Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 7, nº 1 (1980), p. 1-27

<http://www.numdam.org/item?id=ASNSP_1980_4_7_1_1_0>

Twisted Sheaves on Complex Spaces.

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Let $(X, \mathcal{O})$ be a complex space, let $\mathcal{L}$ be an invertible $\mathcal{O}$-module and let $\mathcal{F}$ be a coherent $\mathcal{O}$-module. Define $\mathcal{L}^m = \mathcal{L}^{\otimes m} = \mathcal{L} \otimes \ldots \otimes \mathcal{L}$ (m-times) for $m \geqslant 0$ ($\mathcal{L}^0 = \mathcal{O}$) and set $\mathcal{L}^m = (\mathcal{L}^{-1})^{-m}$ for $m \leqslant 0$, where $\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$ is the invertible dual sheaf of $\mathcal{L}$. The *twisted sheaves of* $\mathcal{F}$ *relative to* $\mathcal{L}$ are the sheaves $\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{L}^m$. Our purpose is the study of these sheaves.

In [7], if X fulfils some convexity or concavity assumptions and if $\mathcal{L}$ is associated to a positive or negative line bundle, one obtains theorems concerning the behaviour of the cohomology of these sheaves for $m \to +\infty$ similar to the vanishing theorem of Serre ([24], n. 74, th. 1). In a first paragraph we complete these results for the case $m \to -\infty$ using an idea that goes back to [24]. The paragraph also contains an attempt to formulate a dual theorem to theorem A of Cartan-Serre by giving a meaning to the « cogeneration of the cofibres ».

Let us consider the graded ring $\mathcal{A}(X, \mathcal{L}) = \bigoplus_{m=0}^{\infty} \Gamma(X, \mathcal{L}^m)$. In a second paragraph we study the finite generation over $\boldsymbol{C}$ of this ring and also the finite generation of the $\mathcal{A}(X, \mathcal{L})$-modules $\oplus H^q(X, \mathcal{F}(m))$, as well as the asymptotic behaviour of the function $m \rightsquigarrow \dim H^q(X, \mathcal{F}(m))$.

These results are established for X compact complex space and $\mathcal{L}$ such that a convenient power $\mathcal{L}^r$ of $\mathcal{L}$ has no fixed points (partial results are also given in the pseudoconcave case). These questions were started by Zariski in connection with the generalized 14-th Hilbert problem (see for example [28]). The ring $\mathcal{A}(X, \mathcal{L})$ was used in [5] and [6] to obtain compactification of pseudoconcave spaces.

In a third paragraph we consider the particular case $X = \boldsymbol{P}^n$ and $\mathcal{L} = \mathcal{O}_{\boldsymbol{P}^n}(1)$. In this case the freeness of the coherent sheaf $\mathcal{F}$ can be traced through the Hilbert polynomial $m \rightsquigarrow \Sigma(-1)^q \dim H^q(X, \mathcal{F}(m))$ and the

(†) Scomparso il 21 Febbraio 1980.

(**) Increst Bd. Păcii 220, Bucharest - Romania.

Pervenuto alla Redazione il 6 Luglio 1978.

question is connected with the so called Castelnuovo Lemma ([19], lecture 14) and with the dual version of it.

Almost all results could be generalized replacing $\mathcal{L}$ by a locally free coherent sheaf, substituting the powers with the corresponding symmetric tensor powers, but we confine ourselves to give only one remark in paragraph 2.

1. – The sheaves $\mathcal{F}(m)$ for $m \to \pm\infty$.

a) Dualizing sheaves. Let X be a complex space and let $\mathcal{F}$ be an analytic coherent sheaf on X (we write shortly $\mathcal{F} \in \operatorname{Coh} X$). For all Stein open sets U in X, the space $H^q_k(U, \mathcal{F})$ has a natural topology of DFS space and we can consider the strong dual of it. For an inclusion $V \subset U$ we can consider the transposed of the natural continuous extension map $H^q_k(V, \mathcal{F}) \to$ $\to H^q_k(U, \mathcal{F})$. In this way we define a presheaf whose associated sheaf is denoted $\mathfrak{D}^q \mathcal{F}$ and is called the *q-dualizing sheaf* of $\mathcal{F}$. In [2] and [4] the following statements are proved:

1) $\mathfrak{D}^q \mathcal{F}$ is a coherent analytic sheaf on X for any q;

2) for all Stein open sets U, $\Gamma(U, \mathfrak{D}^q \mathcal{F})$ equals the strong dual of $H^q_k(U, \mathcal{F})$;

3) for any embedding $i\colon U \to W$ of the open subset U of X into an n-dimensional manifold W there exist natural isomorphisms

$$i_*(\mathfrak{D}^q \mathcal{F}|U) \simeq \mathfrak{D}^q i_*(\mathcal{F}|U) \simeq \mathcal{E}xt^{n-q}_{\mathcal{O}_W}(i_*(\mathcal{F}|U), \Omega_W);$$

4) if X is of finite dimension and if $\mathcal{K}^\bullet_X$ denote the dualizing complex of X [22], then there are isomorphisms

$$\mathfrak{D}^q \mathcal{F} \simeq \mathcal{E}xt^{-q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}^\bullet_X);$$

5) if $\mathcal{F} \neq 0$ then $\mathfrak{D}^q \mathcal{F} = 0$ if $q \notin [\operatorname{depth} \mathcal{F}, \dim \mathcal{F}]$ and $\mathfrak{D}^q \mathcal{F} \neq 0$ when $q = \operatorname{depth} \mathcal{F}$ or $q = \dim \mathcal{F}$;

6) $\dim \mathfrak{D}^q \mathcal{F} \leqslant q$ for every q and $\mathfrak{D}^{\dim \mathcal{F}} = \mathcal{F}$.

We will need also the following

LEMMA. *If $\mathcal{F}$ is a Cohen-Macaulay coherent sheaf (i.e. $\mathcal{F} \neq 0$ and* depth $\mathcal{F} = \dim \mathcal{F}$*) then $\mathfrak{D}^q \mathcal{F} = 0$ for $q \neq q_0 = \dim \mathcal{F}$* and *$\mathfrak{D}^{q_0} \mathcal{F}$ is also Cohen Macaulay.*

PROOF. From the facts stated above we get $\mathcal{D}^q \mathcal{F} = 0$ for $q \neq q_0$ and $\dim \mathcal{D}^{q_0} \mathcal{F} = q_0$. It will be enough to show that depth $\mathcal{D}^{q_0} \mathcal{F} \geqslant q_0$.

