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## On Nontrivial Solutions of a Semilinear Wave Equation (\*).

PAUL H. RABINOWITZ

The question of the existence of nontrivial time periodic solutions of autonomous or forced semilinear wave equations has been the object of considerable recent interest [1-12]. These papers study the equation

$$(1.1) \quad u_{tt} - u_{xx} + f(x, u) = 0, \quad 0 < x < l$$

(or its analogue where  $f$  also depends on  $t$  in a time periodic fashion) together with boundary conditions in  $x$  and periodicity conditions in  $t$ . In particular the following result was proved in [11, Theorem 3.37 and Corollary 4.14]:

**THEOREM 1.2.** *Let  $f \in C([0, l] \times \mathbf{R}, \mathbf{R})$  and satisfy*

( $f_1$ )  $f(x, 0) = 0$  and  $f(x, r)$  is strictly monotone increasing in  $r$ ,

( $f_2$ )  $f(x, r) = o(|r|)$  at  $r = 0$ ,

( $f_3$ ) there are constants  $\bar{r} > 0$  and  $\mu > 2$  such that

$$0 < \mu F(x, r) \leq r f(x, r)$$

for  $|r| \geq \bar{r}$  and  $x \in [0, l]$  where

$$F(x, r) = \int_0^r f(x, s) ds.$$

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Then for any  $T$  which is a rational multiple of  $l$ , equation (1.1) possesses a non trivial continuous weak solution  $u$  satisfying

$$(1.3) \quad \begin{cases} u(0, t) = 0 = u(l, t), \\ u(x, t + T) = u(x, t). \end{cases}$$

Furthermore  $f \in C^k$  implies  $u \in C^k$ .

As part of the proof of Theorem 1.2, it was shown that the functional

$$(1.4) \quad I(u) = \int_0^T \int_0^l [\frac{1}{2}(u_t^2 - u_x^2) - F(x, u)] dx dt$$

defined on the class of functions satisfying (1.3) (and of which (1.1) is formally the Euler equation) has a positive critical value. Therefore  $(f_1)$  and the form of  $I$  imply that  $u_t \neq 0$  for the corresponding critical point  $u$ . Thus  $u$  is nonconstant in  $x$  and must depend explicitly on  $t$ . It was further observed in [11] (Theorem 5.24 and Remark 5.25) that if  $g$  satisfies  $(f_1)$ - $(f_3)$  the equation

$$(1.5) \quad u_{tt} - u_{xx} - g(x, u) = 0, \quad 0 < x < l$$

together with (1.3) also possesses a nontrivial weak solution. Indeed the arguments of Theorem 1.2 go through with minor modifications to establish this fact. However the functional one studies for this case is

$$(1.6) \quad J(u) = \int_0^T \int_0^l [\frac{1}{2}(u_x^2 - u_t^2) - G(x, u)] dx dt$$

where  $G$  is the primitive of  $g$ . Again the positivity of  $J(u)$  for a critical point  $u$  implies  $u$  is nonconstant but we can no longer conclude that  $u$  depends explicitly on  $t$ . In fact it is known [13, 14] that as a consequence of  $(f_2)$ - $(f_3)$ , the ordinary differential equation boundary value problem

$$(1.7) \quad -\frac{d^2u}{dx^2} = g(x, u), \quad u(0) = 0 = u(l)$$

has an unbounded sequence of solutions which can be characterized by the number of zeros they possess in  $(0, l)$ .

Our goal in this paper is to show that if  $(f_3)$  is strengthened somewhat, (1.5), (1.3) possesses infinitely many time dependent solutions. More precisely we will prove:

**THEOREM 1.8.** *Let  $g \in ([0, l] \times \mathbf{R}, \mathbf{R})$  and suppose  $g$  satisfies  $(f_1)$ - $(f_2)$  and  $(\bar{f}_3)$ . There is a constant  $\mu > 0$  such that*

$$0 < \mu F(x, r) \leq rf(x, r)$$

for all  $r \neq 0$ .

