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Group Law on the Intersection of Two Quadrics (*).

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Introduction.

In [7] Reid studied linear subspaces contained in the intersection X of two quadratic hypersurfaces Q, Q' generally situated in a complex projective space of arbitrary dimension. His main result can be described as follows: the maximal linear subspaces contained in $X = Q \cap Q' \subset \mathbb{P}^{2n+1}$ are $(n-1)$-dimensional; the variety A parametrizing all subspaces $\mathbb{P}^{n-1} \subset X$ is an n-dimensional abelian variety, in fact the Jacobian of a hyperelliptic curve E (of genus n) which arises naturally from the data in the pencil of quadrics containing X. His proof gives a birational map from $S^n E$, the n-th symmetric product of E, to A, identifying A birationally as an abelian variety; and concludes by computing a numerical invariant which forces A to be an abelian variety biregularly. The subject is also treated briefly in [8], en route to a study of the intersection of 3 quadrics.

The case $n = 2$ (the « quadric line complex ») is studied thoroughly in the old literature (Kummer, Klein, C. Segre, ...) in connection with Kummer surfaces, line geometry, cyclides ... More recently it appeared (*e.g.* [6]) as a moduli space for vector bundles on an algebraic curve. The isomorphism of A with $J(E)$ $(n = 2)$ is also proved there.

The purpose of this paper is to exhibit the group law on A explicitly, using geometric constructions analogous to the standard addition on an elliptic cubic plane curve. Following preliminaries on linear spaces in quadrics (Section 1) the construction is carried out in two steps in Section 2. First a partial addition is defined (where one of the terms is in special position) using the inclusion relations of quadrics and linear subspaces (Section 2.2).

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It is extended over all of $A\times A$ by a decomposition lemma (2.6) allowing to break a general subspace into sum of ones in special position. A surprisingly important role in making this work is played by the right choice of origin for the group, as explained in Section 2.

In Section 3 we reprove the standard isomorphisms of A with $J(E)$ and $J(X)$, the middle-dimension intermediate Jacobian of X. The explicit group-law allows us to construct directly an isomorphism

$$\varphi\colon J(E)\to A$$

(without passing through S^nE which is birationally equivalent to both) and to factor the isomorphism

$$j\colon A\to J(X)$$

through rational equivalence, showing that on X all cycles (in the middle dimension) are generated by subspaces (mod rational equivalence) and that on the set of subspaces, rational equivalence is generated by the equations of the group-law. We conclude with an analogue of Torelli's theorem.

The direct geometric approach to the problem developed out of my thesis [3] which dealt with a skew-symmetric analogue of the intersection of quadrics. I would like to thank Joe Harris and my advisor Phil Griffiths for introducing me to the subject and discussing some of the questions involved. My thanks also to H. Knörrer who read an earlier version and made some valuable remarks, and to the referee for pointing out some inaccuracies and suggesting improvements, especially in §§ 1.3, 2.3.

1. – Spaces in quadrics.

The material in this section is rather standard. Most of it can be found, from slightly differing viewpoints, in either [1], [5] or [7].

Let V be an n-dimensional vector space over the field $\mathbb{C}$ of complex numbers. A quadric Q in $\mathbb{P}^{n-1}=\mathbb{P}(V)$ is given by a symmetric $n\times n$ matrix $M(Q)$. The rank r of Q is that of $M(Q)$, and we let

$$\operatorname{corank}(Q)=n-r\,.$$

Q is a cone over a smooth quadric in $\mathbb{P}^{r-1}$, with a $\mathbb{P}^{n-r-1}$ for vertex. We shall be interested mainly in quadrics which are smooth ($r=n$) or have a unique ordinary double point ($r=n-1$), so we shall call these quadrics « general ».

The symmetric quadratic form determined by $M(Q)$ defines a linear map $V\to V^*$ which is an isomorphism for Q nonsingular. After projectivizing, this associates to a point $p \in \mathbb{P}(V)$ its *polar* $p^\perp$, or $(p^\perp)_Q$, which is a point of $\mathbb{P}(V^*)$ or a hyperplane in $\mathbb{P}(V)$. For an arbitrary Q this is still defined for p not in the vertex of Q. The hyperplane $(p^\perp)_Q$ contains the point p if and only if $p \in Q$, in which case $(p^\perp)_Q = T_pQ$, the projective tangent space to Q at p. We note that $(p^\perp)_Q$ depends linearly on Q, so that when Q varies in a pencil (and p is fixed) the points $(p^\perp)_Q \in \mathbb{P}(V^*)$ trace out either a line (general case) or a fixed point. The latter happens if and only if p is in the vertex of one of the quadrics, as can easily be verified.

1.1. *One quadric.*

We discuss the existence of linear spaces contained in a quadric Q, generalizing the two families of lines on a smooth quadric surface in $\mathbb{P}^3$.

LEMMA 1.1. *Let V be a vector space of dimension $2n+2$, respectively $2n+1$. Let Q be a general quadric in $\mathbb{P}(V)$. The maximal linear subspaces in Q have dimension n (respectively $n-1$). Their collection can naturally be made into the underlying set of an algebraic variety S of dimension $(n+1)n/2$.*

PROOF. *Q smooth.* Take $p \in Q$. Any linear subspace $s \subset Q$ through p is contained in the intersection

$$Q' = Q \cap T_pQ$$

of Q with its projective tangent space at p. Q' is a corank-one quadric in T_pQ with vertex at p, so it is a point-cone over a smooth quadric Q'' in $\mathbb{P}^{2n-1}$ (respectively, $\mathbb{P}^{2n-2}$). Thus the maximal linear subspaces in Q have dimension one larger than those in Q'', proving inductively the first assertion.