The question is of local nature on X and using suitable embedding we may as well assume that X is a manifold of dimension n. As $\operatorname{dh} \mathcal{F} = n - q_0$, $\mathcal{F}$ has locally a resolution

$$0 \to \mathcal{E}^{n-q_0} \to \ldots \to \mathcal{E}^0 \to \mathcal{F} \to 0$$

where the $\mathcal{E}^i$ are free $\mathcal{O}_X$-modules of finite rank. Now $\mathcal{D}^q \mathcal{F} = \mathcal{E}xt^{n-q}_{\mathcal{O}_X}(\mathcal{F}, \Omega_X)$ and thus the cohomology of the complex

$$(*) \qquad 0 \to \mathcal{H}om(\mathcal{E}^0, \Omega_X) \to \ldots \to \mathcal{H}om(\mathcal{E}^{n-q_0}, \Omega_X) \to 0$$

is trivial but in the dimension $n - q_0$ where it is just $\mathcal{D}^{q_0}\mathcal{F}$. This means that $\mathcal{D}^{q_0}\mathcal{F}$ has locally a resolution of length $n - q_0$ by free sheaves of finite rank. Thus $\operatorname{dh} \mathcal{D}^q \mathcal{F} \leqslant n - q_0$ i.e. $\operatorname{depth}(\mathcal{D}^{q_0} \mathcal{F}) \geqslant q_0$.

b) *q-pseudoconvex spaces.* Let X be a complex space, let L be a holomorphic line bundle on X and let $\mathcal{F} \in \operatorname{Coh} X$. *We agree* to denote by $\mathcal{F}(m)$ the twisted sheaves relative to the invertible $\mathcal{O}_X$-module $\mathcal{L}$ of germs of holomorphic sections of L. We shall write $m \gg 0$ to mean « for m sufficiently large ».

THEOREM 1. *Let X be a strongly q-pseudoconvex space of finite dimension, let L be a positive holomorphic line bundle on X and let $\mathcal{F} \in \operatorname{Coh} X$. For the associated twisted sheaves $\mathcal{F}(m)$ we have*

(i) $H^r(X, \mathcal{F}(m)) = 0$ *for* $r > q$ *and* $m \gg 0$.

(ii) $H^r_k(X, \mathcal{F}(-m)) = 0$ *for* $r < \operatorname{depth} \mathcal{F} - q$ *and* $m \gg 0$.

PROOF. Statement (i) is theorem 1 of [7]. So we need to prove only (ii). For any r and any m the space $H^r_k(X, \mathcal{F}(-m))$ has a natural QDFS topology and its associated separated space is isomorphic to the strong dual of $\operatorname{Ext}^{-r}(X; \mathcal{F}(-m), \mathcal{K}^\bullet_X)$, this last space being endowed with its natural QFS topology [22]. For any m there exists a spectral sequence

$$E_2^{\alpha,\beta}(m) = H^\alpha\big(X, \mathcal{E}xt^\beta(\mathcal{F}(-m), \mathcal{K}^\bullet_X)\big)$$

which converge to $\operatorname{Ext}^{\alpha+\beta}(X; \mathcal{F}(-m), \mathcal{K}^\bullet_X)$.

For any three $\mathcal{O}_X$-modules $\mathcal{M}$, $\mathcal{N}$, $\mathcal{S}$ one gets a natural morphism

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N}) \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{S}, \mathcal{O}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{M} \otimes \mathcal{S}, \mathcal{N})$$

given by the map

$$\varphi \otimes \psi \to \big(m \otimes p \to \psi(p)\,\varphi(m)\big)\,.$$

Moreover, this is an isomorphism if $\mathcal{F}$ is locally free of finite rank.

Let $\mathcal{K}_X^\bullet \to \mathcal{I}^\bullet$ be an injective resolution of the dualizing complex. We have the isomorphism

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}om(\mathcal{L}^{-m}, \mathcal{O}) \simeq \mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes \mathcal{L}^{-m}, \mathcal{I}^\bullet)\,.$$

Taking cohomology we get the isomorphism

$$\mathcal{E}xt^{\beta}\big(\mathcal{F}(-m), \mathcal{K}_X^\bullet\big) \simeq \mathcal{E}xt^{\beta}(\mathcal{F}, \mathcal{K}_X^\bullet)(m) \simeq (\mathcal{D}^{-\beta}\mathcal{F})(m)\,.$$

As $\mathcal{D}^{\beta}\mathcal{F} = 0$ for all but finitely many β's, by means of (i) we can find m_0 such that

$$H^{\alpha}\big(X, \mathcal{D}^{\beta}\mathcal{F}(m)\big) = 0\,, \qquad \forall\alpha > q,\ \forall\beta,\ \forall m \geqslant m_0\,.$$

On the other hand $\mathcal{D}^{\beta}\mathcal{F} = 0$ if $\beta < \operatorname{depth}\mathcal{F}$. Therefore

$$E_2^{\alpha,\beta}(m) = 0 \qquad \text{for } \alpha + \beta = -r \text{ when } r < \operatorname{depth}\mathcal{F} - q \text{ and } m \gg 0\,.$$

As $H_k^r\big(X, \mathcal{F}(m)\big)$ is separated iff $\operatorname{Ext}^{-r+1}\big(X; \mathcal{F}(-m), \mathcal{K}_X^\bullet\big)$ is separated [22], we conclude with the assertion (ii).

REMARK. The proof shows that $H_k^{\operatorname{depth}\mathcal{F}-q}\big(X, \mathcal{F}(-m)\big)$ is separated in agreement with the general statement of separation of these groups on strongly q-pseudoconvex spaces [3].

Let us denote by H_r and H_r^{∞} the homology groups (with compact supports) and respectively the homology groups with closed supports. We will denote by the suffix $*$ the associated dual cosheaf [4]. In virtue of the previous theorem and the separation of $H_k^{\operatorname{depth}\mathcal{F}-q}\big(X, \mathcal{F}(-m)\big)$ we get

THEOREM 2. *Let X be a strongly q-pseudoconvex space of finite dimension. Let $\mathcal{F} \in \operatorname{Coh} X$, let L be a positive holomorphic line bundle on X and denote by $\mathcal{F}(m)$ the corresponding twisted sheaves. Then*:

(i) $H_r\big(X, \mathcal{F}(m)_*\big) = 0$ *for* $r > q$ *and* $m \gg 0$,

(ii) $H_r\big(X, \mathcal{F}(-m)_*\big) = 0$ *for* $r < \operatorname{depth}\mathcal{F} - q$ *and* $m \gg 0$.

c) *The pseudoconvex case* (i.e. $q = 0$). We first recall the following definition [15]:

Let A be a local ring with maximal ideal $\mathfrak{m}$ and let M an A-module; we denote by

$$\gamma(M) = (0\colon \mathfrak{m})_M = \{s \in M | \mathfrak{m}s = 0\}$$

and we call it the socle of M.

Let $(X, \mathcal{O})$ be a complex space and $\mathcal{F} \in \operatorname{Coh} X$. We say that « *$\mathcal{F}$ fulfils the dual of theorem* A *in dimension* r » or that « *the space* $H^r_k(X, \mathcal{F})$ *cogenerate its cofibres* » if for every $x \in X$ the canonical map $H^r_x(X, \mathcal{F}) \xrightarrow{i} H^r_k(X, \mathcal{F})$ is injective on the socle of $H^r_x(X, \mathcal{F})$ (i.e. if $\xi \in H^r_x(X, \mathcal{F})$ is such that $i(\xi) = 0$ and $\mathfrak{m}_x \xi = 0$ then $\xi = 0$).

