ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

G. D'ESTE

The $\oplus_c$-topology on abelian p-groups

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 7, nº 2 (1980), p. 241-256

<http://www.numdam.org/item?id=ASNSP_1980_4_7_2_241_0>

© Scuola Normale Superiore, Pisa, 1980, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (http://www.sns.it/it/edizioni/riviste/annaliscienze/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

The $\oplus_c$-Topology on Abelian p-Groups (*).

G. D'ESTE

Introduction.

In this paper we investigate the topology of an abelian p-group G which admits as a base of neighborhoods of 0 all the subgroups X of G such that G/X is a direct sum of cyclic groups. We call this topology the $\oplus_c$-*topology* of G. If G with the $\oplus_c$-topology is a complete Hausdorff topological group, then G is said to be $\oplus_c$-*complete*. The Hausdorff completion of G with respect to the $\oplus_c$-topology is called the $\oplus_c$-*completion* of G and is denoted by $\check{G}$.

In section 1 we prove that the $\oplus_c$-completion $\check{G}$ of a p-group G is a $\oplus_c$-complete group; moreover the completion topology of $\check{G}$ and its own $\oplus_c$-topology are the same. The group $\check{G}$ coincides with the completion of G with respect to the inductive topology if and only if G is thick.

In section 2 we study the class of $\oplus_c$-complete groups. This class of separable p-groups is very large, containing the groups which are direct sums of torsion-complete p-groups, as well as the groups which are the torsion part of direct products of direct sums of cyclic p-groups. But the most interesting result in this direction perhaps is that every separable p^σ-projective p-group is $\oplus_c$-complete. There are a lot of these groups: in fact Nunke proved in [12] that, for every ordinal σ, there exists a p^σ-projective p-group which fails to be p^τ-projective for every $\tau < \sigma$. Moreover the class of $\oplus_c$-complete groups has many closure properties typical of both the classes of p^ω-projective and p^ω-injective p-groups.

In section 3 we study the $\oplus_c$-completion with respect to basic subgroups and we prove the inadequacy of the socle in determining the $\oplus_c$-complete groups; finally we give some applications in connection with the class of thick groups.

(*) Lavoro eseguito nell'ambito dei Gruppi di Ricerca Matematica del C.N.R.
Pervenuto alla Redazione il 6 Febbraio 1979 ed in forma definitiva il 18 Giugno 1979.

I would like to express my gratitude to Dr. L. Salce for his many helpful suggestions.

1. – The $\oplus_c$-completion.

All groups considered in the following are abelian groups. Notations and terminology are those of [4]. In particular p is a prime number and the symbol $\oplus_c$ denotes a direct sum of cyclic p-groups. If G is any group and G' is a pure subgroup of G, then we write $G' \underset{*}{\leqslant} G$. A p-group G may be equipped with various topologies. The p-adic topology has the subgroups $p^n G$ with $n \in \mathbb{N}$ as a base of neighborhoods of 0; the inductive topology has the family of large subgroups as a base of neighborhoods of 0. Throughout the paper, for every p-group G, the group $\bar{G}$ stands for the completion of G with respect to the inductive topology. If λ is a limit ordinal, then the generalization of the p-adic topology is the λ-adic topology. This topology, studied by Mines in [11], has the subgroups $p^\sigma G$ with $\sigma < \lambda$ as a base of neighborhoods of 0. In [13] Salce has studied the λ-inductive topology introduced by Charles in [3]; a base of neighborhoods of 0 for this topology consists of all subgroups $G(\boldsymbol{u})$ where $G(\boldsymbol{u}) = \{x \in G : h(p^n x) \geqslant \sigma_n,\ n \in \mathbb{N}\}$ and $\boldsymbol{u} = (\sigma_n)_{n \in \mathbb{N}}$ is an increasing sequence of ordinals $\sigma_n < \lambda$ for all $n \in \mathbb{N}$. In the following, unless otherwise indicated, every p-group G is endowed with the $\oplus_c$-topology. If we are dealing with some other topology, then the group G equipped with its $\oplus_c$-topology is denoted by $(G, \oplus_c)$.

Let G be a p-group and let L be a large subgroup of G. Since $G/L = \oplus_c$ ([4] Proposition 67.4), L is open with respect to the $\oplus_c$-topology of G and so the $\oplus_c$-topology is finer than the inductive topology. The next statement immediately follows from this result and the fact that a p-group G is thick if and only if $G/X = \oplus_c$ implies $L \leqslant X$ for some large subgroup L of G.

PROPOSITION 1.1. *Let G be a p-group. Then G is thick if and only if the $\oplus_c$-topology coincides with the inductive topology and a thick group G is $\oplus_c$-complete if and only if it is torsion-complete.*

Since quasi-complete groups are thick ([4] Theorem 74.1, Corollary 74.6; [1] Theorem 3.2), the quasi-complete and non torsion-complete group constructed by Hill and Megibben in ([7] Theorem 7) is an example of a group which is not $\oplus_c$-complete. Let us note the following facts.

1) A p-group G is discrete in the $\oplus_c$-topology if and only if $G = \oplus_c$ and G is Hausdorff if and only if $p^\omega G = 0$.

2) Every homomorphism $f: G \to H$ with G and H p-groups is continuous with respect to the $\oplus_c$-topologies. In fact if $H/X = \oplus_c$, the same holds for $G/f^{1-}(X)$.

3) For every p-group G the $\oplus_c$-topology of $G/p^\omega G$ coincides with the quotient topology of the $\oplus_c$-topology of G. By property 2, it is enough to observe that the natural homomorphism $G \to G/p^\omega G$ is open.

Therefore in the study of the $\oplus_c$-completion it is not restrictive to confine ourselves to separable non thick groups. In order to show that the $\oplus_c$-completion of a p-group is $\oplus_c$-complete, we need two lemmas.