Then for any  $T \in l\mathbf{Q}$  there is a  $k_0 \in \mathbf{N}$  such that for all  $k \geq k_0$ , (1.5), (1.3) possesses a solution  $u_k$  which is  $kT$  periodic in  $t$  and  $\partial u_k / \partial t \neq 0$ . Moreover infinitely many of the functions  $u_k$  are distinct.

**REMARK 1.9.** We have no estimate for the size of  $k_0$  and do not know if the result is false in general for  $k = 1$ . Note also that since (1.5) is an autonomous equation with respect to  $t$ , whenever  $u(x, t)$  is a solution, so is  $u(x, t + \theta)$  for any  $\theta \in \mathbf{R}$ . The above statement about the  $u_k$ 's being distinct means in particular that they do not differ by merely a translation in time.

The proof of Theorem 1.8 draws on several results from [11] and ideas from [12]. For convenience we will take  $l = \pi$  and  $T = 2\pi$ . Choosing  $k \in \mathbf{N}$ , we seek a solution  $u_k$  of (1.5) which is  $2\pi k$  periodic in  $t$  and  $\partial u_k / \partial t \neq 0$ . Making the change of time scale  $\tau = t/k$ , the period becomes  $2\pi$  again and the problem to be solved is

$$(1.10) \quad \begin{cases} U_{\tau\tau} - k^2(U_{xx} + g(x, U)) = 0, & 0 < x < \pi \\ U(0, \tau) = 0 = U(l, \tau) \\ U(x, \tau + 2\pi) = U(x, \tau) \end{cases}$$

with  $U(x, \tau) = u(x, t)$ .

For the convenience of the reader and to set the stage for a key estimate, the argument used in [11] to establish the existence of nontrivial solutions of (1.10) will be sketched quickly. Solutions are obtained by an approximation argument. To begin (1.10) is modified in two ways. The wave operator  $\partial^2 / \partial \tau^2 - k^2(\partial^2 / \partial x^2)$  possesses an infinite dimensional null space in the class of functions satisfying the boundary and periodicity conditions of (1.10) and given by

$$N = \text{span} \{ \sin jx \sin kj\tau, \sin jx \cos kj\tau \mid j \in \mathbf{N} \}.$$

The fact that  $N$  is infinite dimensional complicates the analysis of (1.10) and to introduce some compactness to the problem in  $N$ , we perturb (1.10) by adding a  $\beta V_{tt}$  term to the left hand side of the equation. Here  $\beta > 0$  and  $V$  is the ( $L^2$  orthogonal) projection of  $U$  onto  $N$ . A second difficulty in treating (1.10) arises due to the unrestricted rate of growth of  $g(x, r)$  as

$|r| \rightarrow \infty$ . We get around this by truncating  $g$ . More precisely  $g(x, r)$  is replaced by  $g_K(x, r)$  which coincides with  $g$  for  $|r| \leq K$  and grows cubically at  $\infty$  [11]. Thus (1.10) is replaced by the modified problem

$$(1.11) \quad \begin{cases} U_{\tau\tau} + \beta V_{\tau\tau} - k^2(U_{xx} + g_K(x, U)) = 0, & 0 < x < \pi \\ U(0, \tau) = 0 = U(l, \tau) \\ U(x, \tau + 2\pi) = U(x, \tau) \end{cases}$$

where  $g_K$  satisfies  $(f_1), (f_2), (\tilde{f}_3)$  with a new constant  $\bar{\mu} = \min(\mu, 4)$ .

