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Commensurability Classes and Volumes of Hyperbolic 3-Manifolds.

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The first purpose of this paper is to answer some questions raised by W. Thurston [24: 6.7.6] about families of commensurable hyperbolic 3-manifolds. Two hyperbolic 3-manifolds M, M' are commensurable if they have two diffeomorphic finite coverings. M is said to be minimal if it does not properly cover any other hyperbolic 3-manifold. We shall see that if $\mathcal{M}$ is a full commensurability class of orientable hyperbolic 3-manifolds of finite volume, then $\mathcal{M}$ contains infinitely many non-isomorphic minimal elements if $\pi_1(M)$ is definable arithmetically, but only one if it is not (and V-manifolds are allowed). Moreover the volumes of the elements in $\mathcal{M}$ are all integral multiples of some number (which is not necessarily one of the volumes, though).

An orientable hyperbolic 3-manifold M is the quotient of the hyperbolic 3-space H^3 by a discrete torsion-free subgroup Γ of the identity component $I(H^3)^0$ of the group if isometries of H^3 (which is isomorphic to $\boldsymbol{PGL}_2(\boldsymbol{C})$). We shall also allow Γ to have torsion, and then M is a V-manifold or an « orbifold » in the terminology of [24] (i.e., it looks locally as the quotient of euclidean space by a finite linear group). $M = H^3/\Gamma$ and $M' = H^3/\Gamma'$ are commensurable if Γ and Γ' are commensurable up to conjugacy, i.e., if Γ and a conjugate ${}^g\Gamma' = g\Gamma' g^{-1}$ of Γ' by some $g \in I(H^3)^0$ are commensurable (their intersection has finite index in both). Moreover, M is minimal if and only if Γ is maximal in its commensurability class. Given two commensurable subgroups Γ, Γ' of a group, let us define the generalized index of Γ' in Γ by

(1) $$[\Gamma : \Gamma'] = [\Gamma : \Gamma \cap \Gamma'] \cdot [\Gamma' : \Gamma \cap \Gamma']^{-1}\,.$$

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Then clearly

$$\mu(\Gamma') = [\Gamma:\Gamma']\cdot\mu(\Gamma)\,, \tag{2}$$

where $\mu(\Gamma)$ denotes the volume of H^3/Γ with respect to the hyperbolic metric. Therefore the above results are equivalent to the following theorem:

THEOREM. *Let Γ be a discrete subgroup of $\boldsymbol{PGL}_2(\boldsymbol{C})$ of finite covolume and $\mathcal{A}_\Gamma$ the set of subgroups of $\boldsymbol{PGL}_2(\boldsymbol{C})$ commensurable with Γ.*

(i) *If Γ is arithmetic, then $\mathcal{A}_\Gamma$ contains infinitely many nonconjugate elements which are maximal, or maximal among torsion-free elements of $\mathcal{A}_\Gamma$. If Γ is non-arithmetic, then $\mathcal{A}_\Gamma$ has a biggest element.*

(ii) *The indices $[\Gamma:\Gamma']$ $(\Gamma' \in \mathcal{A}_\Gamma)$ are integral multiples of some number.*

In fact, we shall prove this more generally when $\boldsymbol{PGL}_2(\boldsymbol{C})$ is replaced by a product

$$G_{a,b} = \boldsymbol{PGL}_2(\boldsymbol{R})^a \times \boldsymbol{PGL}_2(\boldsymbol{C})^b\,, \qquad (a, b \in \boldsymbol{N},\ a + b \geqq 1)\,, \tag{3}$$

and, for convenience, Γ is assumed to be « irreducible » (cf. [17: 5.20, 5.21]) [i.e., it is not possible to write $G_{a,b}$ as a direct product $G_{a,b} = H \cdot H'$ (H, H' closed connected, non-trivial) such that $(\Gamma \cap H)\cdot(\Gamma \cap H')$ has finite index in Γ. This is equivalent to $\Gamma \cap N = \{1\}$ for each proper normal closed subgroup N, or also to the fact that the projection of Γ on any infinite non-trivial quotient of $G_{a,b}$ is non-discrete]. Geometrically, this means that we consider irreducible discrete groups of automorphisms of the product $\boldsymbol{H}_{a,b}$ of a copies of the upper-half plane H^2 by b copies of the hyperbolic 3-space H^3. Our initial case of interest is then $a = 0$, $b = 1$. Also included are Fuchsian groups ($a = 1, b = 0$) or Hilbert-Blumenthal groups. The notion of arithmetic groups in the present case will be recalled in § 3. One commensurability class of such groups is associated to a number field k with b complex places and a quaternion algebra B over k which is unramified at a set of a real places of k (3.3). We denote it by $\mathcal{C}(k, B)$. If $a + b \geqq 2$, Γ is automatically arithmetic (3.4).

In the non-arithmetic case, these results are trivial consequences of a theorem announced by G. A. Margoulis in [13], as will be seen in § 1, so that we shall be mainly concerned with the arithmetic case. There we shall get a hold of maximal elements in $\mathcal{A}_\Gamma$ by looking at their closures in the groups of p-adic points of the form of $\boldsymbol{PGL}_2$ underlying the definition of Γ. For this we shall need some facts on the Bruhat-Tits building of $\boldsymbol{SL}_2$ and on the maximal compact subgroups of $\boldsymbol{SL}_2$ and $\boldsymbol{PGL}_2$ over a p-adic field which are reviewed in § 2. The assertion (i) above is proved in § 4, and (ii) in § 5. To a maximal order $\mathfrak{O}$ of B, we associate a maximal element $\Gamma_{\mathfrak{O}}$ of $\mathcal{C}(k, B)$ such that the minimum of the volume $\mu(\Gamma)$ $(\Gamma \in \mathcal{C}(k, B))$ is achieved on $\Gamma_{\mathfrak{O}}$. The g.c.d. of the volumes in the class are multiples of $2^{-c}\cdot\mu(\Gamma_{\mathfrak{O}})$ where c is at most equal to the number of primes dividing two in k. If for instance $a = 0$, $b = 1$, $k = \boldsymbol{Q}(\sqrt{-3})$, and H^3/Γ is not compact, then $c = 1$.

§ 7 gives an expression of the value of $\mu(\Gamma^1_{\mathfrak{O}})$, where $\Gamma^1_{\mathfrak{O}}$ is the image of the group of elements of reduced norm 1 in $\mathfrak{O}$, in terms of data depending only on the field k and on the places of k at which the quaternion algebra B is ramified (7.3). This formula is deduced here from the fact that the Tamagawa number of a k-form of $\boldsymbol{SL}_2$ is one [29] and from local computations of volumes made in § 6. It includes formulae of G. Humbert (see [24: § 7]) when k is imaginary quadratic and of C. L. Siegel [23] and Shimizu [22] when k is totally real, (7.5). The volumes in the class $\mathcal{C}(k, B)$ are all rational multiples of a number depending only on k and on the number of real places at which B is ramified. From this and a result of R. Baer [1], it follows that, given an arithmetic subgroup Γ_1 of $\boldsymbol{G}_{a,b}$, there exists an infinite set F of arithmetic subgroups of $\boldsymbol{G}_{a,b}$, which are not pairwise commensurable up to conjugacy, such that however $\mu(\Gamma)$ is a rational multiple of $\mu(\Gamma_1)$ for all $\Gamma \in F$ (7.6).

If $b = 0$, it is well-known that all volumes are commensurable. It is widely expected that this is not so when $b \neq 0$, but as far as I know, this has not been proved. This raises some questions on values of zeta functions at two (7.7).

If $a + b \geqq 2$, it is known that the set of all volumes is discrete ([28], cf. 8.3), but this is not so for $a + b = 1$. However, we shall prove that the set of covolumes of arithmetic subgroups is discrete (8.2). For this, we shall use estimates of Odlyzko's on the discriminant of number fields [15] and the fact that, given a constant c, there exists an integer $n(c)$ such that if $\mu(\Gamma) \leqq c$, then Γ is generated by $n(c)$ elements. (For $a = 1$, this is standard; for $b = 1$, this is proved in [24: Chap. 13], cf. 8.1) For $a = 0$, $b = 1$, the results of [24] imply then that the arithmetic subgroups are comparatively rare among all discrete subgroups of finite covolume of $\boldsymbol{PGL}_2(\boldsymbol{C})$. In 8.4 to 8.6, we give an arithmetic expression for the index $[\Gamma_{\mathfrak{O}} : \Gamma^1_{\mathfrak{O}}]$. It involves the class group of k, which makes it difficult to estimate it. Therefore we also single out a subgroup Γ_{R_f} intermediary between $\Gamma_{\mathfrak{O}}$ and $\Gamma^1_{\mathfrak{O}}$, whose covolume is independent of the class group, and which is equal to $\Gamma_{\mathfrak{O}}$ if k has class number one.

Finally, §§ 9 and 10 contain some remarks on, and examples of, groups operating on hyperbolic 3-space or products of upper half-planes.

The study of maximal arithmetic subgroups proceeds along rather standard lines and can be (has been) carried out in much greater generality.

In fact, part of what we prove here in the non-cocompact case is contained in [9; 18; 19]. However the case of forms of $\boldsymbol{PGL}_2$ has some peculiar features. I have therefore preferred to limit myself to it, but give a rather complete treatment.

I thank E. Bombieri for his computations of small volumes in the arithmetic case, R. P. Langlands for some helpful remarks on the local measures and volumes and W. Thurston for suggesting 8.2 when $a = 0$, $b = 1$ and for many useful conversations.

1. – The non-arithmetic case.

Given a subgroup H of a group G, we let C_H be the *commensurability subgroup* of H in G, i.e., the set of elements $x \in G$ such that ${}^xH = xHx^{-1}$ is commensurable with H. It is obviously a subgroup, which contains all subgroups of G commensurable with H. A result announced by G. Margoulis [13: Theorem 9] implies that if $G = \boldsymbol{G}_{a,b}$ and Γ (always assumed to be irreducible) has finite covolume in G, then either C_Γ is dense and Γ is arithmetically definable (see § 3 for this notion), or C_Γ is discrete and Γ is not arithmetically definable. [In deducing this from Margoulis' theorem, we also use the fact that if C_Γ is not discrete, then it is dense, as follows from [17: 5.13].] Therefore, in the latter case C_Γ is the biggest element in the commensurability class of Γ. The assertion (ii) is then clear in that case. Also, the commensurability class of Γ has only one maximal element, at any rate if we allow torsion. However, it is conceivable that $\mathcal{A}_\Gamma$ may contain infinitely many conjugacy classes of subgroups of finite index which are maximal among torsion-free subgroups, in which case there would again be infinitely many non-isomorphic minimal manifolds (in the strict sense, i.e., without V-singularities) in the commensurability class of H/Γ. It was pointed out to me by R. Griess and J-P. Serre, independently, that $L = \boldsymbol{SL}_2\boldsymbol{Z}/\{\pm 1\}$ indeed contains infinitely many non-conjugate subgroups of finite index which are maximal among torsion-free subgroups of L. However I do not know whether this is the rule or the exception in the case under consideration.

2. – Maximal compact subgroups and buildings for $\boldsymbol{SL}_2$ and $\boldsymbol{PGL}_2$ over a local field.

2.1. In this section F denotes a non-archimedean local field with finite residue field and $\mathfrak{o}_F$ its ring of integers. In fact, only the case where F is a finite extension of the field $\boldsymbol{Q}_p$ of p-adic numbers (p prime) will be needed, but the facts recalled here are also valid in the equal characteristic case. Let p be the characteristic and q the order of the residue field. Let $| \, |$ be the normalized valuation of F and $v(\ )$ the order of an element. Thus

$$(1) \qquad |x| = q^{-v(x)}, \quad \text{with } v(x) \in \boldsymbol{Z} .$$

2.2. For the contents of the section, see e.g. [21: Chap. II] and [10. Prop. 2.30, 2.31]. Let $\mathcal{T}$ be the Bruhat-Tits building of $\boldsymbol{SL}_2(F)$. It is a tree. $\boldsymbol{SL}_2(F)$ is transitive on the edges, and the vertices form two orbits O_1, O_2 under $\boldsymbol{SL}_2(F)$. The stability groups of the vertices are the maximal compact subgroups of $\boldsymbol{SL}_2(F)$ and form two conjugacy classes, represented by $K_1 = \boldsymbol{SL}_2(\mathfrak{o}_F)$, and the group K_2 of matrices of determinant 1 of the form

$$\begin{pmatrix} a & \pi\cdot b \\ \pi^{-1}c & d \end{pmatrix}, \qquad (a, b, c, d \in \mathfrak{o}) .$$

There are $q + 1$ edges with a given vertex $P \in \mathcal{T}_v$, and the stability group of P operates transitively on them. In fact, by reduction mod v, the group K_1 identifies to $\boldsymbol{SL}_2(\boldsymbol{F}_q)$ and the set of edges incident to the fixed point of K_1 identifies to the projective line $\boldsymbol{P}^1(\boldsymbol{F}_q)$ over $\boldsymbol{F}_q$. The stability groups of the edges are the Iwahori subgroups of $\boldsymbol{SL}_2(F)$. They form one conjugacy class under $\boldsymbol{SL}_2(F)$, represented by $K_1 \cap K_2$.

