shows that

$$(4.1) \quad \nabla_{ij}u + g_{ij}u = \tilde{u}_{ij}(1 + \varrho^2)^{\frac{1}{2}},$$

where $\tilde{u}_{ij} = \partial^2 \tilde{u} / \partial x^i \partial x^j$, and

$$\frac{\det(\nabla_{ij}u + g_{ij}u)}{\det(g_{ij})} = \det(\tilde{u}_{ij})(1 + \varrho^2)^{(n/2)+1}.$$

It follows from (4.1) and (1.3) that if F is a (strictly) convex hypersurface then \tilde{u} is a (strictly) convex function in D' .

4.4. THEOREM (Pogorelov [14], p. 73). *Let $\tilde{u}(x)$ be a strictly convex solution of class $C^4(R) \cap C^1(\bar{R})$ of the equation*

$$(4.2) \quad \det(\tilde{u}_{ij}) = p(x) > 0$$

in a domain $R \subset T'$ and $\tilde{u}|_{\partial R} = 0$. Then the second derivatives of \tilde{u} can be estimated in any interior point $x \in R$ in terms of the $\max_R |\tilde{u}|$, $\max_R |\text{grad } \tilde{u}|$, the function p and its first and second derivatives, and the distance from x to ∂R .

4.4.1. THEOREM (Pogorelov [14], p. 76). *Let $\tilde{u}(x)$ be a strictly convex solution of class $C^5(R) \cap C^2(R)$ of equation (4.2) and $\tilde{u}|_{\partial R} = 0$. Then the third derivatives of \tilde{u} can be estimated at $x \in R$ by the C^2 -norm of \tilde{u} , the C^3 -norm of p , and the distance from x to ∂R .*

4.5. Let $\bar{\omega}$ now be the the spherical image of the generalized solution S of (1.7), (1.8), and ν its area function. In the following we always assume that ν is precisely the area function constructed in section 3, that is, S is contained in a closed convex hypersurface \tilde{S} a part of which consists of a portion of the boundary cone P . In order to avoid confusion, we will denote by $\tilde{\nu}$ the area function of \tilde{S} .

Our immediate objective is a construction of a smooth approximation of \tilde{S} with special properties.

First of all observe that by Theorem 3.1 for any Borel subset $G \subset \omega$, $\tilde{\nu}(G) = \int_G \varphi \, d\mu$. Since $\tilde{\nu}$ is an area function of a closed convex hypersurface \tilde{S} ,

it satisfies the condition (see, for example, [2], p. 63)

$$(4.3) \quad \int_{\Sigma} \xi \tilde{\nu}(dG) = 0,$$

where dG has an obvious meaning.

Suppose now that φ satisfies not only the hypothesis of Theorem 3.1 but also is of class C^k , $k \geq 3$, inside ω . This assumption will be assumed to be in effect throughout the rest of this section.

Construct a sequence of functions φ_m on Σ with the following properties:

- (a) $\varphi_m \in C^{k+1}(\Sigma)$;
- (b) $\varphi_m > 0$ on Σ ;
- (c) φ_m converges uniformly together with its derivatives up to the third order to φ on any compact subdomain of ω ;
- (d) the measures

$$\nu_m(G) = \int_G \varphi_m(\xi) d\mu$$

converge weakly to ν .

Let us show that if the condition

$$(4.4) \quad \int_{\Sigma} \xi \varphi_m(\xi) d\mu = 0$$

is not satisfied then we can replace φ_m by $\tilde{\varphi}_m$ satisfying (a), (b), (c), (d) and (4.4). The construction here is similar to that in [4].

Suppose for some m

$$\int_{\Sigma} \xi \varphi_m(\xi) d\mu = a_m \bar{\xi}_m,$$

where $a_m \neq 0$, and $\bar{\xi}_m \in \Sigma$. Choose an orthonormal frame $\eta_1, \dots, \eta_{n+1}$ in \mathbf{R}^{n+1} . Let $\bar{\xi}_m = \sum_i \alpha_m^i \eta_i$, $\xi = \sum_i \beta^i(\xi) \eta_i$, $b^i = \left(\int_{\Sigma} [\beta^i(\xi)]^2 d\mu \right)^{-1}$, $i = 1, \dots, n+1$.

Now we « distribute » a_m over the hypersphere Σ . In order to do that consider the function $f_m(\xi) = \langle \bar{\xi}_m, \xi' \rangle$, where $\xi' = \sum_i b^i \beta^i(\xi) \eta_i$.

