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Asymmetric unbounded liquid bridges

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Asymmetric Unbounded Liquid Bridges.

THOMAS I. VOGEL (*)

Introduction.

An unbounded liquid bridge is that surface formed when an object is withdrawn a small distance from an infinite pool of liquid (see figure 1).

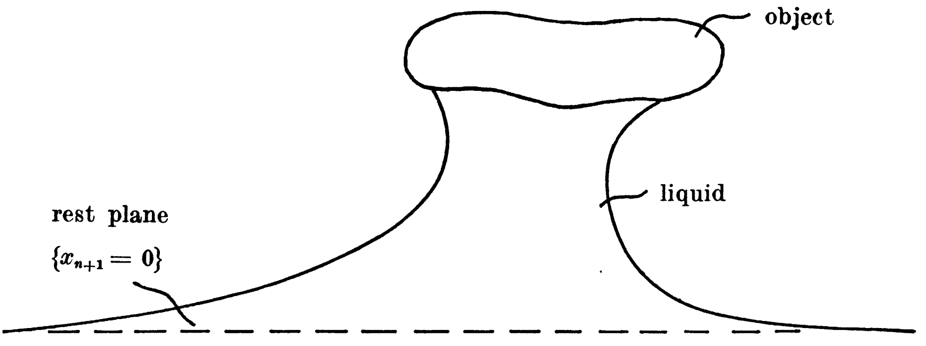

Figure 1. Unbounded liquid bridge.

In this paper I study the surfaces formed in $\boldsymbol{R}^n \times \boldsymbol{R}$ when the object withdrawn is a planar figure parallel to the rest plane of the pool $\{x_{n+1} = 0\}$. Let $\mathfrak{D}$ satisfy an internal sphere condition of radius R. I will show that in the complete wetting case, if $\mathfrak{D}$ is at a height $h < h_n(R)$, then the bridge whose existence is shown by a certain limiting procedure is a graph over $\{x_{n+1} = 0\}$. (Completely wetting means that the angle between the normal of the free surface and the normal to the surface of the object is zero at

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contact when the object is smooth. One case of this is when the object is glass and the liquid is water.) The fact that the bridge surface is a graph implies that it is the graph of a smooth function. The method of proof of the main theorem will be to use the symmetric bridge surfaces (see Vogel [10]) as barriers.

1. – Formulation of the problem and existence of a solution.

In approaching capillary problems using the theory of functions of bounded variation, the complete wetting case is modeled by using an obstacle formulation. In the case of unbounded liquid bridges, when a solid object $\mathfrak{D}$ is withdrawn from a pool of liquid, I seek a set Ω with $\mathfrak{D} \subset \Omega$, $\{x_{n+1} < 0\} \subset \Omega$, and

$$\mathcal{E}(\Omega) = \int |D\chi_\Omega| + \int_\Omega x_{n+1}$$

minimized over compact perturbations over the sets containing $\mathfrak{D} \bigcup \{x_{n+1} < 0\}$, where $\int |D\chi_\Omega|$ is the perimeter in the sense of De Giorgi (see De Giorgi, Colombini, Piccinini [1]), and χ_Ω is the characteristic function of Ω. Here the set Ω-$\mathfrak{D}$ corresponds to the liquid. The constants involved: surface tension, wetting energy, and g, have been normalized to be 1 without loss of generality. When $\mathfrak{D}$ is a solid body in $\boldsymbol{R}^{n+1}$, the existence of an unbounded liquid bridge with obstacle $\mathfrak{D}$ is the result of a more general theorem (Vogel [11]).

However, when $\mathfrak{D}$ is a plate, i.e. a region in $\boldsymbol{R}^n \times \{x_{n+1} = h\}$, where h is the height of the plate, the above approach is insufficient. The problem is that $\mathfrak{D}$ would have $\boldsymbol{H}_{n+1}$ (Hausdorff $(n+1)$ dimensional) measure zero, and hence perimeter zero. There would be nothing to hold the liquid bridge up, so the only solution would be the trivial one $\Omega = \mathfrak{D} \bigcup \{x_{n+1} \leqslant 0\}$. This leads to the notion of a flat obstacle.