Let $G(k, l)$ denote the Grassmannian of subspaces $\mathbb{P}^{k-1}$ in a fixed $\mathbb{P}^{l-1}$. The subset

$$S(Q) \subset G(n+1, 2n+2)$$

(respectively, $S(Q) \subset G(n, 2n+1)$) of the subspaces on which the quadratic form associated to the matrix $M(Q)$ is identically zero, is clearly algebraic. We find its dimension from the following diagram:

$$\begin{array}{ccc} & I & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ S(Q) & & Q \end{array}$$

where $I \subset S \times Q$ is the incidence correspondence

$$I = \{(s, p) | s \in S,\ p \in Q,\ p \in s\}.$$

mapping onto either factor. The fiber of p_1 is a space $\mathbb{P}^n$ (respectively, $\mathbb{P}^{n-1}$), the fiber of p_2 is $s(Q'')$, of dimension $n(n-1)/2$ by an induction hypothesis, and $\dim Q = 2n$ (respectively, $2n-1$). Putting these together we find:

$$\dim S(Q) = \dim Q + \text{fiber}\dim p_2 - \text{fiber}\dim p_1 = \frac{n(n+1)}{2}.$$

Q of corank 1. Q is a point-cone over a smooth Q', with vertex at the singularity p. A maximal subspace in Q is the join of p with a maximal subspace in Q'. Q.E.D.

LEMMA 1.2. *$S(Q)$ has one or two components, according as rank (Q) is odd or even. Each of these is a unirational variety.*

PROOF. As in the second part of the previous proof, we may reduce to the case where Q is smooth. p_1 has irreducible fibers so that the components of $S(Q)$ correspond to components of I. p_2 is a fiber-bunble map with simply-connected base Q (By the Lefschetz hypersurface theorem, cf. [4]) and thus the components of I correspond to those of $S(Q'')$, the fiber. By induction we are reduced to a smooth quadric in $\mathbb{P}^1$ (two points) or $\mathbb{P}^2$ (an irreducible conic).

For the unirationality, let s_0 be a generic point of the irreducible component α of $S(Q)$, and fix a generic complementary subspace $\mathbb{P}^{n+1} \subset \mathbb{P}(V)$ intersecting s_0 in a unique point. Setting

$$Q_0 = \mathbb{P}^{n+1} \cap Q$$

we obtain a rational map

$$\Pi\colon \alpha \to Q_0$$

sending $s \in \alpha \subset S(Q)$ to its unique point of intersection with $\mathbb{P}^{n+1}$ (which is in Q_0). (Π is defined outside the proper algebraic subvariety of subspaces s intersecting $\mathbb{P}^{n+1}$ non-transversely.) Π is surjective since the map p_2 (in Lemma 1.1) is. (This also follows from the existence of a group of automorphisms of the configuration, transitive on the quadric Q_0.) The generic fiber of Π is (a component of) $S(Q'')$ which is unirational by induction. Thus $S(Q)$ is, birationally, a bundle with unirational fiber and base, and is therefore unirational. Q.E.D.

LEMMA 1.3. *Let $Q \subset \mathbb{P}^{2n+1}$ be a general quadric, $x \subset Q$ a linear subspace of dimension $n-1$, not through the vertex if Q is singular. For each irreducible* *component $\alpha \subset S(Q)$ there is a unique n-dimensional subspace $s = s(\alpha, x)$ such that*

$$s \in \alpha\,, \qquad s \supset x\,.$$

PROOF. If Q is singular, s is the join of x with the vertex. So assume Q smooth, and let $T = T_x Q$ be the projective tangent space to Q along x,

$$T = \bigcap_{q \in x} T_q Q\,.$$

We first note that $T \cap Q$ contains x, in fact it is a cone with vertex x since each $T_q Q \cap Q$ is. Let the points $q_i \in x$, $(1 \leqslant i \leqslant n)$ span x. Then

$$T = \bigcap_{i=1}^{n} T_{q_i}\,, \qquad Q = \bigcap_{i=1}^{n} q_i^{\perp} \qquad \text{(polar hyperplanes)}$$

and since the q_i are linearly independent, so are $q_i^{\perp}$, so T is a $\mathbb{P}^{n+1}$. By Lemma 1.1,

$$T \not\subset Q$$

so that $T \cap Q$ is a quadric cone in T with vertex x of codimension 2, so rank $(T \cap Q) \leqslant 2$ and it is the union of two subspaces s_1, $s_2 \approx \mathbb{P}^n$ in T. Applying the proof of Lemma 1.2 successively to the n points q_i shows that $T \cap Q$ is actually a cone over a non-singular quadric in $\mathbb{P}^1$ (thus union of two distinct hyperplanes in T) and that its two components s_i belong one to each component of $S(Q)$. Q.E.D.

1.2. *Pencil of quadrics.*

We consider a generic pencil of quadrics in $\mathbb{P}^N = \mathbb{P}(V)$ generated by Q, Q' and consisting of $\langle Q_\lambda \rangle$, $\lambda \in \mathbb{P}^1$ where

$$M(Q_\lambda) = \lambda M(Q) + M(Q')\,.$$

By « generic » we mean that

$$\det\big(M(Q_\lambda)\big)$$

which is a polynomial in λ of degree $N + 1 = \dim V$, has $N + 1$ distinct roots

$$\lambda_0, \ldots, \lambda_N \in \mathbb{P}^1\,.$$

This requirement immediately implies (and in fact is equivalent to) either of the following:

a) Each Q_λ is general (in the sense of 1.1) and the Q_{λ_i} are the only singular ones.

b) The base locus $X = Q \cap Q'$ of the pencil is smooth.

In fact, x is the transversal complete intersection of any two Q_λ in the pencil. This implies that the vertex p_i of the singular Q_{λ_i} is not in X. Moreover, the p_i are the (projectivized) eigenvectors of the matrix

$$-M(Q)^{-1}M(Q')$$

corresponding to the distinct eigenvalues λ_i and therefore are linearly independent. Choosing a basis $v_0, \ldots, v_N$ of V such that p_i is the projectivization of v_i, we find a simultaneous diagonalization of the matrices $M(Q_\lambda)$.

LEMMA 1.4. *In* $\mathbb{P}^{2n}$, *X contains subspaces* $\mathbb{P}^{n-1}$ (*but no* $\mathbb{P}^n$).

