ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

F. BALDASSARRI

On inseparable descent

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 9, nº 3 (1982), p. 443-462

<http://www.numdam.org/item?id=ASNSP_1982_4_9_3_443_0>

© Scuola Normale Superiore, Pisa, 1982, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (http://www.sns.it/it/edizioni/riviste/annaliscienze/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

On Inseparable Descent.

F. BALDASSARRI

Introduction.

Let k be a perfect field of characteristic $p \neq 0$. Put $R = k[\![x]\!] =$ the ring of formal power series in x with coefficients in k, $R' = k[\![x^{p^{-\infty}}]\!] = \bigcup_{n=0}^{\infty} k[\![x^{p^{-n}}]\!]$, and Q, $Q' =$ the quotient fields of R, R', respectively. We also use the notation $W(\cdot)$ to denote the ring of infinite Witt vectors (relative to the prime number p) with components in $\cdot$, and put $K = W(k)$. Let A denote the ring $K[\![X]\!]$ of formal power series in X with coefficients in K, and let B denote the p-adic completion of the ring $A[1/X]$. We will define in section 3 embeddings of rings $A \to W(R')$ and $B \to W(Q')$.

The purpose of this paper is to give a manageable expression for descent data on modules relatively to the extensions $R \to R'$, $Q \to Q'$ and on p-adically separated and complete modules relatively to the extensions $A \to W(R')$, $B \to W(Q')$.

The simple form of the results obtained, say in the case $R \to R'$, depends on the following fact. Let $S = \operatorname{Spec} R$, $X = \operatorname{Spec} R'$, $G =$ the affine S-group Cartier dual to $(\mathbf{Q}_p/\mathbf{Z}_p)_S$ (the standard étale p-divisible group of height 1, viewed over S). Then it is possible to define a morphism of schemes: $G \times_S X \to X$, making $X \to S$ into a principal homogeneous space under G. We do not pursue in the present paper this geometric viewpoint: our aim here is not towards greatest generality but towards a complete understanding of the extensions of rings mentioned above.

We will apply the, results obtained here in subsequent papers to give a generalization of Dieudonné theory for p-divisible groups defined over R or Q.

This paper is essentially self-contained: we send to the references only for the proof of two theorems. Some computations are however left to the reader.

Pervenuto alla Redazione il 12 Settembre 1981.

In writing this paper we have been stronlgy influenced by the work of Barsotti: some of the constructions we use are due to him and others are direct generalizations of the former worked out in the same spirit.

1. – In this paper the word « ring » means « commutative ring with 1 »; a morphism of rings always sends 1 to 1 and « module » means « unitary module ». If k is a ring, a k-algebra will always be associative with a right and left identity element 1, and a morphism of k-algebras (a *representation*) will always send 1 to 1. If A, B are k-algebras, an *antirepresentation* $f\colon A \to B$ is a representation of the opposite k-algebra A^* of A in B.

If A is a linearly topologized (l.t.) ring and M, N are linearly topologized (l.t.) A-modules, the *usual topology* of $\mathrm{Hom}_A(M, N)$ will be the topology of simple convergence on the elements of M. The usual topology of $\mathrm{Hom}_A(M, N)$ is A-linear and a fundamental system of open submodules of $\mathrm{Hom}_A(M, N)$ in that topology is given by the set of the

$$\mathfrak{U}(S, V) = \{f \in \mathrm{Hom}_A(M, N)\colon f(S) \subseteq V\}$$

as S varies among the finite subsets of M and V among the open submodules of N. Unless otherwise specified A, M, N will be equipped with the discrete topology and $\mathrm{Hom}_A(M, N)$ with the usual topology.

If $f\colon A \to B$ is a morphism of rings and $g\colon M \to N$ is a morphism of A-modules, we denote by $g_{(f)}\colon M \otimes B \to N \otimes B$ the morphism of B-modules obtained by the scalar extension f.

(1.1) Definition. Let k be a l.t. ring, separated and complete. A *linearly topologized* (l.t.) *k-hyperalgebra* is a structure $(A, i, \mu, \mathbf{P}, \varepsilon, \varrho)$, that we usually denote simply by A, where:

(1.1.i) A is a separated and complete l.t. k-module;

(1.1.ii) $\mu\colon A \hat{\otimes}_k A \to A$ (« product ») and

$i\colon k \to A$ (« structural morphism »)

are continuous morphisms of k-modules satisfying:

(1.1.ii.1) $\mu(\mathrm{id}_A \hat{\otimes} \mu) = \mu(\mu \hat{\otimes} \mathrm{id}_A)$ (associativity),

(1.1.ii.2) $\mu(i \hat{\otimes} \mathrm{id}_A) = \mu(\mathrm{id}_A \hat{\otimes} i) = \mathrm{id}_A$ (existence of 1_A);

therefore (A, i, μ) is a l.t. k-algebra;

(1.1.iii) $\mathbf{P}\colon A \to A \hat{\otimes}_k A$ (« coproduct »),

$\varepsilon\colon A \to k$ (« augmentation »),

$\varrho\colon A \to A$ (« antipodism »),

are continuous morphisms of k-algebras such that:

(1.1.iii.1) $(\mathrm{id}_A \hat{\otimes} \mathbf{P})\mathbf{P} = (\mathbf{P} \hat{\otimes} \mathrm{id}_A)\mathbf{P}$ (coassociativity),

(1.1.iii.2) $(\mathrm{id}_A \hat{\otimes} \varepsilon)\mathbf{P} = (\varepsilon \hat{\otimes} \mathrm{id}_A)\mathbf{P} = \mathrm{id}_A$,

(1.1.iii.3) $\mu(\varrho \hat{\otimes} \mathrm{id}_A)\mathbf{P} = \mu(\mathrm{id}_A \hat{\otimes} \varrho)\mathbf{P} = i\varepsilon$.

Let $\varkappa\colon A \hat{\otimes}_k A \to A \hat{\otimes}_k A$ be the continuous k-linear map defined by $\varkappa(a \hat{\otimes} b) = b \hat{\otimes} a$. Then the l.t. k-hyperalgebra A is *commutative* if $\mu = \mu\varkappa$, and it is *cocommutative* if $\varkappa\mathbf{P} = \mathbf{P}$.

Morphisms of l.t. k-hyperalgebras are defined in the obvious way. The kernel of $\varepsilon\colon A \to k$ will be denoted by A^+ and will be called the *augmentation ideal* of A.

If k is a ring, a *k-hyperalgebra* is a discrete l.t. k-hyperalgebra, where k is equipped with the discrete topology.

(1.2) Definition. Let $k \to A$ be a morphism of rings and M be an A-module. A *descent datum* on M relatively to $k \to A$ is a homomorphism of $A \otimes_k A$-modules:

(1.2.0) $$\Theta\colon M \otimes_k A \to A \otimes_k M$$

such that:

(1.2.1) $$\Theta_{(\mu_A)} = \mathrm{id}_M;$$

(1.2.2) if $p_{ij}\colon A \otimes A \to A \otimes A \otimes A$ (e.g. $p_{32}(a \otimes b) = 1 \otimes b \otimes a$)

and $\Theta_{ij} = \Theta_{(p_{ij})}$, the diagram:

$$\begin{tikzcd} M \otimes A \otimes A \arrow[rr, "\Theta_{13}"] \arrow[dr, "\Theta_{12}"'] & & A \otimes A \otimes M \\ & A \otimes M \otimes A \arrow[ur, "\Theta_{23}"'] & \end{tikzcd}$$

is commutative.