LEMMA 1.2. *Let G be a p-group. Then the $\oplus_c$-completion $\check{G}$ of G is a p-group.*

PROOF. By definition $\check{G} = \varprojlim G/X$ where X ranges over the subgroups X of G such that $G/X = \oplus_c$. Let $\hat{G}$ denote the p-adic completion of G. Since $\hat{G} = \varprojlim G/p^n G$ where $n \in \mathbb{N}$, there is a canonical homomorphism $\varphi: \check{G} \to \hat{G}$ such that $\varphi((g_X + X)_X) = (g_{p^n} + p^n G)_n$ for all $(g_X + X)_X \in \check{G}$. Since the completion of G in the inductive topology is the group $\bar{G} = \varprojlim G/L$ with L running over the large subgroups of G, there exists a natural homomorphism $\psi: \check{G} \to \bar{G}$ that takes $(g_X + X)_X$ to $(g_L + L)_L$ for all $(g_X + X)_X \in \check{G}$. To show that $\check{G}$ is a p-group, it suffices to check that ψ is an embedding, and this clearly holds if φ is injective. We shall now prove that if $(g_X + X)_X \in \operatorname{Ker} \varphi$, then $g_X \in X$ for all X. To see this, fix X. Let $m \in \mathbb{N}$; if $Y = X \cap p^m G$, then $G/Y = \oplus_c$. By hypothesis $g_{p^m G} \in p^m G$ and, by the choice of Y, $g_Y + p^m G = g_{p^m G} + p^m G$; consequently $g_Y \in p^m G$. On the other hand $g_X + X = g_Y + X$ and so the height of $g_X + X$ in G/X is at least m. Since m is any natural number and $G/X = \oplus_c$, we conclude that $g_X \in X$, as claimed. This completes the proof that $\check{G}$ is a p-group. $\square$

From now on we shall identify $\check{G}$ with the subgroup $\psi(\check{G})$ of $\bar{G}$ and, if G is separable, then we shall view G as a subgroup of $\check{G}$.

LEMMA 1.3. *Direct summands of $\oplus_c$-complete groups are $\oplus_c$-complete.*

PROOF. Let G' be a direct summand of a $\oplus_c$-complete group G. Since the inclusion $G' \to G$ is continuous, every Cauchy net in G' is a Cauchy net in G. Therefore the hypothesis that G is $\oplus_c$-complete and the continuity of the projection of G onto G' assure that G' is $\oplus_c$-complete. $\square$

We are now ready to establish the main result of this section.

THEOREM 1.4. *Let G be a p-group. Then the $\oplus_c$-completion γ of G is $\oplus_c$-complete.*

PROOF. Without loss of generality we may assume that G is separable. For every ordinal λ we define a group G_λ as follows: if $\lambda = 0$, then $G_\lambda = G$; if $\lambda > 0$ and λ is not a limit ordinal, then G_λ is the $\oplus_c$-completion of $G_{\lambda-1}$; if λ is a limit ordinal, then $G_\lambda \bigcup_{\sigma<\lambda} G_\sigma$. To prove the theorem, we shall use three facts:

(i) The $\oplus_c$-topology of $\check{G}$ is finer than the completion topology.

Let $\mathfrak{B}$ be the family of all subgroups X of G such that $G/X = \oplus_c$. Then $\check{G} = \varprojlim_{X\in\mathfrak{B}} G/X$ and $\check{G}$ with the completion topology is a topological subgroup of the group $\prod_{X\in\mathfrak{B}} G/X$ equipped with the product topology of the discrete topologies on every G/X. Thus a base of neighborhoods of 0 for the completion topology of $\check{G}$ consists of all subgroups $U_F = \check{G} \cap \prod_{X\in\mathfrak{B}\setminus F} G/X$ where F is a finite subset of $\mathfrak{B}$. Since

$$\check{G}/U_F \cong \check{G} + \prod_{X\in\mathfrak{B}\setminus F} G/X \Big/ \prod_{X\in\mathfrak{B}\setminus F} G/X \leqslant \prod_{X\in\mathfrak{B}} G/X \Big/ \prod_{X\in\mathfrak{B}\setminus F} G/X = \oplus_c,$$

every U_F is a neighborhood of 0 for the $\oplus_c$-topology of $\check{G}$, and so (i) is proved.

(ii) G_λ is a subgroup of $\bar{G}$ for all λ.

We shall prove by transfinite induction that $G_\lambda \underset{*}{\leqslant} \bar{G}$ for all λ. If $\lambda = 0$ the assertion is obvious. Let $\lambda > 0$ and assume $G_\sigma \underset{*}{\leqslant} \bar{G}$ for every $\sigma < \lambda$. If λ is a limit ordinal, then evidently $G_\lambda \underset{*}{\leqslant} \bar{G}$. If λ is not a limit ordinal and $\lambda = \sigma + 1$, then the hypothesis that $G \leqslant G_\sigma \underset{*}{\leqslant} \bar{G}$ implies that $G_\sigma \leqslant G_\lambda \leqslant \bar{G}_\sigma = \bar{G}$. Since $G_\sigma \underset{*}{\leqslant} G_\lambda$, we get $\bar{G}_\sigma = \bar{G} \leqslant \bar{G}_\lambda$ and therefore $G_\lambda \underset{*}{\leqslant} \bar{G}$, as required.

(iii) G_1 is a direct summand of G_λ for all $\lambda \geqslant 1$.

Assume by transfinite induction that G_1 is a summand of G_σ for all $1 \leqslant \sigma < \lambda$. Write $G_\sigma = G_1 \oplus G'_\sigma$ for all $1 \leqslant \sigma < \lambda$. If λ is a limit ordinal, G_1 is a direct summand of G_λ, because $G_\lambda = \bigcup_{\sigma<\lambda} G_\sigma = G_1 \oplus \Big(\bigcup_{1\leqslant\sigma<\lambda} G'_\sigma\Big)$. If λ is not a limit ordinal and $\lambda = \sigma + 1$ then, by the induction hypothesis, $G_\sigma = G_1 \oplus G'_\sigma$. Let $\pi\colon (G_\sigma, \oplus_c) \to (G_1, \mathcal{T})$ be the canonical projection where $(G_1, \mathcal{T})$ is the $\oplus_c$-completion of G. To check that π is continuous, let U be an open subgroup of $(G_1, \mathcal{T})$. Then, by property (i), there is some $W \leqslant U$ such that $G_1/W = \oplus_c$. Since $G_\sigma/\pi^{-1}(W) = G_1 \oplus G'_\sigma/W \oplus G'_\sigma \cong G_1/W = \oplus_c$, we see that π is continuous. This result guarantees the existence of a homomorphism $\bar{\pi}$ making the following diagram commute

$$\begin{tikzcd}
(G_\sigma, \oplus_c) \arrow[r, "\pi"] \arrow[d] & (G_1, \mathcal{T}) \arrow[d] \\
(G_\lambda, \mathcal{T}) \arrow[r, "\bar{\pi}"] & (G_1, \mathcal{T})
\end{tikzcd}$$

where the vertical maps are the natural ones and $(G_\lambda, \mathcal{T})$ is the $\oplus_c$-completion of $(G_\sigma, \oplus_c)$. Consequently $G_\lambda = G_1 \oplus \operatorname{Ker} \bar{\pi}$ and so G_1 is a direct summand of G_λ, as claimed.