1.1 Definition. Let $V = \boldsymbol{R}^n \times [0, h]$, and let F be a region in $\boldsymbol{R}^n$. A set Ω is said to be a solution to the flat obstacle unbounded bridge problem with obstacle F at height h if Ω minimizes over compact perturbations the energy functional

$$\mathcal{E}_V(\Omega) = \int_V |D_{\chi_\Omega}| + \int_{V \cap \Omega} x_{n+1}$$

over the class of sets equal to $F \times [h, \infty) \bigcup \{x_{n+1} < 0\}$ outside of V. Although the existence theorem in Vogel [11] does not apply immediately, the method of proof can be adapted without difficulty to show that for any $h > 0$ and for any $F \subseteq \boldsymbol{R}^n$, there is a solution to the flat obstacle problem. My formulation is cruder than that in Chapter 4 of De Giorgi, Colombini, Piccinini [1]. However, I use this approach so that the results in Vogel [11] are easily applicable.

If h is large enough, one would expect that the only set minimizing $\mathcal{E}_V$ over compact perturbations would be the obstacle $F\times[h, +\infty) \bigcup \{x_{n+1} \leqslant 0\}$. However, the following remark shows that the theory is not vacuous by eliminating the trivial set as a solution for small h.

1.2 Remark. If $F \subseteq \boldsymbol{R}^n$ is a bounded set with ∂F piecewise C^1 then, for $h > 0$ sufficiently small, $F\times[h, +\infty) \bigcup \{x_{n+1} \leqslant 0\}$ is not a solution to the flat obstacle problem with obstacle F at height h.

Proof. Consider the set $S = F\times[0, +\infty) \bigcup \{x_{n+1} \leqslant 0\}$. This is a compact perturbation of $F\times[h, +\infty) \bigcup \{x_{n+1} \leqslant 0\}$. For some suitably large compact K, I have

$$\mathcal{E}_{V\cap K}(S) = \mathcal{E}_{K\cap V}(F\times[h, +\infty) \bigcup \{x_{n+1} \leqslant 0\}) - 2|F| + h\int_{\boldsymbol{R}^n} |D\chi_F| + \int_{F\times[0,h]} x_{n+1} ,$$

where $|F|$ is the n-dimensional Lebesgue measure of F. As h goes to zero, $h\int_{\boldsymbol{R}^n} |D\chi_F| + \int_{F\times[0,h]} x_{n+1}$ goes also to zero, so $\mathcal{E}_{V\cap K}(S) < \mathcal{E}_{V\cap K}(F\times[h, +\infty) \bigcup \bigcup \{x_{n+1} \leqslant 0\})$ for any sufficiently small positive h. Therefore, there exist non-trivial solutions to the flat obstacle problem.

2. – Some results on symmetric unbounded liquid bridges.

The asymmetric unbounded liquid bridges will be studied using symmetric liquid bridges as barriers. Symmetric liquid bridges have only been studied for the case $n = 2$ (Vogel [10]). Since I want to consider the case $n \geqslant 2$, I will have to extend some of the results proven in 2 dimensions. The first thing that must be shown is that symmetric unbounded liquid bridges have smooth boundaries away from the disc that is holding them up. This follows from more general results for $n \leqslant 7$ (see Massari [5]), but their symmetry must be exploited to prove smoothness in higher dimensions. The proof of smoothness (see Vogel [9]) is roughly the same as Gonzalez's proof of the smoothness of the pendent drop (Gonzalez [3]).