PROOF. We have to show that in $G(n, 2n+1)$ the two subvarieties $S(Q)$, $S(Q')$ do intersect. For this it suffices to check that the class of $S(Q)$ in the middle-homology

$$H = H_{n(n+1)}(G(n, 2n+1), \mathbb{Z})$$

has positive self-intersection. We recall (cf. [3] or [4] for notations and rules of the game) that H is a free abelian group having for basis the Schubert cycles

$$\sigma_{a_1 \ldots a_n}$$

where

$$n+1 \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$$

$$\sum a_i = \frac{n(n+1)}{2}$$

with respect to this basis, the bilinear form given by intersection is normalized, so that

$$\sigma_{a_1 \ldots a_n} \cdot \sigma_{b_1 \ldots b_n} = 0$$

unless $\sigma_{b_1 \ldots b_n}$ is the dual cycle to $\sigma_{a_1 \ldots a_n}$,

$$b_i = n + 1 - a_{n+1-i} \quad \text{for all } i$$

in which case the product is 1. Of particular interest is the self-dual

$$\sigma = \sigma_{n,n-1,\dots,1}\,.$$

Choosing a chain of subspaces

$$A_1 \subset \dots \subset A_n$$

where A_i is a $\mathbb{P}^{2i-1}$ in $\mathbb{P}^{2n}$, σ is

$$\{x \in G(n, 2n+1) \,|\, \dim(x \cap A_i) \geqslant i-1\}\,.$$

(In particular, x intersects the line A_1 and is contained in the hyperplane A_n.)

Define

$$C_n = [S(Q)]\cdot\sigma$$

the intersection number of σ with the class of $S(Q)$. We claim:

$$C_n = 2C_{n-1}\,.$$

For, A_1 intersects Q in two points p_1, p_2, and maximal subspaces through p_i correspond to spaces on a quadric in $\mathbb{P}^{2n-2}$, as in Lemma 1.1, while spaces through p_i in $\sigma_{n,n-1,\dots,1}$ are those that come (via intersection with $T_{p_i}Q$ and projection from p_i) from $\sigma_{n-1,\dots,1}$.

Now $C_1 = 2$ (the intersection of a plane conic with a line), so $C_n = 2^n$ and we have

$$[S(Q)] \geqslant 2^n \sigma$$

by the self-duality of σ. Thus

$$[S(Q)]^2 \geqslant 4^n > 0\,. \qquad \text{Q.E.D.}$$

Note: With a little extra work we could check that:

1) $[S(Q)] = 2^n\sigma$.
2) Thus, $[S(Q)]^2 = 4^n$.
3) The intersection $S(Q) \cap S(Q')$ is transversal and consists of 4^n distinct points. We shall not need any of these facts. (cf. [7]).

COROLLARY 1.5. *In P^{2n+1}, X contains no spaces $\mathbb{P}^n$. The subvariety $A \subset G(n, 2n+2)$ of spaces $x \subset X$ has dimension $\geqslant n$. Moreover, there is an x through an arbitrary point p of X.*

The proof is again by intersection (with $T_pX = T_pQ \cap T_pQ'$) and projection (from p), reducing to an X in $\mathbb{P}^{2n-2}$ as in Lemma 1.4. By the previous observation there are actually 4^{n-1} spaces x through a generic $p \in X$.

1.3. *Hyperelliptic curve.*

We shall construct the hyperelliptic curve E which underlies the geometry of X. From now on we work in projective space of odd dimension $N = 2n + 1$. Let

$$I \subset \mathbb{P}^1 \times G(n+1, 2n+2), \qquad I = \{(\lambda, s) | s \in S(Q_\lambda)\}.$$

The generic fiber of

$$p_1 : I \to \mathbb{P}^1$$

has two components, which define a double cover E of $\mathbb{P}^1$, smooth outside the $2n + 2$ points λ_i. More precisely, we apply the Stein factorization theorem to P_1, obtaining maps

$$p_1' : I \to E, \qquad \pi : E \to \mathbb{P}^1$$

where p_1' has connected fibers, π is a finite morphism, and

$$p_1 = \pi \circ p_1'.$$

LEMMA 1.6. *E is a hyperelliptic curve of genus n. Each λ_i is a simple ramification point for π.*

PROOF. At each λ_i there are only two possibilities: either π has a simple ramification point, and E is locally irreducible there, or else the two sheets of E intersect at λ_i without interchanging, and E is locally reducible. We shall show that the first is the case, thus E is an irreducible double cover of $\mathbb{P}^1$ with $2n + 2$ ramification points, hyperelliptic of genus n.

The nature of π at λ_i can be checked by (fairly complicated) local analysis. Instead we appeal to the homogeneity of the situation, as follows. Let $\mathbb{P} = \mathbb{P}^M$, $M = \binom{2n+3}{2} - 1$, parametrize all quadrics in $\mathbb{P}^{2n+1}$; $D \subset \mathbb{P}$ be the (singular) hypersurface parametrizing singular quadrics (D is given by the vanishing of the determinant. In particular, D is irreducible.) The pencil $\{Q_\lambda\}$ is given by a line L in $\mathbb{P}$ intersecting D transversally, $L \in G(2, M + 1)$.

1) The behavior at all λ_i is the same. For the variety $Y \subset G(2, M+1) \times D$,

$$Y = \{(L, Q) \mid Q \in L, Q \in D, L \in G(2, M+1)\}$$

is irreducible (maps onto D with fiber $\mathbb{P}^{M-1}$) and remains so after deletion of the closed subvariety of non-generic pencils. Thus any two pairs

$$(L, Q), \quad (L', Q')$$

can be deformed into each other through generic pencils, and the local behavior is preserved along the deformation. Apply this to

$$(L, Q_{\lambda_i}), \quad (L, Q_{\lambda_j}).$$

2) E is locally irreducible at λ_i. For otherwise E would be globally reducible, as a nowhere-ramified cover of $\mathbb{P}^1$. But in the global setting we have a double cover $p: F \to \mathbb{P}$, smooth outside of D with fiber over $Q \in \mathbb{P} \setminus D$ corresponding to the two families of spaces on Q. In $\mathbb{P}^{2n+1}$ there is a projective automorphism mapping Q to itself and interchanging the two families of spaces on Q, which can be deformed in $\mathbb{P}GL(2n+2)$ to the identity, thus F is irreducible. Since $E = p^{-1}(L) \subset F$, E has to be irreducible too, for generic L. Q.E.D.