It follows from (1.2.1), (1.2.2) that:

(1.2.3) Θ is an isomorphism.

Let us prove (1.2.3). Let $\mu_{13}\colon A\otimes A\otimes A\to A\otimes A$, $\mu_{13}(a\otimes b\otimes c)=ac\otimes b$ and $\Theta'_{ij}=(\Theta_{ij})_{(\mu_{13})}$ Then: $\Theta'_{13}=\mathrm{id}_{M\otimes A}$, $\Theta'_{12}=\Theta$, $\Theta'_{23}=\varkappa\Theta\varkappa \underset{\text{def.}}{=} \Theta_\varkappa$; therefore $\Theta_\varkappa\Theta=\mathrm{id}_{M\otimes A}$. Analogously $\Theta\Theta_\varkappa=\mathrm{id}_{A\otimes M}$: Q.E.D.

(1.3) GROTHENDIECK'S DESCENT THEOREM. *Let $k\to A$ be a morphism of rings and M_0 be a k-module. Then $M=M_0\otimes_k A$ is automatically equipped with a descent datum $\Theta=\Theta_{M_0}$, relative to $k\to A$, namely:*

$$\Theta_{M_0}\colon (M_0\otimes_k A)\otimes_k A\to A\otimes_k(M_0\otimes_k A) \tag{1.3.1}$$
$$(m\otimes a)\otimes b\mapsto a\otimes(m\otimes b)$$

for $m\in M_0$, $a, b\in A$.

Assume that $k\to A$ is faithfully flat and let M, Θ be as in (1.2). *Then,*

$$M_0=\{m\in M/\ \Theta(m\otimes 1)=1\otimes m\} \tag{1.3.2}$$

is a k-submodule of M, $M=M_0\otimes_k A$, and $\Theta=\Theta_{M_0}$. Moreover, if (M,Θ), (M',Θ') are two data as in (1.2), *if M_0 is given by* (1.3.2) *and M'_0 is given analogously, then an A-linear morphism $f\colon M\to M'$ is obtained by the scalar extension $k\to A$ from a k-linear morphism $f_0\colon M_0\to M'_0$ if and only if the following diagram commutes:*

$$\begin{tikzcd} M\otimes_k A \arrow[r, "f\otimes\mathrm{id}_A"] \arrow[d, "\Theta"'] & M'\otimes_k A \arrow[d, "\Theta'"] \\ A\otimes_k M \arrow[r, "\mathrm{id}_A\otimes f"] & A\otimes_k M' . \end{tikzcd} \tag{1.3.3}$$

(See [2] for the proof).

Let k be a ring, A a commutative faithfully flat k-algebra and D a commutative (but not necessarily cocommutative) k-hyperalgebra. Let

$$u\colon A\to D\otimes_k A \tag{1.4}$$

be a k-algebra morphism such that:

$$(\mathbf{P}_D\otimes\mathrm{id}_A)u=(\mathrm{id}_D\otimes u)u; \tag{1.4.1}$$

(1.4.2) if $p_2\colon A\to D\otimes A$ is defined by $p_2(a)=1\otimes a$, then

$$\chi=u\otimes p_2\colon A\otimes A\to D\otimes A$$

is a k-algebra isomorphism.

From (1.4.1) and (1.4.2) it follows that

$$(\varepsilon_D \otimes \mathrm{id}_A) u = \mathrm{id}_A . \tag{1.4.3}$$

Let us prove (1.4.3). Let $f = (\varepsilon_D \otimes \mathrm{id}_A) u$; we have: $uf = (\varepsilon_D \otimes u) u = (\varepsilon_D \otimes \mathrm{id}_D \otimes \mathrm{id}_A)(\mathrm{id}_D \otimes u) u = (\varepsilon_D \otimes \mathrm{id}_D \otimes \mathrm{id}_A)(\mathbb{P}_D \otimes \mathrm{id}_A) u = u$; but u is injective. Q.E.D.

Let τ denote the χ-linear isomorphism:

$$\begin{aligned} \tau\colon A \otimes_k M &\to D \otimes_k M \\ a \otimes m &\mapsto u(a)(1 \otimes m) . \end{aligned} \tag{1.4.4}$$

Suppose we are given M, Θ as in (1.2). Let us put:

$$\begin{aligned} \varphi\colon M &\to D \otimes_k M \\ m &\to \tau\big(\Theta(m \otimes 1)\big) . \end{aligned} \tag{1.5}$$

Then:

(1.5.1) $\varphi(am) = \tau\big((a \otimes 1)\Theta(m \otimes 1)\big) = u(a)\varphi(m)$;

(1.5.2) Let $m \in M$ and $\Theta(m \otimes 1) = \sum a_i \otimes m_i$: Then: $(\varepsilon_D \otimes \mathrm{id}_M)\varphi(m) = \sum_i (\varepsilon_D \otimes \mathrm{id}_M)\tau(a_i \otimes m_i) = \sum_i (\varepsilon_D \otimes \mathrm{id}_M)\big(u(a_i)(1 \otimes m_i)\big) = \sum_i \big((\varepsilon_D \otimes \mathrm{id}_A) u(a_i)\big) m_i = \sum_i a_i m_i = m$,

(the last equality follows from (1.2.1));

(1.5.3) Let $m \in M$, $\Theta(m \otimes 1) = \sum_i a_i \otimes m_i$, $\Theta(m_i \otimes 1) = \sum_j a_{ij} \otimes m_{ij}$.

According to (1.2.2) we have:

$$\sum_i a_i \otimes 1 \otimes m_i = \sum_{i,j} a_i \otimes a_{ij} \otimes m_{ij} . \tag{1.5.3.1}$$

Therefore:

$$(\mathbb{P}_D \otimes \mathrm{id}_M)\varphi(m) = (\mathbb{P}_D \otimes \mathrm{id}_M) \sum_i u(a_i)(1 \otimes m_i) =$$

$$\sum_i \big((\mathbb{P}_D \otimes \mathrm{id}_A) u(a_i)\big)(1 \otimes 1 \otimes m_i) = \sum_i \big((\mathrm{id}_D \otimes u) u(a_i)\big)(1 \otimes 1 \otimes m_i) =$$

$$(\mathrm{id}_D \otimes \tau) \sum_i u(a_i) \otimes m_i = (\mathrm{id}_D \otimes \tau) \sum_i (\chi \otimes \mathrm{id}_M)(a_i \otimes 1 \otimes m_i) =$$

$$(\mathrm{id}_D\otimes\tau)(\chi\otimes\mathrm{id}_M)\sum_{i,j}a_i\otimes a_{ij}\otimes m_{ij}=\sum_{i,j}\Big((\mathrm{id}_D\otimes u)\big(u(a_i)(1\otimes a_{ij})\big)\Big)(1\otimes 1\otimes m_{ij})=$$
$$\sum_{i,j}\big((\mathrm{id}_D\otimes u)u(a_i)\big)\big(1\otimes\big(u(a_{ij})(1\otimes m_{ij})\big)\big)=$$
$$\sum_i\big((\mathrm{id}_D\otimes u)u(a_i)\big)\big(1\otimes\varphi(m_i)\big)=\sum_i(\mathrm{id}_D\otimes\varphi)\big(u(a_i)(1\otimes m_i)\big)=(\mathrm{id}_D\otimes\varphi)\varphi(m).$$

We conclude from the computations (1.5) that from a descent datum Θ on M relatively to $k\to A$, if the morphism $u\colon A\to D\otimes_k A$, as in (1.4), is given, we obtain a k-linear morphism:

(1.6) $$\varphi\colon M\to D\otimes_k M$$

satisfying:

(1.6.1) $\varphi(am)=u(a)\varphi(m)$, for a in A and m in M;

(1.6.2) $(\varepsilon_D\otimes\mathrm{id}_M)\varphi=\mathrm{id}_M$;

(1.6.3) $(\mathbf{P}_D\otimes\mathrm{id}_M)\varphi=(\mathrm{id}_D\otimes\varphi)\varphi$.

