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Pseudodifferential operators and Hardy kernels on $L^p(\mathbf{R}^+)$

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Pseudodifferential Operators and Hardy Kernels on $L^p(\boldsymbol{R}^+)$ (*).

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Introduction.

We develop an algebra of pseudodifferential operators on $L^p(\boldsymbol{R}^+)$, $1<p<\infty$, which includes H, the Hilbert transform restricted to $\boldsymbol{R}^+$, and some classical Hardy kernels. Such operators arise in the study of singular integral equations in $L^p(\boldsymbol{R}^+)$ since $H^2=-I+K$, where K is a Hardy kernel operator. The algebra of operators described here, called Mellin operators in $OP\Sigma_{1/p}$, are defined via the Mellin transform. As in the case of the Hilbert transform [Sh 1] or a Hardy kernel [FJL 1], the spectrum of the operator depends upon the L^p space on which it acts. As described in the remarks of Section 5, there are operators in $OP\Sigma_{1/p}$ for all p, $1<p<\infty$, which admit a parametrix in $OP\Sigma_{1/p}$ for some values of p, but do not have a parametrix in $OP\Sigma_{1/p}$ for other values of p; the parametrices in $OP\Sigma_{1/p}$, for different values of p, do not necessarily agree on $C_0^\infty(\boldsymbol{R}^+)$.

E. Shamir [Sh 1, Sh 2] studied the spectrum of the Hilbert transform on $L^p(\boldsymbol{R}^+)$. G. I. Ěskin [E 1, E 2] has made an extensive study of operators defined via the Mellin transform and given applications to weighted L^2 spaces and boundary value problems. In [N] J. Nourrigat has defined a class of pseudodifferential operators on $\boldsymbol{R}^+$ defined by the Mellin transform and studied their properties on weighted L^2 spaces. B. A. Plamenevskii in [P] has studied an algebra of pseudodifferential operators in $\boldsymbol{R}^+\times S^{n-1}$ defined using the Mellin transform. H. O. Cordes and E. A. Herman [CH] studied singular integrals on $L^2(\boldsymbol{R}^+)$.

In Section 1 we state the properties of the Mellin transform and Mellin multipliers to be used in the sequel. A representation for variable symbol Mellin operators is studied in Section 2. The space of symbols, $\Sigma_{1/p}$, and

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the space of Mellin operators, $OP\Sigma_{1/p}$, are defined in Section 3; the principal symbol σ_p is defined. The symbolic calculus is developed in Section 4. The operators in $OP\Sigma_{1/p}$ which admit a parametrix in $OP\Sigma_{1/p}$ are characterized by their symbols in Theorem 5; the remarks following Theorem 5 describe a typical situation. The index of an elliptic operator in $OP\Sigma_{1/p}$ is studied in Section 6. An application to an oblique derivative problem for Laplace's equation in a plane sector is given in Section 7.

1. – Preliminaries on the Mellin transform.

We shall deal with functions in $L^p = L^p(\boldsymbol{R}^+)$, $1 < p < \infty$, with the norm

$$\|f\|_p = \left(\int_0^\infty |f(x)|^p dx\right)^{1/p}.$$

It will be convenient to consider functions $g(x) \in L^p_*(\boldsymbol{R}^+)$ where

$$\|g\|_{p,*} = \left(\int_0^\infty |g(x)|^p \frac{dx}{x}\right)^{1/p}.$$

If $f(x) \in L^p$, we define $f_p(x) = x^{1/p} f(x) \in L^p_*(\boldsymbol{R}^+)$ and the functions $F(u) = f(\exp[-u])$, and $F_p(u) = f_p(\exp[-u])$, $u \in \boldsymbol{R}$. Note that

$$\|f\|_p = \|f_p\|_{p,*} = \|F_p\|_{L^p(\boldsymbol{R})} = \left(\int_{-\infty}^{+\infty} \exp[-u]|F(u)|^p du\right)^{1/p}.$$

If $f(x) \in C_0^\infty(\boldsymbol{R}^+)$ we define the Fourier transform of F_p as the function

$$\hat{F}_p(\xi) = \int_{-\infty}^{+\infty} \exp[-iu\xi] F_p(u)\, du = \int_0^\infty x^{1/p+i\xi-1} f(x)\, dx\,, \qquad \xi \in \boldsymbol{R}\,. \tag{1.1}$$

The Mellin transform of a function $f \in C_0^\infty(\boldsymbol{R}^+)$ is defined as

$$\tilde{f}(z) = \int_0^\infty x^{z-1} f(x)\, dx\,, \qquad z \in \boldsymbol{C}\,. \tag{1.2}$$

It follows that for $f \in C_0^\infty(\boldsymbol{R}^+)$, $\tilde{f}(z)$ is an entire function and we have the inversion formula

$$f(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} \tilde{f}(z)\, dz ,$$

where the notation $1/2\pi i \int_{a-i\infty}^{a+i\infty} \ldots dz$ denotes contour integration along the path $z = a + i\xi$, $-\infty < \xi < \infty$. Integration by parts shows that

$$\left(-x \frac{d}{dx} f\right)^{\sim}(z) = z\tilde{f}(z) \qquad \text{and} \qquad ((\log x) f)^{\sim}(z) = \frac{d}{dz} \tilde{f}(z) .$$

For τ real, $\delta > 0$, we use the notation

$$S_{\tau,\delta} = \{z \in \boldsymbol{C} \colon \ \tau - \delta < \operatorname{Re} z < \tau + \delta\} .$$

If f is measurable on $\boldsymbol{R}^+$ and the integral (1.2) is absolutely convergent for all z in some strip $S_{\tau,\delta}$ we shall call the integral $\tilde{f}(z)$ the *Mellin transform* of f; under these conditions $\tilde{f}(z)$ is a holomorphic function in $S_{\tau,\delta}$. We make the following definition.

DEFINITION 1. *Let $b(z)$ be a bounded measurable function on the line $\operatorname{Re} z = 1/p$. Then we say b is a Mellin multiplier on L^p iff the map*

$$Bf(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} b(z) \tilde{f}(z)\, df , \quad x > 0, \qquad f \in C_0^\infty(\boldsymbol{R}^+),$$

is extendable as a bounded linear operator on L^p.

By (1.1), $\hat{F}_p(\xi) = \tilde{f}(1/p + i\xi)$, so that b is a Mellin multipler on L^p iff the function $\xi \to b(1/p + i\xi)$ is a Fourier multiplier on $L^p(\boldsymbol{R})$ [H].

We give the following examples which will be essential ingredients in the algebra of operators to be constructed in Section 3.

1) *The Hilbert transform on $L^p(\boldsymbol{R}^+)$.* The Hilbert transform of a function $f \in L^p$ is defined as

$$\text{(1.3)} \qquad Hf(x) = p.v. \frac{1}{\pi} \int_0^\infty \frac{f(y)}{x - y}\, dy .$$

Following Shamir [Sh 1] and Eskin [E 1] we can represent H as

$$(1.4) \qquad Hf(x)=\frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} x^{-z}\left\{i\,\frac{1+\exp[2\pi iz]}{1-\exp[2\pi iz]}\right\}\tilde f(z)\,dz\,, \qquad f\in C_0^\infty(\boldsymbol{R}^+)\,.$$

It is well known that H is a bounded operator on L^p. Define the function

$$\theta(z)=\frac{1}{1-\exp[2\pi iz]}=\frac{1}{2}\left(1+\frac{1+\exp[2\pi iz]}{1-\exp[2\pi iz]}\right).$$

Then $\theta(z)$ is a Mellin multiplier on L^p, $1<p<\infty$.