REMARK. Let $\mathfrak{D}$ be a precosheaf on X. For a point $x \in X$ we define the cofibre $\mathfrak{D}^x = \varprojlim \mathfrak{D}(U)$, U open neighbourhood of x. When $\mathfrak{D}$ is the dualizing cosheaf $U \to H^r_k(U, \mathcal{F})$ [4], a duality argument shows that its cofibre in x equals $H^r_x(X, \mathcal{F})$, the r-cohomology with supports in $\{x\}$. We say that $\mathfrak{D}$ « verifies the strong dual of theorem A » or « $\mathfrak{D}$ is strongly cogenerated by the global cosections » if the maps $\mathfrak{D}^x \to \mathfrak{D}(X)$ are injective. If $\mathfrak{D}$ is the cosheaf $U \rightsquigarrow H^r_x(U, \mathcal{F})$ this condition is equivalent to the fact that the maps $H^r_x(X, \mathcal{F}) \to H^r_k(X, \mathcal{F})$ are injective. That is true for example when X is a Stein space (by means of a duality argument). This strong formulation of the dual of the theorem A implies the previous formulation, which was inspired by the fact that the theorem A is nothing else but the surjectivity of the map $\Gamma(X, \mathcal{F}) \to \mathcal{F}_x/\mathfrak{m}_x \mathcal{F}_x$.

We have the following useful

LEMMA. $H^r_x(X, \mathcal{F}) = 0$ *iff* $\gamma(H^r_x(X, \mathcal{F})) = 0$.

PROOF. Via an embedding around x we are reduced to the case when X is a manifold of a certain dimension n. Now $H^r_x(X, \mathcal{F})$ has a natural FS topology and its dual is isomorphic to $\operatorname{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}_x, \Omega_x)$, this last space being endowed with the DFS topology given by uniform convergence of germs via the isomorphism

$$\operatorname{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}_x, \Omega_x) \simeq \mathcal{E}xt^{n-r}_{\mathcal{O}}(\mathcal{F}, \Omega)_x .$$

Now $\mathfrak{m}_x \operatorname{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}_x, \Omega_x)$ is a closed subspace of $\operatorname{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{O}_x)$ (as it is analytic submodule). The topological dual of the quotient

$$\operatorname{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}_x, \Omega_x)/\mathfrak{m}_x \operatorname{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}_x, \Omega_x)$$

is just $\gamma(H^r_x(X, \mathcal{F}))$. Thus the assumption $\gamma(H^r_x(X, \mathcal{F})) = 0$ brings

$$\operatorname{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}_x, \Omega_x) = \mathfrak{m}_x \operatorname{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}_x, \Omega_x)$$

and by Nakayama's lemma we get $\operatorname{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}_x, \Omega_x) = 0$ thus, by duality $H^r_x(X, \mathcal{F}) = 0$. And conversely.

We have the following

THEOREM 3. *Let X be a strongly pseudoconvex space of finite dimension, let L be a positive holomorphic line bundle on X, let $\mathcal{F} \in \operatorname{Coh} X$ and let $\mathcal{F}(m)$ be the corresponding twisted sheaf. Then*

(i) *$\mathcal{F}(m)$ verifies theorem* A *if $m \gg 0$,*

(ii) *$\mathcal{F}(-m)$ verifies the dual of theorem A if $m \gg 0$.*

PROOF. Note that in ([11], theorem II) a weaker form of (i) is proved. Here is the general argument.

Let $X \xrightarrow{\pi} Y$ be the Remmert reduction; π is proper and biholomorphic outside a compact set $K \subset X$. For every $\mathcal{G} \in \operatorname{Coh} X$, $\pi_*(\mathcal{G})$ is coherent on Y. By theorem A of Cartan, $\Gamma(Y, \pi_*(\mathcal{G}))$ generated the fibres of $\pi_*(\mathcal{G})$. Consequently $\Gamma(X, \mathcal{G})$ generates the fibres $\mathcal{G}_x$ for all $x \in X \setminus K$. In virtue of this remark it will be sufficient to show the existence of an integer $m_0 = m_0(\mathcal{F})$ such that $\Gamma(X, \mathcal{F}(m))$ generates the fibres $\mathcal{F}(m)_x$ for any $x \in K$ and $m \geqslant m_0$. We follow the argument by which theorem A is deduced from theorem B. First we establish that there is an integer $l_0 > 0$ such that $\Gamma(X, \mathcal{O}(l_0))$ generates the fibres all over K (and hence in all points of X). Let $x \in K$ and let $\mathfrak{m}(x)$ be the maximal ideal sheaf given by x. From the exact sequence $0 \to \mathfrak{m}(x) \to \mathcal{O} \to \mathcal{O}/\mathfrak{m}(x) \to 0$ we get the exact sequence

$$0 \to \mathfrak{m}(x)(m) \to \mathcal{O}(m) \to \big(\mathcal{O}/\mathfrak{m}(x)\big)(m) \to 0 .$$

By theorem 1, $H^1(X, \mathfrak{m}(x)(m)) = 0$ if $m \gg 0$, therefore the maps

$$\Gamma(X, \mathcal{O}(m)) \to \mathcal{O}(m)_x/\mathfrak{m}_x \mathcal{O}(m)_x$$

are surjective. By Nakayama lemma $\Gamma(X, \mathcal{O}(m))$ generates the fibres $\mathcal{O}(m)_x$ if m is large. Let us fix such a m. By coherence there exists a neighbourhood U of x such that $\Gamma(X, \mathcal{O}(m))$ generates the fibres $\mathcal{O}(m)_{x'}$, for all $x' \in U$. Moreover we note that this property is preserved by changing m with a positive multiple of it. By a compacity argument we find an integer l_0 with the required property.

Let $\mathcal{F} \in \operatorname{Coh} X$. We claim that the sheaves $\mathcal{F}(ml_0)$ are spanned by global sections if $m \gg 0$. With the above notation we have the exact sequence

$$0 \to \mathfrak{m}(x)\, \mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathfrak{m}(x)\, \mathcal{F} \to 0 .$$

As $H^1\big(X, (\mathfrak{m}(x)\,\mathcal{F})(ml_0)\big) = 0$ for $m \gg 0$, as before we obtain that $\Gamma\big(X, \mathcal{F}(ml_0)\big)$ generates the fibre $\mathcal{F}(ml_0)_x$ if $m \gg 0$. Let us fix such an integer m. By coherence $\Gamma\big(X, \mathcal{F}(ml_0)\big)$ generates the fibres in a neighbourhood U of x. As $\Gamma\big(X, \mathcal{O}(l_0)\big)$ generates the fibres of $\mathcal{O}(l_0)$, for all $m' \geqslant m$ the space $\Gamma\big(X, \mathcal{F}(m'l_0)\big)$ spans the fibres through the same U and then by a compacity argument we conclude that $\Gamma\big(X, \mathcal{F}(ml_0)\big)$ generates $\mathcal{F}(ml_0)_x$ for for $m \gg 0$.

We can now prove statement (i). We apply the previous assertion to each of the sheaves $\mathcal{F}, \mathcal{F}(1), \dots, \mathcal{F}(l_0 - 1)$. Consequently the shaves $\mathcal{F}(ml_0)$, $\mathcal{F}(ml_0 + 1), \dots, \mathcal{F}(ml_0 + l_0 - 1)$ are spanned by the global sections for $m \gg 0$. For each m we can write $m = m'l_0 + r$ with $0 \leqslant r < l_0$, and if $m \gg 0$ then $m' \gg 0$. From this assertion (i) follows.

