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A Rotation Invariant Differential Equation for Vector Fields.

H. M. REIMANN

1. – Introduction.

In an attempt to transfer properties of analytic functions to vector fields on $\mathbf{R}^n$ Ahlfors ([1], [2]) has studied the differential operator S, which maps vector fields $v\colon \mathbf{R}^n \to \mathbf{R}^n$ into symmetric tensor fields with vanishing trace, $\varphi\colon \mathbf{R}^n \to \mathbf{R}^{n^2}$, $\varphi^{\mathrm{tr}} = \varphi$, trace $\varphi = 0$:

$$Sv = \frac{1}{2}(V + V^{\mathrm{tr}}) - \frac{1}{n}(\operatorname{trace} V)I$$

with the notations $V = (\partial v_i/\partial x_j)$, $I = (\delta_{ij})$. If composed with the adjoint operator S^*

$$(S^*\varphi)_i = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\varphi_{ij}$$

the resulting operator S^*S is a differential operator mapping vector fields into vector fields. Its fundamental property is its invariance under rotations. The induced action of the special orthogonal group $SO(n)$ on vector fields is given by

$$T_g\colon v \to v_g\,, \qquad v_g(x) = gv(g^{-1}x)$$

and the operator S^*S has the property (see Ahlfors [2])

$$S^*S(v_g) = (S^*Sv)_g\,.$$

In this article a solution of the Dirichlet-problem $S^*Sv = 0$ for the unit ball $B^n \subset \mathbf{R}^n$ with L^2-boundary data will be given. The space of square inte-

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grable vector fields on the unit sphere can be decomposed according to the action of $SO(n)$. Explicit solutions of $S^*Sv = 0$ for the invariant parts can then be exposed. So the solution appears as an infinite sum with coefficients taken from the expansion of the preassigned vector field on the sphere. There is an apparent analogy to solving the Dirichlet-problem for the Laplace equation by expanding the functions on the unit sphere into a series of spherical harmonics.

In the case $n = 3$ the differential equation $S^*Sv = 0$ arises as a limiting case of the elasticity equation

$$a \operatorname{grad} \operatorname{div} v - b \operatorname{rot} \operatorname{rot} v = 0$$

with $a = \frac{2}{3}$, $b = \frac{1}{2}$. Solutions for the elasticity equation are classical (see Debye [3], Weyl [8]). In the following treatment the case $n = 3$ appears to be rather exceptional.

The transformation group $SO(n)$ can be replaced by the group of Möbius transformations, mapping the unit ball onto itself. Associated with it there is the invariant metric

$$ds = \frac{2|dx|}{1 - |x|^2} = \varrho dx .$$

The differential operator (for vector fields)

$$\varrho^{-n-2} S^* \varrho^n Sv = 0$$

is then invariant under the Möbius group [1]. Using this invariance Ahlfors has given a solution for the Dirichlet-problem for the unit ball in the form of a Poisson type integral [1]. The construction is based on the so called center formula and on the fact, that any point $x \in B^n$ can be mapped onto the center by a suitable Möbius transformation. The method can however not be applied to the rotation invariant equation $S^*Sv = 0$, since the action of the group $SO(n)$ on the unit ball is not transitive.

I am indebted to R. Coifman and A. Korányi for their constant help and encouragement. In particular, the construction of the solutions q in section 3 is due to A. Korányi.

2. – The decomposition of the vector fields.

Denote by H^k the Hilbertspace of spherical harmonics of degree k. This is the space of restrictions of (complex valued) harmonic polynomials in $\mathbb{R}^n$, homogeneous of degree k, to the unit sphere $\Sigma = \partial B^n$ in $\mathbf{R}^n$ with the scalar product

$$(f, g) = \int f(x)\bar{g}(x)\,d\sigma(x)$$

$d\sigma$ is the normalized invariant measure on Σ, $\int_\Sigma d\sigma = 1$. $SO(n)$ acts on H^k by

$$U_g : p(x) \to p_g(x) = p(g^{-1}x)$$

and U_g is a unitary representation of $SO(n)$. The tensor product $\mathcal{H}^k = H^k \otimes \mathbf{C}^n$ is identified with the vector valued spherical harmonics of degree k or with the vector valued harmonic polynomials, homogeneous of degree k. For this purpose let $e_i, \ldots, e_m$ be an orthonormal basis in H^k and $f_1, \ldots, f_n$ the standard basis in $\mathbf{C}^n$. A basis in $H^k \otimes \mathbf{C}^n$ is then given by the elements

$$e_i \otimes f_j \qquad i = 1, \ldots, m;\ j = 1, \ldots, n$$

and these elements can be given the meaning of a complex linear mapping $H^k \to \mathbf{C}^n$ with the property, that e_i is mapped onto f_j and e_k for $k \neq i$ onto 0. The element