We can now show that $\check{G} = G_1$ is $\oplus_c$-complete. Suppose this were not true. Then, from Lemma 1.3 and property (iii), we deduce that G_λ is not $\oplus_c$-complete for any λ, and therefore the groups G_λ are all distinct. But this is clearly impossible, because, by property (ii), they are all subgroups of $\bar{G}$. This contradiction establishes that $\check{G}$ is $\oplus_c$-complete and the theorem is proved. $\square$

The next proposition describes the topological structure of the $\oplus_c$-completions.

PROPOSITION 1.5. *For every p-group G the $\oplus_c$-topology of $\check{G}$ coincides with the completion topology.*

PROOF. It is not restrictive to assume $p^\omega G = 0$. As before $\mathcal{T}$ denotes the completion topology of $\check{G}$. By property (i) of Theorem 1.4 we know that the $\oplus_c$-topology of $\check{G}$ is finer than $\mathcal{T}$. On the other hand, by a well known result of general topology ([2] Chapter III § 3, No. 4 Proposition 7), a base of neighborhoods of 0 for the completion topology $\mathcal{T}$ is formed by the closures in $\check{G}$ with respect to $\mathcal{T}$ of the neighborhoods of 0 for the $\oplus_c$-topology of G. Therefore, to end the proof, it is enough to show that if U is an open subgroup of $(\check{G}, \oplus_c)$ and $U' = U \cap G$, then the closure V of U' in $(\check{G}, \mathcal{T})$ is a subgroup of U. To prove this, let $\{g_i\}$ be a Cauchy net in $(G, \oplus_c)$ with $g_i \in U'$ for all i. Since the natural embedding $G \to \check{G}$ is continuous with respect to the $\oplus_c$-topologies, $\{g_i\}$ is a Cauchy net in $(\check{G}, \oplus_c)$. Thus, by Theorem 1.4, it converges to some x in $(\check{G}, \oplus_c)$ and clearly $x \in U$, because U is closed in $(\check{G}, \oplus_c)$ and $g_i \in U$ for all i. Since $\mathcal{T}$ is smaller than the $\oplus_c$-topology of $\check{G}$, the given net converges to x in $(\check{G}, \mathcal{T})$; so $x \in V$, by the definition of V. This means that $V \leqslant U$ and therefore the $\oplus_c$-topology of $\check{G}$ coincides with the completion topology, as claimed. $\square$

COROLLARY 1.6. *Let G be a separable p-group. Then G is a pure topological subgroup with divisible cokernel of a $\oplus_c$-complete group.*

PROOF. By Theorem 1.4 and Proposition 1.5, G is a pure dense topological subgroup of the $\oplus_c$-complete group $\check{G}$. Consequently G is a dense subgroup of $\check{G}$ equipped with the p-adic topology. Hence $\check{G}/G$ is divisible and the proof is complete. $\square$

Before comparing the $\oplus_c$-completion and the completion with respect to the inductive topology, we prove the following lemma.

LEMMA 1.7. *Let G be a separable p-group and let $G \leqslant X \leqslant \bar{G}$. Then $\check{G} \leqslant \check{X}$ and $\check{X} \leqslant \bar{G}$.*

PROOF. Since $G \underset{*}{\leqslant} X$, we may assume $\bar{G} \leqslant \bar{X}$. To show that $\check{G} \leqslant \check{X}$, select $\bar{g} \in \check{G}$. Then, by Proposition 1.5, there exists a net $\{g_i\}$ with $g_i \in G$ for all i which converges to $\bar{g}$ in $(\check{G}, \oplus_c)$. Since $\{g_i\}$ is also a Cauchy net in $(X, \oplus_c)$ and all the canonical maps $\check{G} \to \bar{G}$, $\bar{G} \to \bar{X}$, $\check{X} \to \bar{X}$ are continuous with respect to the $\oplus_c$-topologies, $\bar{g}$ is the limit of $\{g_i\}$ in $(\check{X}, \oplus_c)$ and so $\bar{g} \in \check{X}$. This proves the inclusion $\check{G} \leqslant \check{X}$. To see that $\check{X} \leqslant \bar{G}$, take $\bar{x} \in \check{X}$. As before, there is a net $\{x_i\}$ with $x_i \in X$ for all i which converges to $\bar{x}$ in $(\check{X}, \oplus_c)$. Since $\{x_i\}$ is a Cauchy net in $(\bar{G}, \oplus_c)$ and all the natural embeddings $\check{X} \to \bar{X}$, $\bar{G} \to \bar{X}$ are continuous with respect to the $\oplus_c$-topologies, $\bar{x}$ is the limit of $\{x_i\}$ in $(\bar{G}, \oplus_c)$ and so $\bar{x} \in \bar{G}$. Consequently $\check{X} \leqslant \bar{G}$ and the lemma is proved. □

PROPOSITION 1.8. *Let G be a separable p-group. The following facts hold:*

(i) *If G is not thick, then the group $\bar{G}/\check{G}$ has uncountable rank.*

(ii) *If G is not $\oplus_c$-complete, then the group $\check{G}/G$ may have finite rank.*

PROOF (i). We first show that $\check{G} \neq \bar{G}$. Since $\bar{G}$ is thick, it has the same inductive and $\oplus_c$-topologies. Moreover, by ([13] Theorem 2.3), the inductive topology of $\bar{G}$ induces on G its own inductive topology. On the other hand, by Proposition 1.5, the $\oplus_c$-topology of $\check{G}$ induces on G its own $\oplus_c$-topology. Therefore, if G is not thick, then $\check{G}$ must be a proper subgroup of $\bar{G}$. We now prove that $\bar{G}/\check{G}$ is uncountable. Suppose this were not true. Since $\check{G}$ is a pure subgroup of $\bar{G}$ with countable divisible cokernel, we deduce from ([10] Theorem 3.5) that $\check{G}$ is thick, and this is impossible. In fact $\check{G}$ is $\oplus_c$-complete, but it is not torsion-complete. This contradiction shows that $\bar{G}/\check{G}$ is uncountable.