Letting  $G_K$  denote the primitive of  $g_K$ , in a formal fashion (1.11) can be interpreted as the Euler equation arising from the functional

$$(1.12) \quad J(U; k, \beta, K) = \int_0^{2\pi} \int_0^\pi \left[ \frac{k^2}{2} U_x^2 - \frac{1}{2} U_\tau^2 - \frac{\beta}{2} V_\tau^2 - k^2 G_K(x, U) \right] dx d\tau.$$

Let

$$E_m \equiv \text{span} \{ \sin jx \sin n\tau, \sin jx \cos b\tau \mid 0 < j, n < m \}.$$

The strategy pursued in [11] was to find a critical point  $U_{m,k}$  of  $J|_{E_m}$ , let  $m \rightarrow \infty$ , and then let  $\beta \rightarrow 0$  to get a solution  $U_k$  of (1.10) with  $g$  replaced by  $g_K$ . Then  $L^\infty$  bounds for  $U_k$  independent of  $K$  show if we choose  $K(k)$  sufficiently large  $g_K(x, U_k) = g(x, U_k)$  so (1.10) obtains. A separate comparison argument is required to prove that  $U_k \neq 0$ .

The first step in carrying out the details of the above argument involves obtaining an upper bound  $M_k$  for  $c_{m,k} \equiv J(U_{m,k}; k, \beta, K)$  with  $M_k$  independent of  $m, \beta$ , and  $K$ . For the current problem which also depends on  $k$ , it is crucial to know the behavior of  $M_k$  as a function of  $k$ . Thus we will take a closer look at  $c_{m,k}$  and use a variant of an argument of [12]. By Lemma 1.13 of [11],  $c_{m,k}$  can be characterized in a minimax fashion. We will not write down this characterization explicitly but will note a consequence of it which in turn provides an upper bound for  $c_{m,k}$ . Set

$$\Phi_{m,k} = \text{span} \{ \sin jx \sin n\tau, \sin jx \cos n\tau \mid 0 < j, n < m \text{ and } n^2 \geq j^2 k^2 \}$$

and

$$\psi_k = a_k \sin x \sin (k-1)\tau$$

where  $a_k = \sqrt{2}/\pi$  so  $\|\psi_k\|_{L^1} = 1$ . Set  $\Psi_{m,k} = \Phi_{m,k} \oplus \text{span } \psi_k$ . Then by Lemma 1.3. of [11]

$$(1.13) \quad 0 < c_{m,k} \leq \max_{u \in \Psi_{m,k}} J(u; k, \beta, K).$$

Inequality (1.13) will lead to a suitable choice for  $M_k$ . Note that by  $(\bar{f}_3)$  (or even  $(f_3)$ ), there are constants  $\alpha_1, \alpha_2 \geq 0$  and independent of  $K$  such that

$$(1.14) \quad G_K(x, r) \geq a_1 |r|^{\bar{\mu}} - a_2$$

for all  $r \in \mathbf{R}, x \in [0, \pi]$ . Consequently  $J \rightarrow -\infty$  as  $u \rightarrow \infty$  in  $\Psi_{mk}$  (under  $\|\cdot\|_{L^1}$ ) so there is a point  $z \equiv Z_{mk}$  at which the maximum in (1.13) is achieved. Writing

$$(1.15) \quad z = \|z\|_{L^1}(\gamma\xi + \delta\psi_k)$$

where  $\xi \in \Phi_{mk}, \|\xi\|_{L^1} = 1$ , and  $\gamma^2 + \delta^2 = 1$  and substituting (1.15) into (1.13) gives

$$(1.16) \quad k^2 \int_0^{2\pi} \int_0^\pi G(x, z) dx dt \leq \frac{1}{2} \int_0^{2\pi} \int_0^\pi (k^2 z_x^2 - z_t^2) dx d\tau \leq \\ \leq \frac{\delta^2}{2} \|z\|_{L^1}^2 \int_0^{2\pi} \int_0^\pi [k^2 (\psi_k)_x^2 - (\psi_k)_t^2] dx d\tau \leq k \|z\|_{L^1}^2.$$

Combining (1.14) and (1.16) shows that

$$(1.17) \quad k^2(\alpha_1 \|z\|_{L^{\bar{\mu}}}^{\bar{\mu}} - \alpha_2) \leq k \|z\|_{L^1}^2.$$

Applying the Hölder inequality yields

$$(1.18) \quad \|z\|_{L^1} \leq A$$

where  $A$  is a constant independent of  $m, k, \beta, K$ . Hence by (1.13), (1.18), and the form of  $J$ ,

$$(1.19) \quad c_{mk} \leq Mk$$

for a constant  $M$  independent of  $m, k, \beta, K$ .