Since the boundaries of symmetric liquid bridges are smooth, they must satisfy the condition that the mean curvature of the surface is proportional to the height above the base plane, which follows from the pertinent Euler-

Lagrange equations. For symmetric liquid bridges, this can be expressed as follows. Let Ω be a symmetric liquid bridge in $\boldsymbol{R}^n \times \boldsymbol{R}$, and let $r_n(u)$ be the radius of the sphere $\partial\Omega \bigcap \{x_{n+1} = u\}$. Then $r_n(u)$ satisfies:

$$\frac{r_n r_n''}{(1+(r_n')^2)^{\frac{3}{2}}} - \frac{(n-1)}{\sqrt{1+(r_n')^2}} = ur_n\,(n-1)\,, \tag{2.1}$$

with boundary conditions $\lim_{u\to 0} r_n(u) = +\infty$ and $r_n(h) = R$, if Ω is assumed to be lifted by a disc of radius R at height h. From (2.1), it is clear that $r_n(u) > 0$ since $r_n(h) > 0$, so that $r_n''(u) > 0$. If I define the inclination angle ψ to be $\operatorname{arccot}(r_n'(u))$, then $d\psi/du < 0$, so that I can invert and express u_n as a function of the independent variable ψ. Then, of course, $r_n(\psi) = r_n(u_n(\psi))$. In place of (2.1), there holds the pair of equations:

$$\frac{du_n}{d\psi} = \frac{-r_n \sin\psi}{(n-1)(r_n u_n + \sin\psi)}\,, \tag{2.2}$$

and

$$\frac{dr_n}{d\psi} = \frac{-r_n \cos\psi}{(n-1)(r_n u_n + \sin\psi)}\,, \tag{2.3}$$

a generalization of the equations obtained for $n = 2$ (Vogel [10]).

Siegel [7] has shown that given a radius σ, there is a unique profile curve satisfying (2.1) with $\lim_{u\to 0} r_n(u) = +\infty$, and being vertical at the given radius σ. Let $T_n(\sigma)$ be the height of the vertical point of the profile curve which is vertical at radius σ. From a comparison theorem of Siegel, it follows that $T_n(\sigma)$ is strictly increasing. The appropriate initial conditions for (2.2) and (2.3) are then:

$$u_n\left(\frac{\pi}{2};\,\sigma\right) = T_n(\sigma) \tag{2.4}$$

$$r_n\left(\frac{\pi}{2};\,\sigma\right) = \sigma\,, \tag{2.5}$$

where σ serves to parametrize the family of profile curves. Let Γ_σ denote the curve $\{(r_n(\psi;\sigma),\, u_n(\psi;\sigma)),\ 0 < \psi \leqslant \pi\}$.

Turkington [8] has characterized $T_n(\sigma)$ asymptotically as:

$$T_n(\sigma) \sim \begin{cases} c\sigma \log \dfrac{a}{\sigma} & n = 2 \\ c_n \sigma & n > 2 \end{cases} \tag{2.6}$$

as σ tends to zero.

2.1 THEOREM. *$u_n(\psi;\sigma) < 2T_n(\sigma)$ for all ψ, σ, and n.*

PROOF. The proof is entirely analogous to the case $n = 2$ (see Vogel [10]), using the fact that

$$\frac{1}{r^{n-1}}\frac{d}{dr}(r^{n-1}\sin\psi) = -(n-1)u \tag{2.7}$$

on each of the two parts of the profile curves considered as functions of r.

Theorem 2.1 may be used to estimate $r_n(0;\sigma)$.

2.2 THEOREM. $\lim\limits_{\sigma\to 0} r_n(0;\sigma) = 0$.

PROOF. Integrating (2.7) from $r_n(\pi/2;\sigma)$ to $r_n(0;\sigma)$, there holds

$$-r_n^{n-1}\left(\frac{\pi}{2};\sigma\right) = \int\limits_{r_n(\pi/2;\sigma)}^{r_n(0;\sigma)} d(r^{n-1}\sin\psi) = \int\limits_{r_n(\pi/2;\sigma)}^{r_n(0;\sigma)} -(n-1)r^{n-1}u_n(r)\,dr\,,$$

where $u_n(r)$ refers to the upper part of the profile curve. Thus, since $r_n(\pi/2;\sigma) = \sigma$,

$$\left(\frac{n-1}{n}\right)(r_n^n(0;\sigma)-\sigma^n)\,u_n\left(\frac{\pi}{2};\sigma\right)\leqslant\sigma^{n-1}\leqslant\left(\frac{n-1}{n}\right)(r_n^n(0;\sigma)-\sigma^n)\,u_n(0;\sigma)\,.$$