We turn to the proof of (ii). First we remark that for any complex space X (of finite dimension) and for any coherent sheaf $\mathcal{G}$, the canonical maps

$$\operatorname{Ext}^q(X; \mathcal{G}, \mathcal{K}_X^\bullet) \to \Gamma\big(X, \mathcal{E}xt^q(\mathcal{G}, \mathcal{K}_X^\bullet)\big)$$

are continuous when Ext is endowed with the QFS-topology inherited by the duality theory [22] and when the target space is endowed with the natural topology on the sections of a coherent sheaf. For that it suffices to show that, by composition with the restriction maps

$$\Gamma(X, \mathcal{E}xt\, \dots) \to \Gamma(U, \mathcal{E}xt\, \dots)\,,$$

with U open and Stein, we obtain continuous maps. This derives from the commutative diagrams

$$\begin{array}{ccc} \operatorname{Ext}(X, \dots) & \to & \Gamma(X, \mathcal{E}xt\, \dots) \\ \downarrow & & \downarrow \\ \operatorname{Ext}(U, \dots) & \to & \Gamma(U, \mathcal{E}xt\, \dots) \end{array}$$

and the fact that our assertion is already known when $X = U$. The proof of (ii) then proceeds as follows. By theorem 1 $H^\alpha\big(X, (\mathcal{D}^\beta\mathcal{F})(m)\big) = 0$ for every $\alpha \geqslant 1$, every integer β and $m \ll 0$. Since

$$\mathcal{E}xt^{-\beta}\big(\mathcal{F}(-m), \mathcal{K}_X^\bullet\big) \simeq \mathcal{E}xt^{-\beta}(\mathcal{F}, \mathcal{K}_X^\bullet)(m) = (\mathcal{D}^\beta\mathcal{F})(m)$$

one deduces that the maps

$$\operatorname{Ext}^{-r}\big(X; \mathcal{F}(-m), \mathcal{K}_X^\bullet\big) \to \Gamma\Big(X, \mathcal{E}xt^{-r}\big(\mathcal{F}(-m), \mathcal{K}_X^\bullet\big)\Big) \simeq \Gamma\big(X, (\mathcal{D}^r\mathcal{F})(m)\big)$$

are bijective for every r and $m \gg 0$. As these maps are continuous they are topological isomorphisms. By duality we get then $H^r_k(X, \mathcal{F}(-m)) \simeq$ $\simeq$ strong dual of $\Gamma(X, (\mathcal{D}^r\mathcal{F})(m))$ for $r \geqslant 0$ and $m \gg 0$. If we apply statement (i) to any $\mathcal{D}^r\mathcal{F}$ we obtain that the maps

$$(*) \qquad \Gamma(X, \mathcal{D}^r\mathcal{F}(m)) \to (\mathcal{D}^r\mathcal{F})(m)_x/\mathfrak{m}_x(\mathcal{D}^r\mathcal{F})(m)_x$$

are surjective for every r and x if $m \gg 0$. Now $H^r_x(X, \mathcal{F}(-m))$ is isomorphic to the topological dual of

$$\mathrm{Ext}^{-r}_{\hat{\mathcal{O}}_x}(\mathcal{F}(-m), \mathcal{K}^\bullet_X) \simeq \mathcal{E}xt^{-r}(\mathcal{F}(-m), \mathcal{K}^\bullet_X)_x \simeq (\mathcal{D}^r\mathcal{F})(m)_x \, .$$

Hence by transposing $(*)$ we get that the extension map

$$H^r_x(X, \mathcal{F}(-m)) \to H^r_k(X, \mathcal{F}(-m))$$

is injective on $\gamma\big(H^r_x(X, \mathcal{F}(-m))\big)$ for $m \gg 0$. Consequently, $\mathcal{F}(-m)$ verifies the dual of theorem A for $m \ll 0$ in any dimension r.

COROLLARY. *Assume that* $\sup\limits_{x\in X} \dim(\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x) < \infty$. *Then* $\mathcal{F}$ *is globally the quotient of a coherent locally free sheaf.*

PROOF. Let $m > 0$ be so chosen that $\mathcal{F}(m)$ is spanned by the global sections. Consider Remmert's reduction $X \xrightarrow{\pi} Y$ and let K be a compact in X such that π is biholomorphic on $X \setminus K$. There exist sections $s_1, \ldots, s_r \in$ $\in \Gamma(X, \mathcal{F}(m))$ which generate $\mathcal{F}(m)$ on K. The sheaf $\mathcal{G} = \pi_*(\mathcal{F})(m)$ is coherent on a Stein space Y and as $X\setminus K \simeq Y \setminus \pi(K)$ it follows that

$$\sup_{y\in Y} \dim(\mathcal{G}_y/\mathfrak{m}_y\mathcal{G}_y) < \infty \, .$$

By ([10], [12]) $\mathcal{G}$ is spanned by finitely many global sections. Using this fact, we can find $t_1, \ldots, t_q \in \Gamma(X, \mathcal{F}(m))$ which spann $\mathcal{F}(m)$ on $X\setminus K$. Hence the morphism $\mathcal{O}^{p+q}_X \to \mathcal{F}(m)$ given by $(s_1, \ldots, s_p, t_1, \ldots, t_q)$ is an epimorphism and therefore we get an epimorphism $\mathcal{O}^{p+q}_X(-m) \to \mathcal{F} \to 0$.

THEOREM 4. *Let X be a strongly pseudoconvex space of finite dimension, let L be a positive holomorphic line bundle on X, let q be an integer, let $\mathcal{F} \in \mathrm{Coh}\, X$ and let $\mathcal{F}(m)$ denote the associated twisted sheaves. Then*

(i) depth $\mathcal{F} \geqslant q$ *if and only if* $H^r_k(X, \mathcal{F}(-m)) = 0$ *for* $r < q$ *and* $m \gg 0$;

(ii) $\dim \mathcal{F} \leqslant q$ *if and only if* $H^r_k(X, \mathcal{F}(-m)) = 0$ *for* $r > q$ *and* $m \gg 0$;

(iii) *for every* $m \gg 0$ *and for any* $x \in X$ *such that depth* $\mathcal{F}_x = q$ *or* $\dim \mathcal{F}_x = q$ *there exists a cohomology class* $\xi \in H^q_k(X, \mathcal{F}(-m))$ *with* $\operatorname{supp} \xi = \{x\}$ (i.e. $\xi \neq 0$ *and* $\xi \in \operatorname{Im}\big(H^q_x(X, \mathcal{F}(-m)) \to H^q_k(X, \mathcal{F}(-m))\big)\big)$.

PROOF. By the same argument given in the proof of theorem 3, there exists an integer m'_0 such that

$$H^r_k(X, \mathcal{F}(-m)) \simeq \text{strong dual of } \Gamma(X, (\mathcal{D}^r\mathcal{F})(m))$$

for $m \geqslant m'_0$. On the other hand, by statement (i) of theorem 3, there exists an integer m''_0 for which the sheaves $(\mathcal{D}^r\mathcal{F})(m)$ are spanned by their global sections for all r if $m \geqslant m''_0$.