Letting  $m \rightarrow \infty$  and then  $\beta \rightarrow 0$ , and formalizing what we have just shown gives:

LEMMA 1.20. *Under the hypotheses of Theorem 1.8 (with  $l = \pi$  and  $T = 2\pi$ ), for all  $k \in \mathbf{N}$ , there exists a solution  $U_k$  of (1.10) satisfying*

$$(1.21) \quad c_k \equiv J(U_k; k, 0, K) \leq Mk$$

with  $M$  independent of  $k$  and  $K$ .

It remains to show that for all  $k$  sufficiently large,  $\partial U_k/\partial t \neq 0$  and infinitely many of the functions  $u_k(x, t) = U_k(x, \tau)$  are distinct. If  $U_k$  is independent of  $\tau$  for any subsequence of  $k$ 's tending to  $\infty$ ,  $U_k = U_k(x)$  is a classical solution of (1.7). Thus by (1.21) with  $K = K(k)$  suitably large,

$$(1.22) \quad c_k = 2\pi k^2 \int_0^\pi \left[ \frac{1}{2} \left| \frac{dU_k}{dx} \right|^2 - G(x, U_k) \right] dx.$$

By (1.7),

$$(1.23) \quad \int_0^\pi \left| \frac{dU_k}{dx} \right|^2 dx = \int_0^\pi U_k(x) g(x, U_k(x)) dx.$$

Combining (1.21)-(1.23) yields

$$(1.24) \quad \int_0^\pi \left[ \frac{1}{2} U_k g(x, U_k) - G(x, U_k) \right] dx \rightarrow 0$$

as  $k \rightarrow \infty$  along this subsequence. Moreover by  $(\bar{f}_3)$ ,

$$(1.25) \quad \int_0^\pi \left[ \frac{1}{2} U_k g(x, U_k) - G(x, U_k) \right] dx \geq \int_0^\pi \left( \frac{1}{2} - \frac{1}{\bar{\mu}} \right) U_k g(x, U_k) dx.$$

Thus  $U_k g(x, U_k) \rightarrow 0$  in  $L^1$ . From (1.23) again we conclude that  $dU_k/dx \rightarrow 0$  in  $L^2$  which easily implies  $U_k \rightarrow 0$  in  $L^\infty$ . By  $(f_2)$ , for any  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $|r| < \delta$  implies  $|g(x, r)| < \varepsilon r$ . Choosing  $\varepsilon < 1/\pi$  and  $k$  large enough so that  $\|U_k\|_{L^\infty} < \delta$ , (1.23) then shows

$$(1.26) \quad \left\| \frac{dU_k}{dx} \right\|_{L^2}^2 < \varepsilon \|U_k\|_{L^2}^2 < \pi \varepsilon \left\| \frac{dU_k}{dx} \right\|_{L^2}^2 < \left\| \frac{dU_k}{dx} \right\|_{L^2}^2,$$

a contradiction. Consequently  $U_k$  depends on  $\tau$  for all large  $k$ .