An automorphism of $\mathcal{T}$ either leaves O_1, O_2 stable or permutes them. We shall say that it is *even* in the former case, *odd* otherwise. Any (continuous) automorphism of $\boldsymbol{SL}_2(F)$ induces an automorphism of $\mathcal{T}$. In particular $\boldsymbol{GL}_2(F)$ operates on $\mathcal{T}$. The elements which induce even (resp. odd) automorphisms are those $x \in \boldsymbol{GL}_2(F)$ such that $v(\det x)$ is even (resp. odd), whence the terminology. The center Z of $\boldsymbol{GL}_2(F)$ acts trivially on $\mathcal{T}$ so that $\boldsymbol{PGL}_2(F)$, which is the quotient $\boldsymbol{GL}_2(F)/Z$ can (and will) be viewed as a group of automorphisms of $\mathcal{T}$. An element of $\boldsymbol{GL}_2(F)$ or $\boldsymbol{PGL}_2(F)$ will be said to be even (resp. odd) if it defines an even (resp. odd) automorphism of $\mathcal{T}$. The even elements in $\boldsymbol{PGL}_2(F)$ form a subgroup $\boldsymbol{PGL}_2(F)_0$ of index two. Thus $\boldsymbol{PGL}_2(F)$ is transitive on the vertices and on the edges of $\mathcal{T}$. It has two conjugacy classes of maximal compact subgroups, the stability groups of the vertices and the stability groups of the edges (or, equivalently, of the middle points of the edges). A maximal compact subgroup is of the latter kind if and only it contains an odd element.

2.3. LEMMA. *Let $D \supset C$, $D' \supset C'$ be compact subgroups of $\boldsymbol{PGL}_2(F)$. Assume that C fixes a vertex of $\mathcal{T}$, has no other fixed point in $\mathcal{T}$, and that C' contains an odd element. Then D and D' are not conjugate.*

In fact, since C fixes a vertex P_0 it consists of even elements. Moreover, any compact subgroup of $\boldsymbol{PGL}_2(F)$ must fix some point of $\mathcal{T}$, therefore D also fixes P_0. Then D consists of even elements, hence is not conjugate to D'.

3. – Arithmetic subgroups of $G_{a,b}$.

In this section we describe the discrete subgroups of $\boldsymbol{G}_{a,b}$ which are definable arithmetically, to be called arithmetic for short.

3.0. Before doing so, however, we would like to relate $\boldsymbol{G}_{a,b}$ to the full group of isometries of $\boldsymbol{H}_{a,b}$.

The group $\boldsymbol{PGL}_2(\boldsymbol{C})$ is connected, and is also the quotient $\boldsymbol{SL}_2(\boldsymbol{C})/\{\pm 1\}$. It is the group of orientation preserving isometries of H^3. On the other hand, $\boldsymbol{PGL}_2(\boldsymbol{R})$ has two connected components. Its component of the identity is $\boldsymbol{SL}_2(\boldsymbol{R})/\{\pm 1\}$. The group $\boldsymbol{PGL}_2(\boldsymbol{R})$ may be identified to the group of isometries of H^2, the elements of $\boldsymbol{SL}_2(\boldsymbol{R})/\{\pm 1\}$ are holomorphic, orientation preserving, while the others are antiholomorphic, orientation reversing. Therefore $\boldsymbol{G}_{a,b}$ is the group of all isometries of $\boldsymbol{H}_{a,b}$ which preserve each factor and the orientation of the three-dimensional ones. It has 2^a connected components and is of index $2^b\cdot a!\cdot b!$ in the full group of isometries of $\boldsymbol{H}_{a,b}$. We have singled it out since it turns out to be the most convenient to use for the discussion of arithmetic subgroups.

3.1. In the sequel, k is a number field, $\mathfrak{o}_k$ or simply $\mathfrak{o}$ the ring of integers of k, d the degree of $k/\boldsymbol{Q}$, V (resp. V_∞, resp. V_f) the set of places (resp. infinite places, resp. finite places) of k and r_1 (resp. r_2) the number of real (resp. complex) places of k. For $v \in V$, k_v denotes the completion of k at v and, if $v \in V_f$, $\mathfrak{o}_v$ is the ring of integers of k_v, $\mathfrak{p}_v$ the prime ideal at v and Nv the order of the residue field $\mathfrak{o}_v/\mathfrak{p}_v$.

A k-form of $\boldsymbol{PGL}_2$ (or $\boldsymbol{SL}_2$) is a linear algebraic group over k which is isomorphic to $\boldsymbol{PGL}_2$ (or $\boldsymbol{SL}_2$) over some extension of k. If G is a k-form of $\boldsymbol{PGL}_2$, then its universal covering $\tilde{G}$ is a k-form of $\boldsymbol{SL}_2$.

The group $\tilde{G}$ is the group of elements of reduced norm one in a quaternion algebra B over k. Either $B = \boldsymbol{M}_2(k)$ is the 2×2 matrix algebra over k and $\tilde{G} = \boldsymbol{SL}_2$ or B is a division quaternion algebra.

We let $\sigma\colon \tilde{G} \to G$ be the canonical projection. The group G can also be viewed as the quotient of a reductive k-group H with one-dimensional center, derived group $\tilde{G}$, by its center, namely the group defined by the invertible elements of B. We shall also denote by σ the canonical projection

$H \to G$. We let σ_v be the homomorphism $\tilde{G}_v \to G_v$ or $H_v \to G_v$ defined by σ $(v \in V)$, where, as usual, if M is a k-group, we denote by M_v the group $M(k_v)$ of points of M rational over k_v.

We recall that for any field $k' \supset k$, the map σ: $H(k') \to G(k')$ is surjective, because the kernel of σ is the center Z of H, which is isomorphic over k to the one-dimensional split algebraic torus $\boldsymbol{GL}_1$.

3.2. If H is an algebraic group over k, then a subgroup Γ of $H(k)$ is arithmetic if, given an embedding $\varrho: G \to \boldsymbol{GL}_n$ over k, the group $\varrho(\Gamma)$ is commensurable with $\varrho(G) \cap \boldsymbol{GL}_n(\mathfrak{o})$ (where, as usual, for any commutative algebra A, $\boldsymbol{GL}_n(A)$ is the group of $n \times n$ matrices with coefficients in A and determinant invertible in A).

3.3. Let Γ be a discrete subgroup of $\boldsymbol{G}_{a,b}$. It is said to be *definable arithmetically* if the following conditions are met: there exists a number field k with b complex places, at least a real places, a k-form G of $\boldsymbol{PGL}_2$, a set A of a real places such that

$$(1) \qquad G_w = \boldsymbol{PGL}_2(\boldsymbol{R}), \quad (w \in A), \quad G_w = \boldsymbol{SO}_3 \quad (w \text{ real}, w \notin A),$$

and an isomorphism

$$(2) \qquad \iota: \boldsymbol{G}_{a,b} \xrightarrow{\sim} G_{S_1} = \prod_{w \in S_1} G_w,$$

(where $S_1 \subset V_\infty$ is the union of A and of the complex places of k) which maps Γ onto an arithmetic subgroup of $G(k)$. Here, $G(k)$ is diagonally embedded in G_{S_1} by means of the natural inclusions $G(k) \subset G_w$. We note that, since G_w is compact for w real not in A, the arithmetic subgroups of G, viewed as subgroups of G_{S_1} via the diagonal embedding, are indeed discrete. There are two main cases:

(A) Γ is not cocompact in $\boldsymbol{G}_{a,b}$. Then $d = a + 2b$, and $S_1 = V_\infty$. The group G is just $\boldsymbol{PGL}_2$, viewed as a k-group.

(B) Γ is cocompact in $\boldsymbol{G}_{a,b}$. Then k may have any number $\geqslant a$ of real places. The group $\tilde{G}$ is the group defined by the elements of reduced norm one in a division quaternion algebra B over k which is ramified (at least) at all real places not contained in S_1, and H is the group defined by the invertible elements in B.

To simplify notation, identify G with $\varrho(G)$, with ϱ as in 3.2. For almost all (*i.e.*, all but finitely many) $v \in V_f$, the group $G(\mathfrak{o}_v) = G \cap \mathbf{GL}_n(\mathfrak{o}_v)$ is maximal compact in G_v. If Γ is arithmetic then its closure $Cl_v(\Gamma)$ in G_v is compact open, contained in $G(\mathfrak{o}_v)$ for almost all v's. Conversely, given a compact open subgroup L_v of G_v for each $v \in V_f$ which is equal to $G(\mathfrak{o}_v)$ for almost all v's, the group

$$\Gamma_L = \{g \in G(k) | g \in L_v \text{ for all } v \in V_f\} , \qquad L = \prod_{v \in V_f} L_v , \tag{3}$$

is an arithmetic subgroup of G, and every arithmetic subgroup is contained in one of this type. The maximal ones are *among* the groups Γ_L, where L_v is maximal compact for all $v \in V_f$.

3.4. We recall that if $a + b \geqq 2$, then every irreducible discrete subgroup of finite covolume of $\boldsymbol{G}_{a,b}$ is arithmetic. This follows from results of G. A. Margoulis [7] (see also [25]).

4. – Maximal arithmetic subgroups of $\boldsymbol{G}_{a,b}$.

4.1. We let $R(B)$ or R be the set of places at which B is ramified, R_∞ (resp. R_f) be the set of infinite (resp. finite) places in R, and $r_f = |R_f|$. Thus $B \otimes_k k_v$ is a division algebra if $v \in R$ and is isomorphic to $\boldsymbol{M}_2(k_v)$ otherwise (and $|R|$ is even). Let $\mathfrak{O}$ be a maximal order in B. Then $\mathfrak{O}_v = \mathfrak{O} \otimes_k k_v$ is a maximal order of B_v. For $v \in R_f$ it is the unique maximal order of integral elements in B_v. If $\mathfrak{O}'$ is another maximal order, then $\mathfrak{O}_v = \mathfrak{O}'_v$ for almost all v's, $\mathfrak{O}'$ is the intersection of the $\mathfrak{O}'_v$ and the $\mathfrak{O}'_v$ can be prescribed arbitrarily at finitely many places. $\mathfrak{O}$ and $\mathfrak{O}'$ are said to have the same type if there exists $x \in B^*$ such that $x \cdot \mathfrak{O} = \mathfrak{O}' \cdot x$. The number of types of maximal orders is finite (and divides the class number of B). (For all this, see [4: §§ 8, 11].)

For $v \in V_f - R_f$, the group $\tilde{G}$ is isomorphic to $\boldsymbol{SL}_2$ over k_v. We let $\mathcal{T}_v$ be its Bruhat-Tits building and P_v the fixed point of $\tilde{K}_{1v} = \mathfrak{O}_v \cap \tilde{G}_v$. Furthermore, let e_v be an edge of $\mathcal{T}_v$ incident to P_v; let Q_v be the middle point of e_v and P'_v the second end point of e_v. Let $\tilde{K}'_{1v}$ be the isotropy group of P'_v in $\tilde{G}_v$ and

$$C_v = \{P_v, Q_v, P'_v\} . \tag{1}$$

We denote by K_{1v}, K_{2v} and K'_{1v} the isotropy groups of P_v, Q_v and P'_v in G_v. The groups K_{1v} and K'_{1v} are conjugate in G_v, the groups $\tilde{K}_{1v}$ and $\tilde{K}'_{1v}$ (resp. K_{1v} and K_{2v}) represent the two conjugacy classes of maximal compact subgroups in $\tilde{G}_v$ (resp. G_v) ($v \in V_f - R_f$) (cf. § 2). For convenience, we agree that for $v \in R_f$, the building $\mathcal{C}_v$ is reduced to a point and $G_v = K_{1v} = K_{2v} = K'_{1v}$, $\tilde{G}_v = \tilde{K}_{1v} = \tilde{K}'_{1v}$.