Then

$$\int_{\Sigma} f_m(\xi) \xi d\mu = \sum_{i,j} b^i \alpha_m^i \left(\int_{\Sigma} \beta^i(\xi) \beta^j(\xi) d\mu \right) \eta_j = \sum_{i,j} \alpha_m^i \delta^{ij} \eta_j = \bar{\xi}_m.$$

Put

$$q_m = \max_{\xi \in \Sigma} \{0, a_m f_m(\xi)\},$$

$$\bar{\varphi}_m = \varphi_m - a_m f_m(\xi) + 2q_m.$$

Clearly,

$$\int_{\Sigma} \xi \bar{\varphi}_m(\xi) d\mu = 0.$$

Let us verify that the other conditions are also satisfied for $\bar{\varphi}_m$. Conditions (a) and (b) are obviously satisfied. Since the measures ν_m converge to $\tilde{\nu}$, we have for arbitrary vector $\zeta \in \Sigma$,

$$a_m \langle \bar{\xi}_m, \zeta \rangle = \int_{\Sigma} \langle \zeta, \xi \rangle \varphi_m(\xi) d\mu \xrightarrow{m \rightarrow \infty} \int_{\Sigma} \langle \zeta, \xi \rangle \tilde{\nu}(dG) = 0.$$

Therefore, $a_m \xrightarrow{m \rightarrow \infty} 0$. Since $f_m(\xi)$ is analytic on Σ , $\bar{\varphi}_m$ obviously satisfies (c). Finally, let $y(\xi)$ be an arbitrary continuous function on Σ . Then

$$\int_{\Sigma} y(\xi) \bar{\varphi}_m(\xi) d\mu = \int_{\Sigma} y(\xi) \varphi_m(\xi) d\mu - a_m \int_{\Sigma} y(\xi) f_m(\xi) \varphi_m(\xi) d\mu +$$

$$+ 2q_m \int_{\Sigma} y(\xi) d\mu \xrightarrow{m \rightarrow \infty} \int_{\Sigma} y(\xi) \tilde{\nu}(dG),$$

and therefore (d) is also verified. In order not to introduce new notations we will assume that the original sequence φ_m already satisfies (4.4).

The functions $\varphi_m(\xi)$ satisfy all requirements needed to solve the Minkowski problem for closed convex hypersurfaces, that is, for each φ_m there exists a unique closed convex hypersurface \tilde{S}_m of class $C^{k+2,\alpha}$, $0 < \alpha < 1$, such that at the point with the unit normal ξ its Gaussian curvature is $\varphi_m^{-1}(\xi)$ ([14], § 3, or [4]).

Since the area functions of \tilde{S}_m converge weakly to the area function $\tilde{\nu}$ of \tilde{S} , we can show that the diameters of the hypersurfaces \tilde{S}_m are uniformly bounded independently on m . The argument is similar to the one presented in 3.2.2 and we will not repeat it here. Thus, we can select from the sequence \tilde{S}_m a subsequence converging to a convex hypersurface \tilde{S}' . Since the area functions of \tilde{S}_m converge weakly to the area function $\tilde{\nu}$, \tilde{S}' has the same area function as \tilde{S} . Aleksandrov proved (see [2], p. 70) that there exists only one (up to a translation) convex hypersurface with the same surface function. Therefore we can assume that $\tilde{S}' \equiv \tilde{S}$.

Finally we note that the support functions h_m of \tilde{S}_m converge uniformly on Σ to the support function h of \tilde{S} .

4.6. LEMMA. *The support functions h_m and their first derivatives are uniformly bounded on \tilde{S}_m .*

PROOF. It follows from previous discussion that diameters of all \tilde{S}_m are uniformly bounded. By formula (1.3) the position vectors of \tilde{S}_m are given by

$$r_m = \text{grad } h_m + h_m \xi.$$

Therefore, h_m and $|\text{grad } h_m|$ are uniformly bounded.

4.7. LEMMA. *Let Ω be a convex domain on Σ lying strictly inside a hemisphere, and η an arbitrary interior point of $\omega = \Sigma \setminus \bar{\Omega}$. Then there exists a hyperplane A passing through \mathcal{O} which strictly separates η and $\bar{\Omega}$.*

PROOF. Denote by Σ^+ the open hemisphere that contains $\bar{\Omega}$ strictly inside. Three possibilities may occur: (i) η and $\bar{\Omega}$ lie in Σ^+ , (ii) $\eta \in \partial\Sigma^+$, (iii) $\eta \in \Sigma^-$, where $\Sigma^- = \Sigma \setminus \bar{\Sigma}^+$.