Conversely, let φ as in (1.6) be given, and define $\Theta\colon M\otimes A\to A\otimes M$, by:

(1.7) $$\Theta(m\otimes a)=\tau^{-1}\big((1\otimes a)\varphi(m)\big).$$

Then Θ is $A\otimes A$-linear, and:

(1.7.1) if $m\in M$ and $\Theta(m\otimes 1)=\tau^{-1}\varphi(m)=\sum_i a_i\otimes m_i$, then $\varphi(m)=\sum_i u(a_i)(1\otimes m_i)$ so that $\sum_i a_im_i=(\varepsilon_D\otimes\mathrm{id}_M)\sum_i u(a_i)(1\otimes m_i)=(\varepsilon_D\otimes\mathrm{id}_M)\varphi(m)=m$;

(1.7.2) if $\Theta(m\otimes 1)=\sum_i a_i\otimes m_i$ and $\Theta(m_i\otimes 1)=\sum_j a_{ij}\otimes m_{ij}$, by inverting the reasoning used in (1.5.3) one proves that $\sum_i a_i\otimes 1\otimes m_i=\sum_{i,j}a_i\otimes a_{ij}\otimes m_{ij}$ and therefore (1.2.2) for this Θ.

It follows from Grothendieck's theory of descent ((1.3)), that if M, φ are as in (1.6) and one puts:

(1.8) $$M_0=\{m\in M/\ \varphi(m)=1\otimes m\}$$

then $M = A \otimes_k M_0$ and $\varphi(a \otimes m) = u(a)(1 \otimes m) = u(a) \otimes m$. If now we denote φ by φ_M and let (N, φ_N) be a datum analogous to (M, φ_M), a A-linear homomorphism $f\colon M \to N$ is the extension by A-linearity of a k-linear homomorphism $f_0\colon M_0 \to N_0$ iff the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varphi_M"'] & N \arrow[d, "\varphi_N"] \\ D \otimes_k M \arrow[r, "\mathrm{id}_D \otimes f"] & D \otimes_k N \,. \end{tikzcd} \tag{1.8.1}$$

We will assume in the rest of this section that D, as a k-module, is the direct limit of a direct system of finite locally free k-modules, say $D = \varinjlim_\alpha D_\alpha$. Then if k is given the discrete topology and the k-modules $C_\alpha = \mathrm{Hom}_k(D_\alpha, k)$, $C = \mathrm{Hom}_k(D, k)$ are given the usual topology, $C = \varprojlim_\alpha C_\alpha$, with the inverse limit topology (the topology of C_α is the discrete). Besides $\mathrm{Hom}_k(D_\alpha \otimes D_\alpha, k) = C_\alpha \otimes_k C_\alpha$ and $\mathrm{Hom}_k(D \otimes D, k)$, with the usual topology, equals $\varprojlim_\alpha C_\alpha \otimes C_\alpha = C \hat{\otimes}_k C$. Under our assumptions, C, endowed with the operations dualizing those of D, is a l.t. k-hyperalgebra (commutative but not necessarily commutative) called the *Cartier dual* of D. Let us put:

$$\begin{aligned} &S\colon C \to \mathrm{End}_k D \\ &c \mapsto S_c\,, \quad \text{where} \\ &S_c(d) = (c \otimes \mathrm{id}_D)\,\mathbb{P}_D d\,, \quad \text{for } d \in D\,, \end{aligned} \tag{1.9}$$

a continuous injective antirepresentation of k-algebras (the topology of D is the discrete). If $c \in C$ and $d \in D$ we will denote $c(d) \in k$ by $c \circ d$ and $S_c(d) \in D$ by cd; the previous formula then reads:

$$cd = (c\circ \otimes \mathrm{id}_D)\,\mathbb{P}_D d\,. \tag{1.9.1}$$

One can prove that:

$$\begin{aligned} \mathbb{P}_D(cd) &= (c \otimes \mathrm{id}_D)\,\mathbb{P}_D d \\ c(dd') &= \mu_D\big((\mathbb{P}_C c)(d \otimes d')\big) \\ (cc')d &= c'(cd) \\ c\circ d &= \varepsilon_D(cd) \end{aligned} \tag{1.9.2}$$

for $c, c' \in C$ and $d, d' \in D$. The second formula in (1.9.2), as many similar formulas to come, should be interpreted as follows. Suppose $\mathbb{P}_C c = \sum_{j,h} c_j \hat{\otimes} c_h$ (a converging sum in $C \hat{\otimes}_k C$); then $(\mathbb{P}_C c)(d \otimes d') = \sum_{j,h} c_j d \otimes c_k d' = \sum_{j,h} S_{c_j}(d) \otimes \otimes S_{c_h}(d')$ (a finite sum in $D \otimes_k D$). Therefore $c(dd') = \sum_{j,h} (c_j d)(c_k d')$.

Furthermore, if D is free with basis $\{d_i, i \in I\}$ over k and if $\{c_i, i \in I\}$ denotes the dual topological k-basis of C, we have:

$$\mathbb{P}_D d = \sum_{i \in I} d_i \otimes c_i d \,. \tag{1.10}$$

Given $u\colon A \to D \otimes A$ as in (1.4), let us put:

$$\begin{aligned} &T\colon C \to \mathrm{End}_k A \\ &c \mapsto T_c\,, && \text{where} \\ &T_c(a) = (c \circ \otimes \mathrm{id}_A) u(a)\,, && \text{for } a \in A\,. \end{aligned} \tag{1.11}$$

Again, T is a continuous antirepresentation of k-algebras (the topology of A being the discrete). If $T_c(a)$ is denoted by ca, one has, for $c \in C$ and $a, a' \in A$:

$$\begin{aligned} u(ca) &= (c \otimes \mathrm{id}_A) u(a) \\ c(aa') &= \mu_A\big((\mathbb{P}_C c)(a \otimes a')\big)\,. \end{aligned} \tag{1.12}$$

The second formula in (1.12) means that $c(aa') = \sum_{j,h} (c_j a)(c_h a')$, if $\mathbb{P}_C c = = \sum_{j,h} c_j \hat{\otimes} c_h$.