Therefore, for any r and any $m \geqslant m_0 = \sup(m'_0, m''_0)$

$$H^r_k(X, \mathcal{F}(-m)) = 0 \quad \text{if and only if} \quad \mathcal{D}^r\mathcal{F} = 0\,.$$

The statements (i) and (ii) follow now by what has been said in section *a*). For any q and $m \gg 0$ by statement (ii) of theorem 3 the extension map

$$\gamma\big(H^q_x(X, \mathcal{F}(-m))\big) \to H^q_k(X, \mathcal{F}(-m))$$

is injective. It is enough to show that the socle of $H^q_x(X, \mathcal{F}(-m))$ is non zero to find the desired cohomology class ξ. By the above lemma it suffices to show that $H^q_x(X, \mathcal{F}(-m)) \neq 0$. Now, by duality $H^q_x(X, \mathcal{F}(-m))$ is isomorphic to the topological dual of $(\mathcal{D}^q\mathcal{F})(m)_x \simeq (\mathcal{D}^q\mathcal{F})_x$. In accordance with section *a*) the $\mathcal{O}_x$ module $(\mathcal{D}^q\mathcal{F})_x$ is non null provided that $q = \operatorname{depth} \mathcal{F}_x$ or $q = \dim \mathcal{F}_x$.

COROLLARY. *Let X be a normal strongly pseudoconvex space of dimension $\geqslant 2$, let $\mathcal{F} \neq 0$ be a locally free sheaf of finite rank and let $\mathcal{F}(m)$ be the twisted sheaves corresponding to a positive line bundle on X. Then*

$$H^0_k(X, \mathcal{F}(-m)) = H^1_k(X, \mathcal{F}(-m)) = 0 \quad \textit{if } m \gg 0\,.$$

Moreover, let us assume that X is of pure dimension 2. Then for every $m \ll 0$ and for any point $x \in X$ there exist cohomology classes $\xi \in H^2_k(X, \mathcal{F}(-m))$ for which $\operatorname{supp}(\xi) = \{x\}$.

Indeed depth $\mathcal{F} \geqslant 2$ and depth $\mathcal{F} = 2$ when X is of pure dimension 2. We may note also the following consequence of theorem 1, 3 and 4.

COROLLARY. *Let X be a strongly pseudoconvex open subset of a non singular n-dimensional projective variety, let $\mathcal{F}$ be an analytic coherent sheaf on X locally free and let $\mathcal{F}(m)$ denote the twisted sheaves associated to the hyperplane divisor. Then*

(i) $H^r(X, \mathcal{F}(m)) = 0$ *for* $r > 0$ *and* $\Gamma(X, \mathcal{F}(m))$ *generates the fibres of* $\mathcal{F}(m)$ *if* $m \ll 0$;

(ii) $H^r_k(X, \mathcal{F}(-m)) = 0$ *for* $r < n$ *and for any* $x \in X$ *there exists* $\xi \in H^n_k(X, \mathcal{F}(-m))$ *such that* $\operatorname{supp} \xi = \{x\}$ *if* $m \gg 0$.

REMARK. As theorems 3 and 4 show, the strongly pseudoconvex spaces possessing a positive line bundle are simultaneously generalisation of Stein spaces and of projective varieties. Let remind that a complex space X is called after Grauert and Remmert projectively separated (cf. [13] where is used the term « analytically separated »; « projectively separated » has been proposed by H. Cartan) if for any $x \in X$ there is a morphism into a projective space such that x is isolated in the fibre. Then we have the following

STATEMENT. *Let X be a strongly pseudoconvex space of bounded Zariski dimension. The following assertions are equivalent*:

(i) *X admits a positive line bundle,*

(ii) *There exists a closed embedding* $X \hookrightarrow \boldsymbol{C}^N \times \boldsymbol{P}^N$,

(iii) *X is projectively separated.*

PROOF. First of all, let us remark that the implications (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are easy. The implication (i) $\Rightarrow$ (ii) is proved in [11] in the non-singular case, but the argument still works in the general case: using the vanishing theorem [7], one gets an integer m_0 and sections $s_0, \ldots, s_t \in \Gamma(X, \mathfrak{L}^{m_0})$ which give rise to a mapping from a neighbourhood of the exceptional set K of X into $\boldsymbol{P}^t$ and which is injective on K and local embedding in the points of K. Then, one finds sections without common zeros $s^{t+1}, \ldots, s^{t+s}$ of the analytical restriction of $\mathfrak{L}^{m_0}$ to $X^* = \{x \in X | s_j(x) = 0, 1 \leqslant j \leqslant t\}$ (X^* is a closed analytic subset of X, contained in $X \setminus K$, hence is Stein and one makes use of theorems A and B). Using again [7] and replacing eventually m_0 by a multiple, one can extend these sections on the whole of X; the morphism $X \to \boldsymbol{P}^{t+s}$, together with an embedding $Y \to \boldsymbol{C}^p$ of Remmert's reduction gives the required embedding (see [11] for details). The implication (iii) $\Rightarrow$ (ii) is a straightforward consequence of the vanishing theorem of Grauert and Remmert for projective maps as follows. Let $X \xrightarrow{\pi} Y$ be the Remmert reduction, K the maximal compact analytic subset of X

and U a relatively compact open set which contains $\pi(K)$. By the hypothesis, we can find a morphism $X \to \boldsymbol{P}^n$ such that each $x \in \pi^{-1}(U)$ will be isolated in the fibre. Consider the morphism $f\colon X \to \boldsymbol{P}^n \times Y$ given by the product of the previous one and π. Denote by $\mathcal{O}_{\boldsymbol{P}^n \times Y}(1)$ the reciprocal image of $\mathcal{O}_{\boldsymbol{P}^n}(1)$ through the projection $\boldsymbol{P}^n \times Y \to \boldsymbol{P}^n$, and let $\mathcal{L} = f^*\big(\mathcal{O}_{\boldsymbol{P}^n \times Y}(1)\big)$. We claim that for every $\mathcal{F} \in \operatorname{Coh} X$, $H^q\big(X, \mathcal{F}(m)\big) = 0$ for $q \geqslant 1$ when $m \gg 0$: if this is true, then by the proof of (i) $\Rightarrow$ (ii), the conclusion follows. Denote by $p\colon \boldsymbol{P}^n \times Y \to Y$ the projection. Using Leray spectral sequence of π and theorem B on Y we get isomorphisms $H^q\big(X, \mathcal{F}(m)\big) \simeq \Gamma\Big(Y, R^q\pi_*\big(\mathcal{F}(m)\big)\Big)$. As the sheaves $R^q\pi_*\big(\mathcal{F}(m)\big)$ are zero on $Y \setminus \pi(K)$ for $q \geqslant 1$, to finish the proof it suffices to show that $R^q\pi_*\big(\mathcal{F}(m)\big)|_U = 0$ for $q \geqslant 1$ if $m \gg 0$. Since f is finite one obtains (for example, looking at the spectral sequence associated to the composition $\pi = pf$, as $R^q f_* = 0$ for $q \geqslant 1$) the isomorphism

$$R^q\pi_*\big(\mathcal{F}(m)\big) \simeq R^q p_*\Big(f_*\big(\mathcal{F}(m)\big)\Big) .$$