2) *Hardy kernels on L^p.* Let $k(x)$ be a measurable function on $\boldsymbol{R}^+$ such that for some a, b with $0<a<b<1$,

$$\int_0^\infty x^{a-1}|k(x)|\,dx+\int_0^\infty x^{b-1}|k(x)|\,dx<\infty\,.$$

Then for all p, $a<1/p<b$, the Hardy operator with kernel k is defined on L^p by

$$(1.5) \qquad Kf(x)=\int_0^\infty k\left(\frac{x}{y}\right)f(y)\,\frac{dy}{y}\,.$$

Following [FJL 1], for $f\in C_0^\infty(\boldsymbol{R}^+)$,

$$(1.6) \qquad Kf(x)=\frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} x^{-z}\tilde k(z)\tilde f(z)\,dz\,,$$

where $\tilde k(z)$ is the Mellin transform of the kernel k which is defined and holomorphic for $a<\operatorname{Re} z<b$.

3) *The operator* $T_{\zeta,p}$ (a particular Hardy kernel). Let $1<p<\infty$ and $\zeta\in\boldsymbol{C}$, $\operatorname{Re}\zeta\neq 1/p$. For $f\in C_0^\infty(\boldsymbol{R}^+)$, define

$$(1.7) \qquad T_{\zeta,p}f(x)=\frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} x^{-z}\,\frac{1}{\zeta-z}\,\tilde f(z)\,dz\,.$$

The function $b(z)=1/(\zeta-z)$ is a Mellin multiplier on L^p, $1/p\neq\operatorname{Re}\zeta$, and the L^p norm of the operator $T_{\zeta,p}$ is bounded by $C|\mathrm{Re}\,\zeta - 1/p|^{-1}$ (C is independent of p). If $1/p < \mathrm{Re}\,\zeta$, the kernel for $T_{\zeta,p}$ on L^p is given by

$$k_{\zeta,p}(x) = 0\,, \quad 0 < x < 1\,, \quad k_{\zeta,p}(x) = x^{-\zeta}\,, \quad x > 1\,.$$

If $\mathrm{Re}\,\zeta < 1/p$, the kernel for $T_{\zeta,p}$ on L^p is given by

$$k_{\zeta,p}(x) = -\,x^{-\zeta}\,, \quad 0 < x < 1\,, \quad k_{\zeta,p}(x) = 0\,, \quad x > 1\,.$$

The bound for the L^p norm of $T_{\zeta,p}$ is a consequence of Young's inequality.

2. – A class of bounded operators on $L^p(\mathbf{R}^+)$.

We now introduce a class of Mellin integral operators on $L^p(\mathbf{R}^+)$ with variable kernels.

THEOREM 1. *Let $a(x, z)$ be a function defined for $x > 0$ and z in some strip $S_{1/p,\delta}$, $1 < p < \infty$. Suppose that for all x, $a(x, z)$ is holomorphic in $S_{1/p,\delta}$ and that there is an $\varepsilon > 0$ and a constant C such that*

$$\sup_{x>0} |a(x, z)| \leqslant C(1 + |z|)^{-1-\varepsilon}\,, \qquad z \in S_{1/p,\delta}\,.$$

Then the operator defined by

$$Af(x) = \frac{1}{2\pi i} \int\limits_{1/p-i\infty}^{1/p+i\infty} x^{-z} a(x, z)\tilde{f}(z)\, dz\,, \qquad f \in C_0^\infty(\mathbf{R}^+)\,, \tag{2.6}$$

is extendable as a bounded operator on L^p.

PROOF. Let $0 < \delta_1 < \delta$ and let $\Gamma_{1/p,\delta_1}$ denote the contour

$$\left\{z = \frac{1}{p} + \delta_1 + i\xi,\ -\infty < \xi < \infty\right\} \cup \left\{z = \frac{1}{p} - \delta_1 - i\xi,\ -\infty < \xi < \infty\right\}.$$

If $\mathrm{Re}\, z = 1/p$, by the Cauchy integral formula

$$a(x, z) = \frac{1}{2\pi i} \int\limits_{\Gamma_{1/p,\delta_1}} \frac{a(x, \zeta)}{\zeta - z}\, d\zeta\,.$$

Using this representation for $a(x, z)$ in (2.6) and applying Fubini's Theorem, we obtain

$$Af(x) = \frac{1}{2\pi i} \int_{\Gamma_{1/p,\delta_1}} a(x, \zeta) \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} \frac{1}{\zeta - z} \tilde{f}(z)\, dz = \frac{1}{2\pi i} \int_{\Gamma_{1/p,\delta_1}} a(x, \zeta)\, T_{\zeta,p} f(x)\, d\zeta .$$

An application of Minkowski's Integral Inequality gives

$$\|Af\|_p \leqslant \frac{1}{2\pi} \int_{\Gamma_{1/p,\delta_1}} \sup_{x>0} |a(x, \zeta)|\, \|T_{\zeta,p} f\|_p d|\zeta| \leqslant C_1 \delta_1^{-1} \|f\|_p . \qquad q.e.d.$$

THEOREM 2. *Let $a(x, z)$ be a function defined for $x > 0$ and for z in some strip $S_{1/p,\delta}$. Suppose that*

1) *$a(x, z)$ is continuously differentiable in $\mathbf{R}^+ \times S_{1/p,\delta}$ and holomorphic in z,*

2) *For some $\varepsilon > 0$ there is a constant C such that for all x and z*

$$(2.7) \qquad |a(x, z)| \leqslant C \left(\frac{x}{1 + x^2}\right)^{\varepsilon} \left(\frac{1}{(1 + |z|)^{2+\varepsilon}}\right),$$

$$(2.8) \qquad \left| x \frac{\partial}{\partial x} a(x, z) \right| \leqslant \frac{C}{(1 + |z|)^{1+\varepsilon}} .$$

Then the operator A defined by (2.6) *is compact on L^p.*

PROOF. By the proof of Theorem 1 and (2.7) it follows that the operators $f \to \chi_{(0,\lambda)}(x) Af(x)$ and $f \to \chi_{(\lambda^{-1},\infty)}(x) Af(x)$ have small L^p norm if λ is small. The map $f \to -x(d/dx) Af(x) = Tf(x)$ is represented by

$$Tf(x) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} \left\{ a(x, z) z - x \frac{\partial}{\partial x} a(x, z) \right\} \tilde{f}(z)\, dz .$$

By (2.8) and Theorem 1, T is bounded on L^p. From these observations it follows that the family $\{Af : \|f\|_p \leqslant 1\}$ is equicontinuous in $L^p(0, N)$ for every N and that

$$\lim_{N\to\infty} \sup_{\|f\|_p \leqslant 1} \int_{|x|>N} |Af(x)|^p dx = 0 .$$

This establishes the compactness of A on L^p. $\qquad$ *q.e.d.*

3. – Spaces of symbols and Mellin operators.