If D is free we have again:

$$u(a) = \sum_{i \in I} d_i \otimes c_i a\,. \tag{1.13}$$

Analogously, given $\varphi\colon M \to D \otimes M$ as in (1.6), we define:

$$\begin{aligned} &U\colon C \to \mathrm{End}_k M \\ &c \mapsto U_c\,, && \text{where} \\ &U_c(m) = (c \circ \otimes \mathrm{id}_M) \varphi(m)\,, && \text{for } m \in M\,. \end{aligned} \tag{1.14}$$

Once again, U is a continuous antirepresentation of k-algebras (the topology of M being the discrete) and, after writing cm for $U_c(m)$, it satisfies:

$$\begin{aligned} \varphi(cm) &= (c \otimes \mathrm{id}_M) \varphi(m) \\ c(am) &= \mu_{sc}\big((\mathbb{P}_C c)(a \otimes m)\big) \end{aligned} \tag{1.15}$$

for any $c \in C$, $a \in A$, $m \in M$, where $\mu_{sc}\colon A \otimes_k M \to M$ denotes the scalar products. So $c(am) = \sum_{j,h} (c_j a)(c_h m) = \sum_{j,h} T_{c_j}(a)\, U_{c_h}(m)$, if $\mathbf{P}_C c = \sum_{j,h} c_j \hat{\otimes} c_h$. If D is free, we have again:

$$\varphi(m) = \sum_{i \in I} d_i \otimes c_i m\,. \tag{1.16}$$

Conversely, let $U\colon C \to \mathrm{End}_k M$, for a (discrete) A-module M, be a continuous antirepresentation of k-algebras satisfying the second formula in (1.15); assume that D is free. Then $\varphi\colon M \to D \otimes M$ defined by (1.16) satisfies (1.6.1, 2, 3). Formula (1.7) now becomes:

$$M_0 = \{m \in M \,/\, cm = 0, \text{ for all } c \in C^+\}\,. \tag{1.17}$$

Clearly:

$$M_0 = \{m \in M \,/\, cm = (\varepsilon_C c)m, \text{ for all } c \text{ in } C\}\,, \tag{1.18}$$

and, if $c \in C$, $m = m_0 \otimes a \in M = M_0 \otimes_k A$, then:

$$cm = U_c(m) = m_0 \otimes T_c(a) = m_0 \otimes ca\,. \tag{1.19}$$

As a consequence of the considerations above we will say that the map U of (1.14) is a *descent datum* on M relatively to $k \to A$. Let now M and N be two A-modules with descent data relatively to $k \to A$. An A-linear map $f\colon M \to N$ is the extension by A-linearity of a k-linear map $f_0\colon M_0 \to N_0$ iff:

$$f(cm) = cf(m)\,, \quad \text{for all } c \text{ in } C \text{ and } m \text{ in } M\,. \tag{1.20}$$

One verifies immediately that the induced descent data on $M \otimes_A N$ and $\mathrm{Hom}_A(M, N)$ can be respectively expressed as follows:

$$c(m \otimes n) = (\mathbf{P}c)(m \otimes n)\,, \tag{1.21}$$

$$(cf)(m) = \sum_{j,h} c_j\big(f((\varrho_C c_h)m)\big)\,, \tag{1.22}$$

if $c \in C$, $\mathbf{P}c = \sum_{j,h} c_j \hat{\otimes} c_h$ (a converging series in $C \hat{\otimes} C$), $m \in M$, $n \in N$, $f \in \mathrm{Hom}_A(M, N)$. The right-hand term in (1.21) is to be interpreted in the following way. Let $t\colon M \otimes_k N \to M \otimes_A N$ be the canonical map. Then $c(m \otimes n) = c(m \otimes_A n) = t\big((\mathbf{P}c)(m \otimes_k n)\big) = \sum_{j,h} (c_j m)(c_h n)$ (a finite sum in $M \otimes_A N$). Notice that this is a good definition.

2. – Let k be a perfect field of characteristic $p \neq 0$ and, for $n \in \mathbf{N}$, let $K_n = W_n(k)$ be the ring of Witt vectors (relative to the prime number p) of length n with components in k; in particular $K_1 = k$. Let $K_n[\![x]\!]$ be the affine algebra of the standard multiplicative formal group G_{K_n} over K_n: $K_n[\![x]\!]$ is endowed with the (x)-adic topology and it is the l.t. K_n-hyperalgebra (K_n is discrete) whose coproduct $\mathbf{P}$ and augmentation ε are given by:

$$\mathbf{P}(x) = 1 \hat{\otimes} x + x \hat{\otimes} 1 + x \hat{\otimes} x\,, \qquad \varepsilon(x) = 0\,. \tag{2.1}$$

For any m in $\mathbf{N}$, the multiplication by p^m of G_{K_n} (in additive notation) is expressed by the continuous morphism of K_n-algebras:

$$\begin{aligned} P_m\colon K_n[\![x]\!] &\to K_n[\![x]\!] \\ x &\mapsto (1+x)^{p^m} - 1\,. \end{aligned} \tag{2.2}$$

In fact P_m is an injective homomorphism of l.t. K_n-hyperalgebras. Let us regard P_m as an embedding of $K_n[\![x]\!]$ in another copy of itself that we denote by $K_n[\![x_m]\!]$; namely we put:

$$\begin{aligned} &P_m\colon K_n[\![x]\!] \hookrightarrow K_n[\![x_m]\!] \\ &P_m(x) = x = (1 + x_m)^{p^m} - 1\,. \end{aligned} \tag{2.3}$$

One immediately checks that $K_n[\![x_m]\!]$ is freely generated as a $K_n[\![x]\!]$-module by $\{1, x_m, x_m^2, \ldots, x_m^{p^m-1}\}$. Let us denote by $K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]$ the group hyperalgebra of the group $p^{-m}\mathbf{Z}_p/\mathbf{Z}_p$ (that is the Cartier dual of its affine algebra). Explicitly we have: $K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]$ is the free K_n-module generated by the symbols $\{Y_g,\ g \in p^{-m}\mathbf{Z}_p/\mathbf{Z}_p\}$, and:

$$\begin{aligned} Y_g Y_h &= Y_{g+h} \\ \mathbf{P} Y_g &= Y_g \otimes Y_g \\ \varepsilon Y_g &= 1 \end{aligned} \tag{2.4}$$

for any g, h in $p^{-m}\mathbf{Z}_p/\mathbf{Z}_p$. It is clear that the K_n-module homomorphism:

$$\begin{aligned} &K_n[\![x_m]\!] \underset{K_n[\![x]\!] \nearrow \varepsilon}{\otimes} K_n \to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \\ &(1 + x_m)^i \otimes 1 \mapsto Y_{ip^{-m}+\mathbf{Z}_p} \end{aligned} \tag{2.5}$$

is an isomorphism of K_n-hyperalgebras (notice that $K_n[\![x_m]\!] \bigotimes_{K_n[\![x]\!] \nearrow \varepsilon} K_n$ is naturally a K_n-hyperalgebra). We deduce from (2.5) a surjective morphism of l.t. K_n-hyperalgebras ($K_n[p^{-m}\,\mathbf{Z}_p/\mathbf{Z}_p]$ is discrete), with kernel $xK_n[\![x_m]\!]$:

$$\sigma_m\colon K_n[\![x_m]\!] \to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \qquad x_m \mapsto Y_{p^{-m}+\mathbf{Z}_p} - 1\,. \tag{2.6}$$

If we denote by ${}_{p^m}G_{K_n}$ the finite multiplicative K_n-group whose affine algebra is $K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]$, we have proved above that the sequence:

$$0 \to {}_{p^m}G_{K_n} \to G_{K_n} \xrightarrow{p^m} G_{K_n} \to 0 \tag{2.7}$$

is exact (in the category of (faithfully) flat sheaves of abelian groups on finite K_n-algebras).