Consider first the case (i). A central projection of Σ^+ from \mathcal{O} onto the hyperplane A tangent to Σ^+ at the pole obviously will preserve convexity. Since $\bar{\Omega} \cap \partial\Sigma^+ = \emptyset$, the image $\bar{\Omega}'$ of $\bar{\Omega}$ will be a finite convex set on A disjoint from the point η' —the image of η . It is well known that in this case there exists an $(n-1)$ -dimensional plane π in the hyperplane A which strictly separates η' and $\bar{\Omega}'$. The plane π can be selected so that it is perpendicular to the segment realizing the shortest distance between η' and $\bar{\Omega}'$. Now it is clear that the hyperplane A' formed by the straight lines passing through \mathcal{O} and the points of π strictly separates η and $\bar{\Omega}$.

The case (ii) is obviously just the limiting case of (i) and can be treated similarly. In case (iii) the hyperplane determining the hypersphere Σ^+ can be taken as A' . The lemma is proved.

4.8. Let ξ be an arbitrary interior point of ω and α a hyperplane strictly separating ξ and $\bar{\Omega} = \Sigma \setminus \omega$. Let S be the hypersurface constructed in section 3 and h its support function. *In this section we prove the interior regularity of the hypersurface S .*

Let T be the hyperplane tangent to the pole of the hemisphere containing ξ strictly inside. Following the discussion in section 4.3 we associate with h the function $\tilde{h}(x) = (1 + \rho^2)^{\frac{1}{2}} h(x)$, $x \in T'$. The function h is uniformly bounded on Σ , and since the origin \mathcal{O} lies on $\tilde{S} \supset S$, $h \geq 0$ on Σ . Thus, $\tilde{h}(x) \xrightarrow{|x| \rightarrow \infty} +\infty$.

For the sequence h_m of support functions of smooth strictly convex hypersurfaces constructed in 4.5 we have the corresponding strictly convex functions $\tilde{h}_m(x)$. On every compact subset of T the sequence $\tilde{h}_m(x)$ converges

uniformly to $\tilde{h}(x)$. The function $\tilde{h}(x)$ is obviously also convex and for a sufficiently large constant $a > 0$ the set $\Gamma = \{x \in T' \mid \tilde{h}(x) \leq a\}$ is a nonempty compact convex set on T' . The functions $\tilde{h}'_m(x) = \tilde{h}_m(x) - a$ satisfy in Γ the equation

$$(4.5) \quad \det [(\tilde{h}'_m)_{ij}] = \varphi_m(1 + \varrho^2)^{-(n/2+1)}.$$

It follows easily from Lemma 4.6 that the functions \tilde{h}_m and their gradients are uniformly bounded on Γ . Since the functions $\varphi_m \in C^{k+1}$, $k \geq 3$, the functions $\tilde{h}'_m \in C^{k+2,\alpha}$. Shrinking the domain Γ to a domain Γ_1 , if necessary, we conclude from Theorems 4.4 and 4.4.1 that \tilde{h}'_m admit uniform estimates of the second and third derivatives. Then one can select a subsequence still converging to \tilde{h}' in Γ_1 , and $\tilde{h}' \in C^{2,1}$. Since \tilde{h}'_m satisfy (4.5), the function \tilde{h}' will satisfy the equation

$$(4.6) \quad \det (\tilde{h}'_{ij}) = \varphi(1 + \varrho^2)^{-((n/2)+1)}.$$

Put $\vartheta_m = \partial \tilde{h}'_m / \partial x_s$ for some s , $s = 1 \dots, n$. By differentiating (4.5) with respect to x_s we arrive at a linear differential equation for ϑ_m