4.2. The group $\tilde{G}$ is simple, simply connected, not compact at infinity, hence has the *strong approximation property*: the group $\tilde{G}(k)$, embedded diagonally in the restricted product $\tilde{G}(A_f)$ of the $\tilde{G}_v$ ($v \in V_f$) is dense. Without using the notion of restricted product, we can, in our case, express this as follows: let S be a finite subset of $V_f - R_f$. For $v \in V_f - S$, let L_v be a compact open subgroup of $\tilde{G}_v$ which is equal to $\tilde{G}(\mathfrak{o}_v)$ for almost all v, and put

$$\tilde{G}(k)_L = \{g \in \tilde{G}(k) | g \in L_v \text{ for } v \in V_f - S\} . \tag{1}$$

Then, for any set of elements $g_v \in \tilde{G}_v$ ($v \in S$), there exists $g \in \tilde{G}(k)_L$ which is arbitrarily close to g_v for every $v \in S$.

This can also be formulated in the following way: let S be a finite subset of V_f; for $v \in S$, let D_v be a finite subset of $\mathcal{C}_v$ and $E_v = g_v \cdot D_v$ for some $g_v \in \tilde{G}_v$. Then there exists $g \in \tilde{G}(k)$ such that $g \cdot D_v = E_v$ for $v \in S$ and $g \cdot P_v = P_v$ for $v \in V_f - S$.

4.3. The group G has center reduced to the identity. Therefore the commensurability subgroup C_Γ (see § 1) of an arithmetic subgroup Γ is equal to $G(k)$ [2: Thm. 3]. It follows that if two arithmetic subgroups Γ, Γ' of G are conjugate in $G_{S_1} = \boldsymbol{G}_{a,b}$, then they are conjugate under an element of $G(k)$, hence $\mathcal{C}l_v(\Gamma)$ is conjugate to $\mathcal{C}l_v(\Gamma)$ in G_v for all $v \in V_f$. Also, if $\Gamma \subset \boldsymbol{G}_{a,b}$ is mapped onto an arithmetic subgroup of $G(k)$ under the isomorphism ι, then every subgroup of $\boldsymbol{G}_{a,b}$ commensurable with Γ is mapped by ι onto an (arithmetic) subgroup of $G(k)$. This then allows one to transfer the discussion of the commensurability class of Γ in $\boldsymbol{G}_{a,b}$ to that of $\iota(\Gamma)$ in $G(k)$.

4.4. PROPOSITION. *For two finite disjoint subsets S, S' of $V_f - R_f$ set*

$$\Gamma_{S,S'} = \{g \in G(k) | g \in K_{1v} (\text{resp. } K_{2v}, \text{resp. } K'_{1v}) \textit{ for } v \in V_f - (S \cup S'); \ (\text{resp. } S, \text{ resp. } S')\}. \tag{1}$$

(i) *For* $v \in S'$ (resp. $v \in V_f$, $v \notin S \cup S'$), K'_{1v}(resp. K_{1v}) *is the unique maximal compact subgroup of G_v containing $\Gamma_{S,S'}$.*

(ii) *Let Γ be an arithmetic subgroup of G containing an element which is odd at some $v \notin S$. Then Γ is not conjugate to a subgroup of $\Gamma_{S,S'}$.*

(iii) *Given an arithmetic subgroup Γ of G, let $S(\Gamma)$ be the set of v's such that Γ contains an element odd at v. Then there exists S' such that Γ is conjugate to a subgroup of $\Gamma_{S(\Gamma),S'}$, or to $\Gamma_{S(\Gamma),S'}$ itself if Γ is maximal.*

Given S, S' let for $v \in V_f$

(1) $L_v = K_{1v}$ (resp. K_{2v}, resp. K'_{1v}) if $v \notin S \cup S'$ (resp. $v \in S$, resp. $v \in S'$).

(i) Amounts to asserting that P_v or P'_v, as the case may be, is the unique fixed point of $\Gamma_{S,S'}$ in $\mathcal{T}_v$. For any $u \in V_f - \{v\}$, we may find a compact open subgroup M_u of $\tilde{G}_u$, equal to $\tilde{K}_{1u}$ for almost all u's, such that $\sigma(M_u) \subset L_u$ for $u \in V_f - \{v\}$. Set $M_v = \tilde{K}_{1v}$ (resp. $M_v = \tilde{K}'_{1v}$) if $v \notin S \cup S'$ (resp. $v \in S'$) and let

$$\tilde{\Gamma}_M = \{g \in \tilde{G}(k) | g \in M_u \text{ for } u \in V_f - R_f\} .$$

By strong approximation, $\tilde{\Gamma}_M$ is dense in M_v, hence P_v, or P'_v, is the unique fixed point of $\tilde{\Gamma}_M$ in $\mathcal{T}_v$ $(v \in V_f - S)$. Since $\sigma(\tilde{\Gamma}_M) \subset \Gamma_{S,S'}$ by construction, (i) follows.

(ii) Is a consequence of (i) and 2.3.

(iii) By the generalities recalled in 3.3, there exists for every $v \in V_f$ a maximal compact subgroup J_v of G_v, equal to $K_{1,v}$ for almost all v's, consisting of even elements if $v \notin \Gamma(S)$, such that $\Gamma \subset \Gamma_J$, where J is the product of the J_v's. Let $T \subset V_f$ be the union of R_f and of the set of v's for which $J_v \neq K_{1,v}$. It contains $S(\Gamma)$. Using 4.1 and 4.2, we see that there exists $g \in \sigma(\tilde{G}(k))$ with the following properties:

$$g \in K_{1,v} \text{ if } v \notin T;\ {}^gJ_v = K_{1,v} \text{ or } K'_{1,v} \text{ if } v \in T - S(\Gamma),\ {}^gL_v = K_{2,v} \text{ if } v \in S(\Gamma) .$$

Then ${}^g\Gamma \subset \Gamma_{S(\Gamma),S'}$. There is then obviously equality if Γ is maximal. This proves (iii).

REMARK. A group $\Gamma_{S,S'}$ may be non-maximal. The point is that we cannot assert that for $v \in S$ the group $K_{2,v}$ is the unique maximal compact subgroup of G_v containing $\Gamma_{S,S'}$. It could happen that for some $v \in S$ no element of $\Gamma_{S,S'}$ is odd at v, and then $\Gamma_{S,S'}$ would fix pointwise the edge e_v containing P_v, Q_v (notation 4.1) and be a proper subgroup of $\Gamma_{S'',S'}$ where $S'' = S - \{v\}$. In order to show that there are indeed infinitely many non-conjugate maximal subgroups among the groups $\Gamma_{S,S'}$ we need therefore an existence statement. This is provided by the following lemma:

4.5. LEMMA. *Let $v \in V_f - R_f$. Then there exists a torsion-free arithmetic subgroup of G containing an element which is odd at v.*

Let n be such that G is embedded in $\boldsymbol{GL}_n$. If $g \in G(k)$ is of finite order, then its eigenvalues are roots of one, of degree $\leq d\cdot n$, hence there are only finitely many possibilities for the order of g. Therefore, for almost all $v \in V - R_f$ there exists a congruence subgroup K'_v of $K_{1,v}$ such that if $g \in K'_v \cap \cap G(k)$, then g has infinite order or $g = 1$. Choose one such place $v' \neq v$. Then any arithmetic subgroup contained in $K'_{v'}$ is torsion-free. Now we claim

(*) *There exists $g \in G(k)$ which is odd at v and contained in a compact open subgroup L_u of G_u for $u \in V_f$, where $L_u = K'_{v'}$, if $u = v'$.*

Assume this for the moment. We may take $L_u = G(\mathfrak{o}_u)$ for almost all u's. Then the arithmetic group Γ_L is torsion-free, since it is contained in $K'_{v'}$ and has an element odd at v, namely g. We are reduced therefore to proving (*).

By the Chinese remainder theorem, we can find an element $c \in k$ which is the square of a unit in $\mathfrak{o}_{v'}$, has order one at v, and is positive at all $v \in R_\infty$. In case (A), let

$$x = \begin{pmatrix} 0 & c \\ -1 & 0 \end{pmatrix} \in \boldsymbol{GL}_2(k)\,.$$

In case (B), let x be an element of reduced norm c in the quaternion algebra B which underlies the definition of G. Such an element exists by the norm theorem of Hasse-Schilling (see e.g. Prop. 3 in [28: XI, § 3]). The first condition implies the existence of an element $y_{v'} \in \tilde{G}_{v'}$ such that $\sigma_{v'}(x) = = \sigma_{v'}(y_{v'})$. Let T be the set of $v \in V_f$, $v \notin R_f \cup \{v'\}$ such that $\sigma(x) \notin G(\mathfrak{o}_v)$. It is finite. Using strong approximation, we can find $h \in \tilde{G}(k)$ such that

$$\sigma_{v'}(h\cdot y_{v'}) \in K'_{v'}\,, \quad h\cdot x(e_v) = e_v\ (v \in T), \quad h \in \tilde{G}(\mathfrak{o}_v) \quad \text{for } v \in V_f - T\,.$$

Then $\sigma_{v'}(h\cdot x) = \sigma_{v'}(h\cdot y_{v'}) \in K'_{v'}$ and $\sigma_v(h\cdot x)$ belongs to the stability group L_v of e_v for $v \in T$, to $G(\mathfrak{o}_v)$ for the other $v \in V_f - R_f$. Thus $\sigma(h\cdot x)$ satisfies the requirements imposed on g in (*).

4.6. THEOREM. *Let Γ be an arithmetically defined subgroup of $\boldsymbol{G}_{a,b}$. Then the commensurability class of Γ contains infinitely many non-conjugate elements which are maximal among discrete subgroups of $\boldsymbol{G}_{a,b}$ or maximal among torsion-free discrete subgroups of $\boldsymbol{G}_{ab}$.*

Let k and G be as in 3.3. Then, as pointed out in 4.3, it is equivalent to prove the same statement for the commensurability class of arithmetic subgroups of $G(k)$ and conjugacy by elements of $G(k)$. Let $\Gamma_1, \ldots, \Gamma_m$ be non-conjugate maximal (resp. maximal among torsion-free) arithmetic subgroups of $G(k)$. By 4.4, each one is conjugate to some subgroup of a group $\Gamma_{S,S'}$ and, by 4.2, 4.4, for almost all v's in $V_f - R_f$, the point P_v is the unique fixed point of Γ_i in $\mathcal{C}_v$ $(i = 1, \ldots, m)$. Fix one, say v_0. By 4.5, there exists a torsion-free arithmetic subgroup Γ' of G having an element which is odd at v_0. By 2.3, any arithmetic subgroup containing Γ' is not conjugate to any of the Γ_i's. Among those there is a maximal one and one which is maximal among torsion-free arithmetic subgroups, whence the theorem.

4.7. Remark. The first assertion of 4.4 is contained in more general statements of [18; 19]. For $\boldsymbol{PGL}_2$ over a number field, the existence of infinitely many non-conjugate maximal arithmetic subgroups is already proved in [9].

4.8. In 4.7, we proved the existence of an element which is odd at a given place, but it may be odd at other places as well. The proof shows that, in order to produce a group for which $\Gamma(S)$ consists of just one given finite place v, it is enough to find $c \in \dot{k}$ which has odd order at v, even order at all $u \notin R_f \cup \{v\}$, and is positive at all $u \in R_\infty$. This last condition is of course vacuous if $R_\infty = \emptyset$, in particular in case (A). The other two will be fulfilled if the prime ideal $\mathfrak{o} \cap \mathfrak{p}_v$ of $\mathfrak{o}$ has an odd power which is principal, in particular if $\mathfrak{o}$ is a principal ideal domain.

4.9. We shall write $\Gamma_{\mathfrak{O}}$ for $\Gamma_{S,S'}$ when S and S' are empty. Since $\sigma\colon H(k) \to G(k)$ is surjective (3.3) we have

$$\Gamma_{\mathfrak{O}} = \Gamma_{\phi,\phi} = \sigma(\operatorname{Norm} \mathfrak{O})\,, \tag{2}$$

where

$$\operatorname{Norm} \mathfrak{O} = \{x \in B^* | x \cdot \mathfrak{O} = \mathfrak{O} \cdot x\}\,. \tag{3}$$

Given S' finite in V_f consider the set of points (R_v) where $R_v = P_v$ for $v \notin S'$ and $R_v = P'_v$ otherwise. There is a unique maximal order $\mathfrak{O}' = \mathfrak{O}(S')$ such that $\tilde{G}_v \cap \mathfrak{O}'_v$ fixes R_v for all v's. Thus we have

$$\Gamma_{\phi,S'} = \Gamma_{\mathfrak{O}'}\,. \tag{4}$$

By strong approximation, the systems (R_v), for varying S', form a system of representatives for the conjugacy classes of maximal orders with respect to $\tilde{G}(k)$.