Let us define now:

$$u_m\colon K_n[\![x_m]\!] \to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \bigotimes_{K_n} K_n[\![x_m]\!] \qquad f \mapsto (\sigma_m \hat{\otimes}\, \mathrm{id})\,\mathbf{P}f \tag{2.8}$$

(notice that, since $K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]$ is a finite K_n-module and $K_n[\![x_m]\!]$ is complete, we could replace $\hat{\otimes}$ by $\otimes$ in (2.8)). Clearly, u_m is a $K_n[\![x]\!]$-algebra morphism and it is determined, as a $K_n[\![x]\!]$-linear map, by:

$$u_m(1 + x_m)^a = Y_{ap^{-m}+\mathbf{Z}_p} \otimes (1 + x_m)^a\,, \quad \text{for } a \in \mathbf{N}\,. \tag{2.8.1}$$

We would like to prove that u_m satisfies to the properties required for u in (1.4), with the following replacements: (in the left-hand column of (2.9) find the symbols of section 1 while in the right-hand one find the symbols replacing them)

$$\begin{array}{lcl} k & , & K_n[\![x]\!] \\ A & , & K_n[\![x_m]\!] \\ D & , & K_n[\![x]\!] \otimes_{K_n} K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \\ D \otimes_k A\,, & & K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \otimes_{K_n} K_n[\![x_m]\!]\,. \end{array} \tag{2.9}$$

In the first place since $K_n[\![x]\!] \hookrightarrow K_n[\![x_m]\!]$ is free, it is also faithfully flat. (1.4.1) is obvious. Let us check (1.4.2). We observe first that:

$$l\colon K_n[\![x_m]\!] \hat{\bigotimes}_{K_n} K_n[\![x_m]\!] \to K_n[\![x_m]\!] \hat{\bigotimes}_{K_n} K_n[\![x_m]\!] \qquad f \hat{\otimes} g \to (\mathbf{P}f)(1 \hat{\otimes} g) \tag{2.10}$$

is a continuous K_n-algebra isomorphism. The inverse of l is:

$$r\colon f\hat{\otimes} g \to ((\mathrm{id}\hat{\otimes}\varrho)\mathbf{P}f)(1\otimes g)\,. \tag{2.11}$$

Let us now consider the diagram:

$$\begin{tikzcd}
K_n[\![x_m]\!]\hat{\bigotimes}_{K_n} K_n[\![x_m]\!] \arrow[r, shift left, "l"] \arrow[d, "\mathrm{can.}"] & K_n[\![x_m]\!]\hat{\bigotimes}_{K_n} K_n[\![x_m]\!] \arrow[l, shift left, "r"] \arrow[d, "\sigma_m\hat{\otimes}\mathrm{id}"] \\
K_n[\![x_m]\!]\bigotimes_{K_n[\![x]\!]} K_n[\![x_m]\!] \arrow[r, shift left, "\bar{l}"] & K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]\hat{\bigotimes}_{K_n} K_n[\![x_m]\!]\,. \arrow[l, shift left, "\bar{r}"]
\end{tikzcd} \tag{2.12}$$

If we prove that in (2.12) the barred arrows exist in such a way that the resulting diagram is commutative, (1.4.2) will follow for u_m. We would have in fact then $\bar{l}(f\otimes g) = u_m(f)(1\otimes g)$ and $\bar{r}\bar{l} = \mathrm{id}_{K_n[\![x]\!]\bigotimes_{K_n[[x]]} K_n[\![x_m]\!]}$ and $\bar{l}\bar{r} = \mathrm{id}_{K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]\bigotimes_{K_n} K_n[\![x_m]\!]}$. To show the existence of $\bar{l}$ it is enough to prove that $(\sigma_m\hat{\otimes}\,\mathrm{id})\,l(fg\hat{\otimes} h) = (\sigma_m\hat{\otimes}\,\mathrm{id})\,l(g\hat{\otimes} fh)$, for any f in $K_n[\![x]\!]$ and g, h in $K_n[\![x_m]\!]$. Now l is right $K_n[\![x_m]\!]$-linear so that we can put $h = 1$. We have to prove that: $(\sigma_m\hat{\otimes}\,\mathrm{id})(\mathbf{P}f)(\mathbf{P}g) = (1\otimes f)(\sigma_m\hat{\otimes}\,\mathrm{id})\mathbf{P}g$, if $f\in K_n[\![x]\!]$ and $g\in K_n[\![x_m]\!]$. This follows from the fact that the kernel of σ_m is $xK_n[\![x_m]\!]$.

For the existence of $\bar{r}$, it is enough to prove that $r(x\hat{\otimes} 1)$ has zero image in $K_n[\![x_m]\!]\bigotimes_{K_n[\![x]\!]} K_n[\![x_m]\!]$. Now we have $r(x\hat{\otimes} 1) = (\mathrm{id}\hat{\otimes}\varrho)\mathbf{P}x \in K_n[\![x]\!]\hat{\bigotimes}_{K_n} K_n[\![x]\!]$; its image in $K_n[\![x_m]\!]\bigotimes_{K_n[\![x]\!]} K_n[\![x_m]\!]$ coincides with $\mu_{K_n[\![x]\!]}(\mathrm{id}\otimes\varrho)\mathbf{P}x = i\varepsilon x = 0$ ($i = i_{K_n[\![x]\!]}$ is as usual the structural morphism of $K_n[\![x]\!]$).

We conclude that the map u_m defined in (2.8) is a $K_n[\![x]\!]$-algebra homomorphism:

$$u_m\colon K_n[\![x_m]\!] \to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]\hat{\bigotimes}_{K_n} K_n[\![x_m]\!] \tag{2.13}$$

such that:

$$(\mathrm{id}\otimes u_m)\,u_m = (\mathbf{P}\otimes\mathrm{id})\,u_m \tag{2.13.1}$$

and

$$\text{the map: } K_n[\![x_m]\!]\bigotimes_{K_n[\![x]\!]} K_n[\![x_m]\!] \to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p]\bigotimes_{K_n} K_n[\![x_m]\!] \tag{2.13.2}$$
$$f\otimes g \to u_m(f)(1\otimes g)$$

is an isomorphism of $K_n[\![x]\!]$-algebras.

We are exactly in the situation of section 1 with the substitutions indicated in (2.9).

At this point we need to make a typographical specification: the x_m belonging to $K_n[\![x_m]\!]$ will be denoted by $x_m^{(n)}$; the symbol x_m will denote only $x_m^{(1)}$, and we also write x for x_0, so that $x_m = x^{p^{-m}}$, for m in N. We also put $x_0^{(n)} = x^{(n)}$. Let $A_n = \bigcup_{m=0}^{+\infty} K_n[\![x_m^{(n)}]\!]$. We want to prove that $A_n = W_n(A_1)$, where A_1 obviously coincides with the perfectionate $k[\![x^{p^{-\infty}}]\!]$ of $k[\![x]\!]$. Let us denote by $\varphi_{n,m}$ the continuous ring homomorphism $\varphi_{n,m}$: $K_{n+1}[\![x_m^{(n+1)}]\!] \to K_n[\![x_m^{(n)}]\!]$ extending the natural map (reduction modulo p^n) $K_{n+1} \to K_n$, such that $\varphi_{n,m}(x_m^{(n+1)}) = x_m^{(n)}$. Let then φ_n: $A_{n+1} \to A_n$ be defined by $\varphi_n(a) = \varphi_{n,m}(a)$ if $a \in K_{n+1}[\![x_m^{(n+1)}]\!]$. Let us regard A_n as a discrete l.t. ring. Then $A = \varprojlim_n (A_n, \varphi_n)$ is a strict p-ring in the sense of [1], chap. II, sect. 5, and it coincides then with $W(A_1)$. It follows that $A_n = W_n(A_1)$. If the symbol $[a]$ denotes the multiplicative representative in $W_n(A_1)$ of $a \in A_1$, we have $[1 + x] = 1 + x^{(n)}$ and, in general, $[1 + x^{p^{-m}}] = 1 + x_m^{(n)}$, in the identification above. A word of caution on the embedding of $K_n[\![x_m^{(n)}]\!]$ in $W_n(A_1)$. Let us (provisionally) topologize A_1 with the (x)-adic topology and $W_n(A_1)$ (in $1-1$ correspondence with A_1^n) with the product topology (notice that this topology coincides with the $([x])$-adic). Then the embedding above is characterized as a continuous K_n-algebra morphism ($K_n[\![x_m^{(n)}]\!]$ being endowed with the $(x_m^{(n)})$-adic topology) by the assignment $x_m^{(n)} \mapsto$ $\mapsto [1 + x^{p^{-m}}] - 1$. In the sequel, no topology will be given to $K_n[\![x_m^{(n)}]\!]$ or to $W_n(A_1)$, but by « the embedding $x_m^{(n)} \mapsto [1 + x^{p^{-m}}] - 1$ », we will always mean the one described above.