Now, using the fact that $u_n(\pi/2;\sigma) = T_n(\sigma)$ and $u_n(0;\sigma)\leqslant 2T_n(\sigma)$, this becomes:

$$\left(\frac{n-1}{n}\right)(r_n^n(0;\sigma)-\sigma^n)\,T_n(\sigma)\leqslant\sigma^{n-1}\leqslant\left(\frac{n-1}{n}\right)(r_n^n(0;\sigma)-\sigma^n)\,2T_n(\sigma)\,. \tag{2.8}$$

From this inequality, there follows immediately:

$$\left(\frac{\sigma^{n-1}}{2T_n(\sigma)}\left(\frac{n}{n-1}\right)+\sigma^n\right)^{1/n}\leqslant r_n(0;\sigma)\leqslant\left(\frac{\sigma^{n-1}}{T_n(\sigma)}\left(\frac{n}{n-1}\right)+\sigma^n\right)^{1/n}. \tag{2.9}$$

This, along with Turkington's asymptotic estimates on $T_n(\sigma)$ as σ approaches zero, yields the desired result.

3. – Conditions for the bridge surface to be a graph.

I first need the following definition:

3.1 DEFINITION. Given a measurable set $E\subseteq \boldsymbol{R}^n\times\boldsymbol{R}$, let $f(x_1, \ldots, x_n) =$

$= \int_{x_{n+1}>0} \chi_E(x_1, \ldots, x_{n+1})\, dx_{n+1}$: Define the set E^p to be $\{x \in \boldsymbol{R}^n \times \boldsymbol{R};\ 0 < x_{n+1} < f(x_1, \ldots, x_n)\}$. From now on, I will call the first n coordinates $\bar{x}$.

3.2 LEMMA. *Let E be a Caccioppoli set bounded in the positive x_{n+1} direction. Let $\mathcal{A} \subseteq \boldsymbol{R}^n$ be a bounded open set in the base space. Then*

$$\int_{\mathcal{A}\times \boldsymbol{R}^+} |D\chi_{E^p}| \leqslant \int_{\mathcal{A}\times \boldsymbol{R}^+} |D\chi_E| \,. \tag{3.1}$$

PROOF. This is a consequence of Theorem 2.2 of Miranda [6]. Let $\tilde{E} = (E \cap (\boldsymbol{R}^n \times \boldsymbol{R}^+)) \cup$ (the reflection of $E \cap (\boldsymbol{R}^n \times \boldsymbol{R}^+)$), and let $\tilde{E}^p = \{x \in \boldsymbol{R}^n \times \boldsymbol{R};\ |x_{n+1}| < f(\bar{x})\}$, where f is the function from Definition 3.1.

Miranda's theorem shows that

$$\int_{\mathcal{A}\times \boldsymbol{R}} |D\chi_{\tilde{E}^p}| \leqslant \int_{\mathcal{A}\times \boldsymbol{R}} |D\chi_{\tilde{E}}| \,.$$

But this implies

$$2\int_{\mathcal{A}\times \boldsymbol{R}^+} |D\chi_{E^p}| \leqslant 2\int_{\mathcal{A}\times \boldsymbol{R}^+} |D\chi_E| \,.$$

I will be using the symmetric surfaces as barriers, so I must find under what conditions they are graphs.