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ji}(e_i \otimes f_j) \in H^k \otimes \mathbf{C}^n\,, \qquad a_{ji} \in C$$

is then the linear mapping $A: H^k \to \mathbf{C}^n$ described by the matrix (a_{ji})

$$\Big(\sum_{i,j} a_{ji}(e_i \otimes f_j)\Big)\, e_k = \sum_{j=1}^{n} a_{jk} f_j$$

$H^k \otimes \mathbf{C}^n$ is a Hilbert space with the scalar product

$$\Big(\sum_{i,j} a_{ji}(e_i \otimes f_j),\ \sum_{i,j} b_{ji}(e_i \otimes f_j)\Big) = \sum_{i,j} a_{ji}\bar{b}_{ji} = \operatorname{trace} AB^* \qquad (B^* = \bar{B}^{\mathrm{tr}}).$$

The representation U_g on H^k can be expressed by the matrices $\big(u_{ki}(g)\big)$

$$U_g e_i = \sum_{k=1}^{m} u_{ki}(g)\, e_k\,; \quad (U_g e_i)(x) = e_i(g^{-1}x)\,.$$

If R_g denotes the standard representation of $SO(n)$ on $\mathbf{C}^n$

$$R_g f_j = \sum_{l=1}^{n} g_{lj} f_l$$

then the induced representation T_g on $H^k \otimes \mathbf{C}^n$ is given by

$$T_g A = R_g A U_g^{\mathrm{tr}} .$$

The elements $\sum_{i,j} a_{ji}(e_i \otimes f_j) \in H^k \otimes C^n$ can now be identified with the vector fields

$$v(x) = \begin{pmatrix} \sum_i a_{1i}\, e_i(x) \\ \cdots \\ \sum_i a_{ni}\, e_i(x) \end{pmatrix} .$$

This implies the following relations for the representation T_g:

$$(T_g v)_i(x) = \sum_{k,l,j} g_{ij} a_{jl} u_{kl} e_k(x) = \sum_{l,j} g_{ij} a_{jl} e_l(g^{-1}x) , \qquad (T_g v)(x) = g v(g^{-1}x)$$

where $g \in SO(n)$ is identified with the matrix (g_{ij}). This is the way $SO(n)$ acts on vector fields.

The scalar product of the vector fields v and w

$$v_j(x) = \sum_{i=1}^{m} a_{ji} e_i(x) , \qquad w_j(x) = \sum_{i=1}^{m} b_{jk} e_k(x)$$

takes the form

$$(v, w) = \sum_{i,j} a_{ji} \bar{b}_{ji} = \sum_{k,i,j} \int_\Sigma a_{ji} \bar{b}_{jk} e_i(x) \bar{e}_k(x)\, d\sigma(x) = \int_\Sigma \sum_{j=1}^{n} v_j(x) \overline{w}_j(x)\, d\sigma(x)$$

and T_g is unitary with respect to this scalar product.

Consider now the Hilbert space $\mathfrak{L}^2(\Sigma)$ of square integrable vector fields with the scalar product

$$(v, w) = \int_\Sigma \sum_j v_j(x) \overline{w}_j(x)\, d\sigma(x)$$

and the action T_g:

$$T_g v(x) = g v(g^{-1}x) .$$

Clearly, $\mathcal{H}^k = H^k \otimes \mathbf{C}^n$ is an invariant subspace. Furthermore $\mathfrak{L}^2(\Sigma)$ is the orthogonal direct sum of the spaces $\mathcal{H}^k$, $k = 0, 1, 2, \ldots$ This is a consequence of the fact that

$$L^2(\Sigma) = \widehat{\bigoplus_{k=0}^{\infty}} H^k .$$

To be more precise, let $\{e_i^k\}$ $k = 0, 1, 2, \ldots$ be an orthonormal system of spherical harmonics, $e_i^k \in H^k$, $i = 1, 2, \ldots, d_k$ $(d_k = \dim H^k)$. Then for $v_j \in L^2(\Sigma)$

$$v_j(x) = \sum_{i,k} c_{ji}^k e_i^k(x) \qquad |x| = 1$$

with

$$c_{ji}^k = \int_\Sigma v_j(x)\, \bar{e}_i^k(x)\, d\sigma(x)$$

and the expansion converges in $L^2(\Sigma)$. These formulas therefore give the expansion for the vector fields $v \in \mathfrak{L}^2(\Sigma)$ with components $v_j \in L^2(\Sigma)$. The completeness relation takes the form

$$\|v\|^2 = \int_\Sigma |v(x)|^2 d\sigma(x) = \sum_{j=1}^n \sum_{i,k} |c_{ji}^k|^2 .$$