To prove the second assertion of Theorem 1.8, suppose two functions  $U_k(x, \tau)$ ,  $U_j(x, \tau)$  correspond to the same function of  $x, t$  modulo a translation in time (keeping Remark 1.9 in mind). Thus  $U_k(x, \tau) = U_k(x, t/k) \equiv U(x, t)$  and  $U_j(x, \tau) = U_j(x, t/j) = U(x, t + \theta)$  for some  $\theta \in \mathbf{R}$  or  $U_k(x, \tau) = U(x, k\tau)$ ,  $U_j(x, \tau) = U(x, j\tau + \theta)$ . Since  $U$  must be both  $2\pi k$  and  $2\pi j$  periodic in  $t$ , letting  $\sigma$  denote the greatest common divisor of  $j$  and  $k$ , we have  $j = \sigma \bar{j}$ ,

$k = \sigma \bar{k}$  and  $U$  has period  $2\pi\sigma$  in  $t$ . Furthermore

$$\begin{aligned}
 (1.27) \quad c_k &= \int_0^{2\pi} \int_0^\pi \left[ \frac{k^2}{2} U_{kx}^2 - \frac{1}{2} U_{k\tau}^2 - k^2 G(x, U_k) \right] dx d\tau = \\
 &= k \int_0^{2\pi k} \int_0^\pi \left[ \frac{1}{2} (U_x^2 - U_t^2) - G(x, U) \right] dx dt = \\
 &= \frac{k^2}{\sigma} \int_0^{2\pi\sigma} \int_0^\pi \left[ \frac{1}{2} (U_x^2 - U_t^2) - G(x, U) \right] dx dt \equiv \frac{k^2}{\sigma} b
 \end{aligned}$$

and similarly

$$(1.28) \quad c_j = \frac{j^2}{\sigma} b.$$

Consequently if there were a sequence of solutions  $U_{k_i}$  of (1.10) corresponding to the same function  $U$  (up to a translation in  $t$ ), by (1.27)-(1.28) we have

$$(1.29) \quad c_{k_i} = \frac{k_i^2}{\sigma} b$$

and  $c_{k_i} \rightarrow \infty$  like  $k_i^2$  along this sequence contrary to (1.19). Thus at most finitely many functions  $U_k(x, \tau)$  correspond to the same solution  $u_k(x, t)$  of (1.5), (1.3) and infinitely many of the functions  $u_k$  must be time dependent solutions of (1.5), (1.3). The proof of Theorem 1.8 is complete.

**REMARK 1.30.** Both the existence assertions from [11] and the arguments given above use hypothesis  $(f_2)$  which requires that  $g$  vanish more rapidly than linearly at 0. However this condition can be weakened. The simplest such generalization would be to replace  $g(x, r)$  by  $\alpha r + g(x, r)$  with  $\alpha$  a constant and for this case we have:

**THEOREM 1.31.** *Let  $g$  satisfy  $(f_1)$ ,  $(f_2)$ ,  $(\bar{f}_3)$  and let  $\alpha > 0$ . Then for all  $T \in \mathbb{Q}$ , there exists a  $k_0 \in \mathbb{N}$  such that for all  $k \geq k_0$ , the problem*

$$(1.32) \quad \begin{cases} u_{tt} - u_{xx} - \alpha u - g(x, u) = 0 & 0 < x < l \\ u(0, t) = 0 = u(l, t) \\ u(x, t + kT) = u(x, t) \end{cases}$$

has a continuous weak solution  $u_k$  which is  $kT$  periodic in  $t$  and  $\partial u_k / \partial t \neq 0$ . Moreover infinitely many of these functions are distinct.



PROOF. For convenience we again take  $l = \pi$ ,  $T = 2\pi$ . It was shown in [11] that Theorem 1.2 carries over to (1.32) for  $\alpha > 0$ . It is also easy to see that the argument of Lemma 1.20 will give (1.21) for this setting. Likewise (1.27)-(1.29) are unaffected by the  $\alpha$  term. Thus we get Theorem 1.31 provided that we can show  $U_k(x, \tau)$  depends on  $\tau$  for all large  $k$ . If not, the analogues of (1.22)-(1.23) here are

$$(1.33) \quad c_k = 2\pi k^2 \int_0^\pi \left[ \frac{1}{2} \left| \frac{dU_k}{dx} \right|^2 - \frac{1}{2} \alpha U_k^2 - G(x, U_k) \right] dx$$

and

$$(1.34) \quad \int_0^\pi \left| \frac{dU_k}{dx} \right|^2 dx = \int_0^\pi (\alpha U_k^2 + U_k g(x, U_k)) dx .$$