We identify a point $e \in E$ with the corresponding component of $S(Q)$, some $Q \in L$, that is with the fiber of I over e. For $x \in A$ we define a map $\varphi_x: E \to A$ by the formula

$$s(e, x) \cap X = x \cup \varphi_x(e)$$

where $s = s(e, x)$ is the unique $\mathbb{P}^n$ in the family e containing x (Lemma 1.3). If $Q' \neq Q$ is any other quadric in L, then $s \cap X = s \cap Q'$ is a quadric in s containing x and therefore also another hyperplane which we call $\varphi_x(e)$. We can think of φ as a map

$$\varphi: E \times A \to A, \quad \varphi(e, x) = \varphi_x(e)$$

the following is immediate from Corollary 1.5.

LEMMA 1.7. *φ is a morphism. For fixed $e \in E$, the restriction $\varphi_e: A \to A$ is a biregular isomorphism, in fact an involution (hence its own inverse). For fixed $x \in A$, φ_x is injective.*

EXAMPLE. In the case $n = 1$, $X = Q \cap Q'$ is an elliptic curve embedded in $\mathbb{P}^3$ as a quartic curve, base locus for the pencil Q_λ of quadric surfaces.

A generic Q_λ contains two families of lines, each parametrized by $\mathbb{P}^1$. There are 4 singular Q_{λ_i} which are ordinary cones. A is trivially identified with X, and choosing a base point $x \in X$ gives an isomorphism

$$\varphi_x \colon E \xrightarrow{\approx} X$$

(sending a family e to the residual point of intersection with X of the unique line in e through x.)

2. – The group law.

2.1. *Origin*

In principle, any point on an abelian variety can serve as origin, that is, as neutral element for the group law. However, for a « natural » group-law, a propitious choice of origin must be made. To illustrate this, consider the smooth elliptic cubic curve in $\mathbb{P}^2$. The familiar relation:

$$\text{« } x + y + z = 0 \quad \text{if } x, y, z \text{ are collinear »}$$

requires that the origin o be taken at one of the nine flexes, so that $o + o + o = 0$.

Let $X = Q \cap Q'$ where Q is a smooth quadric in $\mathbb{P}^{2n+1}$ and Q' a point-cone with vertex p. Let $e_0 \in E$ be the (unique) family $S(Q')$ of spaces in Q'.

LEMMA 2.1. *There is an n-dimensional subspace $s \in e_0$ intersecting Q (tangentially) in an $(n-1)$-dimensional subspace o, so that*

$$s \cap Q = 2 \cdot o\,.$$

PROOF. We use the facts discussed in the first paragraphs of Chapter 1 and of Section 1.2. The vertex p is not in Q (nor in Q_λ for $Q_\lambda \neq Q'$ in the pencil) so its polar hyperplane

$$H = (p^\perp)_{Q_\lambda}$$

is nowhere tangent to Q_λ. Since H does not depend on λ we can restrict everything to H:

$$\begin{aligned} \bar{Q}_\lambda &= Q_\lambda \cap H\,, \\ \bar{X} &= X \cap H = \bar{Q} \cap \bar{Q}'\,, \\ \bar{L} &= \{\bar{Q}_\lambda \mid \lambda \in \mathbb{P}^1\}\,. \end{aligned}$$

We claim $\bar{L}$ is a generic pencil in H:

1) The generic $\bar{Q}_\lambda$ is smooth, since Q_λ is smooth and H is nowhere tangent to Q_λ.

2) Let Q_{λ_i} $i = 1, \ldots, 2n+1$ be the singular quadrics in L other than Q', with vertices p_i. Then $p_i \in H$. (In fact, for any $q \in \mathbb{P}^{2n+1}$, $(q^\perp)_{Q_{\lambda i}} \ni p_i$). Therefore $\bar{Q}_{\lambda_i}$ is still a point-cone, as the intersection of Q_{λ_i} with a hyperplane through its vertex. Thus, the pencil $\bar{L}$ has $2n+1$ distinct singularities and is therefore generic.

Now we can apply Lemma 1.4 to $\bar{X}$, to obtain an $(n-1)$-dimensional linear subspace o which is contained in $\bar{X}$ (and thus in X). We take s to be the span of o, p. To conclude, we need to verify that s is tangent to Q (and to each Q_λ) at each point of o. But by polarity, if

$$q \in \bar{Q} = Q \cap H$$

then

$$T_q Q \ni p = H^\perp .$$

In particular, p is in

$$T_o Q = \bigcap_{q \in o} T_q Q .$$

Since o is linear, $o \subset T_o Q$, so that

$$s = \langle o, p \rangle \subset T_o Q$$

and s is tangent to Q along o. Q.E.D.

COROLLARY 2.2. *There is a subspace* $\mathbb{P}^{n+1} \subset \mathbb{P}^{2n+1}$ *whose intersection with* X *is four times* o. (*Namely,* $T_s Q'$.)

EXAMPLE. For $n = 1$, this says that the projection of X from o is a smooth plane cubic on which the projection of o itself is a flex. Thus we get a group-law on X (or A, or E) via:

$$« x + y + z = 0 \quad \text{if } x, y, z, o \text{ are coplanar} ».$$

2.2. *Partial addition.*

We want the group law to be induced by rational equivalence. Thus we define, for $e \in E_o$, $x \in A$:

$$e - x = \varphi_x \varphi_o^{-1}(e)$$

so that the cycles

$$[o] + [e]$$

and

$$[x] + [e - x]$$

are cut out on X by spaces $\mathbb{P}^n$ varying in the same unirational family $\varphi_o^{-1}(e) \in E$ (of spaces of complementary dimension on some quadric $Q_\lambda(\lambda = \pi(e))$) and are therefore rationally equivalent. (We use the notation $[\alpha]$ for the cycle represented by a subspace α.)