As a preliminary step we introduce some spaces of functions and their Mellin transforms.

DEFINITION 2. *Let τ be real. If $\delta > 0$, by $\mathcal{F}_{\tau,\delta}$ we denote the class of functions $a(x) \in C^\infty(\boldsymbol{R}^+)$ such that the following property holds: for every δ_1, $0 < \delta_1 < \delta$, and every j there is a constant $C = C(\delta_1, j, a)$ such that*

$$\left|\left(x\frac{d}{dx}\right)^j a(x)\right| \leqslant Cx^{-\tau}\left(\frac{x}{1+x^2}\right)^{\delta_1}.$$

By $\mathcal{F}_\tau$ we denote the space of functions $a(x)$ such that $a \in \mathcal{F}_{\tau,\delta}$ for some δ.

DEFINITION 3. *Let τ be real. If $\delta > 0$, by $\tilde{\mathcal{F}}_{\tau,\delta}$ we denote the class of functions $b(z)$ which are defined and holomorphic in the strip $S_{\tau,\delta}$ and such that the following property holds: for every δ_1, $0 < \delta_1 < \delta$, and every j and k there is a constant $C = C(\delta_1, j, k, b)$ such that*

$$\left|z^j \frac{d^k}{dz^k} b(z)\right| \leqslant C$$

for all $z \in S_{\tau,\delta_1}$. By $\tilde{\mathcal{F}}_\tau$ we denote the space of functions $b(z)$ such that $b \in \tilde{\mathcal{F}}_{\tau,\delta}$ for some δ.

The fact that the functions in $\tilde{\mathcal{F}}_\tau$ are precisely the Mellin transforms of the functions in $\mathcal{F}_\tau$ is consequence of the following result whose proof is contained in the article of A. Avantaggiati [A, Sec. 2].

LEMMA 1. *If $a \in \mathcal{F}_{\tau,\delta}$, then its Mellin transform $\tilde{a} \in \tilde{\mathcal{F}}_{\tau,\delta}$. Conversely, given $b \in \tilde{\mathcal{F}}_{\tau,\delta}$, define the function*

$$a(x) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} x^{-z} b(z)\,dz\,, \qquad x > 0\,.$$

Then $a \in \mathcal{F}_{\tau,\delta}$ and $\tilde{a} = b$.

We define the symbols of the class of smoothing operators to be constructed.

DEFINITION 4. *Let τ be real. If $\varepsilon, \delta > 0$, by $\Phi_{\tau,\delta,\varepsilon}$ we denote the class of functions such that:*

1) $a(x, z) \in C^\infty(\boldsymbol{R}^+ \times S_{\tau,\delta})$,

2) *for all* x, $a(x, z)$ *defines a holomorphic function on* $S_{\tau,\delta}$,

3) *for each* ε_1, $0 < \varepsilon_1 < \varepsilon$, *and each* δ_1, $0 < \delta_1 < \delta$, *and each* M, j, k *there is a constant* $C = C(\varepsilon_1, \delta_1, M, j, k, a)$ *such that*

$$\left| z^M \left(x \frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial z}\right)^k a(x, z) \right| \leqslant C \left(\frac{x}{1 + x^2}\right)^{\varepsilon_1}$$

for $z \in S_{\tau,\delta_1}$.

By Φ_τ *we denote the class of functions* $a(x, z)$ *which belong to* $\Phi_{\tau,\varepsilon,\delta}$ *for some* ε, δ.

We recall that the function $\theta(z) = 1/(1 - \exp[2\pi i z])$ is holomorphic in the strip $0 < \operatorname{Re} z < 1$ and that for all M and j, uniformly in the strip $0 < \delta < \operatorname{Re} z < 1 - \delta < 1$,

$$\lim_{\operatorname{Im} z \to +\infty} \left| z^M \frac{d^j}{dz^j} (\theta(z) - 1) \right| = 0 \tag{3.1}$$

and

$$\lim_{\operatorname{Im} z \to -\infty} \left| z^M \frac{d^j}{dz^j} \theta(z) \right| = 0 . \tag{3.2}$$

It follows that $\theta(z)(1 - \theta(z)) \in \tilde{\mathcal{F}}_{1/p}$ for $1 < p < \infty$.

Finally we are ready for the definition of the space of symbols of an algebra of Mellin operators on $L^p(\boldsymbol{R}^+)$.

DEFINITION 5. *Let* $1 < p < \infty$. *Denote by* $\Sigma_{1/p}$ *the space of functions* $a(x, z) \in C^\infty(\boldsymbol{R}^+ \times S_{1/p,\delta})$ *for some* $\delta = \delta(a) > 0$ *and for which there is a representation of the following form in* $\boldsymbol{R}^+ \times S_{1/p,\delta}$:

$$a(x, z) = a_+(x)\theta(z) + a(x)(1 - \theta(z)) + a(z) + \alpha(x, z) \tag{3.3}$$

where

1) $a_+(x)$ *and* $a_-(x)$ *are extendable as continuous functions on* $\overline{\boldsymbol{R}^+}$ *in such a way that* $a_\pm(x) - a_\pm(0) \in \mathcal{F}_0$,

2) $a(z) \in \tilde{\mathcal{F}}_{1/p}$,

3) $\alpha(x, z) \in \Phi_{1/p}$.

DEFINITION 6. *For each symbol* $a \in \Sigma_{1/p}$ *we define the Mellin operator*

$$Af(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} a(x, z) \tilde{f}(z) \, dz , \qquad f \in C_0^\infty(\boldsymbol{R}^+) . \tag{3.10}$$

The space of all such operators will be denoted by $OP\Sigma_{1/p}$. *The function* $a(x, z) \in \Sigma_{1/p}$ *will be called the symbol of the Mellin operator* A. *If the symbol of the operator* A *is also in the class* $\Phi_{1/p}$, *we shall write* $A \in OP\Phi_{1/p}$ *and shall call* A *a smoothing operator.*

DEFINITION 7. *If* A *is a Mellin operator with symbol*

$$a(x, z) = a_+(x)\theta(z) + a_-(x)\big(1 - \theta(z)\big) + a(z) + \alpha(x, z)\,,$$

the function $a(x, z) - \alpha(x, z)$ *will be called the principal symbol of* A *and be denoted by* $\sigma_p(A)(x, z)$.

From Theorem 1 it follows that if $A \in OP\Sigma_{1/p}$, then A can be extended as a bounded operator on L^p; moreover, if $\sigma_p(A)(x, z) \equiv 0$, A is a compact operator on L^p.

4. – The symbolic calculus for $OP\Sigma_{1/p}$.

We study the compositions and adjoints of operators in $OP\Sigma_{1/p}$.