It was already mentioned that $\mathcal{H}^k$ is an invariant subspace. The representation T_g on $\mathcal{H}^k$ is however not irreducible. Let R^m, $m = (m_1, \ldots, m_l) \in \mathbb{Z}^l$ denote the representation of $SO(n)$ with highest weight m. The integer l is given by $n = 2l$ if n is even and by $n = 2l + 1$ if n is odd. It is well known that $R^{(k,0,\ldots,0)}$ is equivalent to the representation U_g on H^k and $R^{(1,0,\ldots,0)}$ is equivalent to the standard representation R_g. For $n \geqslant 5$ (and $k \geqslant 1$) the tensor product representation decomposes as

$$R^{(k,0,\ldots} \otimes R^{(1,0,\ldots)} = R^{(k+1,0,\ldots)} \oplus R^{(k,1,0,\ldots)} \oplus R^{(k-1,0,\ldots)}$$

into irreducible parts. For $n = 4$ $R^{(k,1)}$ further decomposes such that

$$R^{(k,0)} \otimes R^{(1,0)} = R^{(k+1,0)} \oplus R^{(k,1)} \oplus R^{(k,-1)} \oplus R^{(k-1,0)} .$$

The decomposition into irreducible parts for $n = 3$ is given by the Clebsch-Gordan formula

$$R^{(k)} \otimes R^{(1)} = R^{(k+1)} \oplus R^{(k)} \oplus R^{(k-1)}$$

(for this see e.g. Levine [6], Mihlin [7]).

The invariant subspaces of $\mathcal{H}^r$, consisting of vector fields transforming according to the irreducible representations $R^{(k+1,0,\ldots)}$, $R^{(k-1,0,\ldots)}$ will be denoted by $\mathcal{M}^k$ and $\mathcal{N}^k$ respectively. $\mathfrak{Q}^k$ is the subspace corresponding to $R^{(k,1,0,\ldots)}$ $(n \geqslant 5)$, $R^{(k,1)} \oplus R^{(k,-1)}$ $(n = 4)$ or $R^{(k)}$ $(n = 3)$. Observe that in the decomposition of $R^{(k,0,\ldots)} \otimes R^{(1,0,\ldots)}$ any fixed irreducible representation occurs at most once. Therefore

$$\mathcal{H}^k = \mathcal{M}^k \oplus \mathfrak{Q}^k \oplus \mathcal{N}^k$$

is an orthogonal decomposition of $\mathcal{H}^k$.

Let us summarize the above statements in the following theorem.

A similar decomposition result can be found in Korányi-Vagi [5] (see also Korányi [4]).

THEOREM 1.

$$\mathfrak{L}^2(\Sigma) = \widehat{\bigoplus_{k=0}^{\infty}} \mathcal{H}^k = \widehat{\bigoplus_{k=1}^{\infty}} (\mathcal{M}^k \oplus \mathfrak{Q} \oplus \mathcal{N}^k) \oplus \mathcal{H}_0 .$$

For $n \geqslant 5$, the representation T_g decomposes into the irreducible parts

$$T = \bigoplus_{k=1}^{\infty} (R^{(k+1,0,\ldots)} \oplus R^{(k,1,0,\ldots)} \oplus R^{(k-1,0,\ldots)}) \oplus R^{(1,0,\ldots)}$$

for $n = 4$

$$T = \bigoplus_{k=1}^{\infty} (R^{(k+1,0)} \oplus R^{(k,1)} \oplus R^{(k,-1)} \oplus R^{(k-1,0)}) \oplus R^{(1,0)}$$

and for $n = 3$

$$T = \bigoplus_{k=1}^{\infty} (R^{(k+1)} \oplus R^{(k)} \oplus R^{(k-1)}) \oplus R^{(1)} .$$

The representations $R^{(k,0,\ldots)}$ in this decomposition occur twice (for $k > 0$), or three times in the dimension $n = 3$. This leads to the complicated structure of the solutions p_i^k defined in the next section.

PROPOSITION 1.

$$\mathcal{M}^k = \{h(x) = \operatorname{grad} u(x) \colon u \in H^{r+1}\}$$

$$\mathcal{N}^k = \{h(x) = (n + 2k - 4)xu(x) - |x|^2 \operatorname{grad} u(x) \colon u \in H^{k+1}\} .$$

The functions $u \in H^k$ are considered as harmonic polynomials in $\mathbb{R}^n$, which are homogeneous of degree k.