Thus (1.19), (1.33)-(1.34), and  $(\bar{f}_3)$  show that  $U_k g(x, U_k) \rightarrow 0$  in  $L^1$  as  $k \rightarrow \infty$  as in (1.24)-(1.25). Since

$$(1.35) \quad \|g(x, U_k)\|_{L^1} \leq \pi \max_{0 \leq x \leq \pi, |r| \leq 1} |g(x, r)| + \|U_k g(x, U_k)\|_{L^1}$$

and the right hand side of (1.35) is uniformly bounded in  $k$ , it follows from (1.7) that the functions  $d^2 U_k/dx^2$  are uniformly bounded in  $L^1$ . The boundary conditions  $U_k(0) = 0 = U_k(\pi)$  imply that there is  $x_k \in (0, \pi)$  such that  $(dU_k/dx)(x_k) = 0$ . Hence

$$\frac{dU_k}{dx} = \int_{x_k}^x \frac{d^2 U_k(\xi)}{d\xi^2} d\xi$$

which implies that

$$(1.36) \quad \left\| \frac{dU_k}{dx} \right\|_{L^\infty} \leq \left\| \frac{d^2 U_k}{dx^2} \right\|_{L^1} .$$

Thus the functions  $U_k$ ,  $dU_k/dx$  are bounded in  $L^\infty$  and by (1.7) again, so are  $d^2 U_k/dx^2$ . It follows that a subsequence of  $U_k$  converges (in  $\|\cdot\|_{C^2}$ ) to a solution  $U$  of (1.7) as  $k \rightarrow \infty$ . But  $(f_1)$  and  $\|U_k g(x, U_k)\|_{L^1} \rightarrow 0$  as  $k \rightarrow \infty$  imply  $U \equiv 0$ .

Next observe that (1.7) can be written as

$$(1.37) \quad U_k(x) = \int_0^\pi H(x, y) (\alpha U_k(y) + g(y, U_k(y))) dy$$

where  $H$  is the Green's function for  $-d^2/dx^2$  under the boundary condition  $U(0) = 0 = U(\pi)$ . Dividing (1.37) by  $\|U_k\|_{C^1}$  gives

$$(1.38) \quad \frac{U_k(x)}{\|U_k\|_{C^1}} = \int_0^\pi H(x, y) \left( \alpha \frac{U_k(y)}{\|U_k\|_{C^1}} + \frac{g(x, U_k(y))}{\|U_k\|_{C^1}} \right) dy .$$

By  $(f_2)$ , the arguments of the integral operator are uniformly bounded in  $C^1$ . Hence since this operator is compact from  $C^1$  to  $C^2$ , by  $(f_2)$  again a subsequence of  $U_k/\|U_k\|_{C^1}$  converge to  $V$  satisfying  $\|V\|_{C^1} = 1$  and

$$(1.39) \quad V(x) = \alpha \int_0^\pi H(x, y) V(y) dy$$

or equivalently

$$(1.40) \quad -V'' = \alpha V \quad 0 < x < \pi; \quad V(0) = 0 = V(\pi) .$$

If  $\alpha$  is not an eigenvalue of  $-d^2/dx^2$  under these boundary conditions we have a contradiction and the proof is complete. Thus suppose  $\alpha$  is an eigenvalue. Consider the eigenvalue problems:

$$(1.41) \quad -z'' = \lambda z, \quad 0 < x < \pi; \quad z(0) = 0 = z(\pi)$$

$$(1.42) \quad -y'' = \mu \left( \alpha + \frac{g(x, \varphi)}{\varphi} \right) y, \quad 0 < x < \pi; \quad y(0) = 0 = y(\pi)$$