THEOREM 3. *Let* $A, B \in OP\Sigma_{1/p}$. *Then* $AB \in OP\Sigma_{1/p}$. *Moreover, if*

$$\sigma_p(A)(x, z) = a_+(x)\theta(z) + a_-(x)\big(1 - \theta(z)\big) + a(z)$$

and

$$\sigma_p(B)(x, z) = b_+(x)\theta(z) + b_-(x)\big(1 - \theta(z)\big) + b(z)\,,$$

then

$$\sigma_p(AB)(x, z) = c_+(x)\theta(z) + c_-(x)\big(1 - \theta(z)\big) + c(z)\,,$$

where

(4.1) 1) $c_+(x) = a_+(x)b_+(x)$

(4.2) 2) $c_-(x) = a_-(x)b_-(x)$

(4.3) 3) $c(z) = \sigma_p(A)(0, z)\cdot\sigma_p(B)(0, z) - a_+(0)b_+(0)\theta(z) - a_-(0)b_-(0)\big(1 - \theta(z)\big)\,.$

PROOF. We shall first show that the composition of two smoothing operators is a smoothing operator. Let $a(x, z)$, $b(x, z) \in \Phi_{1/p}$ and let A and B

be the corresponding Mellin operators. For $f \in C_0^\infty(\boldsymbol{R}^+)$,

$$Bf(z) = \int_0^\infty x^{z-1} \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-w} b(x, w) \tilde{f}(w)\, dw\, dx$$
$$= \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} \tilde{b}(z - w, w) \tilde{f}(w)\, dw\,,$$

where $\tilde{b}(z, w)$ denotes the Mellin transform of $b(x, w)$ in the x-variable. Thus

$$ABf(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} a(x, z) \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} \tilde{b}(z - w, w) \tilde{f}(w)\, dw\, dz$$
$$= \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-w} c(x, w) \tilde{f}(w)\, dw\,,$$

where

$$c(x, w) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-(z-w)} a(x, z) \tilde{b}(z - w, w)\, dz$$
$$= \frac{1}{2\pi i} \int_{0 - i\infty}^{0 + i\infty} x^{-v} a(x, v + w) \tilde{b}(v, w) dv\,.$$

In the above calculations, the absolute convergence of the integrals justifies the use of Fubini's Theorem.

To verify that $c(x, w) \in \Phi_{1/p}$ we apply repeatedly the observation that if $a(x, z) \in \Phi_{1/p}$, then

$$z^M (\log x)^N \left(x \frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial z}\right)^j a(x, z) \in \Phi_{1/p}\,.$$

We next show that if $A \in OP\Sigma_{1/p}$, $B \in OP\Phi_{1/p}$, then $AB \in OP\Phi_{1/p}$. Let the symbol of A be

$$a(x, z) = a_+(x)\theta(z) + a_-(x)\big(1 - \theta(z)\big) + a(z) \quad \text{and let } b(x, z) \in \Phi_{1/p}\,.$$

In this case we argue as above and again use Fubini's Theorem to obtain that

$$ABf(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-w} c(x, w) \tilde{f}(w)\, dw\,,$$

where

$$c(x, w) = \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} a(x, v+w) \tilde{b}(v, w)\, dv .$$

Consider, e.g., the contribution of

$$\frac{a_+(0)}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} \theta(v+w) \tilde{b}(v, w)\, dv .$$

Modulo $\Phi_{1/p}$, this integral is $c_+(x, w)$ where

$$c_+(x, w) = \frac{a_+(0)}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} [\theta(v+w) - \theta(w)] \tilde{b}(v, w)\, dv , \tag{4.4}$$

Now

$$c_+(x, w) = \frac{a_+(0)}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} \int_0^1 \frac{\partial \theta}{\partial w} (w + tv)\, v \tilde{b}(v, w)\, dt\, dv .$$

To show that $c_+(x, w) \in \Phi_{1/p}$, we observe that for some $\delta > 0$, it is possible to shift the integral $(1/2\pi i) \int_{0-i\infty}^{0+i\infty} \ldots dv$ to either of the integrals

$$\frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \ldots dv \qquad \text{or} \qquad \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \ldots dv .$$

Hence $x^{-\delta} w^N (x(\partial/\partial x))^j (\partial/\partial w)^k c_+(x, w)$ is a linear combination of integrals of the form

$$I(x, w) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta-i\infty} x^{-(v+\delta)} (\log x)^s \int_0^1 t^m (w + tv)^M \theta^{(1+q)} (w + tv) \tilde{b}_r(v, w)\, dt\, dv , \tag{4.5}$$

where $b_r(x, w) \in \Phi_{1/p}$. The integrals $I(x, w)$ are absolutely convergent and for w in some strip around $\operatorname{Re} w = 1/p$ are bounded by $C|\log x|^s$. Repeating the argument with δ replaced by $-\delta$ shows that $c_+(x, w) \in \Phi_{1/p}$. In the same manner one shows that

$$\frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} \{a_-(x)(1 - \theta(v+w)) + a(v+w)\} \tilde{b}(v, w)\, dv$$

gives a function in $\Phi_{1/p}$.

The next step is to show that if $A \in OP\Phi_{1/p}$ and $B \in OP\Sigma_{1/p}$ then $AB \in OP\Phi_{1/p}$. Let $a(x, z)$ be the symbol of A and suppose that $b(x, z) = b_+(x)\theta(z) + b_-(x)(1-\theta(z)) + b(z)$ is the symbol of B. Denote by B_+ the operator

$$B_+ f(x) = \frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} x^{-z} b_+(x)\theta(z)\tilde{f}(z)\,dz .$$

We have that

$$AB_+ f(x) = \frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} x^{-w} c(x, w)\tilde{f}(w)\,dw + C_1 f(x)$$

where

$$c(x, w) = \frac{\theta(w)}{2\pi i}\int_{0-i\infty}^{0+i\infty} x^{-v} a(x, v+w)\big(b_+(x) - b_+(0)\big)^{\sim}(v)\,dv$$

and C_1 is a smoothing operator. Since $\big(b_+(x) - b_+(0)\big)^{\sim}(v) \in \tilde{\mathcal{F}}_0$, it may be shown that $c(x, w) \in \Phi_{1/p}$. The other terms in the composition can be handled similarly.

As the next step we consider two operators A_+, B_+ with symbols $a_+(x)\theta(z)$ and $b_+(x)\theta(z)$. Let Θ be the operator with symbol $\theta(z)$. Then

$$A_+B_+ = a_+(x)b_+(0)\Theta^2 + a_+(x)\big(\Theta b'_+(x)\Theta\big)$$

where

$$b'_+(x) = b_+(x) - b_+(0) \in \mathcal{F}_0 .$$

The second part can be written as

$$Cf(x) = \frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} x^{-w} c(x, w)\tilde{f}(w)\,dw ,$$

where

$$c(x, w) = a_+(x)\theta(w)\frac{1}{2\pi i}\int_{0-i\infty}^{0+i\infty} x^{-v}\theta(v+w)\tilde{b}'_+(v)\,dv .$$

Using the argument for the integral (4.4), we have that modulo a symbol in $\Phi_{1/p}$, $c(x, w) = a_+(x)b'_+(x)\theta^2(w)$.