We note that the special property of o is that

$$o \in E_o = \mathrm{Im}\,(\varphi_o)$$

and in fact

$$o = \varphi_o(e_0)$$

thus we can restate the case $e = o$ of our definition as

$$- x = o - x = \varphi_x(e_0) \qquad x \in A$$

(treating o as the origin-to-be of A.) This leads to the definition

$$e + x = e - (- x) = \varphi_{-x}(\varphi_o^{-1}(e)) = \varphi_{\varphi_x(e_0)}(\varphi_o^{-1}(e)) \, .$$

Lemma 2.1 says that

$$o = - o = o + o$$

and corollary 2.2 implies that under any group-law $+$ which is consistent with rational equivalence,

$$o + o + o + o = 0$$

where 0 is the origin for $+$.

In the next section we show that this partial addition extends naturally over $A \times A$.

2.3. *Extension.*

LEMMA 2.3. *Let $x, y \in A$ be two subspaces $\mathbb{P}^{n-1}$ intersecting each other in codimension r. Then there are (at least one and at most r) subspaces $y_i \in A$, $i = 1, \ldots, k \leqslant r$ such that*

$$y_i \supset x \cap y$$

$$\mathrm{codim}\,(y_i \cap x) = r - 1 \qquad \mathrm{codim}\,(y_i \cap y) = 1 \, .$$

PROOF. Assume first $x \cap y = \emptyset$. For $e \in E$, recall that $s(e, x)$ denotes the unique $\mathbb{P}^n$ in the family e through x.

Claim: not all the $s(e, y)$ intersect x.

For otherwise, they would all be contained in $v = \operatorname{span}(x, y) = \mathbb{P}^{2n-1}$; but for the singular e_i, $s(e_i, y)$ contains the vertex p_i, contradicting the linear independence of the p_i.

Claim: for generic λ, $T_y Q_\lambda \cap x = \emptyset$. ($T_y Q_\lambda$ is the tangent space to Q_λ along y.)

For this, note that

$$T_y Q_\lambda \cap x = T_y Q_\lambda \cap Q_\lambda \cap x = \big(s(\alpha, y) \cap s(\beta, y)\big) \cap x$$

where α, β are the two families on Q_λ; thus the intersection is empty for all but finitely many λ, by previous claim.

For such a generic λ, let $\bar{Q}_\lambda = Q_\lambda \cap v$ be the restriction to $\operatorname{span}(x, y)$.

Claim: $\bar{Q}_\lambda$ is smooth.

$T_y \bar{Q}_\lambda$ is a linear subspace of v containing y and disjoint (by previous claim) of x, thus

$$T_y \bar{Q}_\lambda = y$$

since a larger subspace would intersect x by dimension argument. This shows that y is a maximal subspace contained in $\bar{Q}_\lambda$, so $\bar{Q}_\lambda$ is general. Since $x \subset \bar{Q}_\lambda$ is another subspace, disjoint of y and of the same dimension, $\bar{Q}_\lambda$ must be smooth.

We restrict the polarity with respect to $\bar{Q}_\lambda$ to the disjoint maximal subspaces x, y. This induces a linear isomorphism.

$$T_\lambda \colon y^* \to x$$

(y^* is the dual space of y) sending a hyperplane $h \in y^*$ to the unique point in $T_h \bar{Q}_\lambda \cap x$. If $\mu \in \mathbb{P}^1$ is also generic, we have another isomorphism

$$T_\mu \colon y^* \to x\,,$$

such that the $\mathbb{P}^{n-1}$ joining $h \subset y$ with $x_0 \in x$ is in X if and only if h is an eigenvector (= fixed point) for

$$T_\mu^{-1} T_\lambda \colon y^* \to y^*$$

and $x_0 = T_\lambda h$. (We can think of y^* as a vector subspace of V, or as its projectivization. Accordingly, x_0 is an eigenvector or a fixed point, etc.)

Over C, at least one eigenvector always exists. We show next that all eigenspaces are one-dimensional, hence we can attach to each fixed point its multiplicity, that is the dimension of its generalized eigenspace, or size of block in the Jordan canonical form of $T_\mu^{-1}T_\lambda$. The sum of the multiplicities is $r = n$.

Claim: there are at most n fixed points, or non-proportional eigenvectors.

Otherwise, there would be a continuous family (a projective subspace) of them, say y_t, $t \in \mathbf{P}^1$.

Now y, y_t are contained in a unique n-dimensional subspace s_t (their span) which in turn is contained in a unique Q_t, belonging to the family $e_t \in E$. This gives a map

$$e: \mathbf{P}^1 \to E$$

which is necessarily constant! We now have a contradiction since in a given family e there is a unique s containing y.

This proves the lemma for $r = n$. In general, we repeat the above proof, replacing x, y by their $(r-1)$-dimensional projections from their intersection $u = x \cap y$ and similarly for $v = \operatorname{span}(x, y)$. Q.E.D.

On A there is the natural involution

$$\mu: A \to A\,, \qquad \mu(x) = -x = \varphi_x(e_0)$$

$\mu(x)$ is the unique $\mathbf{P}^{n-1}$ which is contained together with x in a $\mathbf{P}^n$ of the family e_0.

On E_o there is an involution τ_o, obtained from the hyperelliptic involution $\tau: E \to E$ through the isomorphism φ_o. We need a comparison of these involutions.

LEMMA 2.4. *For* $x \in A$, $e \in E_o$, $\mu(e - x) = \tau_o(e) - \mu(x)$.

PROOF. The fourtuple $e - x$, x, $\mu(x)$, $\tau_o(e) - \mu(x)$ is the complete intersection of the $\mathbf{P}^{n+1}$ spanned by them with X. In fact, $e - x$, x sit on a unique $s \in S(e)$, in the quadric $Q_{\pi(e)}$. The residual intersection of $Q_{\pi(e)}$ with $\mathbf{P}^{n+1}$, $\big($spanned by $e - x, x, \mu(x)\big)$ is an $s' \in S(\tau e)$. s' contains $\mu(x)$, therefore its residual intersection with X is $\tau_o(e) - \mu(x)$.