On the other hand it is easy to see that $f_*\big(\mathcal{F}(m)\big) \simeq \big(f_*(\mathcal{F})\big)(m)$. Now the proof is over, since from the theorem of Grauert and Remmert recalled above, $R^q p_*\big(f_*(\mathcal{F})(m)\big)|_U = 0$ for $q \geqslant 1$ when $m \gg 0$.

d) q-pseudoconcave case. We have the following

THEOREM 5. *Let X be a strongly q-pseudoconcave space of finite dimension, let $\mathcal{F} \in \operatorname{Coh} X$ and let $\mathcal{F}(m)$ be the associated twisted sheaves corresponding to a negative holomorphic line bundle. Then*

(i) $H^r\big(X, \mathcal{F}(m)\big) = 0$ *for* $r < \operatorname{depth} \mathcal{F} - q - 1$ *and* $m \gg 0$,

(ii) $H^r_k\big(X, \mathcal{F}(-m)\big) = 0$ *for* $r > q + 1$, $m \gg 0$, *provided $\mathcal{F}$ is Cohen-Macaulay.*

PROOF. The statement (i) is theorem 2 of [7]. We need only to prove (ii). For any r and m the separated space associated to the QDFS space $H^r_k\big(X, \mathcal{F}(-m)\big)$ is isomorphic to the strong dual of $\operatorname{Ext}^{-r}\big(X;\, \mathcal{F}(-m), \mathcal{K}^\bullet_X\big)$. For any m consider the spectral sequence of term

$$E_2^{\alpha,\beta}(m) = H^\alpha\Big(X, \mathcal{E}xt^\beta\big(\mathcal{F}(-m), \mathcal{K}^\bullet_X\big)\Big) \simeq H^\alpha\big(X, (\mathcal{D}^{-\beta}\mathcal{F})(m)\big)$$

which converge to $\operatorname{Ext}^{\alpha+\beta}\big(X;\, \mathcal{F}(-m), \mathcal{K}^\bullet_X\big)$. Let $\beta_0 = \dim \mathcal{F} = \operatorname{depth} \mathcal{F}$. In virtue of the lemma in section a), $\mathcal{D}^\beta\mathcal{F} = 0$ if $\beta \neq \beta_0$ and $\mathcal{D}^{\beta_0}\mathcal{F}$ is Cohen-Macaulay and of dimension β_0. Accordingly, depth $(\mathcal{D}^{\beta_0}\mathcal{F}) = \beta_0$ and by (i), $H^\alpha\big(X, (\mathcal{D}^{\beta_0}\mathcal{F})(m)\big) = 0$ when $\alpha < \beta_0 - q - 1$ and $m \gg 0$. It follows that $E_2^{\alpha,\beta}(m) = 0$ when $\alpha + \beta < -q - 1$ and $m \gg 0$. Hence $\operatorname{Ext}^{-r}\big(X;\, \mathcal{F}(m), \mathcal{K}^\bullet_X\big) = 0$

when $r > q+1$ and $m \gg 0$. Now $H^r_k(X, \mathcal{F}(-m))$ is separated iff $\operatorname{Ext}^{-r+1}(X; \mathcal{F}(-m), \mathcal{K}^\bullet_X)$ is separated. By duality we deduce that $H^r_k(X, \mathcal{F}(-m)) = 0$ when $r > q+2$ and that the separated space associated to $H^{q+2}_k(X, (\mathcal{F}-m))$ is zero. But $H^{q+2}_k(X, \mathcal{F}(-m))$ is finite dimensional hence separated and hence zero.

COROLLARY. *Let X be a strongly pseudoconcave open subset of a nonsingular n-dimensional projective variety, let $\mathcal{F}$ be a coherent sheaf on X, locally free and let $\mathcal{F}(m)$ denote the twisted sheaves which correspond to the hyperplane section. Then*

(i) $H^r(X, \mathcal{F}(-m)) = 0$ *for* $r < n-1$ *and* $m \gg 0$;

(ii) $H^r_k(X, \mathcal{F}(m)) = 0$ *for* $r > 1$ *and* $m \gg 0$.

For any strongly q-pseudoconcave space and any $\mathcal{G} \in \operatorname{Coh} X$, the space $H^{\operatorname{depth}\mathcal{G}-q-1}(X, \mathcal{G})$ is separated. This is proved in [4] for the nonsingular case and in [21] for the general case. Using this fact, together with theorem 5 and duality, one obtains the following

THEOREM 6. *Let X be a strongly q-pseudoconcave space of finite dimension, let $\mathcal{F} \in \operatorname{Coh} X$ and denote by $\mathcal{F}(m)$ the twisted sheaves associated to a negative holomorphic line bundle on X. Then*

(i) $H_r(X, \mathcal{F}(m)_*) = 0$ *for* $r < \operatorname{depth}(\mathcal{F}) - q - 1$ *and* $m \gg 0$;

(ii) $H^\infty_r(X, \mathcal{F}(m)_*) = 0$ *for* $r > q+1$ *and* $m \gg 0$, *provided $\mathcal{F}$ is Cohen-Macaulay.*

REMARK. We do not know if the Cohen-Macaulay assumptions in theorems 5 and 6 are effectively needed.

2. – The algebra $\mathcal{A}(X, \mathcal{L})$ and some polynomial functions.

a) Definitions. Let X be a complex space and let $\mathcal{L}$ be an invertible sheaf on X. We denote by $\mathcal{A}(X, \mathcal{L})$ the graded ring

$$\mathcal{A}(X, \mathcal{L}) = \bigoplus_{m=0}^{\infty} \Gamma(X, \mathcal{L}^m);$$

tensorial multiplication gives the natural structure of graded $\boldsymbol{C}$-algebra. For an analytic sheaf $\mathcal{F}$ on X and for any integer $q \geq 0$, we denote by $\mathcal{M}^q(X; \mathcal{L}, \mathcal{F})$ or simply $\mathcal{M}^q(X, \mathcal{F})$ the $\mathcal{A}(X, \mathcal{L})$-graded module $\bigoplus_{m=0}^{\infty} H^q(X, \mathcal{F}(m))$ (here, as usual, $\mathcal{F}(m)$ are the associated twisted sheaves).

We say that $\mathcal{A}(X, \mathcal{L})$ is without fixed point if, for any $x \in X$, there exist an integer $m > 0$ and a section $s \in \Gamma(X, \mathcal{L}^m)$ such that $s(x) \neq 0$ (i.e. $s_x \notin \mathfrak{m}_x \mathcal{L}_x^m$). If for all $\mathcal{F} \in \operatorname{Coh} X$ we have a vanishing theorem of the form $H^1(X, \mathcal{F}(m)) = 0$ for $m \gg 0$, then $\mathcal{A}(X, \mathcal{L})$ is without fixed points; this as we have seen occurs in many instances when $\mathcal{L}$ corresponds to a positive holomorphic line bundle. Another example is given by the case of a nonsingular algebraic projective variety X when $\mathcal{L} = \{D\}$ is associated to a divisor D whose linear system $|D|$ has finitely many base points; indeed, by a result due to Zariski ([28], theorem 6.2) the complete linear system $|mD|$ for $m \gg 0$ has no base points.