At this point we observe that $\theta^2 = \theta(\theta - 1) + \theta$ so that the operator θ^2 contains the Hardy kernel operator with symbol $\theta(z)\big(\theta(z) - 1\big) \in \tilde{\mathcal{F}}_{1/p}$. Thus the principal symbol of A_+B_+ is given by

$$\sigma_p(A_+B_+)(x,z)=a_+(x)b_+(x)\theta(z)+a_+(0)b_+(0)\theta(z)\big(\theta(z)-1\big)\,.$$

Since the composition of two Hardy kernel operators with symbols in $\tilde{\mathcal{F}}_{1/p}$ is a Hardy kernel operator with symbol in $\tilde{\mathcal{F}}_{1/p}$, we leave to the reader the proof of the cases not considered explicitly and the calculation of the principal symbol. *q.e.d.*

COROLLARY. *If* $A, B\in OP\Sigma_{1/p}$ *the commutator*

$$[A,B]=AB-BA\in OP\Phi_{1/p}\,.$$

We consider the adjoint of an operator $A\in OP\Sigma_{1/p}$. If $1/p+1/q=1$ we define $A^*\colon L^q\to L^q$ to be the operator such that

$$\int_0^\infty Af(x)\overline{g(x)}\,dx=\int_0^\infty f(x)\overline{A^*g(x)}\,dx\,,\qquad f,g\in C_0^\infty(\boldsymbol{R}^+)\,.$$

THEOREM 4. *Let* $A\in OP\Sigma_{1/p}$ *and* $1/p+1/q=1$. *Then* $A^*\in OP\Sigma_{1/q}$; *moreover, the principal symbol of* A^* *is*

$$\sigma_q(A^*)(x,z)=\overline{\sigma_p(A)(x,1-\bar z)}\,. \tag{4.6}$$

In particular if

$$\sigma_p(A)(x,z)=a_+(x)\theta(z)+a_-(x)\big(1-\theta(z)\big)+a(z)\,, \tag{4.7}$$

then

$$\sigma_q(A^*)(x,z)=\bar a_+(x)\theta(z)+\bar a_-(x)\big(1-\theta(z)\big)+\bar a(1-\bar z)\,, \tag{4.8}$$

$\operatorname{Re} z$ *near* $1/q$.

PROOF. We recall that the Hilbert transform H is representable as $H=i(2\Theta-1)\in OP\Sigma_{1/p}$. Using the kernel representation (1.3) of H, we have that $H^*=-H\in OP\Sigma_{1/q}$. A calculation shows that

$$\sigma_q(H^*)(z)=\overline{\sigma_p(H)(1-\bar z)}\,.$$

Next consider the operator $Af(x)=a(x)f(x)$ where $a(x)-a(0)\in\mathcal{F}_0$. Then $A^*g(x)=\bar a(x)g(x)$. Representing Θ in terms of H and applying Theorem 3 proves the theorem for operators of the form

$$A = a_+(x)\Theta + a_-(x)(1-\Theta).$$

We now consider an operator A which is a Hardy operator with symbol $a(z) \in \tilde{\mathcal{F}}_{1/p}$. There is a kernel $k(x) \in \mathcal{F}_{1/p}$ with $\tilde{k} = a$ such that $Af(x) = \int_0^\infty k(x/y)f(y)(dy/y)$. The adjoint A^* is representable with a kernel $k^*(x) = (1/x)\bar{k}(1/x)$. For $\operatorname{Re} w$ near $1/q$ we have that $\tilde{k}^*(w) = \overline{a(1-\bar{w})} \in \tilde{\mathcal{F}}_{1/q}$.

Finally we show that if $A \in OP\Phi_{1/p}$ then $A^* \in OP\Phi_{1/p}$. If the symbol of A is $a(x, z) \in \Phi_{1/p}$ and $f, g \in C_0^\infty(\mathbf{R}^+)$, represent

$$f(y) = \frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} y^{-z}\tilde{f}(z)\,dz \qquad \text{and} \qquad g(y) = \frac{1}{2\pi i}\int_{1/q-i\infty}^{1/q+i\infty} y^{-w}\tilde{g}(w)\,dw.$$

Using Fubini's Theorem, we obtain that

$$\int_0^\infty Af(y)\overline{g(y)}\,dy = \int_0^\infty f(x)\overline{\frac{1}{2\pi i}\int_{1/q-i\infty}^{1/q+i\infty} x^{-w}c(x, w)\tilde{g}(w)\,dw}\,dx,$$

where

$$\overline{c(x, w)} = \frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} x^{\bar{w}+z-1}\int_0^\infty y^{-(z+\bar{w})}a(y, z)\,dy\,dz.$$

Performing the change of variables $z \to 1 - \bar{w} + v$, $\operatorname{Re} v = 0$, we have that

$$\overline{c(x, w)} = \frac{1}{2\pi i}\int_{0-i\infty}^{0+i\infty} x^{-v}\tilde{a}(v, (1-\bar{w})-v)\,dv.$$

Using arguments similar to those for (4.4) and (4.5), we can show that $c(x, w) \in \Phi_{1/q}$. *q.e.d.*

REMARK. If $A \in OP\Sigma_{1/p}$ the transposed operator ${}^tA$ is defined so that

$$\int_0^\infty Af(x)g(x)\,dx = \int_0^\infty f(x)\,{}^tAg(x)\,dx, \; f, g \in C_0^\infty(\mathbf{R}^+).$$

Then if $1/p + 1/q = 1$, ${}^tA \in OP\Sigma_{1/q}$. If $\sigma_p(A)$ is given by (4.6) then

$$\sigma_q({}^tA)(x, z) = a_+(x)\big(1 - \theta(z)\big) + a_-(x)\theta(z) + a(1 - z),$$

$\operatorname{Re} z$ near $1/q$.

REMARK. We observe that smoothing operators map L^p into $\mathcal{F}_{1/p}$.

LEMMA 2. *Let $A \in OP\Phi_{1/p}$. Then if $f \in L^p$, $Af \in \mathcal{F}_{1/p}$.*

PROOF. Let the symbol of A be $a(x, z) \in \Phi_{1/p}$. Define the function $k(x, t)$ by

$$k(x, t) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} t^{-z} a(x, z)\, dz.$$

Then for some $\delta > 0$ and each i, j there is a $C = C(\delta, i, j, k)$ such that

$$\left| \left(x \frac{\partial}{\partial x}\right)^i \left(t \frac{\partial}{\partial t}\right)^j k(x, t) \right| \leqslant C t^{-1/p} \left(\frac{x}{1 + x^2}\right)^\delta \left(\frac{t}{1 + t^2}\right)^\delta. \tag{4.9}$$

Fix $\xi > 0$ and for $f \in C_0^\infty(\boldsymbol{R}^+)$ let

$$A_\xi f(x) = \int_0^\infty k\left(\xi, \frac{x}{y}\right) f(y) \frac{dy}{y}.$$

The Mellin transform of $A_\xi f$ is $a(\xi, z)\tilde{f}(z) \in \tilde{\mathcal{F}}_{1/p}$. Hence

$$A_\xi f(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} a(\xi, z) \tilde{f}(z)\, dz.$$

Putting $\xi = x$ we then have the representation

$$\begin{aligned} Af(x) &= \int_0^\infty k\left(x, \frac{x}{y}\right) f(y) \frac{dy}{y}. \\ &= \int_0^\infty y^{-1/p} k\left(x, \frac{x}{y}\right) [y^{1/p} f(y)] \frac{dy}{y}. \end{aligned}$$

It follows that $x^{1/p}(x(d/dx))^j Af(x)$ is a linear combination of integrals of the form

$$I(x) = \int_0^\infty \left(\frac{x}{y}\right)^{1/p} k_r\left(x, \frac{x}{y}\right) [y^{1/p} f(y)] \frac{dy}{y},$$

where $k_r(x, t)$ satisfies estimates of the form (4.9). By Hölder's inequality,

$$|I(x)| \leqslant C\left(\frac{x}{1+x^2}\right)^{\delta} \left(\int_0^\infty \left(\frac{t}{1+t^2}\right)^{\delta_q} \frac{dt}{t}\right)^{1/q} \|f\|_p,$$

where $1/p + 1/q = 1$. *q.e.d.*

5. – Elliptic operators in $OP\Sigma_{1/p}$.