Now apply the same logic in reverse to Q_o: It contains a $\mathbf{P}^n$ joining x, $\mu(x)$; its residual intersection with $\mathbf{P}^{n+1}$ is another $\mathbf{P}^n$ which must contain $e - x$, $\mu(e - x)$. (This proof works for generic e, x. The lemma extends to all e, x by continuity.) Q.E.D.

For r elements

$$e_1, \ldots, e_r \in E_o$$

we define their repeated sum

$$e_1 + \ldots + e_r$$

inductively by

$$e_1 + \ldots + e_r = e_1 + (e_2 + \ldots + e_r) .$$

COROLLARY 2.5.

(1) $\mu(e) = \tau_o(e), \; e \in E_o$.

(2) $\mu(e + x) = \mu(e) + \mu(x), \; e \in E_o, \; x \in A$.

(3) *If x is a repeated sum of r elements of E_o, so is $\mu(x)$.*

PROOF.

(1) Take $x = o$ in Lemma 2.4.

(2) $\mu(e + x) = \mu(e - \mu(x))$ by definition of $+$
$= \tau_o(e) - \mu(\mu(x))$ by Lemma 2.4
$= \mu(e) + \mu(x)$ by (1) and the definition of $+$.

(3) For $r = 1, 2$ this follows from parts (1), (2). Use induction and (2). Q.E.D.

DECOMPOSITION LEMMA 2.6. *For $y \in A$ such that*

$$\dim (y \cap o) = n - r - 1$$

there is an r-tuple

$$e_1, \ldots, e_r \in E_o$$

unique up to permutation, such that

$$e_1 + \ldots + e_r = y .$$

Further, the e_i can be arbitrarily permuted.

PROOF. *Existence.* When $r = 0$ then $y = o$ and there is noth'ng to prove. We claim that when $r = 1$, $y \in E_o$. Indeed, let s denote the $\mathbb{P}^n$ spanned by y, o; s is necessarily contained in some Q_λ. (Let $p \in s$ be a point not in o, y. There is a Q_λ containing p; its intersection with s contains the reducible quadric $y \cup o$ and an extra point p, thus $s \subset Q_\lambda$.) Therefore

$$y = s(\alpha, o) \in E_o$$

α being the family on Q_λ containing s. This proves the lemma for $r = 1$.

Note: The above argument actually shows more: If y, $y' \in A$ intersect in codimension 1, then $y' = \varphi_y(\alpha) \in E_y$, for some $\alpha \in E$.

We complete the proof by induction, assuming existence up to $r-1$. By Lemma 2.3 (applied to $x = o$) there is some $y_1 \in A$ such that

$$\operatorname{codim}(y_1 \cap o) = r - 1 \qquad \operatorname{codim}(y_1 \cap y) = 1 .$$

By the above note, there is $\alpha_1 \in E$ and corresponding $e_1 \in E_o$ such that

$$y = e_1 - y_1 = e_1 + \mu(y_1) .$$

By Corollary 2.5 (3), we can decompose

$$\mu(y_1) = e_2 + \dots + e_r$$

as required.

Commutativity. It suffices to show, for e_1, $e_2 \in E_o$, $x \in A$,

$$e_1 + (e_2 + x) = e_2 + (e_1 + x) .$$

The argument parallels Lemma 2.4: The three subspaces

$$\mu(e_2) - x , \qquad x , \qquad \mu(e_1) - x$$

span a $\mathbb{P}^{n+1}$ since the middle one intersects each of the other two in codimension 1. The first two are contained in a unique $s \in S(\mu(e_2))$, in the quadric $Q_{\pi(e_2)}$. The residual intersection of $Q_{\pi(e_2)}$ with $\mathbb{P}^{n+1}$ is an $s' \in S(\tau\mu(e_2)) = S(e_2)$. s' contains $\mu(e_1) - x$, therefore its residual intersection with X is

$$e_2 - (\mu(e_1) - x) = e_2 - \mu(e_1 + x) = e_2 + e_1 + x .$$

By symmetry this also equals $e_1 + e_2 + x$. (Again this proof works generically, whenever no two of the three subspaces coincide; the result holds in all cases by continuity.)

Uniqueness. Let $y = e_1 + \dots + e_r$ satisfy

$$\operatorname{codim}(y \cap o) = r .$$

To show that the e_i are all uniquely determined, we check that the number k_i of times e_i appears in y equals the dimension m_i of the generalized eigenspace (= multiplicity) in the transformation $T_\mu^{-1} T_\lambda$ of Lemma 2.3, corresponding to the eigenvector $y_i = e_i - y$. Since $\sum k_i = \sum m_i = r$, we show $k_i \leqslant m_i$.

When all $k_i = 1$, this is clear from the proof of Lemma 2.3. To handle the general case, we observe that the transformation $T_\mu^{-1} T_\lambda$ (for fixed λ, μ) depends continuously on y, hence on $e_1, \ldots, e_r$. For simplicity, we replace it by its conjugate

$$T_\lambda T_\mu^{-1} \colon o \to o$$

acting on the (fixed) space o. The corresponding eigenvectors are

$$x_i = T_\lambda y_i = T_\lambda(e_i - y) \in o \, .$$

We let $e_1, \ldots, e_r$ vary in a family

$$e_1(t), \ldots, e_r(t)$$

such that for $t \neq 0$, the $e_i(t)$ are distinct, and $e_i(0) = e_i$. If, say, $e_i = e_{j(1)} = \ldots = e_{j(k)}$ where $k = k_i$ and $j(1) < \ldots < j(k)$, then consider the space $\mathbf{P}^{k-1} = \mathbf{P}^{k-1}(0)$ in o which is the limit position of the $\mathbf{P}^{k-1}(t)$ spanned by $x_{j(1)}, \ldots, x_{j(k)}$. (The limit position exists since the Grassmannian is a complete variety.) Clearly $\mathbf{P}^{k-1}(0)$ is contained in the generalized eigenspace of x_i, for it is invariant and contains no fixed point other than x_i. Hence $k_i = k \leqslant m_i$ as required. Q.E.D.

This completes the construction: To find $x + y$, y is decomposed into little pieces which are added one-by-one to x.