Let $|D|$ be a complete linear system on a nonsingular projective surface F over an algebraically closed ground field k. We assume that some multiple $|mD|$ of $|D|$ has no base points. In the same paper of Zariski the following statements are proved (theorem 6.5 and p. 611 (2)):

a) The ring $R^*(D) = \bigoplus \Gamma(F, \{mD\})$ is finitely generated over k;

b) There exists a finite number of polynomials $f_1(t), \ldots, f_n(t)$ of one variable t such that, setting

$$s(mD) = \dim H^1(F, \{mD\})$$

(superabundance of $|mD|$), we have $s(mD) = f_{\lambda(m)}(m)$ for $m \gg 0$, where $\lambda(m) \in \{1, 2, \ldots, n\}$ is a periodic function of m;

c) A counterexample is also given when the graded ring $R^*(D)$ is not finitely generated.

These statements emphasize the interest of the following questions:

1) is the algebra $\mathcal{A}(X, \mathcal{L})$ finitely generated?

2) are the $\mathcal{A}(X, \mathcal{L})$-modules $\mathcal{M}^q(X, \mathcal{F})$ finitely generated?

3) what is the behaviour of the function

$$m \to \dim H^q(X, \mathcal{F}(m))$$

for $m \gg 0$?

Before we examine these questions let us first recall the following theorem of finiteness for graded sheaves proved in [8].

Let $(X, \mathcal{O})$ be a complex space and let $T_1, \ldots, T_N$ denote some indeterminates. The sheaf of polynomials $\mathcal{O}[T_1, \ldots, T_N]$ is a coherent sheaf of rings ([8], lemma 1.2). From any morphism $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between complex spaces one deduces a natural morphism, also denoted by f,

$$f\colon (X, \mathcal{O}_X[T_1, \dots, T_N]) \to (Y, \mathcal{O}_Y[T_1, \dots, T_N]) .$$

One has the following facts:

« If f is a proper morphism, for any graded coherent $\mathcal{O}_X[T_1, \dots, T_N]$-module $\mathcal{M}$, the generalized images $R^q f_*(\mathcal{M})$ are coherent $\mathcal{O}_Y[T_1, \dots, T_N]$-modules ([8], theorem 1) ».

In particular we obtain:

« If $(X, \mathcal{O})$ is a compact complex space and $\mathcal{M}$ is a coherent graded $\mathcal{O}[T_1, \dots, T_N]$-module, then $H^q(X, \mathcal{M})$ is a $\boldsymbol{C}[T_1, \dots, T_N]$-module of finite type for any value of q ».

b) Compact case. We have the following

THEOREM 7. *Let X be a compact complex space, let $\mathcal{L}$ be an invertible sheaf on X and let $\mathcal{F} \in \operatorname{Coh} X$.*

(i) *Assume $\mathcal{A}(X, \mathcal{L})$ without fixed points. Then the $\boldsymbol{C}$-algebra $\mathcal{A}(X, \mathcal{L})$ is finitely generated and for every q, $\mathcal{M}^q(X, \mathcal{F})$ is an $\mathcal{A}(X, \mathcal{L})$-module of finite type.*

(ii) *Assume $\Gamma(X, \mathcal{L})$ without fixed points. Then for every q the function $m \to \dim H^q(X, \mathcal{F}(m))$ is a polynomial of degree $\leqslant \dim \mathcal{F}$ for $m \gg 0$, and the function $m \to \Sigma(-1)^q \dim H^q(X, \mathcal{F}(m))$ is just a polynomial.*

(iii) *Assume that for any two distinct points x, x' there exists a section $s \in \Gamma(X, \mathcal{L})$ such that $s(x) = 0$ and $s(x') \neq 0$. Then the degree of the polynomial $m \to \Sigma(-1)^q \dim H^q(X, \mathcal{F}(m))$ equals $\dim \mathcal{F}$.*

PROOF. (i) Suppose first that the elements of $\Gamma(X, \mathcal{L})$ have no common zero. As X is compact there exist elements $s_1, \dots, s_N \in \Gamma(X, \mathcal{L})$ such that for every $x \in X$ at least one of them s_i has the property $s_i(x) \neq 0$.

By the substitution $T_i \to s_i$ one obtains a natural structure of graded $\mathcal{O}[T_1, \dots, T_N]$-module on $\mathcal{M} = \mathcal{M}(\mathcal{F}) = \mathcal{F} \oplus \mathcal{F}(1) \oplus \dots$. We claim that $\mathcal{M}$ is $\mathcal{O}_X[T_1, \dots, T_N]$-coherent. Let $x \in X$; choose a section s_i such that $s_i(x) \neq 0$. In a neighbourhood U of x the morphism $\mathcal{O}_X \to \mathcal{L}$ given by $\varphi \to \varphi s_i$ is an isomorphism. Then $\mathcal{M}|_U \simeq \mathcal{F}[T]|_U \simeq \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X[T]|_U$ and the structure of $\mathcal{O}_X[T_1, \dots, T_N]$-module is obtained by setting $T_j \to 0$ for $j \neq i$ and $T_i \to T$. There is the identification

$$\mathcal{F}[T] = \mathcal{F}[T_1, \dots, T_N]/(T_1, \dots, \hat{T}_i, \dots, T_N)\, \mathcal{F}[T_1, \dots, T_N] .$$

Now the conclusion follows as $\mathcal{F}[T_1, \ldots, T_N]$ $\left(\simeq \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}[T_1, \ldots, T_N]\right)$ is coherent over $\mathcal{O}[T_1, \ldots, T_N]$ (as one can see taking locally exact sequences of the form $\mathcal{O}_X^\alpha \to \mathcal{O}_X^\beta \to \mathcal{F} \to 0$). By means of the theorem of coherence mentioned above, we derive that $H^q(X, \mathcal{M}(\mathcal{F})) \simeq \bigoplus_{m=0}^{\infty} H^q(X, \mathcal{F}(m))$ is a $C[T_1, \ldots, T_N]$-module of finite type for any q. In particular the C-algebra $\mathcal{A}(X, \mathcal{L})$ is a $C[T_1, \ldots, T_N]$-module of finite type, thus it is finitely generated. Also it follows that $\mathcal{M}^q(X, \mathcal{F}) = \bigoplus H^q(X, \mathcal{F}(m))$ is a module of finite type over $\mathcal{A}(X, \mathcal{L})$.

Let us now assume that $\mathcal{A}(X, \mathcal{L})$ has no fixed points, i.e. its elements have no common zeros. Then as X is compact, there is an integer $m_0 > 0$ such that the elements of $\Gamma(X, \mathcal{L}^{m_0})$ have no common zeros. For every $m > 0$ we can write $\mathcal{F}(m) = \left(\mathcal{F}(r)\right)(hm_0)$, h and r being integers and $0 \leqslant r < m_0$. If we apply the first part of the proof to $\mathcal{L}^{m_0}$ and to each $\mathcal{F}(r)$, $0 \leqslant r < m_0$, it follows that the algebra $\bigoplus \Gamma(X, \mathcal{L}^{mm_0})$ is finitely generated and that for and q and r, $0 \leqslant r < m_0$, $\bigoplus H^q\left(X, \mathcal{F}(r)(mm_0)\right)$ is a module of finite type over $\bigoplus \Gamma(X, \mathcal{L}^{mm_0})$. If one puts together, for $r = 0, \ldots, m_0 - 1$, the generators, one finds that $\bigoplus_{m=0}^{\infty} H^q(X, \mathcal{F}(m))$ is a $\bigoplus \Gamma(X, \mathcal{L}^{mm_0})$-module of finite type and thus of finite type over $\mathcal{A}(X, \mathcal{L})$. In particular $\mathcal{A}(X, \mathcal{L})$ is a module of finite type over $\bigoplus \Gamma(X, \mathcal{L}^{mm_0})$, hence it is finitely generated as C-algebra.