We characterize the operators $A \in OP\Sigma_{1/p}$ which are « elliptic », *i.e.*, for which there exists a parametrix $B \in OP\Sigma_{1/p}$ such that $AB - I$ and $BA - I$ are smoothing operators.

THEOREM 5. *Let $A \in OP\Sigma_{1/p}$ with principal symbol*

$$\sigma_p(A)(x, z) = a_+(x)\theta(z) + a_-(x)\big(1 - \theta(z)\big) + a(z).$$

The following two conditions are equivalent:

1) *There is an operator $B \in OP\Sigma_{1/p}$ such that $AB - I \in OP\Phi_{1/p}$.*

2) *The following three conditions are satisfied by $\sigma_p(A)(x, z)$:*

$$\text{(5.1)} \qquad \begin{cases} \inf\limits_{\xi \in R} \left|\sigma_p(A)\left(0, \dfrac{1}{p} + i\xi\right)\right| > 0, \\ \inf\limits_{x>0} |a_+(x)| > 0, \\ \inf\limits_{x>0} |a_-(x)| > 0. \end{cases}$$

PROOF. Suppose that 1. is satisfied and let

$$\sigma_p(B)(x, z) = b_+(x)\theta(z) + b_-(x)\big(1 - \theta(z)\big) + b(z).$$

By Theorem 3

$$\sigma_p(AB)(x, z) = 1 = 1 \cdot \theta(z) + 1\big(1 - \theta(z)\big).$$

By (4.1) and (4.2) and the observation that

$$\sigma_p(AB)(0, z) = \sigma_p(A)(0, z)\cdot\sigma_p(B)(0, z)\,,$$

we have the identities

$$1 = a_+(x)b_+(x)\,,$$
$$1 = a_-(x)b_-(x)\,,$$
$$1 = \sigma_p(A)(0, z)\sigma_p(B)(0, z)\,.$$

Condition 2 follows.

Conversely, suppose that 2. is satisfied. Note that for some $\delta > 0$,

$$\inf_{z\in S_{1/p,\delta}} |\sigma_p(A)(0, z)| > 0\,.$$

We define an operator B with symbol

$$b(x, z) = \frac{1}{a_+(x)}\theta(z) + \frac{1}{a_-(x)}\big(1-\theta(z)\big) + b(z)\,, \tag{5.2}$$

where

$$b(z) = \frac{1}{\sigma_p(A)(0, z)} - \frac{1}{a_+(0)}\theta(z) - \frac{1}{a_-(0)}\big(1-\theta(z)\big)\,, \tag{5.3}$$

$z \in S_{1/p,\delta}$. It may be shown, using the properties of $a_+(x)$, $a_-(x)$ and (3.1), (3.2), that $b(x, z) \in \Sigma_{1/p}$. A direct calculation using (4.1), (4.2), and (4.3) shows that B is a parametrix for A. *q.e.d.*

DEFINITION 8. *If $A \in OP\Sigma_{1/p}$ and A satisfies condition* 1. *or* 2. *of Theorem* 5, *we shall say that A is an elliptic operator in $OP\Sigma_{1/p}$.*

REMARK. We emphasize that the definition of ellipticity in $OP\Sigma_{1/p}$ depends on p. The following situation is typical.

Consider an operator A with principal symbol

$$\sigma_p(A)(x, z) = a(x, z) = a_+(x)\theta(z) + a_-(x)\big(1-\theta(z)\big) + a(z)\,.$$

Suppose that $\inf|a_+(x)| > 0$, $\inf|a_-(x)| > 0$, and that $a(z) \in \tilde{\mathcal{F}}_{1/p}$ for all p, $1 < p < \infty$. Then the function $\psi(z) = \big(a(0, z)\big)^{-1}$ is meromorphic in the strip $0 < \operatorname{Re} z < 1$; moreover, in any strip $0 < \delta < \operatorname{Re} z < 1-\delta < 1$, $\psi(z)$ has only a finite number of poles. Hence for all p outside a discrete set, $N(A)$, A is elliptic in $OP\Sigma_{1/p}$. Define $b(x, z)$ by (5.2) where $b(z)$ is defined by (5.3). Then $b(z)$ has the same poles and residues as $\psi(z)$. Then if $p \notin N(A)$, $b(x, z)$ is the principal symbol of a parametrix for A in $OP\Sigma_{1/p}$. If p_1 and $p_2 \notin N(A)$, let $B_{p_i} \in OP\Sigma_{1/p_i}$ be parametrices for A in $OP\Sigma_{1/p_i}$, $i = 1, 2$. The calculation in [FJL 1] shows that for $f \in C_0^\infty(\boldsymbol{R}^+)$,

$$B_{p_1}f(x) - B_{p_2}f(x) = \int_0^\infty k\left(\frac{x}{y}\right) f(y)\,\frac{dy}{y}\,,$$

where the kernel k is given by

$$k(x) = \frac{1}{2\pi i}\int_{1/p_1 - i\infty}^{1/p_1 + i\infty} x^{-z} b(z)\,dz - \frac{1}{2\pi i}\int_{1/p_2 - i\infty}^{1/p_2 + i\infty} x^{-z} b(z)\,dz\,.$$

If $a(0, \zeta) = 0$ for some ζ, $\operatorname{Re}\zeta$ between $1/p_1$ and $1/p_2$, then $k(x) \not\equiv 0$ (see [FJL 1]).

REMARK. If A is elliptic in $OP\Sigma_{1/p}$ and $Af = 0$, $f \in L^p$, then $f \in \mathcal{F}_{1/p}$. This follows from Lemma 2.

6. – The index of an elliptic operator in $OP\Sigma_{1/p}$.

We will relate the index of an elliptic operator in $OP\Sigma_{1/p}$ to the winding numbers of the coefficients $a_\pm(x)$.

LEMMA 3. *For ν an integer define*

$$\varphi_\nu(x) = \exp\left([2\pi i\nu]\frac{x}{x+1}\right).$$

Then

1) $\varphi_\nu \colon \overline{\boldsymbol{R}^+} \to S^1 = \{|z| = 1\}$ *and the winding number of* φ_ν *is* ν.
2) $\varphi_\nu(x) - 1 \in \mathcal{F}_0$.
3) *If $a(x)$ is a function mapping $\overline{\boldsymbol{R}_+} \to \boldsymbol{C}\setminus\{0\}$ such that $a(x) - a(0) \in \mathcal{F}_0$, and the winding number of a is ν, then there is a continuous homotopy*

   $F\colon [0, 1]\times\overline{\boldsymbol{R}^+} \to \boldsymbol{C}\setminus\{0\}$ *such that*

   (i) $F(0, x) = a(0)\varphi_\nu(x)$,

   (ii) $F(1, x) = a(x)$,

   (iii) $F(t, 0) = a(0)$, $0 \leqslant t \leqslant 1$,

   (iv) $F(t, x) - a(0) \in \mathcal{F}_0$, $0 \leqslant t \leqslant 1$.

