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Stability of the Polynomial Hull of T^2.

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1. – Introduction.

As is well known, the polynomial hull of the standard 2-torus $\boldsymbol{T}^2 = \{(z_1, z_2) \in \boldsymbol{C}^2 \colon |z_1| = |z_2| = 1\}$ is the closure of the unit polydisk $\Delta^2 = \{z \in \boldsymbol{C}^2 \colon |z_1|, |z_2| < 1\}$. The existence of the hull is seen because the topological boundary of Δ^2 consists of two 1-parameter families of disks, namely $\partial\Delta \times \Delta$ and $\Delta \times \partial\Delta$, whose boundaries lie in $\boldsymbol{T}^2$. Alexander [1] has shown that under a small deformation of $\boldsymbol{T}^2$, there continue to exist two 1-parameter families of (deformed) disks, whose boundaries lie in the deformed torus. Here we extend that work somewhat and show that these deformed disks give exactly the polynomial hull. Our method is similar in spirit to that of [1]: we solve a functional equation that is analogous to the one used by Bishop [3] to construct disks near a «vanishing» disk. We also adopt the functional analysis approach to this problem developed by Hill and Taiani [5].

Given complex functions $\psi = (\psi_1, \psi_2) \in C^{m+3}(\boldsymbol{T}^2)^2$, we may extend them to $(\boldsymbol{C}\setminus\{0\})^2$ by making them constant in the variables $|z_1|$ and $|z_2|$. Let us define the map $\Psi = (\boldsymbol{C}\setminus\{0\})^2 \to \boldsymbol{C}^2$ by setting $\Psi = (\Psi_1, \Psi_2)$ with

$$\Psi_j(z) = z_j\big(1 + \psi_j(z)\big)\,. \tag{1}$$

Let $\boldsymbol{T}^2_\psi$ denote the image of $\boldsymbol{T}^2$ under Ψ.

We will prove that the polynomial hull of $\boldsymbol{T}^2$ is stable in the following sense.

THEOREM. *For $m > 1$ and $1 > \delta > 0$, there exists an $\varepsilon > 0$ such that if $\psi \in C^{m+3}(\boldsymbol{T}^2)^2$ and $\|\psi\|_{C^{m+3}} < \varepsilon$, then there exist*

$$F^{(1)}_\psi = F^{(1)} \quad \textit{and} \quad F^{(2)}_\psi = F^{(2)}\,, \quad F^{(1)} \colon \partial\Delta \times \bar{\Delta} \to \boldsymbol{C}^2\,, \quad F^{(2)} = \bar{\Delta} \times \partial\Delta \to \boldsymbol{C}^2$$

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with the following properties:

(2a) $\|F^{(j)}(z) - z\|_{C^m} < \delta$ *(and thus F is a diffeomorphism) for* $j = 1, 2$.

(2b) $F^{(j)}$ *is holomorphic in* $\zeta \in \Delta$ *for fixed* $\tau \in \partial\Delta$, $j = 1, 2$.

(2c) $F^{(1)}(\partial\Delta \times \partial\Delta) = F^{(2)}(\partial\Delta \times \partial\Delta) = \boldsymbol{T}^2_\psi$.

(2d) *If Ω is the bounded component of* $\boldsymbol{C}^2 \setminus (F^{(1)}(\partial\Delta \times \bar{\Delta}) \cup F^{(2)}(\bar{\Delta} \times \partial\Delta))$, *then $\bar{\Omega}$ is the polynomial hull of* $\boldsymbol{T}^2_\psi$.

2. – Construction of disks.

It will be more convenient for us to use Ψ^{-1} rather than Ψ in (1). If $\|\psi\|_{C^1}$ is small, then Ψ gives a diffeomorphism between $\boldsymbol{T}^2$ and $\boldsymbol{T}^2_\psi$. If we write $z^* = \Psi(z)$, then there exists $\varphi \in C^{m+3}(\boldsymbol{T}^2)^2$ such that $\Phi(z^*) = z$ for $z \in \boldsymbol{T}^2$. For points $z^* \in \boldsymbol{T}^2_\psi$, we may write φ as

$$\varphi_j(z^*) = \frac{-\psi_j(z)}{1 + \psi_j(z)}\,.$$

Thus there is an open set $U \subset C^1(\boldsymbol{T}^2)^2$ containing 0 such that we may define

$$S\colon U \cap C^{m+3}(\boldsymbol{T}^2)^2 \to C^{m+2}(\boldsymbol{T}^2)^2$$

given by $S(\psi) = \varphi$, where $\Phi|_{\boldsymbol{T}^2_\psi} = \Psi^{-1}$. By Lemma 5.1 of [5] S is a mapping of class C^1. We note also that at $\psi = 0$, the differential of this Banach space mapping is the negative of the identity i.e. $dS(0, \chi) = -\chi$.