$$(4.7) \quad A_m^{ij}(\vartheta_m)_{ij} = (\tilde{\varphi}_m)_s,$$

where A_m^{ij} is the cofactor of the element $(\tilde{h}'_m)_{ij}$ in the corresponding Hessian matrix of \tilde{h}'_m , and $(\tilde{\varphi}_m)_s$ the derivative of the right hand side of (4.5). Since the second derivatives of \tilde{h}'_m are uniformly bounded in Γ , φ is uniformly bounded away from zero (hence, so are the φ_m), and the equation (4.5) is satisfied, the equation (4.5) is elliptic in Γ_1 for all sufficiently large m . Moreover, the first derivatives of A_m^{ij} are Lipschitz continuous with the Lipschitz constant independent on m . Applying Schauder's estimates ([8], section 35) we can conclude that the $C^{2,1}$ -norms of ϑ_m are uniformly bounded. The same is true for any $s = 1, \dots, n$. Therefore, the $C^{3,1}$ -norms of \tilde{h}'_m are uniformly bounded. Under such circumstances we can select a subsequence of \tilde{h}'_m converging to \tilde{h}' in the $C^{3,\alpha}$ -norm for any $\alpha \in (0, 1)$. Hence, the solution \tilde{h}' of (4.6) is in $C^{3,\alpha}$. Differentiating the equation (4.5) twice and repeating the same arguments as before (with some obvious modifications) we conclude that \tilde{h}' belongs to $C^{4,\alpha}$, $0 < \alpha < 1$. In a similar fashion one establishes that if $\varphi \in C^k$, $k \geq 3$, then $\tilde{h}' \in C^{k+1,\alpha}$ for any α less than one; one only needs to require the sequence φ_m constructed in (4.5) to approximate φ in C^k . If φ is analytic then \tilde{h}' is analytic as it follows from the results in [8], section 44.

Finally, it should be pointed out that while shrinking the domain Γ to Γ_1 we could have left out the point ξ that we started with. But because

the set I' is defined for any sufficiently large a , we can select I' so that the image of ξ under the central projection on T and then on T' will be inside I'_1 . Since ξ was an arbitrary point of ω , the proof of the Theorem 4.2 is now complete.

REMARK. As the referee correctly pointed out, the Theorem 4.2 could have been also proved with the use of Theorem 3 in [5]. However, it seems to me that the proof presented above is more direct and more appropriate in this particular setting of the problem.

LITERATURE

- [1] A. D. ALEKSANDROV, *Convex Polyhedrons*, GITTL, M.-L., 1950.
- [2] H. BUSEMANN, *Convex Surfaces*, Interscience, New York, 1958.
- [3] E. CALABI, *Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens*, Michigan Math. J., **5** (1958), pp. 105-126.
- [4] S. Y. CHENG - S. T. YAU, *On the regularity of the solution of the n -dimensional Minkowski problem*, Comm. Pure Appl. Math., **29** (1976), pp. 499-516.
- [5] S. Y. CHENG - S. T. YAU, *On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial x_j) = F(x, u)$* , Comm. Pure Appl. Math., **30** (1977), pp. 41-68.
- [6] H. GLUCK, *The generalized Minkowski problem in differential geometry in the large*, Ann. of Math., **96** (1972), no. 2, pp. 245-276.
- [7] H. LEWY, *On differential geometry in the large, I (Minkowski problem)*, Trans. Amer. Math. Soc., **43** (1938), pp. 258-270.
- [8] C. MIRANDA, *Partial Differential Equations of Elliptic Type, Second Edition*, Ergebnisse der Mathematik, Band 2, Springer, New York, 1970.
- [9] L. NIRENBERG, *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure Appl. Math., **6** (1953), pp. 337-394.
- [10] V. I. OLIKER, *On the linearized Monge-Ampère equations related to the boundary value Minkowski problem and its generalizations, preprint*, Conference on Monge-Ampère Equations, Florence, 1980.
- [11] V. I. OLIKER, *On certain elliptic differential equations on a hypersphere and their geometric applications*, Indiana Univ. Math. J., **28** (1979), pp. 35-51.
- [12] A. V. POGORELOV, *Regularity of a convex surface with given Gaussian curvature*, Mat. Sb., **31** (73) (1952), pp. 88-103 (see also [13], chapter 7, § 3).
- [13] A. V. POGORELOV, *Extrinsic Geometry of Convex Surfaces*, Translations of mathematical monographs, V. **35**, Amer. Math. Soc., 1973.
- [14] A. V. POGORELOV, *Multidimensional Minkowski Problem*, Engl. transl.: John Wiley and Sons, New York, 1978.
- [15] A. V. POGORELOV, *An analogue of Minkowski problem for complete infinite convex hypersurfaces*, DAN USSR (1980), pp. 553-556.

The University of Iowa
 Department of Mathematics
 Iowa City, IA 52242