PROOF. Parts 1 and 2 follow by a calculation. To prove 3, we remark that if we replace $a(x)$ by $a(x)\varphi_{-\nu}(x)$ it is sufficient to construct the homotopy when $\nu = 0$. In this case it is well known that there is a homotopy $G(t, x)$ which satisfies (i)-(iii) and such that $G(t,\cdot) \in C^\infty(\boldsymbol{R}^+)$ for every t. Let $\varepsilon = \frac{1}{2}|a(0)| > 0$ and choose $\delta > 0$ such that

$$|G(t, x) - a(0)| < \varepsilon \quad \text{for } 0 \leqslant t \leqslant 1,\ 0 < x < 4\delta \text{ or } (4\delta)^{-1} < x < \infty .$$

Construct a nonnegative partition of unity on $\overline{\boldsymbol{R}^+}$ as $1 = \alpha_1(x) + \alpha_2(x) + \alpha_3(x)$ where $\alpha_1(x) = 1$ if $0 \leqslant x \leqslant \delta$, $\alpha_1(x) = 0$ if $x > 2\delta$, $\alpha_3(x) = 1$ for $x > \delta^{-1}$, $\alpha_3(x) = 0$ for $x < \frac{1}{2}\delta^{-1}$ and $\alpha_2(x) = 1 - \alpha_1(x) - \alpha_3(x)$. Define the homotopy F as

$$F(t, x) = \big(\alpha_1(x) + \alpha_3(x)\big)\Big(a(0) + t\big(a(x) - a(0)\big)\Big) + \alpha_2(x)G(t, x) . \tag{6.1}$$

The verification of (i)-(iv) is left to the reader. *q.e.d.*

As an application of the previous lemma we have the following theorem.

THEOREM 6. *Let A be elliptic in $OP\Sigma_{1/p}$ and let*

$$\sigma_p(A)(x, z) = a_+(x)\theta(z) + a_-(x)\big(1 - \theta(z)\big) + a(z) . \tag{6.2}$$

Suppose that ν_+ and ν_- are the winding numbers of a_+ and a_-. Then A as an operator on L^p has index $\nu = \nu_+ - \nu_-$.

The proof of Theorem 6 is accomplished by a sequence of lemmas.

LEMMA 4. *With the notation of Theorem 6, A has the same index as the operator $A^\#$ with symbol*

$$a^\#(x, z) = a_+(0)\varphi_{\nu_+}(x)\theta(z) + a_-(0)\varphi_{\nu_-}(x)\big(1 - \theta(z)\big) + a(z) . \tag{6.3}$$

PROOF. By Lemma 3 there are homotopies $F_\pm(t, x)$ which connect $a_\pm(x)$ to the functions $a_\pm(0)\varphi_{\nu_\pm}(x)$ in such a way that the operators A_t with symbols

$$a_t(x, z) = F_+(t, z)\theta(z) + F_-(t, x)\big(1 - \theta(z)\big) + a(z)$$

are elliptic in $OP\Sigma_{1/p}$. Then $A_0 = A$ and $A_1 = A$ mod $OP\Phi_{1/p}$. Hence index $A^\# =$ index A in L^p. *q.e.d.*

LEMMA 5. *With the notation of Theorem 6 and Lemma 4, the operator A has the same index as the operator A_ν with symbol*

$$a_\nu(x, z) = \varphi_\nu(x)\theta(z) + \big(1 - \theta(z)\big) . \tag{6.4}$$

PROOF. Let A_0 be the operator with symbol $a_0^{\#}(z) \equiv a^{\#}(0, z)$ and B_0 be the operator with symbol $b_0(z) = (a_0^{\#}(z))^{-1}$. Then $B_0 A_0^{\#} = I = A_0^{\#} B_0$ so that $A^{\#}$ has the same index on L^p as the operator $B_0 A^{\#} = I + B_0(A^{\#} - A_0^{\#})$. Since $\sigma_p(A^{\#} - A_0^{\#})(0, z) = 0$, Theorem 3 yields that

$$\sigma_p(B_0 A^{\#}) = \varphi_{\nu_+}(x)\theta(z) + \varphi_{\nu_-}(x)(1 - \theta(z)) .$$

Then index $(A_\nu) =$ index $(\varphi_{-\nu_-} B_0 A) =$ index $(A^{\#})$. *q.e.d.*

PROOF OF THEOREM 6. It remains to calculate the index of A_ν on L^p. Since $\sigma_p(A_\nu)(0, z) \equiv 1$, A_ν is elliptic in $OP\Sigma_{1/r}$ for all r, $1 < r < \infty$. If $f \in L^p$, $A_\nu f = 0$, then $f \in \mathcal{F}_{1/r}$, $1 < r < \infty$. If $g \in L^q$, $1/p + 1/q = 1$, $A_\nu^* g = 0$, then $g \in \mathcal{F}_{1/r}$, $1 < r < \infty$. Thus the index of A_ν on L^p is the index of A_ν on L^2. As an operator on L^2, A_ν is in the algebra considered by Cordes and Herman [CH], and its symbol σA_ν, as defined in [CH], has winding number ν and hence index ν. (The particular operator considered in [CH] was $K_0 = \Theta + [(\log x - 2i)/(\log x + 2i)](I - \Theta))$. *q.e.d.*

7. – Application to an oblique derivative problem in a plane sector.

Operators in $OP\Sigma_{1/p}$ arise naturally in the oblique derivative problem in a plane sector.

Let $\Omega = \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\}$. We seek a solution of the following problem:

$$(7.1) \qquad \begin{cases} \Delta u = 0 \quad \text{in } \Omega \\[2mm] \lim\limits_{y \to 0^+} \left(\alpha_1(x) \dfrac{\partial u}{\partial y}(x, y) + \beta_1(x) \dfrac{\partial u}{\partial x}(x, y)\right) = \varphi_1(x) \in L^p(\mathbf{R}^+) , \\[2mm] \lim\limits_{x \to 0^+} \left(\alpha_2(y) \dfrac{\partial u}{\partial x}(x, y) + \beta_2(y) \dfrac{\partial u}{\partial y}(x, y)\right) = \varphi_2(y) \in L^p(\mathbf{R}^+) , \end{cases}$$

where α_j and β_j are real functions such that $\alpha_j^2(t) + \beta_j^2(t) \equiv 1$.