(ii) Assume the rank of $\check{G}/G$ is not finite. Choose a pure subgroup H of $\check{G}$ such that $G \leqslant H$ and $\check{G}/H \cong \mathbb{Z}(p^\infty)$. Then Lemma 1.7 tells us that $\check{H} = \check{G}$. Since the rank of $\check{H}/H$ is 1, the proof is complete. □

2. – $\oplus_c$-complete groups.

In this paragraph we study the $\oplus_c$-complete groups. As the results of section 1 suggest, the class of $\oplus_c$-complete groups is very large.

First we prove a statement that we shall often use.

PROPOSITION 2.1. *Direct sums of $\oplus_c$-complete groups are $\oplus_c$-complete.*

PROOF. Let $G = \bigoplus G_i$ where G_i is $\oplus_c$-complete for all i. To show that G is $\oplus_c$-complete, we notice the following properties:

(i) The groups $X = \bigoplus_{i\in I} X_i$ where $X_i \leqslant G_i$ and $G_i/X_i = \oplus_c$ for every i are a base of neighborhoods of 0 for the $\oplus_c$-topology of G.

This assertion is obvious.

(ii) G is a closed topological subgroup of the group $\prod_{i\in I} G_i$ equipped with the box topology of the $\oplus_c$-topology on each component.

We recall that the box topology considered on $\prod_{i\in I} G_i$ admits the subgroups of the form $\prod_{i\in I} X_i$ with $X_i \leqslant G_i$ and $G_i/X_i = \oplus_c$ for all i as a base of neighborhoods of 0. Thus the conclusion that G is a topological subgroup of $\prod_{i\in I} G_i$ follows from (i). To complete the proof, let $\bar{g} = (g_i)_{i\in I}$ with $g_i \in G_i$ for every i be an element of the closure of G in $\prod_{i\in I} G_i$. Let S be the support of $\bar{g}$, that is let $S = \{i \in I : g_i \neq 0\}$. Then for each $i \in S$ we can choose a subgroup X_i of G_i such that $g_i \notin X_i$ and $G_i/X_i = \oplus_c$. Our assumption on $\bar{g}$ assures that $\bar{g} \in G + \left(\prod_{i\in S} X_i + \prod_{i\in I\setminus S} G_i\right)$; consequently S is finite and so $\bar{g} \in G$. This proves that G is a closed subgroup of $\prod_{i\in I} G_i$, as required.

The hypothesis that every G_i is $\oplus_c$-complete implies that $\prod_{i\in I} G_i$ with the box topology is complete ([4] Proposition 13.3). Hence, by property (ii), G is $\oplus_c$-complete. $\square$

COROLLARY 2.2. *Direct sums of torsion-complete p-groups are $\oplus_c$-complete.*

PROOF. Since torsion-complete p-groups are $\oplus_c$-complete, the corollary follows from Proposition 2.1. $\square$

We shall obtain another large class of $\oplus_c$-complete groups by means of the next lemmas.

LEMMA 2.3. *Let G be a separable p-group and let G' be a subgroup of G with bounded cokernel. Then G is $\oplus_c$-complete if and only if G' is $\oplus_c$-complete.*

PROOF. We first show that G' is a topological subgroup of G. Let X be a subgroup of G' such that $G'/X = \oplus_c$. Since $(G/X)/(G'/X) \simeq G/G'$ is bounded and $G'/X = \oplus_c$, we have $G/X = \oplus_c$. This proves that the restriction to G' of the $\oplus_c$-topology of G is finer than the $\oplus_c$-topology of G'. Therefore the two topologies coincide, because the natural injection $G' \to G$

is continuous. Assume now that G is $\oplus_c$-complete. Since G' is a closed topological subgroup of G, we conclude that G' is $\oplus_c$-complete. Conversely, suppose G' is $\oplus_c$-complete. Since G' is an open complete topological subgroup of G, evidently G is $\oplus_c$-complete and the proof is finished. □

LEMMA 2.4. *Let G be a separable p-group and let P be a bounded subgroup of G with separable cokernel. Then G is $\oplus_c$-complete if and only if G/P is $\oplus_c$-complete.*

PROOF. Assume first G/P is $\oplus_c$-complete and choose $n \in \mathbb{N}$ such that $p^n P = 0$. Let us verify that $\check{G} \leqslant G + \bar{G}[p^n]$. Take $\bar{g} \in \check{G}$; then there is a net $\{g_i\}$ in G which converges to $\bar{g}$ in $\check{G}$. The hypothesis that G/P is $\oplus_c$-complete guarantees that $\{g_i + P\}$ has a limit $g + P \in G/P$. Since the canonical homomorphisms $G/P \to p^n G$ and $p^n G \to G$ are continuous, $\{p^n g_i\}$ converges to $p^n g$ in G and obviously $p^n \bar{g} = p^n g$. Thus $\bar{g} \in G + \bar{G}\,[p^n]$ and therefore $\check{G} \leqslant G + \bar{G}[p^n]$. By Theorem 1.4 and Lemma 2.3, this implies that G is $\oplus_c$-complete. Conversely, suppose G is $\oplus_c$-complete; then Lemma 2.3 says that $p^n G$ is $\oplus_c$-complete. Since $(G/P)/(G[p^n]/P) \cong p^n G$, the first part of the proof assures that G/P is $\oplus$-complete and the lemma follows. □

Observe that the class of $\oplus_c$-complete groups is a full p^ω-class in the sense of [6]. Indeed, by Proposition 2.1 and Lemma 2.3, the class of $\oplus_c$-complete groups is a p^ω-class. Moreover, if G is separable and G/P is $\oplus_c$-complete for some $P \leqslant G[p]$, then, by Lemma 2.4, G is $\oplus_c$-complete.