For the proof of the first formula consider the linear mapping $L\colon H^{k+1} \to \mathcal{H}^k$ given by $Lu = \operatorname{grad} u$. The representation $U = R^{(k+1,0,\ldots)}$ on H^{k+1} is given by

$$(U_g u)(x) = u(g^{-1}x)$$

and the representation T on $\mathcal{H}^k$ is

$$(T_g h)(x) = g h(g^{-1}x) .$$

It suffices to show that $LU_g = T_g L$ for all $g \in SO(n)$, since we already know that $R^{(k+1,0,\dots)}$ appears only once in the representation T on $\mathcal{H}^k$. Setting

$$y_k = (g^{-1}x)_k = \sum_j g_{jk} x_j$$

we have

$$\begin{aligned}(LU_g u)(x) = \operatorname{grad}(U_g u)(x) = \left(\frac{\partial}{\partial x_1} u(g^{-1}x), \dots, \frac{\partial}{\partial x_n} u(g^{-1}x)\right) = \\ = \left(\sum_k u_{,k}(g^{-1}x) g_{1k}, \dots\right) = g(\operatorname{grad} u)(g^{-1}x) = (T_g Lu)(x) .\end{aligned}$$

For the second formula note, that the components of the vector fields are harmonic and homogeneous of degree k:

$$\Delta(x_j u) = \sum_j (x_j u_{,j} + u)_{,j} + x_j \sum_{i \neq j} u_{,ij} = 2u_{,j}$$

$$\begin{aligned}\Delta(|x|^2 u_{,j}) = \sum_j (2u_{,j} + 4x_i u_{,ji} + |x|^2 u_{,jii}) = \\ = n\,2u_{,j} + 4(\operatorname{grad} u_{,j}, x) = 2(n + 2k - 4)\, u_{,j} .\end{aligned}$$

The linear mapping to be considered in this case is

$$Lu = (n + 2k - 4)xu - |x|^2 \operatorname{grad} u$$

and it has to be shown that $LU_g = T_g L$ for all $g \in SO(n)$:

$$(LU_g u)(x) = (n + 2k - 4)\, g g^{-1} x u(g^{-1}x) - |x|^2 g(\operatorname{grad} u)(g^{-1}x) = (T_g Lu)(x) .$$

We finally observe that $\operatorname{div} Lu = cu$ for some constant c depending on k and n.

PROPOSITION 2. *The vector fields $h \in \mathfrak{Q}^k$, considered as vector fields of homogenous harmonic polynomials satisfy the relations* $\operatorname{div} h = 0$

$$(h, x) = \sum_j h_j(x) x_j = 0 .$$

For the proof of the first relation consider the linear mapping $Lh = \operatorname{div} h$ from $\mathfrak{Q}^k$ into H^{k-1}. As above it suffices to establish that $U_g L = L T_g$. Since

T_g restricted to (any irreducible subspace of) Q^k is not equivalent to $R^{(k-1,0,\dots)}$, the mapping L then maps $\mathfrak{Q}^k$ onto 0.

$$(U_g Lh)(x) = \operatorname{div} h(g^{-1}x)$$

$$(LT_g h)(x) = \sum_{i,j} \frac{\partial}{\partial x_i}\big(g_{ij}h_j(g^{-1}x)\big) = \sum_{i,j,k} g_{ij}h_{j,k}(g^{-1}x)\,g_{ik} = \sum_j h_{j,j}(g^{-1}x)\,.$$

Finally, consider the mapping $(Lh)(x) = \big(h(x), x\big)$ from $\mathfrak{Q}^k$ into the space P^{k+1} of polynomials, homogeneous of degree $k+1$. The representation U_g on P^{k+1} decomposes into

$$R^{(k+1,0,\dots)} \oplus R^{(k-1,0,\dots)} \oplus \dots \oplus R^{(1,0,\dots)} \qquad k \text{ even}$$

$$R^{(k+1,0,\dots)} \oplus R^{(k-1,0,\dots)} \oplus \dots \oplus R^{(0,0,\dots)} \qquad k \text{ odd}$$

since any polynomial $p \in P^{k+1}$ can be represented as a sum

$$p(x) = a_{k+1}u_{k+1}(x) + \dots + a_1|x|^k u_1(x) \qquad k \text{ even}$$

$$= a_{k+1}u_{k+1}(x) + \dots + a_0|x|^{k+1} \qquad k \text{ odd}$$

with u_j harmonic, $a_j \in \mathbf{C}$. Because none of the representations occurring in the decomposition of U on P^{k+1} is equivalent to the representation T, restricted to $\mathfrak{Q}^k$, the second relation in the proposition follows from $LT_g = U_g L$:

$$(U_g Lh)(x) = U_g\big(h(x), x\big) = \big(h(g^{-1}x), g^{-1}x\big) = \big(gh(g^{-1}x), x\big) = (LT_g h)(x)\,.$$