Now let us consider a fixed mapping $G(\tau, \zeta) = (g_1, g_2) \in C^{m,\alpha}(\partial\Delta \times \bar{\Delta})^2$ such that G is holomorphic in $\zeta \in \Delta$. We assume also that $G(\partial\Delta \times \partial\Delta) \subset \boldsymbol{T}^2_0$. For small ψ we wish to find $F_\psi \in C^{m,\alpha}(\partial\Delta \times \bar{\Delta})^2$ such that $F_\psi(\partial\Delta \times \partial\Delta) \subset \boldsymbol{T}^2_\psi$, and $F_\psi(\tau, \zeta)$ is holomorphic in $\zeta \in \Delta$. With our notation, this means that we want

$$\Phi(F_\psi) \in \boldsymbol{T}_0\,. \tag{3}$$

Following [2], we write $F_\psi = (f_1, f_2)$ in «polar» coordinates as

$$f_j(\phi, \theta) = g_j(\phi, \theta) \exp{(I + iY)\varrho_j(\phi, \theta)}$$

where $\tau = |\tau| e^{i\phi}$, $\zeta = |\zeta| e^{i\theta}$, and Y denotes the harmonic conjugate operator acting in the ζ-variable. Thus for all functions ϱ_j, $(I + iY)\varrho_j$ is holomorphic in ζ. We let $C_{\boldsymbol{R}}^{m,\alpha}(\boldsymbol{T}^2)$ denote the real valued functions in $C^{m,\alpha}(\boldsymbol{T}^2)$. It is well known that Υ is a bounded operator from $C_{\boldsymbol{R}}^{m,\alpha}(\boldsymbol{T}^2)$ to itself for $0<\alpha<1$.

We consider the mapping

$$E\colon C_{\boldsymbol{R}}^{m,\alpha}(\boldsymbol{T}^2)^2\times C^{3m+3}(\boldsymbol{T}^2)^2\to C_{\boldsymbol{R}}^{m,\alpha}(\boldsymbol{T}^2)^2$$

given by

$$E_j(\varrho,\psi)=\log|\Phi_j(F_\psi)|=\log|f_j(1+\varphi_j(F_\psi))|$$

or

$$E_j(\varrho,\psi)=\varrho_j+\log|1+\varphi_j(g\exp(I+i\Upsilon)\varrho)|\,. \tag{4}$$

Thus (3) is equivalent to

$$E(\varrho,\psi)=0\,. \tag{5}$$

It follows from Lemma 5.1 of [5] that E is a Banach space mapping of class C^1. When $\psi=0$, $\varphi=0$, so $d_1E(d\varrho,0)=d\varrho$. It follows by the implicit function theorem that for ψ near 0 in C^{m+3} there is a $\varrho_\psi\in C^{m,\alpha}$ solving (5), and the correspondence $\psi\to\varrho_\psi$ is continuous.

Proof of (2*a*), (2*b*), *and* (2*c*) *of the Theorem.*

Let us set $G\colon \partial\Delta\times\Delta\to\boldsymbol{C}^2$ equal to the identity mapping $G(\tau,\zeta)=(\tau,\zeta)$. By the argument above, we have $F_\psi^{(1)}\colon \partial\Delta\times\Delta\to\boldsymbol{C}^2$. To obtain $F_\psi^{(2)}\colon \Delta\times\partial\Delta\to\boldsymbol{C}^2$, we replace Υ_θ in (4) by Υ_ϕ, the conjugate operator in in the τ-variable. By the continuity of $\psi\to(F_\psi^{(1)},F_\psi^{(2)})$, we have (2*a*). Now (2*b*) follows from the definition of f_j, and (2*c*) follows from (3).

3. – Polynomial hull.

We write $M_1=M_1(\psi)=F_\psi^{(2)}(\partial\Delta\times\Delta)$ and $M_2=M_2(\psi)=F_\psi^{(1)}(\Delta\times\partial\Delta)$. Since $\overline{M}_1(0)$ and $\overline{M}_2(0)$ intersect transversally at $\boldsymbol{T}_0^2$, it follows from (2*a*) and (2*c*) that $\overline{M}_1(\psi)$ and $\overline{M}_2(\psi)$ are transverse at $\boldsymbol{T}_\psi^2$. It follows from the transversality, then, that $\overline{M}_1(\psi)\cap\overline{M}_2(\psi)=\boldsymbol{T}_\psi^2$. Thus $M_1(\psi)\cup M_2(\psi)\cup\boldsymbol{T}_\psi^2$ is homeomorphic to S^3, and it separates $\boldsymbol{C}^2$ into two components. Let $\Omega(\psi)$ denote the bounded component of $\boldsymbol{C}^2\setminus(\overline{M}_1(\psi)\cup\overline{M}_2(\psi))$. Clearly $M_1(\psi)$ and $M_2(\psi)$ are Levi flat surfaces, and thus they are pseudoconvex from both sides. It follows that $\overline{M}_1(\psi)\cup\overline{M}_j(\psi)$ is pseudoconvex from the «acute» side of the intersection, and so $\Omega(\psi)$ is pseudoconvex.