Let us fix n and put $x_m^{(n)} = X_m$, $X_0 = X$. By taking direct limits in (2.13) we get a morphism of $K_n[\![X]\!]$-algebras:

$$u\colon W_n(A_1) \to K_n[\mathbf{Q}_p/\mathbf{Z}_p] \bigotimes_{K_n} W_n(A_1)\,, \tag{2.14}$$

where $K_n[\mathbf{Q}_p/\mathbf{Z}_p]$ is the free K_n-module generated by $\{Y_g,\ g \in \mathbf{Q}_p/\mathbf{Z}_p\}$, endowed with the hyperalgebra operations defined by formulas (2.4). The morphism of $K_n[\![X]\!]$-algebras u is determined by the relations:

$$u([1 + x^{p^{-m}}]) = Y_{p^{-m}+\mathbf{Z}_p} \otimes [1 + x^{p^{-m}}]\,, \tag{2.14.0}$$

for $m \in \mathbf{N}$. It satisfies:

$$(\mathbf{P} \otimes \mathrm{id})\, u = (\mathrm{id} \otimes u)\, u\,; \tag{2.14.1}$$

$$\begin{gathered} W_n(A_1) \bigotimes_{K_n[\![X]\!]} W_n(A_1) \to K_n[\mathbf{Q}_p/\mathbf{Z}_p] \bigotimes_{K_n} W_n(A_1) \\ f \otimes g \mapsto u(f)(1 \otimes g)\,, \end{gathered} \tag{2.14.2}$$

is an isomorphism of $K_n[\![X]\!]$-algebras.

Moreover the morphism $K_n[\![X]\!] \hookrightarrow W_n(A_1)$ is faithfully flat. (The facts just stated follow from the properties of direct limits). We are then in a position to apply the theory of section 1 to get information on the descent relatively to $K_n[\![X]\!] \hookrightarrow W_n(k[\![x^{p^{-\infty}}]\!])$, $X \mapsto [1+x]-1$.

Let us carry out in detail the constructions of section 1 for the present data. The next diagram indicates the replacements to be operated (as before the right-hand column replaces the left-hand one):

$$
\begin{array}{rcl}
k & , & K_n[\![X]\!] \\
A & , & W_n(k[\![x^{p^{-\infty}}]\!]) \\
k \to A & , & X \mapsto [1+x]-1 \\
D & , & K_n[\![X]\!] \bigotimes\limits_{K_n} K_n[\mathbb{Q}_p/\mathbb{Z}_p] \\
D \bigotimes\limits_{k} A & , & K_n[\mathbb{Q}_p/\mathbb{Z}_p] \bigotimes\limits_{K_n} W_n(k[\![x^{p^{-\infty}}]\!]) \,.
\end{array}
\tag{2.15}
$$

Let $F_n = \mathrm{Hom}_{K_n}(K_n[\mathbb{Q}_p/\mathbb{Z}_p], K_n)$ be the l.t. K_n-hyperalgebra Cartier dual to $K_n[\mathbb{Q}_p/\mathbb{Z}_p]$. We can obviously identify F_n with the l.t. K_n-hyperalgebra of functions defined on the group $\mathbb{Q}_p/\mathbb{Z}_p$ taking values in K_n, endowed with the topology of simple convergence on $\mathbb{Q}_p/\mathbb{Z}_p$ with respect to the discrete topology of K_n. (We recall that $\mathbb{P}\colon F_n \to F_n \hat{\otimes} F_n$ is defined by identifying $F_n \hat{\bigotimes\limits_{K_n}} F_n$ with the K_n-algebra of functions from $\mathbb{Q}_p/\mathbb{Z}_p \times \mathbb{Q}_p/\mathbb{Z}_p$ to K_n, endowed with the topology of simple convergence, and by putting $(\mathbb{P}f)(a, b) = f(a+b)$, for f in F_n and a, b in $\mathbb{Q}_p/\mathbb{Z}_p$, and $\varepsilon f = 0$.)

Such an identification is obtained by interpreting $f\colon \mathbb{Q}_p/\mathbb{Z}_p \to K_n$ as the K_n-linear map: $\sum\limits_{i \in \mathbb{Q}_p/\mathbb{Z}_p} a_i Y_i \mapsto \sum\limits_{i \in \mathbb{Q}_p/\mathbb{Z}_p} a_i f(i)$, from $K_n[\mathbb{Q}_p/\mathbb{Z}_p]$ to K_n. Notice that F_n can naturally be identified with $W_n(F_1)$ as a K_n-algebra: $f \in F_n$ is identified with $(f_0, \ldots, f_{n-1}) \in W_n(F_1)$ if $f_i = c_i f$, where for $i = 0, \ldots, n-1$, $c_i\colon K_n = W_n(k) \to k$ is the function « i-th component » of a Witt vector. The topology of F_n corresponds then to the product topology of the topology of F_1, in the natural bijection $W_n(F_1) \leftrightarrow F_1^n$. We can also identify $W_n(F_1 \hat{\bigotimes\limits_{K_n}} F_1)$ with $W_n(F_1) \hat{\bigotimes\limits_{K_n}} W_n(F_1)$, since they are both isomorphic to the K_n-algebra of functions $f\colon \mathbb{Q}_p/\mathbb{Z}_p \times \mathbb{Q}_p/\mathbb{Z}_p \to K_n$ endowed with the topology of simple convergence. The coproduct of F_n then corresponds to the map:

$$
\begin{aligned}
W_n(\mathbb{P}_{F_1})\colon W_n(F_1) &\to W_n\big(F_1 \hat{\bigotimes\limits_{K}} F_1\big) = W_n(F_1) \hat{\bigotimes\limits_{K_n}} W_n(F_1) \\
(f_0, \ldots, f_{n-1}) &\mapsto (\mathbb{P}_{F_1} f_0, \ldots, \mathbb{P}_{F_1} f_{n-1}) \,.
\end{aligned}
\tag{2.16}
$$

To pursue the correspondence with section 1, we see that the l.t. k-hyperalgebra C is now replaced by $K_n[\![X]\!] \hat{\bigotimes}_{K_n} F_n$ (or $K_n[\![X]\!] \hat{\bigotimes}_{K_n} W_n(F_1)$) where $\hat{\bigotimes}_{K_n}$ is taken with respect to the discrete topology of $K_n[\![X]\!]$. The representation S of (1.9) is now the extension by $K_n[\![X]\!]$-linearity of the K_n-linear continuous representation:

$$\begin{aligned} &S' \colon F_n \to \mathrm{End}_{K_n} K_n[\mathbf{Q}_p/\mathbf{Z}_p] \\ &\qquad f \mapsto S'_f\,, \qquad \text{where} \\ &S'_f(Y_g) = f(g)\, Y_g\,, \qquad \text{for } g \text{ in } \mathbf{Q}_p/\mathbf{Z}_p\,. \end{aligned} \tag{2.17}$$