3.3 LEMMA. *Given a disc of radius R, there is a height $h_n(R)$ so that if $h < h_n(R)$, then $\partial\Omega_h$ is a graph over the base plane. Here Ω_h is the symmetric bridge formed for the obstacle problem.*

PROOF. From inequality (2.9), if Γ_σ is a profile curve which is horizontal at radius R, then σ is bounded away from zero. Let σ_R be this bound, so that $\sigma_1 < \sigma_R$ implies $r_n(0; \sigma_1) < R$. I claim that any profile curve which crosses the line $r = R$ at a height less than $h_n(R) \equiv T(\sigma_R)$ has an inclination angle greater than $\pi/2$ at that crossing.

To see this, let Γ_σ pass through the point (R, h) with $h < h_n(R)$, and suppose that it has an inclination less than $\pi/2$ there. It follows then that $r_n(0; \sigma) \geqslant R$, so $\sigma \geqslant \sigma_R$. But since the angle at (R, h) is less than $\pi/2$, it also follows that $h \geqslant T(\sigma)$. By the monotonicity of $T(\sigma)$, there would hold $h \geqslant T(\sigma_R)$, a contradiction. Hence the inclination of Γ_σ is greater than or equal to $\pi/2$ at (R, h), from which one can conclude that for $0 < u \leqslant h$, Γ_σ is a function of r, so $\partial\Omega_h$ is a graph over the base plane.

3.4 THEOREM. *Let $F \subseteq \boldsymbol{R}^n$ be a region satisfying an internal sphere condition of radius R. Let Ω_h be a set solving the flat obstacle problem with ob-*

stacle F at height h (always assuming that the capillary constant $\varkappa$ is 1). If $h < h_n(R)$, then $F \times (0, h) \subseteq \Omega_h$.

PROOF. Let B be a disc of radius R contained in F. Let A be the symmetric set which minimizes $\mathcal{E}_V$ over compact perturbations with flat obstacle B at height h. (Since ∂A must be a graph, there can be only one such A from Theorem 1 of Siegel [7].) I claim that $A \subseteq \Omega_h$.

To see this, let $\{S_i\}$, $i = 1, 2, \ldots$, be a sequence of concentric discs of radius i in the base plane $\{x_{n+1} = 0\}$ with centers directly below that of B. Let A_i be a set which minimizes $\mathcal{E}_V$ over the family of sets equal to $B \times [h, +\infty) \bigcup S_i \times (-\infty, 0]$ outside of $V = \mathbf{R}^n \times (0, h)$. From Vogel [11], $A_i \cap \Omega_h$ is also an energy minimum. $A_i \cap \Omega_h$ is symmetric, since symmetrization reduces perimeter (Gonzalez [4]).

Now consider $A = \bigcup_{i=1} (A_i \cap \Omega_h)$. A is contained in Ω_h, and since each $A_i \cap \Omega_h$ is symmetric about a common axis, A is symmetric. From Vogel [11], A is a solution of the flat obstacle problem with obstacle B at height h. From Lemma 3.3, A contains $B \times (0, h)$, therefore Ω_h does also. But B was an arbitrary ball of radius R in F, so it follows that Ω_h contains $F \times (0, h)$.

3.5 REMARK. Even if F does not satisfy an internal sphere condition, I may still draw a conclusion. Given an arbitrary Caccioppoli set F, let F_R be the union of all discs of radius R contained in F. Then the proof of Theorem 3.4 shows that if $h < h_n(R)$, then $F_R \times (0, h) \subseteq \Omega_h$.

3.6 REMARK. The above remark shows that if F contains a disc of any radius, then for h small enough, Ω_h is non-trivial. This is an improvement on Remark 1.2.