We could now verify directly that A becomes an abelian group under addition. Instead, we shall interpret the proposed group-law via rational equivalence of cycles. Let

$$\mathcal{A} = \bigoplus_{x \in A} \mathbf{Z}$$

and let $\sim$ be the subgroup in $\mathcal{A}$ generated by o (*i.e.* the function on A which is one on o and zero elsewhere) and by the relations

$$(*) \qquad (x + \varphi_x(e)) - (y + \varphi_y(e)) \qquad e \in E_o;\ x, y \in A \, .$$

(Interpretation as above.) Consider the natural map

$$i \colon A \to \mathcal{A}/\sim \, .$$

The results in § 2.3 imply the surjectivity of i. By construction, addition is preserved by i; to show that A is a group it will therefore suffice to check that i is injective.

Clearly $\sim$ is consistent with rational equivalence (cf. § 2.2) so there is a group-homomorphism

$$i' \colon \mathcal{A}/\!\sim \to \mathcal{A}/(o, \text{ rational equivalence}) .$$

We shall prove in Chapter 3 that the composition

$$\bar{i} = i' \circ i$$

is injective, thus an isomorphism. We obtain our main result:

THEOREM 2.7. *The variety of $(n-1)$-dimensional linear subspaces of $\mathbb{P}^{2n+1}$ contained in the smooth intersection X of two quadratic hypersurfaces is an n-dimensional abelian variety with a natural group-law. With a proper choice of origin in A, the group-law is consistent with rational equivalence of cycles in X, and can be executed explicitly using a family of embeddings*

$$\varphi_x \colon E \to A \qquad x \in A$$

of the hyperelliptic curve E in A, as in §§ 2.2, 2.3.

We conclude with a recipe for adding two lines x, y on the quadric line complex: The $\mathbb{P}^k$ spanned by o, y contains two other lines l_1, l_o. Translate them from o to x (*i.e.* construct $l_1 - x$, $l_2 - x$) and find the fourth line in the $\mathbb{P}^3$ these three span. This involves solving quadratic equations at worst (finding eigenvectors of $T_\mu^{-1} T_\lambda$) so can be done in the plane with compass and ruler, given *e.g.* the entries of $M(Q)$, $M(Q')$, and the coordinates of o.

3. – Isomorphisms.

In this chapter we compare A with $J(E)$, the Jacobian of the hyperelliptic curve E, and with $J(X)$, the intermediate Jacobian of X. (cf. [2] for definition and properties.)

3.1. *Hyperelliptic Jacobian.*

THEOREM 3.1. *The map*

$$\varphi_o \colon E \to A$$

extends naturally to an isomorphism

$$\varphi: J(E) \xrightarrow{\approx} A .$$

PROOF. We construct φ by steps as follows. In $J(E)$ we have the filtration

$$E = W_1 \subset W_2 \subset \ldots \subset W_n = J(E)$$

where W_i is the locus of sums (in the group $J(E)$) of i points of E (representing a divisor on E of degree $\leqslant i$.) Analogously we define

$$A_i = \{x \in A \,|\, \mathrm{codim}\,(x \cap o) \leqslant i\}$$

so that we have

$$E_o = A_1 \subset \ldots \subset A_n = A .$$

We construct inductively isomorphisms

$$\varphi_i: W_i \to A_i$$

starting with $\varphi_1 = \varphi_o$. Let D_0 be the hyperelliptic class on E.

LEMMA 3.2. *If D is a divisor on E such that*

$$d = \deg D \leqslant n = \text{genus}\,(E) \qquad h_0(E, D) \geqslant 2$$

then

$$h^0(E, D - D_0) \geqslant 1 .$$

(*In other words, any divisor of degree $\leqslant n$ which moves on E is the sum of D_0 and an effective divisor.*)

PROOF. By Riemann-Roch,

$$2 \leqslant h^0(E, D) = d - n + 1 + h^0(E, K - D)$$

K being the canonical class on E. Therefore

$$h^0(E, K - D) \geqslant 1$$

and $K - D$ is effective. Since $h^0(E, D_0) = 2$ we have

$$\begin{aligned} h^0(E, K - D + D_0) &\geqslant h^0(E, K - D) + h^0(E, D_0) - 1 \\ &\geqslant h^0(E, K - D) + 1 \end{aligned}$$

so by Riemann-Roch again,

$$\begin{aligned} h^0(E, D - D_0) &= d - 2 - n + 1 + h^0(E, K - D + D_0) \\ &\geqslant d - n + h^0(E, K - D) \\ &= h^0(E, D) - 1 \geqslant 1 . \qquad \text{Q.E.D.} \end{aligned}$$

Note that as a point of $J(E)$, $D_0 = 0$, since the base point e_0 was chosen as a ramification point for $\pi: E \to \mathbf{P}^1$, so that $2e_0 \in |D_0|$. In terms of decomposition, the lemma says therefore that if $D \in W_i$ can be written in more than one way as a sum of i points then actually $D \in W_{i-2}$.

We define φ_i initially on $W_i \setminus W_{i-2}$: a point $D \in W_i \setminus W_{i-2}$ is a sum, in a unique way, of i points of E: we define $\varphi_i(D)$ to be the corresponding sum in A. This clearly defines a biregular map

$$\varphi_i: W_i \setminus W_{i-2} \xrightarrow{\approx} A_i \setminus A_{i-2} .$$

By induction, $\varphi_{i-1}: W_{i-1} \to A_{i-1}$ is a well-defined isomorphism, restricting to $\varphi_{i-2}: W_{i-2} \to A_{i-2}$. We claim

$$\varphi_i|_{W_{i-1} \setminus W_{i-2}} = \varphi_{i-1}|_{W_{i-1} \setminus W_{i-2}} .$$

Indeed, for

$$D = p_1 + \ldots + p_{i-1} \in W_{i-1} \setminus W_{i-2}$$

the unique interpretation of D as a point of W_i is

$$D' = p_1 + \ldots + p_{i-1} + e_0$$

by the lemma; therefore

$$\varphi_i(D') = \varphi_{i-1}(D) + o = \varphi_{i-1}(D) .$$

Thus, the restriction of φ_i to $W_{i-1} \setminus W_{i-2}$ extends over W_{i-1}. By standard extension theorems, φ_i extends to a biregular map

$$\varphi_i: W_i \to A_i$$

Extending φ_{i-1}. Q.E.D.