We can now prove the following

THEOREM 2.5. *Let σ be any ordinal. If G is a p^σ-projective separable p-group, then G is $\oplus_c$-complete.*

PROOF. The proof is by induction on σ. If $\sigma \leqslant \omega$ the assertion is obvious, because $G = \oplus_c$. Let $\sigma > \omega$ and assume the assertion is true for all $\lambda < \sigma$. By ([4] § 82 Ex. 13), G is a summand of the group $\operatorname{Tor}(H_\sigma, G)$, where H_σ is the generalized Prüfer group of length σ. To see that G is $\oplus_c$-complete, we first suppose σ is a limit ordinal. Then, by ([4] § 82 Ex. 2 and 8; Lemma 64.1) and by the induction hypothesis, G is a summand of a direct sum of $\oplus_c$-complete groups. Hence the conclusion that G is $\oplus_c$-complete follows from Lemma 1.3 and Proposition 2.1. Assume now σ is not a limit ordinal. From the exact sequence

$$0 \to p^{\sigma-1} H_\sigma \cong \mathbb{Z}(p) \to H_\sigma \to H_\sigma / p^{\sigma-1} H \cong H_{\sigma-1} \to 0\,,$$

one obtains the long exact sequence

$$0 \to \operatorname{Tor}(\mathbb{Z}(p), G) \cong G[p] \to \operatorname{Tor}(H_\sigma, G) \xrightarrow{\varphi} \operatorname{Tor}(H_{\sigma-1}, G) \xrightarrow{\psi}$$
$$\to \mathbb{Z}(p) \otimes G \cong G/pG \to H_\sigma \otimes G \to H_{\sigma-1} \otimes G \to 0 .$$

Thus the following sequences are exact:

$$(1) \qquad 0 \to G[p] \to \operatorname{Tor}(H_\sigma, G) \to \operatorname{Im}\varphi \to 0 ,$$

$$(2) \qquad 0 \to \operatorname{Im}\varphi \to \operatorname{Tor}(H_{\sigma-1}, G) \to \operatorname{Im}\psi \to 0 .$$

Evidently in (2) the group $\operatorname{Tor}(H_{\sigma-1}, G)$ is $\oplus_c$-complete, by the induction hypothesis, and $\operatorname{Im}\psi$ is bounded; therefore, by Lemma 2.3, $\operatorname{Im}\varphi$ is $\oplus_c$-complete. From Lemma 2.4 and the exactness of (1), we deduce that $\operatorname{Tor}(H_\sigma, G)$ is $\oplus_c$-complete and, by Lemma 1.3, the same applies to its summand G. $\square$

Proposition 2.1 indicates that the class of $\oplus_c$-complete groups has a closure property analogous to a closure property of the class of direct sums of cyclis groups. This projective property can be regarded as dual of the following injective property, which is similar to a closure property of the class of torsion-complete groups ([4] Corollary 68.6).

PROPOSITION 2.6. *The torsion part of a direct product of $\oplus_c$-complete groups is $\oplus_c$-complete.*

PROOF. Let $G = t\left(\prod_{i\in I} G_i\right)$ where G_i is $\oplus_c$-complete for all i. Since $G_i \underset{*}{\leqslant} \bar{G}_i$ for every i, it is easy to check that G is a pure subgroup of the torsion-complete group $T = t\left(\prod_{i\in I} \bar{G}_i\right)$. Therefore, by the first part of Lemma 1.7, we may assume $\check{G} \leqslant T$. Let now $t = (t_i)_{i\in I} \in \check{G}$ with $t_i \in \bar{G}_i$ for all i. Then t is the limit of a net $\{g_j\}_{j\in J}$ where $g_j = (g_{ij})_{i\in I} \in G$ and $g_{ij} \in G_i$ for all $i \in I$, $j \in J$. Fix $i \in I$; to end the proof, it is enough to show that $t_i \in G_i$. Since $\check{G} \leqslant T$ and the canonical projection $T \to \bar{G}_i$ is continuous, $\{g_{ij}\}_{j\in J}$ converges to t_i in $\bar{G}_i$. From the hypothesis that G_i is $\oplus_c$-complete and $g_{ij} \in G_i$ for all j, we get $t_i \in G_i$. This completes the proof. $\square$

As the next corollary shows, Proposition 2.6 gives some information about $\oplus_c$-complete groups which is not contained in Corollary 2.2 and Theorem 2.5.

COROLLARY 2.7. *There is a $\oplus_c$-complete group which is not a direct sum of torsion-complete p-groups and p^σ-projective separable p-groups.*

PROOF. Let $G = t\left(\prod_{n\in\mathbb{N}} G_n\right)$ where $G_n = \bigoplus_{k>n} \mathbb{Z}(p^k)$ for all n. By Proposition 2.6, G is $\oplus_c$-complete. We observe now the following facts:

(i) G is not a direct sum of torsion-complete p-groups.

This can be easily proved.

(ii) A proper p^σ-projective separable p-group G' with $\sigma > \omega$ cannot be a direct summand of G.

Assume the contrary. Then $G' = t\left(\prod_{n\in\mathbb{N}} C_n\right)$ where $C_n = \oplus_c$ for all n ([9] Theorem 3). Since $\sigma > \omega$, there is no $k \in \mathbb{N}$ such that $p^k C_n = 0$ for almost all n. Hence, by ([8] Proposition 1.6), G' has an unbounded torsion-complete group T as a summand, but this is impossible. Indeed, by ([12] Proposition 6.7), a p^σ-projective p-group cannot contain an unbounded torsion-complete group. This contradiction shows that (ii) holds.

The corollary is now obvious. □

Let us note that the group G defined in the proof of Corollary 2.7 is pure-complete ([9] Theorem 2). Another application of Proposition 2.6 enables us to characterize all the $\oplus_c$-complete groups.

THEOREM 2.8. *Let G be a p-group. The following statements are equivalent*:

(i) *G is $\oplus_c$-complete.*

(ii) *G is a closed topological subgroup of the torsion part of a direct product of a direct sums of cyclic p-groups.*

PROOF (i) $\Rightarrow$ (ii). By hypothesis $G = \varprojlim_{X\in\mathcal{B}} G/X$ where $\mathcal{B}$ is a base of neighborhoods of 0 for G and $G/X = \oplus_c$ for all $X \in \mathcal{B}$. Let $\mathbf{\Pi} = \prod_{X\in\mathcal{B}} G/X$ and let $T = t(\mathbf{\Pi})$. If G and T are equipped with the $\oplus_c$-topology and $\mathbf{\Pi}$ is regarded as the topological product of the discrete groups G/X, then all the natural inclusions in the commutative diagram

$$\begin{tikzcd} & G \arrow[dl, "i"'] \arrow[dr, "j"] & \\ T \arrow[rr] & & \mathbf{\Pi} \end{tikzcd}$$

are continuous. Evidently the groups of the form $j^{-1}(U)$ where U ranges over the open subgroups of $\mathbf{\Pi}$ are a base of neighborhoods of 0 for G. Thus the same holds for the groups $i^{-1}(V)$ with V running over the open subgroups

of T. Hence G is a topological subgroup of T. Since G is $\oplus_c$-complete, G must be closed in T and (ii) is proved.