4.10. PROPOSITION. *Fix $S \subset V_f$. Then the groups $\Gamma_{S,S'}$ form finitely many conjugacy classes in $G(k)$, as S' varies through the finite subsets of $V_f - S$.*

Let $\mathfrak{D}'$ be as above and $\mathfrak{D}'' = \mathfrak{D}(S'')$ be similarly associated to S''. We claim that $\Gamma_{\mathfrak{D}'}$ and $\Gamma_{\mathfrak{D}''}$ are conjugate if and only if there exists $g \in B^*$ which is odd at exactly $S' \cup S'' - (S' \cap S'')$. Moreover, if there exists such an element, then there exists also $x \in B^*$ such that

$$x \cdot e_v = e_v \, (v \in S)\,, \quad x \cdot P_v = P_v\,, \quad x \cdot P'_v = P'_v \quad (v \in S' \cap S'')\,, \tag{1}$$

$$x \cdot P_v = P_v\,, \quad (v \in V_f - (S \cup S' \cup S''))\,, \tag{2}$$

$$x \cdot P'_v = P_v \quad (v \in S' - (S' \cap S''))\,, \quad x \cdot P_v = P'_v \quad (v \in S'' - (S' \cap S''))\,. \tag{3}$$

The necessity of the condition is clear. If there exists such an element, say y, then by strong approximation (see end statement in 4.2), we can find $z \in \tilde{G}(k)$ such that $x = \sigma(zy)$ satisfies (1), (2), (3). But then $\Gamma_{S,S'}$ and $\Gamma_{S,S''}$ are conjugate under x. The proposition now follows from this and the finiteness of the type number of B (4.1).

5. – Comparison of the volumes in a commensurability class.

5.1. We consider the commensurability class $\mathcal{C}(k, B)$ of arithmetically defined subgroups of $\boldsymbol{G}_{a,b}$ defined by a k-form G of $\boldsymbol{PGL}_2$. We keep the notation of 3.3, 4.1, 4.4 and identify $\boldsymbol{G}_{a,b}$ with G_{S_1}. For $v \in V_f$, let Nv be the order of the residue field at v.

5.2. LEMMA. *Let S, S' be finite disjoint subsets of $V_f - R_f$ and S'' be a finite subset of $V_f - (R_f \cup S)$. For $v \in S''$ let $\mathcal{E}_v$ be the set of edges of $\mathcal{T}_v$ having P'_v* (resp. P_v) *as a vertex if $v \in S'$* (resp. $v \notin S'$). *Then, given $f_v, f'_v \in \mathcal{E}_v$* ($v \in S''$), *there exists $\gamma \in \Gamma_{S,S'}$ such that $\gamma \cdot f_v = f'_v$ for $v \in S''$.*

This is again a consequence of strong approximation: Let M_v be the isotropy group in $\tilde{G}_v$ of P_v (resp. P'_v) if $v \in S''$, $v \notin S'$ (resp. $v \in S' \cap S''$). As recalled in 2.2, it is transitive on $\mathcal{E}_v$. Let then $g_v \in M_v$ be such that $g_v \cdot f_v = f'_v$ ($v \in S''$). By strong approximation, we can find $g \in \tilde{G}(k)$ which is arbitrarily close to g_v for $v \in S''$, fixes the point P_v for $v \notin S' \cup S''$, the point P'_v for $v \in S'$ and P_v, P'_v for $v \in S$. Then $\sigma(g) \in \Gamma_{S,S'}$ and $\sigma(g) \cdot f_v = f'_v$ for $v \in S''$.

5.3. THEOREM. *Let $\Gamma_{\mathfrak{D}}$ be as in 4.9. Let S, S' be finite disjoint subsets of $V_f - R_f$. Then there exists an integer m $(0 \leqq m \leqq |S|)$ such that*

$$[\Gamma_{\mathfrak{D}} : \Gamma_{S,S'}] = 2^{-m} \prod_{v \in S} (Nv + 1)\,. \tag{1}$$

In particular $[\Gamma_{\mathfrak{D}}: \Gamma_{S,S'}] \geqq 1$ *and* $[\Gamma_{\mathfrak{D}}: \Gamma_{S,S'}] = 1$ *if and only if* S *is empty. Given* $c > 0$, *the groups* $\Gamma_{S,S'}$ *for which* $[\Gamma_{\mathfrak{D}}: \Gamma_{S,S'}] \leqq c$ *are contained in finitely many conjugacy classes.*

In this proof, it is understood that $v \in V_f$. We have

$$[\Gamma_{\mathfrak{D}}: \Gamma_{S,S'}] = [\Gamma_{\mathfrak{D}}: \Gamma_{\phi,S'}]\cdot[\Gamma_{\phi,S'}: \Gamma_{S,S'}] \tag{2}$$

and (1) is equivalent to the following equalities

$$[\Gamma_{\mathfrak{D}}: \Gamma_{\phi,S'}] = 1 \tag{3}$$

$$[\Gamma_{\phi,S'}: \Gamma_{S,S'}] = 2^{-m} \prod_{v \in R_f} (Nv + 1)\,, \qquad (0 \leqq m \leqq |S|)\,. \tag{4}$$

Let $\Gamma_1 = \Gamma_{\mathfrak{D}} \cap \Gamma_{\phi,S'}$. This is the subgroup of $G(k)$ which fixes the edges e_v $(v \in S')$ pointwise and the points P_v for $v \notin S'$. Lemma 5.2 implies:

$$[\Gamma_{\mathfrak{D}}: \Gamma_1] = [\Gamma_{\phi,S'}: \Gamma_1] = \prod_{v \in S'} (Nv + 1)\,, \tag{6}$$

which proves (3).

Let now $\Gamma_2 = \Gamma_{\phi,S'} \cap \Gamma_{S,S'}$. This is the subgroup of $G(k)$ which fixes P_v, P'_v for $v \in S$, the vertex P'_v for $v \in S'$ and P_v otherwise. By 5.2 we have

$$[\Gamma_{\phi,S'}: \Gamma_2] = \prod_{v \in S} (Nv + 1)\,. \tag{7}$$

On the other hand, if $g \in G_v$ stabilizes e_v, then g_v^2 fixes P_v and P'_v. As a consequence

$$[\Gamma_{S,S'}: \Gamma_2] = 2^m \qquad \text{for some } m \in [0, |S|]\,. \tag{8}$$

(4) now follows from (7) and (8).

The right-hand side of (1) can be written as a product of factors indexed by $v \in S$, each of which is $\geqq (Nv + 1)/2$, hence tends to infinity as v varies. Therefore, given $c > 0$, there exist only finitely many S such that

$$[\Gamma_{\mathfrak{D}}: \Gamma_{S,S'}] \leqq c\,, \qquad \text{for some } S'\,.$$

Since for fixed S, the groups $\Gamma_{S,S'}$ are contained in finitely many conjugacy classes (4.10) the last assertion is proved.

5.4. COROLLARY. *Let e be the number of places of k dividing 2 and not contained in R_f. Let Γ be a subgroup of $G_{a,b}$ commensurable with $\Gamma_{\mathfrak{O}}$. Then the volume $\mu(\Gamma)$ is an integral multiple of $2^{-e}\cdot\mu(\Gamma_{\mathfrak{O}})$. It is equal to $\mu(\Gamma_{\mathfrak{O}})$ if Γ is conjugate to a subgroup $\Gamma_{\phi,S'}$, and $>\mu(\Gamma_{\mathfrak{O}})$ otherwise.*

The group Γ is arithmetic in $G(k)$ (3.3). It is enough to consider the case where it is maximal. Γ is then conjugate to a group $\Gamma_{S(\Gamma),S'}$ (see 4.4). By 5.3, $[\Gamma_{\mathfrak{O}}:\Gamma_{S(\Gamma),S'}]$ is an integral multiple of the number

$$m_{S(\Gamma)}=\prod_{v\in S(\Gamma)}(Nv+1)/2\ .$$

If v divides the rational prime p, then Nv is a power of p, hence $Nv+1$ is even except when v divides 2, Therefore all the factors in $m_{S(\Gamma)}$ are integers, except for those v which divide 2. All of them are >1. Since $\mu(\Gamma)= =[\Gamma_{\mathfrak{O}}:\Gamma]\mu(\Gamma_{\mathfrak{O}})$, the corollary follows.

5.5. Let S_2 be the set of primes of k dividing 2 and not contained in R. Let us denote by q_G the g.c.d of the numbers $\mu(\Gamma)$, where Γ runs through the arithmetic subgroups of G. We have just seen that

$$q_G=2^{-c}\cdot\mu(\Gamma_{\mathfrak{O}})\ ,\qquad \text{for some integer } c\in[0,\,|S_2|]\ . \tag{1}$$

If S_2 is empty, then $c=0$. Assume now S_2 to be not empty. If there exists Γ such that $\Gamma(S)$ is not empty, contained in S_2, then $c\geqq 1$. By 4.5, we know there exists Γ such that $\Gamma(S)$ contains any prescribed element of S_2 but this is not enough to insure that $c\geqq 1$, because if $\Gamma(S)$ contains some v dividing an odd prime, then the factor $(Nv+1)/2$ might still contribute a power of two which might compensate for the one stemming from a place dividing two at which Γ has an odd element. The remarks in 4.8 show that, in order to prove that $c=e$, it suffices to show that, given $u\in S_2$, there exists $c\in k$ which has an odd valuation at u, an even valuation at all other places in V_f-R_f and is >0 at the real places at which B is ramified. But such existence theorems do not seem easy to prove in general.

5.6. Assume we are in case (A) and that $a=0$, $b=1$. There exists then a square free negative rational integer m such that $k=\mathbf{Q}(\sqrt{m})$. We have $c=1,2$ and more precisely $c=2$ if and only if $m\equiv 1 \bmod 8$ (see e.g. [3]). For $-m=1,2,3,7,19$ for instance, $\mathfrak{o}$ is a principal ideal domain. Therefore the exponent c in 5.4 (1) is given by:

$$c=1\quad \text{if } -m=1,2,3,19;\qquad c=2\quad \text{if } -m=7\ . \tag{1}$$

6. – Some local computations of volumes.

6.1. Let $\mathfrak{g}$ be the Lie algebra of $\boldsymbol{SL}_2(\boldsymbol{R})$. Then $\mathfrak{g}_c = \mathfrak{g} \otimes_{\boldsymbol{R}} \boldsymbol{C}$ is the Lie algebra of $\boldsymbol{SL}_2(\boldsymbol{C})$. We view it as a 6-dimensional real Lie algebra. Then $\theta\colon x \mapsto -{}^t\bar{x}$ is an automorphism of $\mathfrak{g}_c$ whose fixed point set is the Lie algebra $\mathfrak{su}_2$ of $\boldsymbol{SU}_2$, i.e., the set of skew hermitian matrices. The orthogonal complement $\mathfrak{p}$ of $\mathfrak{su}_2$ with respect to the Killing form is the space of hermitian symmetric 2×2 matrices of trace zero. θ is the Cartan involution of $\mathfrak{g}_c$ associated to su_2. We have $H^3 = \boldsymbol{SL}_2(\boldsymbol{C})/\boldsymbol{SU}_2$ and the canonical projection identifies $\mathfrak{p}$ to the tangent space $T(H^3)_0$ to H^3 at the origin. On $\mathfrak{g}_c$ consider the hermitian form $g_0(x, y) = -2Tr(x\cdot\theta(y))$, where Tr refers to the trace in the standard representation. It is hermitian positive non-degenerate, invariant under inner automorphisms of $\boldsymbol{SU}_2$. Then the hyperbolic metric is the left invariant Riemannian metric whose value at $\mathfrak{p} = T(H^3)_0$ is the inner product defined by the restriction of g_0 to $\mathfrak{p}$.