3.7 THEOREM. *Assume the same hypotheses as in Theorem 3.4, and in addition assume that there is an open disc B_1 in the base space so that outside of $B_1 \times \mathbf{R}$, $\Omega_h^p = \Omega_h$. Then $\Omega_h^p = \Omega_h$ in all of $\mathbf{R}^{n+1}$.*

PROOF. Consider the set Ω_h^p. The map $\Omega_h \to \Omega_h^p$ can only change Ω_h in $B_1 \times (0, h)$ by the assumption on Ω_h, so that Ω_h^p is a compact perturbation of Ω_h. Moreover, Ω_h^p still contains $F \times [h, +\infty)$, since from Theorem 3.4, $F \times (0, +\infty) \subseteq \Omega_h$. Therefore Ω_h^p is a member of the family of sets over which Ω_h minimizes $\mathcal{E}_V$. By Lemma 3.2,

$$\int_{(B_1 - \bar{F}) \times \mathbf{R}^+} |D\chi_{\Omega_h^p}| \leqslant \int_{(B_1 - \bar{F}) \times \mathbf{R}^+} |D\chi_{\Omega_h}| \,.$$

Also, if $\mu_{n+1}(\Omega_h^v \Delta \Omega_h) \neq 0$ (where Δ is the symmetric difference), I must have

$$\int_{\Omega_h^v \cap [(B_1 - F) \times \mathbf{R}^+]} x_{n+1} < \int_{\Omega_h \cap [(B_1 - F) \times \mathbf{R}^+]} x_{n+1} ,$$

since Ω_h^v must have lower potential energy than Ω_h. Hence if $\mu_{n+1}(\Omega_h^v \Delta \Omega_h) \neq 0$, then $\mathcal{E}_{K\cap V}(\Omega_h^v) < \mathcal{E}_{K \cap V}(\Omega_h)$, where $K = \overline{(B_1 - F) \times (0, h)}$, (since $\Omega_h = \Omega_h^v$ on $\partial B_1 \times \mathbf{R}$), which contradicts the choice of Ω_h. Hence $\Omega_h = \Omega_h^v$.

I will next show that for $h < h_n(R)$, that particular liquid bridge Ω_h obtained by the limiting procedure in Vogel [11] satisfies $\Omega_h^v = \Omega_h$. (It is not known that all liquid bridges are obtainable in this manner, making the above claim merely different from Theorem 3.7, not stronger.) I need another technical lemma, whose statement is similar to Lemma 3.3, but which applies to the finite symmetric sets $\{A_i\}$.

3.8 LEMMA. *Let the sequence $\{A_i\}$ be as defined in the proof of Theorem 3.4, with $A_i \subseteq A_j$ if $i < j$. If $h < h_n(R)$, then $\partial A_i \cap (\mathbf{R}^n \times (0, h))$ is a function of radius, for i sufficiently large.*

PROOF. Since A_i is symmetric, $\partial A_i \cap (\mathbf{R}^n \times (0, h))$ is smooth (see Vogel [9], Gonzalez [3]). Therefore, if $r_i(u)$ denotes the radius of ∂A_i at height u, then $r_i(u)$ satisfies (2.1) with boundary conditions $r_i(h) = R$, $r_i(0) = i$. (I am assuming that A_i is non-trivial, but this assumption must be true for sufficiently large i, since A is non-trivial.)

Since the A_i's are nested, it follows that $\{r_i'(h)\}$ is an increasing sequence, and has a limit $s \leqslant r'(h)$, where $r(u)$ is the radius of ∂A. Consider the function $\tilde{r}(u)$ satisfying (2.1) with $\tilde{r}(h) = R$ and $\tilde{r}'(h) = s$. By the theorem on continuous dependence of solutions of O.D.E.'s on their parameters, the sequence $\{r_i(u)\}$ converges almost uniformly to $\tilde{r}(u)$ on some interval $[\delta, h]$. However, the r_i's are already increasing pointwise to $r(u)$ (since $A = \bigcup_{i=1}^{\infty} A_i$), so $\tilde{r}(u) = r(u)$ near h, and $\lim_{i\to\infty} r_i'(h) = r'(h)$.

However, for $h < h_n(R)$, $r'(h) > 0$ from Lemma 3.3. Hence for large enough i, $r_i'(h) > 0$. From (2.1) it is clear that $r_i''(u) > 0$, so $r_i'(u) > 0$ on $[0, h]$. Thus I may invert $r_i(u)$ to obtain u as a function of r on $[0, h]$, as desired.