(ii) $\Rightarrow$ (i). This immediately follows from Proposition 2.6. $\square$

It is now clear that the class of $\oplus_c$-complete groups is the smallest class of separable p-groups $\mathcal{C}$ with the following properties:

(1) $0 \in \mathcal{C}$ and a group isomorphic to a member of $\mathcal{C}$ belongs to $\mathcal{C}$.

(2) If $S \leqslant G[p]$ and $G/S \in \mathcal{C}$, then $G \in \mathcal{C}$.

(3) $\mathcal{C}$ is closed under direct sums and the torsion part of a direct product of groups of $\mathcal{C}$ belongs to $\mathcal{C}$.

(4) $\mathcal{C}$ contains every group that, endowed with its $\oplus_c$-topology, is a closed topological subgroup of a group determined by the above conditions.

3. – Some applications.

In this last section we discuss some consequences of the preceding results.

The next proposition investigates the connection between $\oplus_c$-complete groups and basic subgroups.

PROPOSITION 3.1. *The following facts hold:*

(i) *If two separable p-groups have isomorphic $\oplus_c$-completions, then they have isomorphic basic subgroups.*

(ii) *There exist $2^{\aleph_0}$ pairwise nonisomorphic $\oplus_c$-complete groups with isomorphic basic subgroups.*

PROOF (i). Let G and H be separable p-groups such that $\check{G} \cong \check{H}$. Since $\bar{G}$ is isomorphic to $\bar{H}$, we conclude that G and H have isomorphic basic subgroups.

(ii) Let $B = \bigoplus_{n \geqslant 1} \mathbb{Z}(p^n)$. We want to prove that there exist $2^{\aleph_0}$ pairwise nonisomorphic $\oplus_c$-complete subgroups of $\bar{B}$ whose basic subgroup is B. To see this, let I be a set of cardinality $2^{\aleph_0}$ and let $\{X_i\}_{i \in I}$ be a family of subsets of positive integers such that if $i \neq j$ then $(X_i \setminus X_j) \cup (X_j \setminus X_i)$ is not finite. Let $G_i = t\Big(\prod_{n \in X_i} \mathbb{Z}(p^n)\Big) \oplus \Big(\bigoplus_{\substack{n \geqslant 1 \\ n \notin X_i}} \mathbb{Z}(p^n)\Big)$ for all i; then every G_i is a $\oplus_c$-complete group admitting B as a basic subgroup. To complete the proof, it is enough to show that if $|X_i \setminus X_j| = \aleph_0$, then G_i is not isomorphic

to G_j. Suppose this were not true. Then, by ([4] Theorem 73.6; Lemma 71.1), there exist an isometry $\varphi\colon G_i[p] \to G_j[p]$, a finite subset $F \subseteq \mathbb{N} \setminus X_j$ and some $k \in \mathbb{N}$ such that $\varphi\Big(p^k\Big(\prod_{n\in X_i} \mathbb{Z}(p^n)\Big)[p]\Big) \leqslant t\Big(\prod_{n\in X_j} \mathbb{Z}(p^n)\Big) \oplus \Big(\bigoplus_{n\in F} \mathbb{Z}(p^n)\Big)$. Consequently there is a finite subset $F' \subseteq X_i$ such that $X_i \setminus F' \subseteq F \cup X_j$, while, by hypothesis, $X_i \setminus X_j$ is not finite. This contradiction proves that G_i is not isomorphic to G_j, as claimed. $\square$

The following statement shows that socles, viewed as valued vector spaces, do not give much information in the study of $\oplus_c$-complete groups.

PROPOSITION 3.2. *The following facts are true*:

(i) *There exists a $\oplus_c$-complete group whose socle is isometric to the socle of a non $\oplus_c$-complete group.*

(ii) *There exist nonisomorphic $\oplus_c$-complete groups with isometric socles.*

PROOF (i). Let G be a separable p-group which is neither $\oplus_c$-complete nor thick (for instance, let G be an infinite direct sum of quasi-complete non torsion-complete p-groups) and let $S = \check{G}[p]$. Singe $\check{G} \underset{*}{\leqslant} \bar{G}$, we can choose $x \in \bar{G}[p] \setminus \check{G}$. Let y be an element of order p^2 of $\check{G}$ and let $z = x + y$. Take a subgroup A of $\bar{G}$ such that $\langle G, z\rangle \leqslant A$ and $A/G \simeq \mathbb{Z}(p^\infty)$. Since $A \underset{*}{\leqslant} \bar{G}$ and $A[p] \leqslant S$, there exists a pure subgroup H of $\bar{G}$ such that $A \leqslant H$ and $H[p] = S$. We want to prove that H is not $\oplus_c$-complete. Assume the contrary. Since $G \leqslant H \leqslant \bar{G}$ and, by hypothesis, H is $\oplus_c$-complete, Lemma 1.7 implies that $\check{G}$ is a pure subgroup of H. Using this fact and the equality $H[p] = S = \check{G}[p]$, one obtains $\check{G} = H$. This is a contradiction, because $x \notin \check{G}$, $y \in H$ and $z = x + y \in H$. Hence H is not $\oplus_c$-complete and (i) is proved.

(ii) A result of ([5] Corollary 1) guarantees the existence of two nonisomorphic groups G_1 and G_2 with isometric socles such that G_1 is a direct sum of torsion-complete p-groups and G_2 is a $p^{\omega+1}$-projective p-group. On the other hand, by Corollary 2.2 and Theorem 2.5, G_1 and G_2 are $\oplus_c$-complete. Therefore (ii) holds and the proof is finished. $\square$

Now we point out some relations between $\oplus_c$-complete groups and thick groups. As Proposition 1.8 suggests and as we shall see in the following, these two classes have completely different properties.

PROPOSITION 3.3. *Let G be a separable p-group which is neither $\oplus_c$-complete nor thick. The following facts hold*:

(i) *There exists a thick group K such that $K + \check{G} = \bar{G}$; $K \cap \check{G} = G$ and $K \underset{*}{\leqslant} \bar{G}$.*

(ii) *Condition* (i) *does not necessarily determine the group K up to isomorphisms.*

PROOF (i). By Corollary 1.6 we can choose a subgroup K of $\bar{G}$ such that $\check{G}/G \oplus K/G = \bar{G}/G$. Therefore $K + \check{G} = \bar{G}$; $K \cap \check{G} = G$ and $K \underset{*}{\leqslant} \bar{G}$. It remains to check that K is thick. Since $G \leqslant K \leqslant \bar{G}$, Lemma 1.7 implies that $\check{G} \leqslant \check{K}$ and clearly $\bar{G} = \bar{K}$. Consequently $\check{K} = \bar{K}$ and so, by Proposition 1.8, K is thick, as required.