[To check this, note first that $2Tr = \frac{1}{4}B$, where B is the Killing form $B(x, y) = \mathrm{tr}\,(\mathrm{ad}\, x\circ \mathrm{ad}\, y)$, of $\mathfrak{g}_c$, viewed as real Lie algebra, and that, for the metric defined by the Killing form on $\mathfrak{p}$, the sectional curvature on the plane spanned by x, y is $B([x, y], [x, y])\cdot A(x, y)^{-2}$, where $A(x, y)$ is the area of that plane. Then compute this expression for some choice of x and y, for instance h and u below.] Let

$$(1)\qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad u = e + f\,, \quad v = i(e - f)\,.$$

Then $h/2, u/2$ and $v/2$ form an orthonormal base of $\mathfrak{p}$ with respect to $\mathfrak{g}_0$. In particular, if $d\mu$ denotes the volume element of the hyperbolic metric, then

$$(2)\qquad d\mu(h\wedge u\wedge v) = 8\,.$$

In the real case, we have $H^2 = \boldsymbol{SL}_2(\boldsymbol{R})/\boldsymbol{SO}_2$. We identify $\mathfrak{p} = T(H^2)_0$ with the subspace of $\mathfrak{g}$ spanned by h and u. The computation sketched above also shows that the restriction of $-2Tr(x\cdot\theta(y))$ to $\mathfrak{p}$ defines the hyperbolic metric, and that $h/2$, $u/2$ is an orthonormal basis of $\mathfrak{p}$.

6.2. Let $\omega_0^1, \omega_0^2, \omega_0^3$ be the left invariant 1-forms on $\boldsymbol{SL}_2$ whose values at the identity form a basis dual to the basis (h, e, f) of $\mathfrak{sl}_2$ given by 6.1(1), and $\omega_0 = \omega_0^1\wedge\omega_0^2\wedge\omega_0^3$. Let $\tilde{G}$ be a k-form of $\boldsymbol{SL}_2$. The group $\tilde{G}$ is then isomorphic to $\boldsymbol{SL}_2$ over some algebraic extension of k. Fix such an isomorphism φ and let $\omega = \varphi^*\omega_0$. Then ω is defined over k (see pp. 475-476 in [11]). It is then a « gauge form », which can be used to define first a local measure ω_v on $\tilde{G}_v$ for every v and then a Tamagawa measure $d\tau = |D_k|^{-\frac{3}{2}} \prod_v \omega_v$ on the adelic group G_A. In this section, we are concerned with the local measures ω_v for $v \in V_\infty$. We have

$$\omega_v = \omega\,, \quad \text{if } v \text{ is real}\,, \tag{1}$$

$$\omega_v = \pm\, i^3 \cdot \omega \wedge \bar{\omega}\,, \quad \text{if } v \text{ is complex}\,. \tag{2}$$

As usual, let ω_∞ be the product measure of the ω_v on $\tilde{G}_\infty$. Let again S_1 be the set of real v's such that $\tilde{G}$ is isomorphic to $\boldsymbol{SL}_2$ over k_v. Set

$$\begin{aligned} K_v &= \boldsymbol{SO}_2 && \text{if } v \text{ is real, } v \in S_1\,,\\ K_v &= \tilde{G}_v = \boldsymbol{SU}_2 && \text{if } v \text{ is real, } v \notin S_1\,,\\ K_v &= \boldsymbol{SU}_2 && \text{if } v \text{ is complex}\,. \end{aligned}$$

Then $H_v = G_v/K_v$ is H^2 (resp, a point, resp. H^3) if v is real, $v \in S_1$ (resp. $v \notin S_1$, real resp. v complex). Let $d\mu_v$ be the hyperbolic volume elemnet on H_v in the first and last cases, the point measure in the second case.

LEMMA. *Let dk_v be the measure on K_v such that $\omega_v = d\mu_v \cdot dk_v$ $(v \in V_\infty)$. Then the volume $v(K_v)$ of K_v with respect to dk_v is equal to π (resp. $4\pi^2$, resp. $8\pi^2$) if v is real in S_1 (resp. real not in S_1, resp. complex).*

Assume first v to be complex. Let

$$\sigma^2 = \omega^2 + \omega^3\,, \quad \sigma^3 = \omega^2 - \omega^3\,. \tag{1}$$

Then $\omega^1, \sigma^2, \sigma^3$ is the basis of $\mathfrak{g}^*$ dual to h, $u/2$, $-iv/2$. For a C-linear 1-form τ on $\mathfrak{g}_c$ let $R\tau$ and $I\tau$ be its real and imaginary part. We have

$$\omega^1\sigma^2\sigma^3 = 2\omega\,, \tag{2}$$

$$4\omega \wedge \bar{\omega} = \omega^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \overline{\omega^1} \wedge \overline{\sigma^2} \wedge \overline{\sigma^3} = \pm\, 8 i^3 R\omega^1 \wedge I\omega^1 \wedge R\sigma^2 \wedge I\sigma^2 \wedge R\sigma^3 \wedge I\sigma^3 \tag{3}$$

$$\omega_v = \pm\, (I\omega^1 \wedge I\sigma^2 \wedge R\sigma^3) \wedge (2R\omega^1 \wedge R\sigma^2 \wedge I\sigma^3)\,. \tag{4}$$

The elements ih, iu, iv form a basis of $\mathfrak{su}_2$, and h, u, v a basis of $\mathfrak{p}$. We have

$$(2R\omega^1 \wedge R\sigma^2 \wedge I\sigma^3)(h \wedge u \wedge v) = 8\,, \tag{5}$$

which, in view of 6.1(2), shows that the second factor in the right-hand side of (4) is $d\mu_v$, up to sign.

Identify $\boldsymbol{SU}_2$ to the standard unit sphere in $\boldsymbol{R}^4$ by using the real and imaginary parts of the matrix entries in the first row. The volume for the standard metric is then $2\pi^2$. It is readily seen that ih, iu, iv is an orthonormal basis for the standard metric. If dv_0 is the corresponding volume element on $\mathfrak{su}_2$, we have then

$$|2\omega| = |I\omega^1 \wedge I\sigma^2 \wedge R\sigma^3| = 4\, dv_0 ,$$

and our first assertion for v complex follows. This also shows that volume of K_v for the positive measure defined by the restriction of ω is $4\pi^2$ if v is real, not in S_1.

In the first case, we have $d\mu_v = 2\omega^1 \wedge \sigma^2$, since $h/2$ and $u/2$ then form an orthonormal basis of $\mathfrak{p}$ in that case, as remarked above: therefore $dk_v = \sigma^3/4$, whence our assertion in that case.

6.3. Lemma. *For $v \in R_f$, let $\nu(\tilde{G})$ be the volume of $\tilde{G}_v$ with respect to ω_v. Then*

$$\nu(\tilde{G}_v) = (Nv + 1)\cdot Nv^{-2} , \quad (v \in R_f) . \tag{1}$$

For comparison, let us recall that if $v \in V_f - R_f$, and K_v is a maximal compact subgroup of $\tilde{G}_v$, then ω_v is the standard measure on $\boldsymbol{SL}_2(k_v)$ and K_v is conjugate, by an element of $\boldsymbol{GL}_2(k_v)$, to $\boldsymbol{SL}_2(\mathfrak{o}_v)$. We have then

$$\omega_v(K_v) = Nv^{-3}\cdot \operatorname{Card} \boldsymbol{SL}_2(F_v) , \tag{2}$$

where F_v is the residue field at v, hence

$$\omega_v(K_v) = Nv^{-2}\cdot(Nv^2 - 1) , \quad (v \in V_f - R_f) . \tag{3}$$

(cf. e.g. [16; 29]). In particular, the volumes given by (1) and (3) are rational numbers, but this follows from a general fact (see e.g. [16: 4.2.5]).

6.4. Proof of 6.3. This is a local statement, also valid in the equal characteristic case. We change notation and shift to a purely local situation. Let then E be a $\mathfrak{p}$-field [28: I, § 3], F its unique unramified quadratic extension k_E and k_F the residue fields of E and F, and q the order of k_E. Then k_F is the quadratic extension of k_E and has order q^2. Let π be a uniformizing variable in E. It is then also one in F and $\mathfrak{p}_E = \pi\cdot\mathfrak{o}_E$, $\mathfrak{p}_F = \pi\cdot\mathfrak{o}_F$ are the maximal ideals of $\mathfrak{o}_E$ and $\mathfrak{o}_F$ respectively. We take as multiplicative representative system S' of k_F in F, the set $S = 0 \cup \langle w\rangle$ where w is a primitive $(q^2 - 1)$-st root of one in F. Let $x \mapsto x'$ be the non-trivial automorphism of F over E. We have $w' = w^q$.

Let B be the division quaternion algebra over E. It splits over F and contains F as a maximal subfield. We can write B as a cyclic algebra

$$B = F \oplus F\cdot u \tag{1}$$

and may assume π to be equal to u^2. The map $x \mapsto x'$ extends to an involution of B which sends u to $-u$. We have

$$(xy)' = y'\cdot x' \quad (x, y \in B)\,, \qquad x\cdot u = u\cdot x' \quad (x \in F) \tag{2}$$

$$(x + yu)' = x' - y\cdot u \quad (x, y \in F)\,. \tag{3}$$

The reduced norm will just be denoted by N. We have

$$Nb = xx' - \pi\cdot yy'\,, \quad (b = x + yu, x, y \in F)\,. \tag{4}$$

Let $\mathfrak{o}_B$ be the maximal order of B and $\mathfrak{p}_B$ its maximal ideal. We have

$$\mathfrak{p}_B = \mathfrak{p}_F + \mathfrak{o}_F\cdot u\,, \quad \mathfrak{p}_B^2 = \pi\cdot\mathfrak{o}_B\,, \quad \mathfrak{o}_B/\mathfrak{p}_B = \mathfrak{o}_F/\mathfrak{p}_F = k_F\,. \tag{5}$$

We let B^1 (resp. K) be the subgroup of elements in B^* whose reduced norm is one (resp. a unit). The group K is the biggest compact subgroup of B^*. The reduced norm yields a surjective homomorphism of K onto $\mathfrak{o}_E^*$, with kernel B^1. It follows from the definitions that the measure ν, multiplied by the standard measure on E (which gives volume 1 to $\mathfrak{o}_E$) is the measure on B^* introduced in [11: p. 475]. Denote also by $\nu(\ )$ the corresponding volumes. We have then

$$\nu(B^1) = \nu(K)\cdot\nu(\mathfrak{o}_E^*)^{-1} = \nu(K)\cdot q\cdot(q-1)^{-1}\,. \tag{6}$$

The natural projection of $\mathfrak{o}_B$ onto k_F, with kernel $\mathfrak{p}_B$, maps K onto k_F^*, hence

$$\nu(K) = (q^2 - 1)\cdot\nu(\mathfrak{p}_B)\,. \tag{7}$$

We write $F = E(\beta)$, where β is integral, and the reduction mod π of β generates k_F over k_E. Then

$$\mathfrak{o}_F = \mathfrak{o}_E + \mathfrak{o}_E\cdot\beta\,, \quad \mathfrak{p}_F = \mathfrak{p}_E + \mathfrak{p}_E\cdot\beta\,. \tag{8}$$

We represent B as the set of matrices

$$b = \begin{pmatrix} x & y \\ \pi y' & x' \end{pmatrix}, \qquad (x, y \in F) . \tag{9}$$

Then $Nb = \det b$. Write

$$x = r + \beta s , \quad y = u + \beta v . \tag{10}$$

On $GL_2(F)$ we take as usual as coordinates the matrix entries a, b, c, d. On B^*, we use r, s, u, v. On $GL_2(F)$ we have the standard invariant 4-form

$$\omega = (\det)^{-1}\cdot da \wedge db \wedge dc \wedge dd . \tag{11}$$

On B^*,

$$a = r + \beta s , \quad b = u + \beta v , \quad c = \pi(u + \beta' v) , \quad d = r + \beta' s . \tag{12}$$

Therefore

$$\omega|_{B^*} = \pm (\beta - \beta')^2 \pi \cdot N^{-1} dr \wedge ds \wedge du \wedge dv . \tag{13}$$

$(\beta - \beta')^2$ is the discriminant of F over E, hence is a unit. This form is defined over E, and the measure ν is the one associated to it. On K, the norm is one in absolute value, therefore our measure ν is given by

$$\nu = |\pi| dr \wedge ds \wedge du \wedge dv = q^{-1} \cdot dr \wedge ds \wedge du \wedge dv . \tag{14}$$

But now, by (5) and (8), the element b belongs to $\mathfrak{p}_B$ if and only if

$$r, s \in \mathfrak{p}_E , \quad u, v \in \mathfrak{o}_E . \tag{15}$$

Therefore

$$\nu(\mathfrak{p}_B) = q^{-3} . \tag{16}$$

Together with (6) and (7), this yields

$$\nu(B^1) = (q + 1)\cdot q^{-2} , \tag{17}$$

as was to be proved.

6.5. Remark. A different proof of 6.4 (17), or rather of an obviously equivalent equality, has been given by W. Casselman (Proc. Symp. Pur. Math. **33**, A.M.S. 1978, part 2, p. 155, lemma 5.2.1).