PROPOSITION 3. *If $n = 3$, then*

$$\mathfrak{Q}^k = \big\{h(x) = x \times \operatorname{grad} u(x) \colon u \in H^k\big\}\,.$$

Consider the mapping $L\colon H^k \to \mathcal{H}^k$ given by

$$Lu(x) = x \times \operatorname{grad} u(x) \qquad \text{(vector product)}$$

$$(LU_g u)(x) = x \times \operatorname{grad}(U_g u)(x) = x \times \big(g \operatorname{grad} u(g^{-1}x)\big) =$$

$$= g\big(g^{-1}x \times \operatorname{grad} u(g^{-1}x)\big) = (T_g Lu)(x)\,.$$

It should of course be noted, that the components of $x \times \operatorname{grad} u$ are harmonic, if u is harmonic.

3. – Solutions for $S^*Sv = 0$.

Let $e_1^k, e_2^k, \ldots, e_{d_k}^k$ be an orthonormal basis for the spherical harmonics H^k ($d_k = \dim H^k$). The functions $e_i^k(x)$ can be chosen to be real valued. According to the context, the spherical harmonics are either considered as homogeneous harmonic polynomials in $\mathbb{R}^n$ or as their restrictions onto the unit sphere. We first exhibit an orthonormal basis for the spaces $\mathcal{M}^k$ and $\mathcal{N}^k$.

PROPOSITION 4.

$$\{m_i^k(x) = \big((k+1)(n+2k)\big)^{-\frac{1}{2}} \operatorname{grad} e_i^{k+1}(x)\}_{i=1}^{d_{k+1}}$$

is an orthonormal basis in $\mathcal{M}^k$

$$\{n_i^k(x) = (n+2k-4)^{\frac{1}{2}}(n+k-3)^{-\frac{1}{2}} x e_i^{k-1} - \\ - |x|^2\big((n+2k-4)(n+k-3)\big)^{-\frac{1}{2}} \operatorname{grad} e_i^{k-1}(x]\}_{i=1}^{d_{k-1}}$$

is an orthonormal basis in $\mathcal{N}^k$.

If u is a harmonic homogeneous polynomial of degree $k+1$, it then follows from Green's formula that

$$\int_{<1} |\operatorname{grad} u|^2\, dx = \int_{\Sigma} u(x)\big(\operatorname{grad} u(x), x\big)\, d\omega(x) = (k+1)\int_{\Sigma} |u(x)|^2\, d\omega(x)\,.$$

On the other side, the homogeneity implies

$$\int_{j|x|<} |\operatorname{grad} u|^2\, dx = \int_0^1 r^{n+2k-1}\, dr \int_{\Sigma} |\operatorname{grad} u|^2\, d\omega(x) = \frac{1}{n+2k}\int_{\Sigma} |\operatorname{grad} u|^2\, d\omega(x).$$

Replacing the surface measure $d\omega$ on Σ by the normalized measure $d\sigma$ it follows that

$$\int_{\Sigma} |\operatorname{grad} u|^2\, d\sigma(x) = (n+2k)(k+1)\int_{\Sigma} |u(x)|^2\, d\sigma(x)\,.$$

Similar calculations lead to the normalization for the functions $n_i^k(x)$ and to the orthogonality relations.

PROPOSITION 5. *If* $n = 3$,

$$\{q_i^k(x) = (k^2 + k)^{-\frac{1}{2}}(x \times \operatorname{grad} e_i^k(x))\}_{i=1}^{d_k}$$

is an orthogonal basis in $\mathfrak{Q}^k$.

Note that $|x \times \operatorname{grad} u(x)|$ is the length of the projection of grad u into the tangent plane to Σ at x.

$$\int_\Sigma |x \times \operatorname{grad} e_i^k(x)|^2 \, d\sigma(x) = \int_\Sigma \left(|x|^2 |\operatorname{grad} e_i^k(x)|^2 - |(\operatorname{grad} e_i^k(x), x)|^2\right) d\sigma(x) =$$
$$= k(3 + 2k - 2) - k^2 = k^2 + k \, .$$

The orthogonality relations follow from the identity

$$|x \times y|^2 = |x|^2 |y|^2 - (x, y)^2 \qquad x, y \in \mathbf{R}^3 .$$

For $n \geqslant 4$ we choose an arbitrary orthonormal basis q_i^k $i = 1, 2, \ldots, s_k$ in every space $\mathfrak{Q}^k$.