(ii) Let $G = G_1 \oplus G_2$ where $G_1 = \bigoplus_{n \geqslant 1} \mathbb{Z}(p^n)$ and G_2 is a quasi-complete non torsion-complete group whose basic subgroup is G_1 ([4] § 74 Example). Then G is neither $\oplus_c$-complete nor thick, $\check{G} = G_1 \oplus \bar{G}_2$ and a suitable choice for K is that of $K = \bar{G}_1 \oplus G_2$. We shall prove that there exist $2^{\aleph_0}$ pairwise nonisomorphic subgroups of $\bar{G}$ satisfying condition (i). To see this, let I be a set of cardinality $2^{\aleph_0}$. Take a subgroup H of $\bar{G}$ with $G \leqslant H$ and elements x_n, $y_{in} \in \bar{G}/G$, where $n \in \mathbb{N}$, with the following properties:

(1) $\check{G}/G = \langle x_n : n \in \mathbb{N} \rangle \oplus H/G$ where $\langle x_n : n \in \mathbb{N} \rangle \cong \mathbb{Z}(p^\infty)$, $px_0 = 0$ and $px_{n+1} = x_n$ for all $n \in \mathbb{N}$.

(2) $K/G = \bigoplus_{i \in I} \langle y_{in} : n \in \mathbb{N} \rangle$ where $\langle y_{in} : n \in \mathbb{N} \rangle \cong \mathbb{Z}(p^\infty)$, $py_{i0} = 0$ and $py_{in+1} = y_{in}$ for all $n \in \mathbb{N}$; $i \in I$.

For every $J \subseteq I$, let K_J be the subgroup of $\bar{G}$ such that $G \leqslant K_J$ and

$$K_J/G = \bigoplus_{i \in J} \langle y_{in} + x_n : n \in \mathbb{N} \rangle \oplus \bigoplus_{i \in I \setminus J} \langle y_{in} : n \in \mathbb{N} \rangle .$$

We claim that every K_J satisfies condition (i). In fact let J be any subset of I. Since $K_J/G + \check{G}/G = \langle x_n, y_{in} : n \in \mathbb{N}, i \in I \rangle + H/G = K/G \oplus \check{G}/G = \bar{G}/G$, evidently $K_J + \check{G} = \bar{G}$. The definition of K_J guarantees that $K_J \underset{*}{\leqslant} \bar{G}$, because $G \underset{*}{\leqslant} K_J$ and K_J/G is a divisible subgroup of $\bar{G}/G$. To verify that $K_J \cap \check{G} = G$, select $z \in K_J/G \cap \check{G}/G$. Then there exist $v \in H/G$; n, $n_i \in \mathbb{N}$ and a, $a_i \in \mathbb{Z}$ where $0 \leqslant a$, $a_i < p$ and $a_i = 0$ for almost all i such that

$$z = \sum_{i \in J} a_i (y_{in_i} + x_{n_i}) + \sum_{i \in I \setminus J} a_i y_{n_i} = a x_n + v .$$

Since $K/G \cap \check{G}/G = 0$, we get $a_i = 0$ for all i; hence $z = 0$. Therefore $K_J/G \cap \check{G}/G = 0$ and so $K_J \cap \check{G} = G$. Consequently every K_J satisfies condition (i) and it is easy to show that the groups K_J are all distinct. To end the proof, we apply an argument similar to that used in the last part of ([4] Theorem 66.4). Let B be a basic subgroup of G; then B is countable.

Since $|\mathrm{Hom}(K, \bar{G})| \leqslant |\mathrm{Hom}(B, \bar{G})| = 2^{\aleph_0}$, the subgroups of $\bar{G}$ isomorphic to K are at most $2^{\aleph_0}$. This means that $2^{2^{\aleph_0}}$ groups of the form K_J are pairwise nonisomorphic and so (ii) holds. $\square$

From Propositions 1.8 and 3.3 we deduce that, if G is neither $\oplus_c$-complete nor thick, then there exist a lot of thick groups between G and $\bar{G}$. The following result indicates that, under the same hypotheses on G, there exist also a lot of $\oplus_c$-complete groups between G and $\bar{G}$.

PROPOSITION 3.4. *Let G be a separable group which is neither $\oplus_c$-complete nor thick and let K be as in Proposition 3.3. The following are true*:

(i) *There exists an increasing sequence of $\oplus_c$-complete non thick groups $\{X_n\}$ with $\check{G} \leqslant X_n$ for all n such that $\bar{G} = \bigcup_{n \in \mathbb{N}} X_n$.*

(ii) *There exists an increasing sequence of non $\oplus_c$-complete non thick groups $\{Y_n\}$ with $G \leqslant Y_n$ for all n such that $K = \bigcup_{n \in \mathbb{N}} Y_n$.*

PROOF (i). Let $X_0 = \check{G}$ and let $X_n = \check{G} + \bar{G}[p^n]$ for all $n \geqslant 1$. Then, by Lemma 2.3, every X_n is $\oplus$-complete and the other properties clearly hold.

(ii) Under the hypotheses of (i), let $Y_n = K \cap X_n$ for all n. Then $\{Y_n\}$ is an increasing sequence, $G \leqslant Y_n$ for every n and $K = \bigcup_{n \in \mathbb{N}} Y_n$. To prove (ii), fix $n \in \mathbb{N}$. Since $G \underset{*}{\leqslant} Y_n$ and $\check{G} \nleqslant Y_n$, Lemma 1.7 assures that Y_n is not $\oplus_c$-complete. Moreover, since Y_n/G is bounded and, by hypothesis, G is not thick, it is easily seen that Y_n is not thick. This completes the proof. $\square$

The next statement gives some properties of $\oplus_c$-complete groups and thick groups with respect to intersection and group union.