where  $\varphi$  is  $C^1$  on  $[0, \pi]$ . Let  $\lambda_j$  (resp.  $\mu_j(\varphi)$ ) denote the  $j$ -th eigenvalue of (1.41) (resp. (1.42)), the eigenvalues being ordered according to increasing magnitude. As is well known any eigenfunction corresponding to  $\lambda_m$  or  $\mu_m(\varphi)$  belongs to

$$S_m = \{ \varphi \in C^1([0, \pi], \mathbf{R}) \mid \varphi(0) = 0 = \varphi(\pi), \varphi \text{ has exactly } m - 1 \text{ zeros in } (0, \pi), \text{ and } \varphi' \neq 0 \text{ at all zeros of } \varphi \text{ in } [0, \pi] \} .$$

(Indeed the eigenvalues of (1.41) are  $\lambda_m = m^2\alpha^{-1}$  and corresponding eigenfunctions are multiples of  $\sin mx$ ). Since  $g(x, \varphi)\varphi^{-1} \geq 0$  via  $(f_1)$ , we have  $\lambda_j \geq \mu_j(\varphi)$  for all  $j \in \mathbf{N}$  and  $\varphi \in C^1, \varphi \neq 0$  via a standard comparison theorem [15, Chapter 6]. By (1.40), 1 is an eigenvalue of (1.41), say  $1 = \lambda_m$  and  $V \in S_m$ . Thus  $\mu_m(\varphi) \leq 1$  and since  $S_m$  open (in the  $C^1$  topology) and  $U_k/\|U_k\|_{C^1} \rightarrow V$  in  $C^1$  along some subsequence, it follows that  $U_k/\|U_k\|_{C^1}$  and therefore  $U_k$  belongs to  $S_m$  for all large  $k$  in this subsequence. Writ-

ing (1.7) as

$$(1.43) \quad -U_k'' = \left( \alpha + \frac{g(x, U_k)}{U_k} \right) U_k, \quad 0 < x < \pi; \quad U_k(0) = 0 = U_k(\pi),$$

we see  $\mu_m(U_k) = 1$ . By  $(f_1)$  again,  $g(x, U_k)U_k^{-1} > 0$  except at the  $m + 1$  zeros of  $U_k$ . An examination of the proof of the Sturm Comparison Theorem [16, pp. 208-209] then shows  $U_k$  has a zero between each pair of successive zeros of  $V$ . Consequently  $U_k \in S_{m+1}$ , a contradiction. Thus Theorem 1.31 is established.

**REMARK 1.44.** In [5], Brezis, Coron and Nirenberg study (1.1), (1.3) replacing  $(f_3)$  by

$$(f_4) \quad \frac{1}{2}rf(r) - V(r) \geq \beta|f(r)| - \gamma$$

and

$$(f_5) \quad f(r)/r \rightarrow \infty \quad \text{as } |r| \rightarrow \infty$$

(and with no analogue of  $(f_2)$ ). If we use  $(f_4)$ - $(f_5)$  with  $x$  dependent  $f$  in place of  $(f_3)$ , it is not difficult to see that the proof of [11] carries over for this case as does Lemma 1.20 and (1.27)-(1.29). Thus we obtain a variant of Theorem 1.8 for this case once it is established that  $U_k(x, \tau)$  depends on  $\tau$  for large  $k$ . To do this, we argue as in the proof of Theorem 1.8. Assume  $(f_4)$  holds with  $\gamma = 0$ . Then by (1.25) and  $(f_4)$ ,  $\|g(x, U_k)\|_{L^1} \rightarrow 0$  as  $k \rightarrow \infty$ . This in turn implies  $\|U_k\|_{L^\infty} \rightarrow 0$  via (1.7) and (1.36). Hence (1.26) again provides a contradiction.

It is also possible for us to drop  $(f_2)$  and even the requirement that  $f(x, 0) = 0$  in  $(f_1)$  but then a new existence mechanism is required and we shall not carry out the details here.

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