7. – Volumes and values of zeta functions at 2.

7.1. If Γ is a group, we let $\Gamma^{(2)}$ denote the subgroup generated by the squares of the elements of Γ. It is normal, and $\Gamma/\Gamma^{(2)}$ is a group of exponent 2. If Γ is finitely generated, then $\Gamma/\Gamma^{(2)}$ is a (finite) elementary abelian group of type $(2, 2, \ldots, 2)$. Its $\boldsymbol{F}_2$-rank is at most equal to smallest integer m such that Γ is generated by m elements. The group $\Gamma^{(2)}$ is the smallest normal subgroup of Γ such that $\Gamma/\Gamma^{(2)}$ has exponent 2.

7.2. We return to the commensurability class $\mathcal{C}(k, B)$. We let $N_{B/k}$ or simply N denote the reduced norm from B to k. Fix a maximal order $\mathfrak{O}$ of B, and let $\mathfrak{O}^*$ (resp. $\mathfrak{O}^1$) be the set of elements of $\mathfrak{O}$ whose reduced norm is a unit (resp. one). It is a group, and an arithmetic subgroup of H (resp. $\tilde{G}$). It is known that

$$\mathfrak{O}^* = \{x \in \operatorname{Norm} \mathfrak{O} \mid Nx \in \mathfrak{o}^*\} , \qquad \mathfrak{O}^1 = \operatorname{Norm} \mathfrak{O} \cap \tilde{G} . \tag{1}$$

Not knowing of a good reference, we sketch the proof: For $v \in R_f$, we have $\mathfrak{O}^1_v = \tilde{G}(k_v)$; for $v \in V_f - R_f$, we have $\mathfrak{O}^1_v = \boldsymbol{SL}_2(\mathfrak{o}_v)$; hence in any case $\mathfrak{O}^1_v$ is equal to its normalizer in $\tilde{G}(k_v)$. The group $\mathfrak{O}^1_v$ is the v-adic closure of $\mathfrak{O}^1$ in $\tilde{G}(k_v)$. Consequently, if $x \in \operatorname{Norm} \mathfrak{O}$ has reduced norm one, it belongs to $\mathfrak{O}^1_v$ for all $v \in V_f$, hence to $\mathfrak{O}^1$. This proves the second equality of (1). By a theorem of Eichler [5], the map $x \mapsto Nx$ maps $\mathfrak{O}^*$ onto the group $\mathfrak{o}^*_{R_\infty}$ of units which are positive at R_∞. Let now $x \in \operatorname{Norm} \mathfrak{O}$ be such that $Nx \in \mathfrak{o}^*$. Since Nx has to be positive at R_∞, there exists then $y \in \mathfrak{O}^*$ such that $N(y \cdot x) = 1$. We have then $y \cdot x \in \mathfrak{O}^1$ and $x \in \mathfrak{O}^*$, whence the first equality of (1). Set

$$\Gamma_{\mathfrak{O}^*} = \sigma(\mathfrak{O}^*) = \sigma(k^* \cdot \mathfrak{O}^*) , \qquad \Gamma^1_{\mathfrak{O}} = \sigma(\mathfrak{O}^1) = \sigma(k^* \cdot \mathfrak{O}^1) . \tag{2}$$

Both are arithmetic subgroups of G, normal in $\Gamma_{\mathfrak{O}}$. Let $x \in \operatorname{Norm} \mathfrak{O}$. Then $N(x)^{-1} \cdot x^2$ has reduced norm 1, hence $x^2 \in k^* \cdot \mathfrak{O}^1$ by (1) and therefore

$$\Gamma^1_{\mathfrak{O}} \subset \Gamma^{(2)}_{\mathfrak{O}} . \quad \textit{The group } \Gamma_{\mathfrak{O}}/\Gamma^1_{\mathfrak{O}} \textit{ has exponent two} . \tag{3}$$

If $\tilde{G}$ is isomorphic to $\boldsymbol{SL}_2$ over k, we may choose a k-isomorphism which maps $\mathfrak{O}$ onto $\boldsymbol{M}_2(\mathfrak{o})$. We have then

$$\begin{cases} \mathfrak{O}^1 = \boldsymbol{SL}_2(\mathfrak{o}) , & \mathfrak{O}^* = \boldsymbol{GL}_2(\mathfrak{o}) , \\ \Gamma^1_{\mathfrak{O}} = \boldsymbol{SL}_2(\mathfrak{o})/\{\pm 1\} , & \Gamma_{\mathfrak{O}^*} = \boldsymbol{GL}_2(\mathfrak{o})/\{\pm 1\} . \end{cases} \tag{4}$$

7.3. THEOREM. *Let D_k denote the discriminant of k over $\mathbf{Q}$ and ζ_k the Dedeking zeta function of k. Let G be the k-form of $\mathbf{PGL}_2$ associated to a quaternion algebra B over k. Then*

$$(1)\qquad \mu(\Gamma_{\mathfrak{D}}^1)=\prod_{v\in R_f}(Nv-1)\cdot\frac{2|D_k|^{\frac{3}{2}}\cdot\zeta_k(2)}{2^{2r_1+3r_2-2a}\cdot\pi^{2r_1+2r_2-a}}\,.$$

In particular the volumes $\mu(\Gamma)$, where Γ is arithmetic in G, are all rational multiples of $\pi^{-d-r_1+a}\cdot|D_k|^{\frac{3}{2}}\cdot\zeta_k(2)$.

(Cf. 3.1, 4.1 for the notation.) Let

$$(2)\qquad K_\infty=\prod_{v\in V_\infty}K_v\,,\qquad \nu(K_\infty)=\prod_{v\in V_\infty}\nu(K_v)$$

with $\nu(K_v)$ as in 6.2. We want to prove:

$$(3)\qquad \nu(\tilde{G}_\infty/\mathfrak{D}^1)=\mu(\Gamma_{\mathfrak{D}}^1)\cdot\nu(K_\infty)/2\,,$$

where $\nu(\;)$ on the left-hand side refers to the volume computed with ω_∞ and $\mathfrak{D}^1$ is diagonally embedded in $\tilde{G}_\infty$.

Let Γ be a torsion-free subgroup of finite index of $\mathfrak{D}^1$. Then $\sigma(\Gamma)\tilde{\to}\Gamma$ and $\sigma(\mathfrak{D}^1)=\mathfrak{D}^1/\{\pm 1\}$, hence

$$(4)\qquad [\mathfrak{D}^1\colon\Gamma]=2\cdot[\sigma(\mathfrak{D}^1)\colon\sigma(\Gamma)]\,.$$

We have clearly

$$(5)\qquad \nu(\tilde{G}_\infty/\Gamma)=[\Gamma_{\mathfrak{D}}^1\colon\Gamma]\cdot\nu(\tilde{G}_\infty/\Gamma_{\mathfrak{D}}^1)\,.$$

$$(6)\qquad \mu(\Gamma)=[\Gamma_{\mathfrak{D}}^1\colon\sigma(\Gamma)]\mu(\Gamma_{\mathfrak{D}}^1)\,.$$

Since Γ is torsion-free, $\tilde{G}_\infty/\Gamma$ is fibered over H_∞/Γ, with fiber K_∞, therefore

$$(7)\qquad \nu(\tilde{G}_\infty/\Gamma)=\nu(K_\infty)\cdot\mu(\Gamma)$$

and (3) follows from (4) to (7). By 6.2 we have

$$(8)\qquad \nu(K_\infty)=(8\pi^2)^b\cdot(4\pi^2)^{r_1-a}\cdot\pi^a\,,$$

whence

$$(9)\qquad \nu(\tilde{G}_\infty/\mathfrak{D}^1)=2^{3r_2+2r_1-2a-1}\pi^{2r_2+2r_1-a}\mu(\Gamma_{\mathfrak{D}}^1)\,.$$

The Tamagawa number of $\tilde{G}$ is one [29]. This translates to

$$(10) \qquad \nu(\tilde{G}_\infty/\mathfrak{O}^1) = \Big(\prod_{v\in R_f}(1 - Nv^{-2})\cdot\nu(\tilde{G}_v)^{-1}\Big)\cdot|D_k|^{\frac{3}{2}}\cdot\zeta_k(2)\,.$$

Together with 6.3 and (4), this yields (1).

7.4. *The case of matrix algebras.* 7.3 applies in particular to the case where R is empty, i.e., where B is the matrix algebra $\boldsymbol{M}_2(k)$ and $\tilde{G}$ is k-isomorphic to $\boldsymbol{SL}_2$. The proof of 7.4(1) is then slightly simpler, since 6.3 is not needed. 7.3(1) specializes to

$$(1) \qquad \mu(\boldsymbol{SL}_2\mathfrak{o}/\{\pm 1\}) = 2^{1-3b}\cdot\pi^{-d}\cdot|D_k|^{\frac{3}{2}}\cdot\zeta_k(2)\,.$$

7.5. Remarks. (1) The formula 7.4(1) is due to G. Humbert for k imaginary quadratic ($d = 2, b = 1$) (see [24: § 7]), and to C. L. Siegel [23] when k is totally real. The equality 7.3(1) for totally real fields in general follows from results of Shimizu [22: p. 193].

(2) To get the smallest covolume in $\mathcal{C}(k, B)$, we have to divide the right-hand side of 7.3(1) by the index of $\Gamma^1_{\mathfrak{O}}$ in $\Gamma_{\mathfrak{O}}$. At this point, all we know is that this index is $\leq 2^m$, where m is the smallest cardinality of a generating set for $\Gamma_{\mathfrak{O}}$ (7.1, 7.2(3)). In § 8, we shall give an expression for it in terms of data depending only on k and R.

7.6. Proposition. *Let Γ_1 be an arithmetically defined subgroup of $\boldsymbol{G}_{a,b}$. Then there exist infinitely many commensurability classes of arithmetically defined subgroups of $\boldsymbol{G}_{a,b}$ such that the volumes $\mu(\Gamma)$ are all rational multiples of $\mu(\Gamma_1)$ when Γ runs through these classes.*

From 7.3, we see that, given a and b, the volumes $\mu(\Gamma)$ for the arithmetic subgroups defined by a k-form G of $\boldsymbol{PGL}_2$ (satisfying A) or B) of 3.3, of course) are all rational multiples of a number which depends only on k and a. Therefore, given k with b complex places and at least a real places, we need only to show that there are infinitely many k-forms of $\boldsymbol{PGL}_2$ associated to quaternion algebras over k which, at infinity, are ramified at exactly a places, such that two arithmetic subgroups of $\boldsymbol{G}_{a,b}$ associated to any two of them are not commensurable up to conjugacy.

Let A be the set of automorphisms of k. It is finite, of order $\leq d$. We can choose an infinite sequence of quaternion algebras B_i over k ($i = 1, 2, \ldots$)

such that B_i is not isomorphic to any conjugate ${}^\sigma B_j$ ($\sigma \in A$) of B_j for $i \neq j$ and, at infinity, B_i is ramified at exactly a places of k. [This follows immediately from the fact that a quaternion algebra is determined by its local invariants and that the only conditions imposed on those are to be zero almost everywhere and to have a sum $\equiv 0 \bmod 1$ (cf. [4: VII, § 5])]. Changing the notation slightly, we are then reduced to showing that if G and G' are k-forms of $\boldsymbol{PGL}_2$ associated to two such quaternion algebras B, B', where B' is not isomorphic to a conjugate of B, then an arithmetic subgroup Γ of G is not commensurable up to conjugacy to any arithmetic subgroup of $\boldsymbol{G}_{a,b}$ (3.3). Let $g \in \boldsymbol{G}_{a,b}$ be such that ${}^g\Gamma$ is commensurable with Γ'. Then ${}^g C_\Gamma = C_{\Gamma'}$ (where C_Γ, $C_{\Gamma'}$ denote the commensurability groups, cf. § 1). But $C_\Gamma = G(k)$, $C_{\Gamma'} = G(k')$. Therefore $G(k)$ would be isomorphic to $G'(k)$ as an abstract group. By a theorem of R. Baer [1: Thm. 2, p. 272] (see also [30: Thm. 4.1]), this would imply that G is isomorphic, as an algebraic k-group, to ${}^\sigma G'$ for some $\sigma \in A$, hence that B is isomorphic to ${}^\sigma B'$, a contradiction.