If $\Phi_1(t)$, $\Phi_2(t) \in C_0^\infty(\mathbf{R}^+)$ we study the single layer potential with density Φ_1 along the x-axis and density Φ_2 along the positive y-axis, namely,

$$\begin{aligned} u(x, y) &= \frac{1}{2\pi} \int_0^\infty \log\left((x - t)^2 + y^2\right) \varphi_1(t)\, dt \\ &\quad + \frac{1}{2\pi} \int_0^\infty \log\left(x^2 + (y - t)^2\right) \varphi_2(t)\, dt \\ &= u_1(x, y) + u_2(x, y) . \end{aligned}$$

Then it is known [St, FJL2] that in $L^p(\boldsymbol{R}^+)$,

$$\lim_{y\to 0^+}\frac{\partial u_1}{\partial y}(x, y) = \varphi_1(x)$$

$$\lim_{y\to 0^+}\frac{\partial u_1}{\partial x}(x, y) = p.v.\frac{1}{\pi}\int_0^\infty \frac{1}{x-t}\varphi_1(t)\,dt = H\varphi_1(x),$$

$$\lim_{x\to 0^+}\frac{\partial u_1}{\partial x}(x, y) = -\frac{1}{\pi}\int_0^\infty \frac{t}{y^2+t^2}\varphi_1(t)\,dt = -K_n\varphi_1(y),$$

$$\lim_{x\to 0^+}\frac{\partial u_1}{\partial y}(x, y) = \frac{1}{\pi}\int_0^\infty \frac{y}{y^2+t^2}\varphi_1(t)\,dt = K_\tau\varphi_1(y).$$

Note that K_n and K_τ are Hardy kernel operators with kernels

$$k_n(t) = \frac{1}{\pi}\frac{1}{1+t^2} \qquad \text{and} \qquad k_\tau(t) = \frac{1}{\pi}\frac{t}{1+t^2}.$$

Since $k_n(t)$ and $k_\tau(t)\in\mathcal{F}_{1/p}$, $1<p<\infty$, we have that K_n and K_τ are operators in $OP\Sigma_{1/p}$ and that

$$\sigma_p(K_\tau)(z) = \frac{-\cos((\pi/2)z)}{\sin(\pi z)} \qquad \text{and} \qquad \sigma_p(K_n)(z) = \frac{\sin((\pi/2)z)}{\sin(\pi z)}.$$

Recall that the symbol of the operator H may be written as

$$\sigma_p(H)(z) = -\frac{\cos(\pi z)}{\sin(\pi z)}.$$

Similar formulas hold for the boundary values of the gradient of u_2.

We make the following assumptions on the coefficients of the boundary operators in (7.1):

$$\alpha_j(t)-\alpha_j(0), \quad \beta_j(t)-\beta_j(0)\in\mathcal{F}_0, \qquad j=1,2.$$

Then the boundary operators applied to u give functions $\psi_1(t)$, $\psi_2(t)\in L^p(\boldsymbol{R}^+)$ where

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\alpha_1(t)I+\beta_1(t)H & -\alpha_1(t)K_n+\beta_1(t)K_\tau\\ -\alpha_2(t)K_n+\beta_2(t)K_\tau & \alpha_2(t)I+\beta_2(t)H\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}. \tag{7.2}$$

We write the system (7.2) as $\vec{\psi} = A\vec{\varphi}$ where A is a matrix of operators in $OP\Sigma_{1/p}$. The matrix of principal symbols is given by

$$\sigma_p(A)(t,z)=$$

$$= \begin{pmatrix} v_1(t)\theta(z) + \overline{v_1(t)}(1-\theta(z)) & \dfrac{\alpha_1(0)\cos\big((\pi/2)z\big)+\beta_1(0)\sin\big((\pi/2)z\big)}{\sin(\pi z)} \\ -\dfrac{\alpha_2(0)\cos\big((\pi/2)z\big)+\beta_2(0)\big(\sin(\pi/2)z\big)}{\sin(\pi z)} & v_2(t)\theta(z) + \overline{v_2(t)}(1-\theta(z)) \end{pmatrix}$$

where $v_j(t) = \alpha_j(t) + i\beta_j(t)$.

THEOREM 7. *For $j = 1, 2$, let ν_j be the winding numbers of $v_j(t)$. Suppose that*

$$\inf_{\operatorname{Re} z=1/p} |\det \sigma_p(A)(0, z)| > 0\,.$$

Then

1) *There is a matrix B of operators in $OP\Sigma_{1/p}$ such that $AB - I$ and $BA - I$ are matrices of smoothing operators.*

2) *As an operator on $L^p\times L^p$, the index of A is $2\nu_1 + 2\nu_2$.*

PROOF. Since $\alpha_j^2 + \beta_j^2 = 1$, the function $\det \sigma_p(A)(t, z)$ is the symbol of an elliptic operator D in $OP\Sigma_{1/p}$. Denote by E a parametrix for D and let B be the matrix of operators

$$B = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \alpha_2(t)I + \beta_2(t)H & \alpha_1(0)K_n - \beta_1(0)K_\tau \\ \alpha_2(0)K_n - \beta_2(0)K_\tau & \alpha_1(t)I + \beta_1(t)H \end{pmatrix}.$$

The symbolic calculus establishes that, modulo a matrix of functions in $\Phi_{1/p}$, $\sigma_p(A)(t, z)\cdot\sigma_p(B)(t, z) = I$ and the first conclusion is established.

Let A_0 be the matrix of operators whose symbol is $\sigma_p(A)(0, z)$ and let B_0 be the matrix of operators whose symbol is $\sigma_p(B)(0, z) = [\sigma_p(A_0)(z)]^{-1}$. Then $B_0A_0 = A_0B_0 = I$ on $L^p\times L^p$ so that the index of A is the index of $B_0A = I + B_0(A - A_0)$. The matrix of principal symbols of B_0A is

$$\sigma_p(B_0A)(t, z) = \tag{7.3}$$

$$= \begin{pmatrix} \dfrac{v_1(t)}{v_1(0)}\theta(z) + \dfrac{\overline{v_1(t)}}{\overline{v_1(0)}}(1-\theta(z)) & 0 \\ 0 & \dfrac{v_2(t)}{v_2(0)}\theta(z) + \dfrac{\overline{v_2(t)}}{\overline{v_2(0)}}(1-\theta(z)) \end{pmatrix}.$$

By Theorem 6, the index of B_0A is $2\nu_1 + 2\nu_2$.

REMARK. For the operator $\boldsymbol{A}_0$ there are always values of p for which $\det \sigma_p(\boldsymbol{A}_0)(z) = 0$ for some z, $\operatorname{Re} z = 1/p$.

Suppose that $\alpha_j(0) + i\beta_j(0) = \cos\gamma_j + i\sin\gamma_j$, $j = 1, 2$. Another representation of $\sigma_p(\boldsymbol{A}_0)(z)$ is

$$\sigma_p(\boldsymbol{A}_0)(z) = \frac{1}{\sin(\pi z)}\begin{pmatrix} \sin(\pi z - \gamma_1) & \sin\left(\frac{\pi}{2} z - \left(\frac{\pi}{2} - \gamma_1\right)\right) \\ \sin\left(\frac{\pi}{2} z - \left(\frac{\pi}{2} - \gamma_2\right)\right) & \sin(\pi z - \gamma_2) \end{pmatrix}.$$

Then $\det \sigma_p(\boldsymbol{A}_0)(z) = 0$ when $z = (2k+1)/3$ or $z = (2/\pi)(\gamma_1 + \gamma_2) + (2k+1)$. In particular $\det \sigma_p(\boldsymbol{A}_0)(\frac{1}{3}) = 0$, and $\boldsymbol{A}_0$ is not a Fredholm operator on $L^3 \times L^3$. This is in accordance with the results of [FJL 2] for double layer potentials for the Dirichlet problem for which the operators were not Fredholm for $p = \frac{3}{2}$.

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