Similarly the T of (1.11) is the extension by $K_n[\![X]\!]$-linearity of the K_n-linear continuous representation:

$$\begin{aligned} &T' \colon F_n \to \mathrm{End}_{K_n[\![X]\!]} W_n(k[\![x^{p^{-\infty}}]\!]) \\ &\qquad f \mapsto T'_f\,, \qquad\qquad \text{where} \\ &T'_f((1+X)^{ap^m}) = f(ap^m + \mathbf{Z}_p)(1+X)^{ap^m}\,, \qquad \text{for } a, m \text{ in } \mathbf{Z}\,. \end{aligned} \tag{2.18}$$

We leave to the reader the verification of the formulas in (2.17) and (2.18). We then conclude from section 1, that a descent datum on a $W_n(k[\![x^{p^{-\infty}}]\!])$-module M relatively to $K_n[\![X]\!] \hookrightarrow W_n(k[\![x^{p^{-\infty}}]\!])$, $X \mapsto [1+x]-1$, is equivalent to a continuous representation of K_n-algebras:

$$\begin{aligned} U' \colon F_n &\to \mathrm{End}_{K_n} M \\ f &\mapsto U'_f \end{aligned} \tag{2.19}$$

such that (after skipping the symbols U', T' and denoting by

$$\mu_{sc} \colon W_n(k[\![x^{p^{-\infty}}]\!]) \bigotimes_{K_n} M \to M$$

the scalar product):

$$d(rm) = \mu_{sc}\Big((\mathbf{P}d)\Big(r \bigotimes_{K_n} m\Big)\Big), \tag{2.20}$$

for any d in F_n, r in $W_n(k[\![x^{p^{-\infty}}]\!])$, m in M. Notice that each U'_f, for f in F_n, is then in fact $K_n[\![X]\!]$-linear.

Since $K_n[\![X]\!] \hookrightarrow W_n(k[\![x^{p^{-\infty}}]\!])$ is faithfully flat, all descent data with respect to it are effective and therefore if one puts:

$$M_0 = \{m \in M \,/\, dm = 0 \text{ for } d \in F_n^+\}\,, \tag{2.21}$$

one concludes that M_0 is a $K_n[\![X]\!]$-module, that $M = W_n(k[\![x^{p^{-\infty}}]\!]) \bigotimes_{K^n[\![X]\!]} M_0$, and that $d(r \otimes m) = dr \otimes m$ for each d in F_n, r in $W_n(k[\![x^{p^{-\infty}}]\!])$, and m in M_0.

Since u_m in (2.13) is a $K_n[\![X]\!]$-algebra morphism, we can extend it by $K_n[\![X]\!][1/X]$-linearity to:

$$(2.22) \qquad u'_m\colon K_n[\![X_m]\!][1/X] \to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \bigotimes_{K_n} K_n[\![X_m]\!][1/X]$$

satisfying:

$$(2.22.1) \qquad (\mathrm{id} \otimes u'_m)\, u'_m = (\mathbf{P} \otimes \mathrm{id})\, u'_m;$$

(2.22.2) the map:

$$K_n[\![X_m]\!][1/X] \bigotimes_{K_n[\![X]\!][1/X]} K_n[\![X_m]\!][1/X] \to K_n[p^{-m}\mathbf{Z}_p/\mathbf{Z}_p] \bigotimes_{K_n} K_n[\![X_m]\!][1/X]$$
$$f \otimes g \mapsto u'_m(f)(1 \otimes g),$$

is an isomorphism of $K_n[\![X]\!][1/X]$-algebras.

Moreover $K_n[\![X]\!][1/X] \hookrightarrow K_n[\![X_m]\!][1/X]$ is free and therefore faithfully flat. Notice that $1/X_m = \big(1 + (1 + X_m) + \ldots + (1 + X_m)^{p^m - 1}\big)/X$ so that $K_n[\![X_m]\!][1/X] = K_n[\![X_m]\!][1/X_m]$. By passing to the direct limit for m going to infinity, we get: $\varinjlim_n K_n[\![X_m]\!][1/X] = A_n[1/X] = W_n\big(k((x^{p^{-\infty}}))\big)$, where $k((x^{p^{-\infty}}))$ denotes the perfect closure of the field $k((x))$. We obtain again a morphism of $K_n[\![X]\!][1/X]$-algebras:

$$(2.23) \qquad u'\colon W_n\big(k((x^{p^{-\infty}}))\big) \to K_n[\mathbf{Q}_p/\mathbf{Z}_p] \bigotimes_{K_n} W_n\big(k((x^{p^{-\infty}}))\big);$$

given by:

$$(2.23.0) \qquad u'([1 + x^{p^{-m}}]) = Y_{p^{-m} + \mathbf{Z}_p} \otimes [1 + x^{p^{-m}}] \quad \text{for } m \text{ in } \mathbf{N},$$

and satisfying:

$$(2.23.1) \qquad (\mathbf{P} \otimes \mathrm{id})\, u' = (\mathrm{id} \otimes u')\, u';$$

(2.23.2) the map:

$$W_n\big(k((x^{p^{-\infty}}))\big) \bigotimes_{K_n[\![X]\!][1/X]} W_n\big(k((x^{p^{-\infty}}))\big) \to K_n[\mathbf{Q}_p/\mathbf{Z}_p] \bigotimes_{K_n} W_n\big(k((x^{p^{-\infty}}))\big)$$
$$f \otimes g \mapsto u'(f)(1 \otimes g)$$

is an isomorphism of $K_n[\![X]\!][1/X]$-algebras.

Moreover $K_n[\![X]\!][1/X] \hookrightarrow W_n\big(k((x^{p^{-\infty}}))\big)$ is faithfully flat. We can therefore apply to the descent relatively to that extension the same criteria we proved for $K_n[\![X]\!] \hookrightarrow W_n(k[\![x^{p^{-\infty}}]\!])$.

3. – We keep the notation of section 2. Let us denote by R the ring $k[\![x]\!]$ and by R' its perfectionate $k[\![x^{p^{-\infty}}]\!]$. Let Q, Q' denote the quotient fields of R, R', respectively. We also put $K = W(k) =$ the ring of infinite Witt vectors with components in k. Let $K[\![X]\!]$ be the ring of formal power series in X with coefficients in K; there is a unique morphism of K-algebras:

$$K[\![X]\!] \to W(R') \tag{3.1}$$

sending X to $[1+x]-1$, which is continuous for, say, the (p, X)-adic topology in $K[\![X]\!]$ and the $(p, [x])$-adic one in $W(R')$. We will always regard $K[\![X]\!]$ as embedded in $W(R')$ by means of (3.1). The embedding (3.1) can obviously be uniquely extended, as a ring homomorphism, to give an embedding:

$$K[\![X]\!][1/X] \to W(Q')\,, \tag{3.2}$$

that we will always use in the sequel. Notice that (3.2) can again be extended by p-adic continuity, to an embedding of the p-adic completion B of $K[\![X]\!][1/X]$ in $W(Q')$:

$$B \to W(Q')\,. \tag{3.3}$$

The embeddings (3.1) and (3.3) reduce modulo p^n, to the embeddings used in section 2, for which we were able to give simple descent criteria. Let us restate those results in a more manageable form.