In order to describe the constant vector fields $\mathcal{H}^0(\Sigma)$ we use the notation $f_i = m_i^0$ for the standard basis in $\mathbf{C}^n$ $(i = 1, \ldots, n)$. With these definitions theorem 1 can be reformulated:

THEOREM 1′. *Any vector field* $v \in \mathfrak{L}^2(\Sigma)$ *has a unique expansion*

$$v = \sum_{k=0}^{\infty} \sum_{i=1}^{d_{k+1}} a_{ik} m_i^k + \sum_{k=1}^{\infty} \sum_{i=1}^{d_{k-1}} b_{ik} n_i^k + \sum_{k=1}^{\infty} \sum_{i=1}^{s_k} c^{ik} q_i^k \qquad |x| = 1$$

with $a_{ik} = (v, m_i^k)$, $b_{ik} = (v, n_i^k)$ *and* $c_{ik} = (v, q_i^k)$.

The expansion converges in $\mathfrak{L}^2(\Sigma)$ *and*

$$\|v\|^2 = \sum_{i,k} |a_{ik}|^2 + \sum_{i,k} |b_{ik}|^2 + \sum_{i,k} |c_{ik}|^2 .$$

THEOREM 2. *The vector fields* m_i^k $(i = 1, \ldots, d_{k+1})$, $q_i^k (i = 1, \ldots, s_k)$ *and*

$$p_i^k(x) = x e_i^{k+1}(x) + c(n, k)(1 - |x|^2) \operatorname{grad} e_i^{k-1}(x) \qquad i = 1, \ldots, d_{k-1}$$

with

$$c(n, k) = \frac{1}{2}\,\frac{(k-1)(1-2/n)+n}{(k-1)(1-2/n)+2k+n-4}$$

are solutions of the equation $S^*Sv = 0$.

Observe that the vector fields p_i^k are not homogeneous.

The proof for theorem 2 consists in a rather tedious but straightforward verification. Note in particular, that the vector fields m_i^k and q_i^k both have harmonic components and vanishing divergence (proposition 2). The solutions p_i^k have been found through experimentation. We omit the calculations.

Since $p_i^k(x) = xe_i^{k-1}(x)$ for $|x| = 1$, the system $\{p_i^k(x)\}_{i=1}^{d_{k-1}}$ restricted to Σ is an orthonormal system. A simple calculation shows that on Σ

$$p_i^k(x) = (n+k-3)^{\frac{1}{2}}(n+2k-4)^{-\frac{1}{2}}\, n_i^k(x) + \\ + (k-1)^{\frac{1}{2}}(n+2k-4)^{-\frac{1}{2}}\, m_i^{k-2}(x) \qquad k \geqslant 2$$

$$p_1^1(x) = n_1^1(x) = x\,.$$

If in the orthonormal basis of $\mathfrak{L}^2(\Sigma)$ the vectors n_i^k are replaced by the vectors p_i^k then $v \in \mathfrak{L}^2(\Sigma)$ has an expansion of the form

$$v = \sum_{k,i} a'_{ik} m_i^k + \sum_{k,i} b'_{ik} p_i^k + \sum_{k,i} c_{ik} q_i^k \qquad |x| = 1$$

with

$$a_{i,k-2} m_i^{k-2} + b_{ik} n_i^k = a'_{i,k-2} m_i^{k-2} + b'_{ik} p_i^k = \\ = \left(a'_{i,k-2} + \left(\frac{k-1}{n+2k-4}\right)^{\frac{1}{2}} b'_{ik}\right) m_i^{k-2} + \left(\frac{n+k-3}{n+2k-4}\right)^{\frac{1}{2}} b'_{ik} n_i^k$$

$k = 2, 3, \ldots, (|x| = 1)$ and

$$b_{11} = b'_{11}\,.$$

We conclude that

$$\left(1 - \left(\frac{k-1}{n+2k-4}\right)^{\frac{1}{2}}\right)\left(|a'_{i,k-2}|^2 + |b'_{ik}|^2\right) \leqslant \\ \leqslant |a_{i,k-2}|^2 + |b_{ik}|^2 \leqslant \left(1 + \left(\frac{k-1}{n+2k-4}\right)^{\frac{1}{2}}\right)\left(|a'_{i,k-2}|^2 + |b'_{ik}|^2\right)\,.$$

The coefficients therefore satisfy the double inequality

$$(1-2^{-\frac{1}{2}})\|v\|^2 = \sum_{i,k}|a'_{ik}|^2 + \sum_{i,k}|b'_{ik}|^2 + \sum_{i,k}|c_{ik}|^2 = (1+2^{-\frac{1}{2}})\|v\|^2 .$$

4. – The solution of the Dirichlet problem.

The vector fields m_i^k, n_i^k, p_i^k, q_i^k have been defined in section 3. A complete orthonormal system in $\mathfrak{L}^2(\Sigma)$ is given by the restrictions of m_i^k, n_i^k and q_i^k to Σ. The vector fields m_i^k, p_i^k and q_i^k are solutions of the equation $S^*Sv = 0$. Restricted to Σ, these vector fields constitute a normalized basis of $\mathfrak{L}^2(\Sigma)$, however this basis is not orthogonal.