7.7. *Commensurability questions.* Let $b = 0$. In this case, all volumes are commensurable. In fact, if $\chi(\Gamma)$ is the Euler-characteristic of Γ, in the sense of C. T. C. Wall if Γ has torsion (cf. [20: p. 99]), then

$$\chi(\Gamma) = \mu(\Gamma)/(-2\pi)^a, \tag{1}$$

in agreement with the fact that $|D_k|^{\frac{3}{2}} \cdot \zeta_k(2)\pi^{-2d}$ is rational for k totally real. Let now $b \neq 0$. Then $\chi(\Gamma)$ is always zero. Although it is not expected that all volumes are commensurable, this has not been checked to far. In view of 7.3, to produce an example, it would be enough to exhibit two number fields k, k' of the same degree and the same non-zero number of complex places such that

$$|D_k|^{\frac{3}{2}}\zeta_k(2) \notin \boldsymbol{Q} \cdot |D_{k'}|^{\frac{3}{2}} \cdot \zeta_{k'}(2)\,. \tag{2}$$

Apparently, nothing is known about this question. Of course, the truth of Milnor's conjectures about the Lobatshevski function [24: § 7] would provide many examples of quadratic imaginary fields k, k' satisfying (2).

8. – Discreteness of the set of arithmetic volumes.

In this section, we want to prove that the set of volumes $\mu(\Gamma)$, when Γ runs through the arithmetically defined subgroups of $\boldsymbol{G}_{a,b}$, is discrete (with finite multiplicities, see 8.2 for the precise statement). If $a+b\geqq 2$, these subgroups are all the irreducible discrete subgroups of finite covolume, as already pointed out (3.4), and the discreteness has been proved by H. C. Wang ([27], see 8.3]). For the sake of uniformity we shall also include this case, although this is not really a new proof, since the idea of the proof of 8.1 in that case is taken from [26, 27].

8.1. LEMMA. *Let $a, b \in \boldsymbol{N}$ be given. Let $c>0$. There exists an integer $m(c)$ such that if Γ is an irreducible discrete subgroup of $\boldsymbol{G}_{a,b}$ and $\mu(\Gamma)\leqq c$ then Γ is generated by $m(c)$ elements.*

Let first $a=1$, $b=0$. Since Γ contains a subgroup Γ' of index two which preserves the orientation, we may assume $\Gamma\subset \boldsymbol{SR}_2(\boldsymbol{R})/\{\pm 1\}$. In this case our assertion follows from the standard formula for $\mu(\Gamma)$: let m be the number of cusps of H^2/Γ, $\{\gamma_1, \ldots, \gamma_s\}$ a set of representatives of the classes of elliptic elements of Γ, e_j the order of γ_j $(1\leqq j\leqq s)$ and g the genus of the standard compactification of H^2/Γ. Then Γ is generated by $2g+m+r-1$ elements and we have

$$\mu(\Gamma)=2g-2+m+\sum_{j=1}^{j=s}(1-e_j^{-1})\,. \tag{1}$$

Since $e_j\geqq 2$, we see that $2g+m+r-1\leqq 2\mu(\Gamma)+2$.

If $a=0$ and $b=1$, 8.1 follows from the construction of all H^3/Γ with volume $\leqq c$ by means of Dehn surgery applied to finitely many orbifolds, given in Chap. 13 of [24]; it will be proved explicitly in the final version of these Notes. For torsion-free Γ, all we shall need to know is that $H_1(\Gamma; \boldsymbol{Z}/2\boldsymbol{Z})$ has dimension bounded by some constant $n(c)$, and this follows directly from [24: Chap. 5]; in fact, it is shown there that the hyperbolic 3-manifolds of volume $\leqq c$ are obtained by gluing some solid tori or cusps to finitely many compact manifolds with boundary a union of 2-dimensional tori.

Let now $a+b\geqq 2$. Assume there is a sequence of irreducible subgroups Γ_n of $\boldsymbol{G}_{a,b}$ such that $\mu(\Gamma_n)\leqq c$ and that the smallest cardinality g_n of a generating system of Γ_n tends to infinity. By the argument of [26: p. 137], recalled in [27: p. 480], there exists a subgroup Γ, which is a limit of the Γ_n, in the topology of the space of closed subgroups, such that $\mu(\Gamma)\leqq c$, and moreover a homomorphism $r_n\colon \Gamma\to\Gamma_n$, for n big enough, defining a deformation of Γ which tends to the identity as $n\to\infty$. The group Γ is also irreducible, since r_n has to be trivial on any subgroup of Γ which is contained in a proper factor of $\boldsymbol{G}_{a,b}$. But then Γ is rigid (since $a+b\geqq 2$), hence Γ_n is conjugate to Γ for n big enough, a contradiction with the assumption $g_n\to\infty$.

8.2. THEOREM. *Fix a and b. Let $c > 0$. Then there exist finitely many arithmetic subgroups $\Gamma_1, \ldots, \Gamma_{q(c)}$ of $G_{a,b}$ such that any arithmetic subgroup Γ of $G_{a,b}$ with covolume $\mu(\Gamma) \leqq c$ is conjugate to one of the Γ_i's $(1 \leqq i \leqq q(c))$. In particular the set of volumes $\mu(\Gamma)$, where Γ runs through the arithmetically defined subgroups of $G_{a,b}$, is a discrete subset of the real line.*

In view of 5.3, 5.4, it suffices to prove this theorem for the set of arithmetic subgroups of the form $\Gamma_{\mathfrak{D}}$ defined in 4.9. We first show it for the groups $\Gamma^1_{\mathfrak{D}}$. Consider 7.3(1). Since $\zeta_k(2) \geqq 1$, we have

$$\mu(\Gamma^1_{\mathfrak{D}}) \geqq 2 \cdot |D_k|^{\frac{3}{2}} \cdot (2\pi)^{-2r_1-2r_2} \cdot 2^{-r_2} . \tag{1}$$

$$\mu(\Gamma^1_{\mathfrak{D}}) \geqq 2|D_k|^{\frac{3}{2}} \cdot (4\pi^2)^{-r_1} \cdot (2\sqrt{2} \cdot \pi)^{-2r_2} . \tag{2}$$

Since there are only finitely many number fields with a given discriminant, it suffices to show that the right-hand side of (2) tends to infinity with the degree of k. But this follows from known estimates on the discriminant, e.g., from

$$|D_k| \geqq 50^{r_1} \cdot 19^{2r_2} , \qquad \text{for } d \text{ large enough} \tag{3}$$

which implies

$$|D_k|^{\frac{3}{2}} \geqq 353^{r_1} \cdot 82^{2r_2} , \qquad \text{for } d \text{ large enough} , \tag{4}$$

and follows from 1.8 in [15].

Let now $c > 0$. By 8.1, there exists a constant $m(c)$ such that if $\mu(\Gamma_{\mathfrak{D}}) \leqq c$, then $\Gamma_{\mathfrak{D}}$ has a generating set of cardinality $\leqq m(c)$. Since $\Gamma_{\mathfrak{D}}/\Gamma^1_{\mathfrak{D}}$ has exponent two $(7.2(3))$, we have then

$$[\Gamma_{\mathfrak{D}} : \Gamma^1_{\mathfrak{D}}] \leqq 2^{m(c)} , \tag{5}$$

(7.1), hence

$$\mu(\Gamma^1_{\mathfrak{D}}) \leqq c \cdot 2^{-m(c)} . \tag{6}$$

The possible $\Gamma^1_{\mathfrak{D}}$ form then finitely many conjugacy classes by the first part of the proof. In view of 5.3, the same is then true for the groups $\Gamma_{\mathfrak{D}}$.

8.3. REMARK. Let L be a connected semi-simple Lie group with center reduced to the identity and no compact factor. Theorem 8.1 in [27] asserts that the covolumes $\mu(L/\Gamma)$ (Γ discrete in L) form a discrete set (with finite multiplicities) if L has no three-dimensional factor. This should in particular apply to $L = \boldsymbol{PGL}_2(\boldsymbol{C})$, but there it is contradicted by the results of Thurston-Jorgensen [24: Chap. 5]. The mistake in [27] comes from a misunderstanding of rigidity in that group: the author uses a result he attributes to H. Garland and M. S. Raghunathan, but he misquotes it. However, the proof, as it stands, is valid for the irreducible subgroups of L, provided L is not locally isomorphic to $\boldsymbol{SL}_2(\boldsymbol{R})$ or $\boldsymbol{SL}_2(\boldsymbol{C})$, since in all those cases the rigidity theorem used by Wang is indeed available. In particular, this covers the case of our groups $\boldsymbol{G}_{a,b}$ for $a + b \geqq 2$.

8.4. In this proof we have used discriminant estimates to handle the groups $\Gamma^1_{\mathfrak{D}}$ and then a geometric argument to go over to $\Gamma_{\mathfrak{D}}$. One can of course ask whether it would not be possible also to give an arithmetic proof for $\mu(\Gamma_{\mathfrak{D}})$, using a good estimate of $[\Gamma_{\mathfrak{D}}:\Gamma^1_{\mathfrak{D}}]$. We shall see that this is unlikely since this index depends in part on the class group of k. First we want to give an arithmetic description of it.

We denote by $\mathfrak{o}^*_{R_f}$ the group of R_f-units of k (elements which are integral at all finite places outside R_f) and by $\mathfrak{o}^*_{R_f,R_\infty}$ the group of elements of $\mathfrak{o}^*_{R_f}$ which are positive at R_∞.

We have

$$(1) \qquad [\mathfrak{o}^*_{R_f,R_\infty}:\mathfrak{o}^{*2}_{R_f}] \leqq [\mathfrak{o}^*_{R_f}:\mathfrak{o}^{*2}_{R_f}] \leqq 2^{r_1+r_2+r_f}, \qquad (r_f = |R_f|),$$

where the last inequality follows from the unit theorem. Let now

$$(2) \qquad B^*_{R_f} = \{b \in B^* | N(b) \in \mathfrak{o}^*_{R_f}\}, \qquad \Gamma_{R_f} = \sigma(k^* \cdot B^*_{R_f}).$$

We have $Nb \in \mathfrak{o}^*_{R_f,R_\infty}$ for $b \in B^*_{R_f}$. Moreover, the results of [5] imply that $B^*_{R_f} \in \mathrm{Norm}\ \mathfrak{D}$. We have then the inclusions

$$(3) \qquad \Gamma_{\mathfrak{D}} \supset \Gamma_{R_f} \supset \Gamma_{\mathfrak{D}^*} \supset \Gamma^1_{\mathfrak{D}}.$$

8.5. LEMMA. *The group $\Gamma_{R_f}/\Gamma^1_{\mathfrak{D}}$ is isomorphic to $\mathfrak{o}^*_{R_f,R_\infty}/\mathfrak{o}^{*2}_{R_f}$. In particular $[\Gamma_{R_f}:\Gamma^1_{\mathfrak{D}}] \leqq 2^{r_1+r_2+r_f}$.*

Eichler's theorem implies that $b \mapsto Nb$ maps $B^*_{R_f}$ onto $\mathfrak{o}^*_{R_f,R_\infty}$. If now $Nb = c^2$, with $c \in \mathfrak{o}^*_{R_f}$, then $N(c^{-1} \cdot b) = 1$, hence $b \in k^* \cdot \mathfrak{D}^1$, and the first assertion is proved. The second follows from 8.4(1).

8.6. Let $D_\infty = D_\infty(B)$ (resp. $D_f = D_f(B)$) be the product of the primes in R_∞ (resp. R_f). Thus the ideal D_f is the square root of the discriminant of B. Let $I(k)$ (resp. $P(k)$) be the group of fractional (resp. principal) ideals of k and $P(k, D_\infty)$ the group of principal ideals generated by elements which are $\equiv 1 \bmod^* D_\infty$, i.e, which are positive at R_∞.

LEMMA. *Let M_1 (resp. M_2) be the subgroup of $I(k)$ generated by $P(k, D_\infty)$ (resp. $P(k)$) and the $\mathfrak{o} \cap \mathfrak{p}_v$ ($v \in R_f$). Let $J_1 = I(k)/M_1$ and let J_2 be the image of M_2 in J_1. Then $[\Gamma_{\mathfrak{O}} : \Gamma_{R_f}] = [{}_2J_1 : J_2]$, where ${}_2J_1$ is the kernel of the map $y \mapsto y^2$ in J_1. If k has class number one, then $\Gamma_{\mathfrak{O}} = \Gamma_{R_f}$.*

Let L_1 be the subgroup of $I(k)$ generated by the $\mathfrak{o} \cap \mathfrak{p}_v$ ($v \in R_f$) and the squares of all ideals. It follows from the description of an ideal as an intersection of local ideals that the elements of L_1 are the norms of the two-sided $\mathfrak{O}$-ideals. By a theorem of Eichler [5], an element of L_1 is the norm of a principal $\mathfrak{O}$-ideal if and only it belongs to $P(k, D_\infty)$.