Now, define $\Omega_{h,i}$ to be a set which minimizes $\mathcal{E}_V$ over the family of sets equal to $F \times [h, +\infty) \bigcup S_i \times (-\infty, 0]$, where the sequence $\{S_i\}$ consists of concentric discs in the base plane of radius i with center some fixed point beneath F.

3.9 LEMMA. *Suppose F is a bounded set in $\mathbf{R}^n$ satisfying an internal sphere condition of radius R. If $h < h_n(R)$ then, for large enough i, $\Omega_{h,i}^v = \Omega_{h,i}$.*

PROOF. Let $I(h)$ be such that $i > I(h)$ implies that $r_i(u)$ is a function of radius for $u \in (0, h)$, where $r_i(u)$ is as in Lemma 3.8. Pick $i > I(h) + \operatorname{diam}(F) + 1$. Given any ball B_R of radius R in F, consider the set $A_{I(h)+1}$ minimizing $\mathcal{E}_V$ with respect to the flat obstacles B_R and $S'_{I(h)+1}$, where $S'_{I(h)+1}$ is the disc of radius $I(h)+1$ in the base plane with center directly below that of B_R. By Vogel [11], $A_{I(h)+1} \cap \Omega_{h,i}$ is another energy minimizing set for the flat obstacles B_R and $S'_{I(h)+1}$, so $A_{I(h)+1} \cap \Omega_{h,i}$ is symmetric. Thus, by Lemma 3.8, $B_R \times (0, h) \subseteq A_{I(h)+1} \cap \Omega_{h,i}$. Since B_R was arbitrary, $F \times (0, h) \subseteq \Omega_{h,i}$.

It is now obvious that if $\Omega_{h,i} \neq \Omega^v_{h,i}$, then $\Omega^v_{h,i}$ will have a strictly lower energy and will still contain the same obstacles. From this contradiction, it follows that $\Omega_{h,i} = \Omega^v_{h,i}$ (at least up to sets of measure zero).

3.10 THEOREM. *With F, $\Omega_{h,i}$ as in Lemma 3.9, the set* $\Omega_h = \bigcup_{i=1}^{\infty} \Omega_{h,i}$, *which will be a liquid bridge by the existence theorem in Vogel* [11], *has the property that* $\Omega_h = \Omega^v_h$.

PROOF. This is clear, since $\Omega_{h,i} = \Omega^v_{h,i}$ for i sufficiently large, and the $\Omega_{h,i}$'s may be assumed to be nested from Theorem 5 of Vogel [11].

3.11 REMARK. When $\Omega_h = \Omega^v_h$ and Ω_h is locally bounded in the positive x_{n+1} direction, it follows by a result of Gerhardt ([2], Theorem 2) that the function f in definition 3.1 is Lipschitz continuous. Therefore, f is analytic. That the set Ω_h in Theorem 3.10 is bounded in the positive x_{n+1} direction away from $F \times \mathbf{R}$ is clear, since each $\Omega_{h,i}$ lies beneath $\{x_{n+1} = h\}$ away from $F \times \mathbf{R}$. The Ω_h in Theorem 3.7 is not as well controlled, being more general. An additional assumption must be made if $\partial\Omega_h$ is to be the graph of an analytic function.

From this remark the previous theorem may be interpreted in a classical P.D.E.'s light.

3.12 THEOREM. *Given a region $F \subseteq \mathbf{R}^n$ satisfying an internal sphere condition of radius R, and given $h < h_n(R)$, there is a function $f \in C^\infty(-\bar{F}) \cup C^0(-F^0)$ satisfying*

$$\operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) = nf \qquad \text{in } -\bar{F}$$

and taking the value h continuously on ∂F.

PROOF. The only thing left to prove is the continuous assumption of boundary values. But this follows by using the symmetric surfaces as barriers, since $\partial\Omega_h$ must lie below the plane $\{x_{n+1} = h\}$.

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