In order to show that $\overline{\Omega(\psi)}$ is polynomially convex, we will construct a family $\{\Omega^t\}$ of pseudoconvex domains with the properties:

(6a) $$\Omega^0 = \Omega(\psi)\,, \qquad \bar{\Omega}(\psi) = \bigcap_{s>0} \Omega^s.$$

(6b) $$\Omega^s \subset \Omega^t \quad \text{if } s < t$$

(6c) $$\bigcup_{s<t} \Omega^s = \Omega^t \quad \text{and} \quad \bigcup_{t>\infty} \Omega^t = \boldsymbol{C}^2$$

(6d) $$\operatorname{int} \bigcap_{s>t} \Omega^s = \Omega^t\,.$$

When the family $\{\Omega^t\}$ has been constructed, the polynomial convexity of $\bar{\Omega}^0$ will follow from the Docquier-Grauert Theorem [4].

To construct the domain Ω^t, we set

$$\psi_t = \psi + t$$

and

$$\tilde{\psi}_t = (1 + t)^{-1}\psi\,.$$

Thus $\|\tilde{\psi}_t\|_{C^{3m+3}} \leqslant \|\psi\|_{C^{3m+3}}$ so by Section 2, we have a domain $\Omega(\tilde{\psi}_t)$ satisfying (2a), (2b), and (2c) of the Theorem. Now we set

$$\Omega^t = (1 + t)\,\Omega(\tilde{\psi}_t)\,.$$

In order to verify (6 a-d) we need only verify (6b).

To show (6b) we look at the behavior of $M_1(\psi_s)$ as s increases. It will suffice to show that for $t > s$, $|t - s|$ small, $F_t^{(1)} - F_s^{(1)}$ is transverse to $\bar{M}_1(\psi_s)$ and points outward.

Let us compute the differential $dE(\varrho, \psi; \chi_\varrho, \chi_\psi)$ of E with respect to the variables ϱ and ψ in the directions χ_ϱ and χ_ψ, respectively. We find that $dF(\varrho; \chi_\varrho) = F(I + i\Upsilon)\chi_\varrho$ and thus

(7) $$dE_j(\varrho, \psi; \chi_\varrho, \chi_\psi) = \chi_\varrho + |1 + \varphi_j(F)|^{-1} \times$$
$$\Big[(1 + \operatorname{Re}\varphi_j(F)(\operatorname{Re} dS(\psi; \chi)(F) + \operatorname{Im}\varphi_j(F) \operatorname{Im} dS(\psi; \chi_\psi)(F) + \nabla|1 + \varphi_j(F)| \cdot \\ \cdot (F(I + i\Upsilon)\chi_\varrho)\Big]$$

where ∇ denotes the gradient of φ_j as a function on $\boldsymbol{R}^4 = \boldsymbol{C}^2$, and $\cdot$ is the Euclidean inner product on $\boldsymbol{R}^4$, where we have identified (a_1, b_1, a_2, b_2) with $(a_1 + ib_1, a_2 + ib_2)$.

Now let us set (7) equal to zero and consider the mapping $R\colon \chi_\psi \to \chi_\varrho$. For $\psi = 0$, we have $\varphi = 0$ and $dS(0, \chi_\psi) = -\chi_\psi$; thus $\chi_\varrho = \operatorname{Re}\chi_\psi$. Now for $\varepsilon > 0$ small, we have $|\nabla\varphi|\,\|\Upsilon\| \ll 1$, and so R is a small perturbation of the real part operator. We wish to compare the sets

$$\Omega^s = (1+s)\,\Omega(\tilde{\psi}_s)$$

and

$$\Omega^t = (1+s)\,\Omega\left(\tilde{\psi}_s + \frac{t-s}{1+s}\right),$$

so we take

$$\chi_\psi = \tilde{\psi}_s + \frac{t-s}{1+s} - \tilde{\psi}_s = \frac{t-s}{1+s}.$$

Since R is a small perturbation of the identity operator it follows that $\chi_\varrho > 0$ when $\chi_\psi = (t-s)/(1+s)$.

Thus to first order in χ_ϱ we have

$$\begin{aligned} F^{(1)} - F_s^{(1)} &\approx \tau(\exp(I+i\Upsilon)(\varrho_s+\chi_\varrho) - \exp(I+i\Upsilon)\varrho_s)\\ &\approx \tau(\exp(I+i\Upsilon)\varrho_s((I+i\Upsilon)\chi_\varrho\,. \end{aligned}$$

When $\psi = 0$, the outward normal to $M_1 = \partial\Delta\times\Delta$ at (τ, ζ) is τ. Since $\Upsilon\chi_\varrho$ is bounded in terms of χ_ϱ and since $\chi_\varrho > 0$, it follows that $|\mathrm{Arg}\,(I+i\Upsilon)\chi_\varrho| < < \varkappa < \pi/2$. Thus by (2*a*), $F_t^{(1)} - F_s^{(1)}$ is transverse to $M_1(\psi_s)$, and so (6*b*) holds, which completes the proof.

Remark added in proof. The polynomial convexity of $\overline{\Omega(\psi)}$ may also be seen because it is starshaped with respect to 0.

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