PROPOSITION 3.5. *Let G be a separable p-group. The following facts hold*:

(i) *Let X be a pure subgroup of G and let $X = \bigcap_i X_i$ where $X_i \leqslant G$ for all i. If every X_i is $\oplus_c$-complete, then X is $\oplus_c$-complete; if every X_i is thick, then X is not necessarily thick.*

(ii) *Group unions of $\oplus_c$-complete or thick subgroups of G are not necessarily $\oplus_c$-complete or thick.*

PROOF (i). Suppose X_i is $\oplus_c$-complete for every i. Then, by the first part of Lemma 1.7, $\check{X} \leqslant X_i$ for all i; hence X is $\oplus_c$-complete, as claimed. Let now X be a $\oplus_c$-complete group which is not thick and let $G = \bar{X}$. Evidently X is the intersection of all $X_i \underset{*}{\leqslant} G$ such that $X \leqslant X_i$ and $G/X_i \cong \mathbb{Z}(p^\infty)$. Since all these groups are thick ([10] Theorem 3.5), (i) is proved.

(ii) First assume G is not $\oplus_c$-complete. Then $G = \bigcup_{n\geqslant 1} G[p^n]$ and every $G[p^n]$ is $\oplus_c$-complete. Finally let $G = \bigoplus_{n\geqslant 1} \mathbb{Z}(p^n)$. Since $G[p^n]$ is thick for all $n \geqslant 1$ and G is not thick, (ii) follows. $\square$

This last proposition shows that in a $\oplus_c$-complete group the cardinality of nondiscrete $\oplus_c$-complete subgroups and that of non $\oplus_c$-complete thick subgroups may be as large as possible.

PROPOSITION 3.6. *There exists a $\oplus_c$-complete group G with the following properties*:

(i) *G has $2^{|G|}$ $\oplus_c$-complete nondiscrete non thick subgroups.*

(ii) *G has $2^{|G|}$ non $\oplus_c$-complete thick subgroups.*

PROOF. We claim that the $\oplus_c$-complete group $G = t\Big(\prod_{n\in\mathbb{N}} G_n\Big)$ where $G_n = \bigoplus_{k\geqslant 1} \mathbb{Z}(p^k)$ for all n satisfies the above conditions.

(i) View G as the group of all bounded maps $f\colon \mathbb{N} \to \bigcup_{n\in\mathbb{N}} G_n$ such that $f(n) \in G_n$ for every $n \in \mathbb{N}$. If $f \in G$, let $Z(f) = \{n \in \mathbb{N}\colon f(n) = 0\}$. For every free ultrafilter φ on $\mathbb{N}$, let G_φ be the $\oplus_c$-completion of the group $\mathbf{\Sigma}_\varphi = \{f \in G\colon Z(f) \in \varphi\}$. Fix φ; since $\mathbf{\Sigma}_\varphi \underset{*}{\leqslant} G$, the first part of the proof of Lemma 1.7 guarantees that $G_\varphi \leqslant G$ and clearly $G_\varphi \neq \oplus_c$, because $\mathbf{\Sigma}_\varphi \neq \oplus_c$. Also note that every G_n is a summand of G_φ; consequently G_φ is not thick. The next step is to show that if $\varphi \neq \psi$, then $G_\varphi \neq G_\psi$. To this end, pick $F \in \varphi \setminus \psi$. Let g be an element of $G[p]$ with the property that $Z(g) = F$ and $g(n) \in G_n \setminus pG_n$ for every $n \in \mathbb{N} \setminus F$. Obviously $g \in G_\varphi$, but we claim that $g \notin G_\psi$. Suppose this were not true. Then, from the hypothesis that $g \in G_\psi$, we deduce that g belongs to the closure of $\mathbf{\Sigma}_\psi$ with respect to the $\oplus_c$-topology of G. Hence $g = g_1 + pg_2$ for some $g_1 \in \mathbf{\Sigma}_\psi$ and $g_2 \in G$. Since $Z(g) \notin \psi$ and $Z(g_1) \in \psi$, there exists $n \in Z(g_1) \setminus Z(g)$ and so $0 \neq g(n) = pg_2(n)$, contrary to the choice of g. This contradiction proves that $g \notin G_\psi$; thus the groups G_φ are all distinct. Since $\mathbb{N}$ has $2^{2^{\aleph_0}}$ free ultrafilters and $|G| = 2^{\aleph_0}$, (i) holds.

(ii) This property immediately follows from the proof of (ii) in Proposition 3.3.

Indeed all the groups K_J used in that proof can be embedded in the group $T = t\Big(\prod_{n\geqslant 1} (\mathbb{Z}(p^n) \oplus \mathbb{Z}(p^n))\Big)$ and it is easy to see that G has a direct summand isomorphic to T. $\square$

REFERENCES

[1] K. Benabdallah - R. Wilson, *Thick groups and essentially finitely indecomposable groups*, Canad. J. Math., **30**, no. 3 (1978), pp. 650-654.

[2] N. Bourbaki, *Topology Générale*, Chapitre 1 à 4, Hermann, Paris, 1971.

[3] B. Charles, *Sous-groupes functoriels et topologies*. Studies on abelian groups, Paris, 1968, pp. 75-92.

[4] L. Fuchs, *Infinite Abelian Groups*, Vol. 1 and 2, London - New York, 1971 and 1973.

[5] L. Fuchs - J. M. Irwin, *On $p^{\omega+1}$-projective p-groups*, Proc. London Math. Soc., (3), **30** (1975), pp. 459-470.

[6] L. Fuchs - L. Salce, *Abelian p-groups of not limit length*, Comment. Math. Univ. St. Pauli, **26** (1976), pp. 25-33.

[7] P. D. Hill - C. K. Megibben, *Quasi-closed primary groups*, Acta Math. Acad. Sci. Hungar., **16** (1965), pp. 271-274.

[8] J. M. Irwin - J. D. O'Neill, *On direct products of abelian groups*, Canad. J. Math., **22**, no. 3 (1970), pp. 525-544.

[9] E. L. Lady, *Countable torsion products of abelian p-groups*, Proc. Amer. Math. Soc., **37**, no. 1 (1973), pp. 10-16.

[10] C. Megibben, *Large subgroups and small homomorphisms*, Michigan Math. J., **13** (1966), pp. 153-160.

[11] R. Mines, *A family of functors defined on generalized primary groups*, Pacific J. Math., **26** (1968), pp. 349-360.

[12] R. J. Nunke, *Purity and subfunctors of the identity*, Topics in Abelian Groups, Chicago, 1963, pp. 121-171.

[13] L. Salce, *The λ-inductive topology on abelian p-groups*, Rend. Sem. Mat. Univ. Padova, **59** (1978), pp. 167-177.

Seminario Matematico
Università di Padova
via Belzoni 7
35100 Padova