THEOREM 3. *Given $v \in \mathfrak{L}^2(\Sigma)$, $n \geqslant 3$, there exists a unique solution u of $S^*Su = 0$ in the unit ball $B^n \subset \mathbb{R}^n$ with L^2-boundary values v:*

$$\lim_{r\to 1}\int_{\Sigma}|u(rx) - v(x)|^2\, d\sigma(x) = 0 .$$

This solution is given by the formula

$$u = \sum_{k=0}^{\infty}\sum_{i=1}^{d_{k+1}} a'_{ik} m_i^k + \sum_{k=1}^{\infty}\sum_{i=1}^{d_{k-1}} b'_{ik} p_i^k + \sum_{k=1}^{\infty}\sum_{i=1}^{s_k} c_{ik}\, q_i^k$$

with

$$a'_{ik} = (v, m_i^k) - \left(\frac{k+1}{n+k-1}\right)^{\frac{1}{2}}(v, n_i^{k+2})$$

$$b'_{ik} = \left(\frac{n+2k-4}{n+k-3}\right)^{\frac{1}{2}}(v, n_i^k)$$

$$c_{ik} = (v, q_i^k) .$$

The uniqueness of the solutions follows from the formula (see Ahlfors [1], [2])

$$\int_{B^n}(S^*Su(x), u(x))\, dx = -\int_{B^n}\operatorname{trace} Su(x)\big(Su(x)\big)^{\mathrm{tr}}\, dx$$

which holds for C^∞-vector fields u in B^n with compact support.

The only solutions u of the equation $Su = 0$ are the vector fields

$$u(x) = a + \lambda x + Bx + 2(c, x)\, x - |x|^2 c$$

with λ constant, a and c constant vectors and B a constant matrix with $B = -B^{\text{tr}}$. Hence if u is a solution of $S^*Su = 0$, which is smooth up to the boundary (bounded derivatives) and if u has zero boundary values, then $u = 0$. In the general situation, if u is a solution of $S^*Su = 0$ with zero L^2-boundary values, then $u_r(x) = u(rx)$, $0 < r < 1$, is also a solution of the differential equation. By the preceding argument, u_r is the unique solution in B^n with boundary values $u(rx)$, $|x| = 1$. This solution is therefore given by the formula in the theorem. The corresponding coefficients $a'_{ik}(r)$, $b'_{ik}(r)$ and $c_{ik}(r)$ satisfy

$$\lim_{r\to 1} \sum_{i,k} |a'_{ik}(r)|^2 + \sum_{i,k} |b'_{ik}(r)|^2 + \sum_{i,k} |c_{ik}(r)|^2 \leqslant (1 + 2^{-\frac{1}{2}}) \lim_{r\to 1} \int_{\Sigma} |u(rx)|^2 d\sigma(x) = 0 .$$

We conclude that $u = \lim\limits_{r\to 1} u_r$ is the zero solution.

In order to complete the proof of the theorem, it remains to be shown that the vector field w defined by

$$w(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{d_{k-1}} b'_{ik} c(n, k)(1 - |x|^2) \operatorname{grad} e_i^{k-1}(x)$$

has zero boundary values in $\mathfrak{L}^2(\Sigma)$. We recall that $\{e_i^k\}_{i=1}^{d_{k-1}}$ is an orthonormal basis in the space H^k of harmonic polynomials in $\mathbb{R}^n$, homogenous of degree k.

We need the following lemma:

Lemma.

$$\sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k-2} \leqslant (1 - r^2)^{-2} \sum_{k=1}^{\infty} |a_k|^2 \qquad 0 < r < 1$$

for any sequence $a_1, a_2, \ldots$ of complex numbers.

If $\sum_{k=1}^{\infty} |a_k|^2$ converges, then

$$f(z) = \sum_{k=1}^{\infty} a_k z^k \qquad |z| < 1$$

is in the Hardy space of the disc and

$$\|f\|^2 = \sup_{|r|<1} (2\pi)^{-1} \int_0^{2\pi} |f(re^{i\varphi})|^2 d\varphi = \sum_{k=1}^{\infty} |a_k|^2 .$$

For the derivative f' the following estimate holds:

$$\int_0^{2\pi} |f'(re^{i\varphi})|^2\, d\varphi = \int_0^{2\pi} \left| |(2\pi i)^{-1} \int_{|\zeta|=1} \frac{f(\zeta)}{(re^{i\varphi}-\zeta)^2}\, d\zeta \right| d\varphi \leqslant$$

$$\leqslant \int_0 \left(\frac{1}{2\pi}\int_0 \frac{|f(e^{i\vartheta})|^2}{|re^{i\varphi}-e^{i\vartheta}|^2}\, d\vartheta\right)\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{d\vartheta}{|re^{i\varphi}-e^{i\vartheta}|^2}\right) d\varphi\,.$$