Let now $x \in \operatorname{Norm} \mathfrak{O}$. There exists then a unique ideal $\mathfrak{m}(x)$ prime to D_f such that the ideal (Nx) is the product of $\mathfrak{m}(x)^2$ by a power product of the divisors of D_f. Let $\tau\colon \operatorname{Norm} \mathfrak{O} \mapsto J_2$ be the map which assigns to x the class of $\mathfrak{m}(x)$ in J_2. By the above, its image belongs to ${}_2J_1$ and every element of ${}_2J_1$ occurs in this way. For $c \in k^*$, we have $N(cx) = c^2 \cdot N(x)$, hence τ is constant on $k^* \cdot x$. Assume now that $\tau(x) \in J_2$. This means that we can write (Nx) as the product of a principal ideal (a^2) ($a \in k$) by a power product of the $\mathfrak{o} \cap \mathfrak{p}_v$ ($v \in R_f$). But then $N(a^{-1} \cdot x) \in \mathfrak{o}^*_{R_f}$, hence $x \in k^* \cdot B^*_{R_f}$. Thus τ defines an isomorphism of $\Gamma_{\mathfrak{O}}/\Gamma_{R_f}$ onto ${}_2J_1/J_2$ and the first assertion is proved. If k has class number one, then $J_2 = J_1$, whence the second assertion.

8.7. The first part of the proof of 8.2 also shows that the $\mu(\Gamma_{R_f})$ form a discrete set, but I do not see how to go from there in the same way to $\mu(\Gamma_{\mathfrak{O}})$. An upper bound of ${}_2J_1/J_2$ is the «narrow» class number $h_+(k)$. It may grow about as fast as $|D_k|^{\frac{1}{2}}$, which is too strong to be absorbed by 8.2(3), or even by the stronger estimates of [15]. Of course, for fields of a given degree, it is easy to show that the $\mu(\Gamma_{\mathfrak{O}})$ form a discrete set.

8.8. Another question raised by W. Thurston is whether the g.c.d. of the volumes in a commensurability class have a strictly positive lower bound. Since the volumes do by a well-known theorem of D. Kazdhan and G. A. Margoulis (see [17: XI, 11.9]), this is clear for non-arithmetically defined classes (cf. § 1). The first part of the proof of 8.2 also shows that the numbers $2^{-d} \cdot \mu(\Gamma_{R_f})$ have a strictly positive lower bound. In view of 5.4, this shows that the g.c.d. of the volumes in the commensurability classes attached to fields of class number one do have a strictly positive lower bound. I do not know whether this is true in general.

9. – Hyperbolic 3-folds.

9.1. In this and the next section, we consider the case of hyperbolic 3-space. We have then $a = 0$, $b = 1$, $r_2 = 1$, $d = r_1 + 2$, and 7.3(1) becomes

$$\mu(\Gamma^1_{\mathfrak{D}}) = \prod_{v \in R_f} (Nv - 1) \cdot |D_k|^{\frac{3}{2}} \cdot \zeta_k(2)(2\pi)^{-2d+2} . \tag{1}$$

Since $a = 0$, the set R_∞ is the set of all real places of k, hence $\mathfrak{o}^*_{R_f,R_\infty}$ is the group $\mathfrak{o}^*_{R_f,+}$ all of totally positive R_f-units. Therefore, in view of 8.4, we get for the smallest volume in the given commensurability class

$$\mu(\Gamma_{\mathfrak{D}}) = [\mathfrak{o}^*_{R_f,+} : \mathfrak{o}^{*2}_{R_f}]^{-1} \mu(\Gamma^1_{\mathfrak{D}}) , \quad \textit{if } k \textit{ has class number one}. \tag{2}$$

Assume now k to be imaginary quadratic. Then R_∞ is empty, $\mathfrak{o}^*_{R_f}$ is just the group of all R_f-units. It is the product of a cyclic group of even order by a free abelian group on r_f generators, hence

$$[\mathfrak{o}^*_{R_f,+} : \mathfrak{o}^*_{R_f}] = [\mathfrak{o}^*_{R_f} : \mathfrak{o}^{*2}_{R_f}] = 2r_f + 1 , \quad \textit{if } k \textit{ is imaginary quadratic}. \tag{3}$$

k is necessarily quadratic imaginary in the non-cocompact case, and then R_f is also empty. We get

$$\mu(\boldsymbol{SL}_2(\mathfrak{o})/\{\pm 1\}) = |D_k|^{\frac{3}{2}} \zeta_k(2)/4\pi^2 , \tag{4}$$

which is G. Humbert's formula. If k has class number one, $\mu(\Gamma_{\mathfrak{D}})$ is one-half of the right-hand side of (4).

9.2. It is not surprising from the general formula that small volumes should be tied up to fields of small discriminants and, in the compact case, to quaternion algebras which are as unramified as possible at the finite places. However, because of the factor $[{}_2J_1 : J_2]$ we can confirm this only for group Γ_{R_f}, hence for $\Gamma_{\mathfrak{D}}$ if k has class number one. Remarks (*a*), (*b*), (*c*) below are due to E. Bombieri.

(*a*) Let $k = \boldsymbol{Q}(\sqrt{-3})$. Then the class number is one and

$$\mu(\boldsymbol{GL}_2(\mathfrak{o})/\{\pm 1\}) = 3^{\frac{3}{2}} \cdot \zeta_k(2)/8\pi^2 = 0.08458 \ldots .$$

Simple estimates show that this is the smallest value of $\mu(\Gamma_{R_f})$, when k runs through all the imaginary quadratic fields. It may also be the minimum of $\mu(\boldsymbol{GL}_2(\mathfrak{o})/\{\pm 1\})$ for those cases. We now consider cocompact groups.

(*b*) Let $d = 2$. Then R_f has at least two elements. In this case the minimum of $\mu(\Gamma_{R_f})$ is realized when $D_k = -3$, $\prod_{v \in R_f} (Nv - 1) = 6$, and then

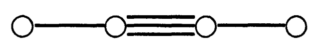
$$\mu(\Gamma_{R_f}) = \mu(\Gamma_{\mathfrak{D}}) = 0.126 \pm 0.0001 \,. \tag{4}$$

(*c*) Let $k = \boldsymbol{Q}(\theta)$, where θ is a root of $x^3 - x - 1$. Then $D_k = -23$, $[\mathfrak{o}_+^* : \mathfrak{o}^{*2}] = 4$. The prime $v = 2 + \theta$ divides 5 and $Nv = 5$. Take then $R_f = \{v\}$. The field k has also class number one. Then

$$\mu(\Gamma_{\mathfrak{D}}) = 0.3536 \ldots \times \zeta_k(2) \leqq 0.47474 \ldots \tag{5}$$

and this $\Gamma_{\mathfrak{D}}$ seems a good candidate for the smallest volume when k has signature $(1, 1)$. At any rate 23 is the minimum of $|D_k|$ for these fields.

(*d*) Let $k = \boldsymbol{Q}(\theta)$ where $\theta = (3 + 2\sqrt{5})^{\frac{1}{2}}$. This is a quartic field of signature $(2, 1)$. Its discriminant is -275 and k is known to have the smallest discriminant in absolute value for fields of signature $(2, 1)$ [8; 14]. Take for B the quaternion algebra over k which is ramified at exactly the two real places of k. Then the group $\Gamma_{\mathfrak{D}}$ is the subgroup of orientation preserving transformations in the Coxeter group:

(6)  ,

as was pointed out by W. Thurston. This appears so far to be the smallest volume known and it seems rather likely to be the smallest obtained from fields of signature $(2, 1)$. Eventually, for fields of high enough degree, the volumes have to become bigger, but it seems well possible that quaternion algebras over fields of relatively small degree with small discriminants might lead to smaller volumes. The next candidate would be the field of signature $(3, 2)$ with discriminant 4511, with R_f consisting of one place dividing a prime with small norm.

10. – Totally real fields. Fuchsian groups.

10.1. Assume now k to be totally real, i.e., $b = 0$. Then the functional equation yields

$$\zeta_k(-1) = |D_k|^{\frac{3}{2}} \zeta_k(2)/(-2\pi^2)^d \,, \tag{1}$$

and 7.3(1) can be written

(2) $$\mu(\Gamma^1_{\mathfrak{D}}) = (-1)^d \zeta_k(-1)\pi^a \cdot 2^{2a+1-d} \cdot \prod_{v \in R_f} (Nv - 1) .$$

The Euler-Poincaré characteristic $\chi(\Gamma^1_{\mathfrak{D}})$ of $\Gamma^1_{\mathfrak{D}}$ is given by

(3) $$\chi(\Gamma^1_{\mathfrak{D}}) = (-2\pi)^a \mu(\Gamma^1_{\mathfrak{D}}) = (-2)^{a-d+1} \zeta_k(-1) \prod_{v \in R_f} (Nv - 1) .$$

The group $\Gamma^1_{\mathfrak{D}}$ is a quotient of $\mathfrak{D}^1$ by a group of order 2, hence

(4) $$\chi(\mathfrak{D}^1) = (-2)^{a-d} \zeta_k(-1) \prod_{v \in R_f} (Nv - 1) .$$

Note that if $a = d$, this is equal to $(-1)^d \chi(\boldsymbol{SL}_2 \mathfrak{o}_{R_f})$, in view of [20: p. 159].

10.2. Consider now the case of Fuchsian groups, where $a = 1$. Using 10.1(1) we can also write 7.3(1) as

(1) $$\mu(\Gamma_{R_f}) = 2^{d-3} \cdot \pi [\mathfrak{o}^*_{R_f, R_\infty} : \mathfrak{o}^{*2}_{R_f}]^{-1} \cdot |\zeta_k(-1)| \prod_{v \in R_f} (Nv - 1) .$$

This can be used in particular for the triangle groups which can be defined arithmetically. Some were already investigated by R. Fricke [6; 7] and a complete determination of those groups and of the associated quaternion algebras has been carried out by K. Takeuchi [12]. Moreover, it is shown there that the underlying groundfields have all class number one, so that $\mu(\Gamma_{R_f})$ realizes the minimum of the volume, and is the triangle group (recall that we have included orientation reversing isometries at the real places (3.0)).

As an example consider the group of the triangle (2, 3, 7). Here $k = \boldsymbol{Q}(\cos 2\pi/7)$ is the maximal totally real subfield of the cyclotomic field of the seventh roots of 1. It is cubic, and we take for B a quaternion algebra ramified at exactly two infinite primes. Thus R_f is empty. It can be checked that $[\mathfrak{o}^*_{R_\infty} : \mathfrak{o}^{*2}] = 2$. Moreover, it is known that $\zeta_k(-1) = -1/21$ [20: p. 163]. We get indeed $\mu(\Gamma_{\mathfrak{D}}) = \pi/42$.

This group is denoted $\bar{\Gamma}_{(63)}$ in [6]. There Fricke also considers commensurable groups $\bar{\Gamma}_{(7)}, \bar{\Gamma}_{(14)}, \bar{\Gamma}$. The group $\bar{\Gamma}_{(7)}$ is the group of the triangle (2, 4, 7), $\bar{\Gamma}_{(7)} = \bar{\Gamma}_{(14)} \cap \bar{\Gamma}_{(63)}$ has index 2 in $\bar{\Gamma}_{(14)}$ and 9 in $\bar{\Gamma}_{(63)}$. These groups can be described as follows in the set-up of §§ 4, 5.

Note first that 2 remains prime in k, and if v_0 is the corresponding place of k, then $Nv_0 = 8$. We may write $\bar{\Gamma}_{63} = \Gamma_{\mathfrak{D}}$, where $\Gamma_{\mathfrak{D}}$ is defined by the vertices (P_v) of the various Bruhat-Tits buildings. Then $\bar{\Gamma}_{(14)}$ is the group which fixes the P_v for $v \neq v_0$ and stabilizes the edge e for $v = v_0$. It indeed contains an element which is odd at exactly 2, namely $z \mapsto (z+1)/(1-z)$ [6: p. 456]. The reduction mod v_0 maps $\bar{\Gamma}_{63}$ onto the projective group of the projective line $\boldsymbol{P}^1(\boldsymbol{F}_8)$, and $\bar{\Gamma}_{(7)}$ on the stability group of a point, i.e., on the affine group of $\boldsymbol{F}_8$. The inverse image of the group of translations has then index 7 in $\bar{\Gamma}_{(7)}$, and is the group $\bar{\Gamma}$.

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