REMARK. φ was constructed as an isomorphism of varieties, but clearly respects addition, thus proving that A is an abelian variety. (and φ becomes a group-isomorphism.)

3.2. *Intermediate Jacobians.*

The variety A was constructed as the base-space of a family of cycles on X, « right below » the middle dimension. By the universal property of the middle-dimension intermediate Jacobian $J(X)$ there is a natural map

$$j: A \to J(X)$$

given by « integration along paths in A, emanating from o ».

THEOREM 3.3. *$j: A \to J(X)$ is an isomorphism.*

REMARKS. *a*) j factors as follows:

$$A \xrightarrow{i} \mathcal{A}/\sim \xrightarrow{i'} \mathcal{A}/(o, \text{rational equivalence}) \xrightarrow{i''} J(X)$$

where i, i' are as in § 2.3, and i'' is the Abel map, well-defined by Abel's theorem. The theorem then implies:

(1) i is injective, therefore an isomorphism by § 2.3, completing the proof of Theorem 2.7.

(2) i' is surjective by definition and injective by surjectivity of i, thus rational equivalence of linear subspaces in X is generated by the relations (∗) in § 2.3.

(3) Finally, i'' is an isomorphism, proving that all cycles on X are generated by linear subspaces modulo equivalence (surjectivity) and that on X rational and Abelian equivalence coincide (injectivity).

b) j fits in a still larger diagram

$$\psi: J(E) \xrightarrow{\varphi} A \xrightarrow{j} J(X)$$

where ψ is the canonical extension to $J(E)$ of

$$\psi_0 = j \circ \varphi_0 : E \to J(X)$$

provided by the universal property of $J(E)$. (Both ψ, $j \circ \varphi$ map $J(E)$ to $J(X)$ extending ψ_0.)

PROOF OF THEOREM. Once we know that A, $J(X)$ are abelian varieties, checking that a given map j is an isomorphism is routine. We sketch this here; for more details see chapter 2 of [3], or [2].

1) First we replace A by $J(E)$:

$j\colon A \to J(X)$ is an isomorphism $\Leftrightarrow$

$\psi\colon J(E) \to J(X)$ is, $\Leftrightarrow$

$\bar{\psi}\colon H_1(E, \mathbb{Z}) \to H_{2n-1}(X, \mathbb{Z})$ is an isomorphism of lattices.

Note that both of these have rank $= 2n$. Verifying this for X involves an easy computation of $C_{2n-1}(X)$, the Euler characteristic; and Lefschetz' hypersurface theorem for discarding all non-middle betti numbers.

Let $(\ ,\ )_E$, $(\ ,\ )_X$ be the (unimodular) intersection forms on the integral middle dimensional homologies of E, X; let $(\ ,\ )_\psi$ be the pull-back form on E. It suffices to show $(\ ,\ )_E = (\ ,\ )_\psi$ *i.e.* that ψ preserves intersection.

2) Let C be a correspondence on E, *i.e.* a curve $C \subset E \times E$. Set

$$C_e = \{f \in E \mid (e, f) \in C\} \qquad e \in E$$

this gives a map

$$C\colon E \to J(E)\,, \qquad C(e) = C_e$$

(ignoring components $e \times E$ of C.) and this extends naturally to $J(E)$:

$$C^*\colon J(E) \to J(E)$$

inducing

$$\psi_C\colon H_1(E, \mathbb{Z}) \to H_1(E, \mathbb{Z})\,.$$

Note that C, C^*, ψ_C depend only on the linear equivalence class of C_e, all $e \in E$.

3) We wish to eliminate $J(X)$ from the picture too, and deal only with $J(E)$. Thus we need the intersection numbers in X of $\varphi_o(\gamma)$, $\varphi_o(\gamma')$ for loops γ, γ' in E; but each of these $(2n-1)$-cycles is a 1-parameter family of spaces $x \in A$, and all of these intersect o in codimension 1, hence each other if $n \geq 3$. To circumvent this difficulty we continuously deform $\varphi_o(\gamma')$ to $\varphi_x(\gamma')$ for a generic $x \in A$, $x \cap o = \emptyset$. Note that φ_o, φ_x induce the same $\psi\colon J(E) \to J(X)$. Now we have a correspondence

$$C \subset E \times E\,, \qquad C = \{(e, f) \mid \varphi_o(f) \cap \varphi_x(e) \neq \emptyset\}\,.$$

(By Lemma 2.3, C_e consists of n points for generic e.) and C induces the pull-back form on E:

$$(\gamma, \gamma')_\psi = (\gamma, \psi_C \gamma')_E \quad \text{for } \gamma, \gamma' \in H_1(E, \mathbb{Z})\,.$$

4) By part 2, to show that ψ_C is the identity it suffices to check that $C(e) - e$ varies in a linear system as e traces E. But by the description of the group law on A in § 2, the sum (in A, or $J(E)$) of the points of $C(e)$ is just $\varphi_x(e) = e - x$, so $C(e) - e$ is linearly equivalent to $-x$ (plus an integral multiple of o.) Thus

$$\psi_C = \psi_{\mathrm{Id}} = \mathrm{Id}\,. \quad \text{Q.E.D.}$$

COROLLARY 3.4. (« *Torelli's theorem* ») *X is determined (biregularly, and even projectively) by $J(X)$.*

PROOF. $J(X)$ is isomorphic to $J(E)$ which determines E by Torelli's theorem for (hyperelliptic) curves. $\psi\colon E \to \mathbb{P}^1$, the hyperelliptic covering, is unique (by Lemma 3.2, for one) determining the λ_i up to an automorphism of $\mathbb{P}^1$. As remarked in § 1.2, the matrices defining the pencil can be brought to the form

$$M(Q) = \mathrm{Id}\,, \qquad M(Q') = \mathrm{Diag}\,(\lambda_i)$$

choosing the vertices p_i of Q_{λ_i} as coordinate-points in $\mathbb{P}^{2n+1}$. Q.E.D.

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