The Poisson kernel satisfies

$$\frac{1-|a|^2}{2\pi}\int_0^{2\pi} \frac{d\vartheta}{|a-e^{i\vartheta}|^2} = 1\,, \qquad |a|<1,$$

therefore

$$\int_0^{2\pi} |f'(re^{i\varphi})|^2\, d\varphi \leqslant \frac{1}{1-r^2}\int_0^{2\pi} |f(e^{i\vartheta})|^2 \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{d\varphi}{|r-e^{i(\vartheta-\varphi)}|^2}\right) d\vartheta = \frac{2}{(1-r^2)^2}\, \|f\|^2\,.$$

On the other side

$$\int_0^{2\pi} |f'(re^{i\varphi})|^2\, d\varphi = \int_0^{2\pi} \left| \sum_{k=1}^{\infty} k a_k r^{k-1} e^{(k-1)i\varphi} \right|^2 d\varphi = 2\pi \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k-2}\,.$$

This completes the proof of the lemma.

We proceed now with the proof of the theorem. Making use of the formula

$$\int_\Sigma |\operatorname{grad} e_i^{k-1}(rx)|^2\, d\sigma(x) = r^{2k-4}\int_\Sigma |\operatorname{grad} e_i^{k-1}(x)|^2\, d\sigma(x) = r^{2k-4}(k-1)(n+2k-4)$$

and of the orthogonality relations, the equality

$$\int_\Sigma |w(rx)|^2\, d\sigma(x) = (1-r^2)^2 \sum_{k=1}^{\infty}\sum_{i=1}^{d_{k-1}} |b'_{ik}|^2 c^2(n,k)\, r^{2k-4}(k-1)(n+2k-4)$$

results. Since $\lim_{k\to\infty} c(n,k) = \frac{1}{2}((1-2/n)/(1-2/n+2))$, it is clear that

$$\sum_{k,i} |b'_{ik}|^2 c^2(n,k) \leqslant c_1 \sum_{k,i} |b'_{ik}|^2$$

($c_1, c_2, \ldots$ denote constants depending on n only). The lemma is now applied to the sequence

$$B_k = c(n, k)\left(\sum_{i=1}^{d_{k-1}} |o'_{ik}|^2\right)^{\frac{1}{2}}$$

and it follows that

$$\sum_{k,i} |b'_{ik}|^2 c^2(n, k)\, r^{2k-4}(k-1)(n+2k-4) \leqslant c_2 \sum_{k=1}^{\infty} r^{2k-4} k^2 B_k^2 \leqslant$$
$$\leqslant c_2 r^{-2}(1-r^2)^{-2} \sum_{k=1}^{\infty} B_k^2 \leqslant c_3 r^{-2}(1-r^2)^{-2} \sum_{k,i} |b'_{ik}|^2$$

and hence

$$\sup_{\frac{1}{2}<r<1} \int_{\Sigma} |w(rx)|^2 \, d\sigma(x) \leqslant c_4 \sum_{k,i} |b'_{ik}|^2 .$$

Consider the partial sums

$$w_N(x) = \sum_{k=1}^{N} \sum_{i=1}^{d_{k-1}} b'_{ik}\, c(n, k)(1-|x|^2) \operatorname{grad} e_i^{k-1}(x) .$$

They satisfy

$$\sup_{<r<1} \int_{\Sigma} |w(rx) - w_N(rx)|^2 \, d\sigma(x) \leqslant c_4 \sum_{k=N+1}^{\infty} \sum_{i=1}^{d_{k-1}} |b'_{ik}|^2$$

according to the preceding result. Since

$$\lim_{r\to 1} \int_{\Sigma} |w_N(rx)|^2 \, d\sigma(x) = 0$$

it follows that

$$\limsup_{r\to 1} \left(\int_{\Sigma} w(rx)|^2 \, d\sigma\right)^{\frac{1}{2}} \leqslant \limsup_{r\to 1} \left(\int_{\Sigma} |w(rx) - w_N(rx)|^2 \, d\sigma\right)^{\frac{1}{2}} +$$
$$+ \limsup_{r\to 1} \left(\int_{\Sigma} |w_N(rx)|^2 \, d\sigma\right)^{\frac{1}{2}} \leqslant c_4 \sum_{k=N+1}^{\infty} \sum_{i=1}^{d_{k-1}} |b'_{ik}|^2$$

for arbitrary N. This then shows that

$$\lim_{r\to 1} \int |w(rx)|^2 \, d\sigma(x) = 0 .$$

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