Formula (2.18) provides us with a map:

$$\begin{cases} F_n \times W_n(R') \to W_n(R') \\ (d, r) \qquad\quad \mapsto T'_d(r) = dr\,. \end{cases} \tag{3.4}$$

Analogously, using (2.22), we get a map (that extends (3.4)):

$$\begin{cases} F_n \times W_n(Q') \to W_n(Q') \\ (d, r) \qquad\quad \mapsto dr\,. \end{cases} \tag{3.5}$$

The map (3.5) can be characterized by the properties:

(3.5.1) $\quad r \mapsto dr \quad$ is $\quad K_n[\![X]\!][1/X]$-linear;

(3.5.2) $\quad d[(1+x^{ap^m})] = d(ap^m + \mathbb{Z}_p)[(1+x)^{ap^m}]\,, \quad$ for a, m in $\mathbb{Z}$.

Moreover (3.5) makes $W_n(Q')$, endowed with the discrete topology, into a l.t. F_n-module and satisfies:

$$(3.5.3) \qquad d(rr') = \mu\big((\mathbb{P}d)(r \otimes r')\big)\,, \qquad \text{for } d \in F_n,\ r, r' \in W_n(Q')\,.$$

The right-hand term in (3.5.3) is to be interpreted in the following way. Suppose $\mathbb{P}d = \sum_{i,j} d_i \hat{\otimes} d_j$, a converging sum in $F_n \hat{\bigotimes}_{K_n} F_n$, then: $d(rr') = \sum_{i,j} (d_i r)(d_j r')$, a finite sum in $W_n(Q')$. A descent datum on a $W_n(R')$-(resp. $W_n(Q')$-) module M, relatively to $K_n[\![X]\!] \hookrightarrow W_n(R')$ (resp. $K_n[\![X]\!] \cdot [1/X] \hookrightarrow W_n(Q')$) is equivalent to a K_n-bilinear map:

$$(3.6) \qquad \begin{cases} F_n \times M \to M \\ (d, m) \mapsto dm \end{cases}$$

making M, endowed with the discrete topology, into a topological F_n-module and satisfying:

$$(3.7) \qquad d(rm) = \mu_{sc}\big((\mathbb{P}_{F_n} d)(r \bigotimes_{K_n} m)\big)\,,$$

for $d \in F_n$, $r \in W_n(R')$ (resp. $W_n(Q')$), $m \in M$. Here, as usual, $\mu_{sc}\colon W_n(R') \bigotimes_{K_n} M \to M$ (resp. $W_n(Q') \bigotimes_{K_n} M \to M$) is the scalar product, and, if $\mathbb{P}d = \sum_{i,j} d_i \hat{\otimes} d_j$ (a converging sum in $F_n \hat{\bigotimes}_{K_n} F_n$), the right-hand term of (3.7) is to be interpreted as $\sum_{i,j} (d_i r)(d_j m)$ (a finite sum in M), through (3.5) and (3.6). Notice that $m \mapsto dm$ is then automatically $K_n[\![X]\!]$-(resp. $K_n[\![X]\!][1/X]$-) linear.

Let F be the l.t. K-hyperalgebra (K being endowed with the p-adic topology) of functions from $\mathbb{Q}_p/\mathbb{Z}_p$ to K, with the topology of simple convergence. A fundamental system of open K-submodules (ideals, in fact) of F is given by the

$$U_{m,n} = \{f \in F /\ f(p^{-m}\mathbb{Z}_p/\mathbb{Z}_p) \subseteq p^n K\}\,,$$

as m, n vary in $\mathbb{N}$. Clearly, $F = \varprojlim_n F_n$, as a topological ring. The identification $F_n \xrightarrow{\sim} W_n(F_1)$ of section 2, now carries over to an identification $F \xrightarrow{\sim} W(F_1)$, the last being equipped with the product topology of the topology of F_1.

By taking inverse limits for $n \to +\infty$ in (3.5), we obtain a map:

$$(3.8) \qquad \begin{cases} F \times W(Q') \to W(Q') \\ (d, r) \qquad \mapsto dr \end{cases}$$

that can be characterized by the following properties (3.8.1) and (3.8.2):

(3.8.1) $\quad r \mapsto dr \quad$ is B-linear;

(3.8.2) $\quad d[(1 + x^{ap^m})] = d(ap^m + \mathbb{Z}_p)[(1 + x)^{ap^m}]\,, \quad$ for a, m in $\mathbb{Z}$.

Moreover, the map (3.8) makes $W(Q')$, endowed with the p-adic topology, into a topological F-module and satisfies:

$$(3.8.3) \qquad d(rr') = \mu\big((\mathbb{P}d)(r \otimes r')\big)\,, \quad \text{for } d \in F,\ r, r' \in W(Q')\,.$$

The right-hand term of (3.8.3) should be interpreted as follows. Let $\mathbb{P}d = \sum_{i,j} d_i \hat{\otimes} d_j$ $\Big($a converging sum in $F \hat{\bigotimes}_K F\Big)$; then $d(rr') = \sum_{i,j} (d_i r)(d_j r')$ (a p-adically convergent sum in $W(Q')$).

Let M be a $W(R')$-(resp. $W(Q')$-) module, p-adically separated and complete. Let

$$(3.9) \qquad \begin{cases} F \times M \to M \\ (d, m) \mapsto dm \end{cases}$$

be a K-bilinear map, making M, endowed with the p-adic topology, into a topological F-module, and satisfying:

$$(3.10) \qquad d(am) = \mu_{sc}\Big((\mathbb{P}_F d)\Big(a \hat{\bigotimes}_K m\Big)\Big)$$

for $d \in F$, $a \in W(R')$ (resp. $W(Q')$), $m \in M$. Here $W(R') \hat{\bigotimes}_K M$ $\Big($resp. $W(Q') \hat{\bigotimes}_K M\Big)$ denotes the p-adic completion of $W(R') \hat{\bigotimes}_K M$ $\Big($resp. $W(Q') \hat{\bigotimes}_K M\Big)$, μ_{sc}: $W(R') \hat{\bigotimes}_K M \to M$ $\Big($resp. $W(Q') \hat{\bigotimes}_K M \to M\Big)$ denotes the scalar product, and, if $\mathbb{P}d = \sum_{i,j} d_i \hat{\otimes} d_j$ $\Big($a converging sum in $F \hat{\bigotimes}_K F\Big)$ the right-hand member of (3.10) is to be interpreted as $\sum_{i,j} (d_i a)(d_j m)$ (a p-adically convergent sum in M) through (3.8) and (3.9).

It is clear that, by reduction modulo p^n, the datum (3.9) satisfying (3.10), provides a series of compatible data on $M/p^n M$ of the type (3.7).

We then easily conclude from the previous section that if we put:

$$M_0 = \{m \in M / \ dm = 0, \text{ if } d(0) = 0\}, \tag{3.11}$$

M_0 is a $K[\![X]\!]$- (resp. B-) submodule of M, M_0 is p-adically separated and complete, $M = M_0 \hat{\bigotimes}_{K[\![X]\!]} W(R')$ $\big($resp. $M_0 \hat{\bigotimes}_B W(Q')\big)$ where $\hat{\otimes}$ means p-adic completion of $\otimes$, and $d(m \hat{\otimes} a) = m \hat{\otimes} da$ for $d \in F$, $m \in M_0$, $a \in W(R')$ (resp. $W(Q')$). Analogous results hold for the descent of morphisms of modules.

REFERENCES

[1] J.-P. Serre, *Corps locaux*, Paris, Hermann, 1968.
[2] W. C. Waterhouse, *Introduction to Affine Group Schemes*, New York, Springer-Verlag, 1979.

Seminario Matematico
Università di Padova
Via Belzoni, 7
35100 Padova