(ii) Under the previous notations we have that $\bigoplus_{0}^{\infty} H^q(X, \mathcal{F}(m))$ is a module of finite type over $C[T_1, \ldots, T_N]$, hence the function $m \to \to \dim H^q(X, \mathcal{F}(m))$ is actually a Hilbert function, hence a polynomial for $m \gg 0$ ([25], Ch. II, th. 2).

We claim that its degree is smaller than $\dim \mathcal{F}$. We prove this fact by induction on $\dim \mathcal{F}$. If $\dim \mathcal{F} = 0$ then the statement is obvious. The general induction step is done as follows. Let $X_1, \ldots, X_k$ be the irreducible components of $\operatorname{supp} \mathcal{F}$ and pick up some points $x_1 \in X_1, \ldots, x_k \in X_k$. There exists a section $s \in \Gamma(X, \mathcal{L})$ such that $s(x_i) \neq 0$ for all i by the assumption. The multiplication by s gives a morphism $\mathcal{F}(-1) \to \mathcal{F}$. We denote by $\mathcal{G}$ and $\mathcal{H}$ its Kernel and Cokernel. In accordance with the choice of s, $\dim \mathcal{G} < \dim \mathcal{F}$ and $\dim \mathcal{H} < \dim \mathcal{F}$. Denote also by

$$\mathcal{K} = \operatorname{Im}\left(\mathcal{F}(-1) \to \mathcal{F}\right).$$

We have the exact sequences

$$0 \to \mathcal{G} \to \mathcal{F}(-1) \to \mathcal{K} \to 0\,, \quad 0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{H} \to 0$$

and also

$$0 \to \mathcal{G}(m) \to \mathcal{F}(m-1) \to \mathcal{K}(m) \to 0\,, \qquad 0 \to \mathcal{K}(m) \to \mathcal{F}(m) \to \mathcal{H}(m) \to 0\,.$$

We have the exact sequences

$$H^q(X, \mathcal{F}(m-1)) \xrightarrow{\varepsilon_1} H^q(X, \mathcal{K}(m)) \xrightarrow{\varepsilon_2} H^{q+1}(X, \mathcal{G}(m))\,,$$
$$H^q(X, \mathcal{K}(m)) \xrightarrow{\varepsilon_3} H^q(X, \mathcal{F}(m)) \xrightarrow{\varepsilon_4} H^q(X, \mathcal{H}(m))\,.$$

$$\begin{aligned}\dim H^q(X, \mathcal{F}(m)) - \dim H^q(X, \mathcal{F}(m-1)) &\leqslant \dim H^q(X, \mathcal{F}(m)) - \\ - \dim \operatorname{Im} \varepsilon_1 = \dim H^q(X, \mathcal{F}(m)) &- \dim H^q(X, \mathcal{K}(m)) + \dim \operatorname{Im} \varepsilon_2 \leqslant \\ \leqslant \dim H^q(X, \mathcal{F}(m)) - \dim H^q(X, \mathcal{K}(m)) &+ \dim H^{q+1}(X, \mathcal{G}(m)) \leqslant \\ \leqslant \dim H^q(X, \mathcal{F}(m)) - \dim \operatorname{Im} \varepsilon_3 &+ \dim H^{q+1}(X, \mathcal{G}(m)) = \dim \operatorname{Im} \varepsilon_4 + \\ + \dim H^{q+1}(X, \mathcal{G}(m)) \leqslant \dim H^q(X, \mathcal{H}(m)) &+ \dim H^{q+1}(X, \mathcal{G}(m))\,.\end{aligned}$$

Under the inductive assumption the degrees of the polynomials associated to the functions $m \to \dim H^q(X, \mathcal{H}(m))$, $m \to \dim H^{q+1}(X, \mathcal{G}(m))$ are $<$ dimension of $\mathcal{F}$. Therefore the polynomial associated to the difference function

$$m \to \dim H^q(X, \mathcal{F}(m)) - \dim H^q(X, \mathcal{F}(m-1))$$

is of degree $< \dim \mathcal{F}$ and from this our contention follows.

To see that the function $m \to \chi(X, \mathcal{F}(m)) = \Sigma(-1)^q \dim H^q(X, \mathcal{F}(m))$ is a polynomial we proceed by the same way, by induction on $\dim \mathcal{F}$. If $\dim \mathcal{F} \leqslant 0$ the assertion is obvious and the general step of induction follows from the relations

$$\chi(X, \mathcal{F}(m)) - \chi(X, \mathcal{F}(m-1)) = \chi(X, \mathcal{H}(m)) - \chi(X, \mathcal{G}(m))\,,$$

with the same notations as before.

(iii) Again we proceed by induction on $\dim \mathcal{F}$. If $\dim \mathcal{F} \leqslant 0$ the statement is clear. Let us prove the general step of induction. Consider the singular set $S_k(\mathcal{F})$ of Scheja [26] and a finite set A of X such that, for every k, A cuts all (if any) k-dimensional irreducible components of $S_k(\mathcal{F})$. Now choose a point x_0 in an irreducible component of supp $\mathcal{F}$ of dimensional equal to $\dim \mathcal{F}$. By hypothesis we can find a section $s \in \Gamma(X, \mathcal{L})$ such that $s(x) \neq 0$ whichever is $x \in A$ but $s(x_0) = 0$. Let $V(s)$ be the locus of zeros of s $\big(V(s) = \operatorname{supp}(\mathcal{L}/\mathcal{O}s)\big)$. Then $\dim \big(V(s) \cap S_{k+1}(\mathcal{F})\big) \leqslant k$ for any k.

By ([26], 1.18) the morphism induced by the multiplication by s, $\mathcal{F}(-1) \to \mathcal{F}$, is injective; let $\mathcal{G}$ be its cokernel. Clearly, $\dim \mathcal{G} = \dim \mathcal{F} - 1$. From the exact sequences

$$0 \to \mathcal{F}(m-1) \to \mathcal{F}(m) \to \mathcal{G}(m) \to 0$$

one gets

$$\chi(X, \mathcal{F}(m)) - \chi(X, \mathcal{F}(m-1)) = \chi(X, \mathcal{G}(m))$$

and the proof is over.

Particular cases of this theorem can be found in ([6], theorem 3) and ([5], proposition 8.2).

c) REMARKS. 1) One could prove the theorem using the proper morphism $X \to P(\Gamma(X, \mathcal{L}))$ (defined when $\mathcal{L}$ has no fixed points), using Grauert coherence theorem [14], and the associated Leray's spectral sequence, together with results of [24] (for the last assertion of (ii) one uses the invariance of Euler-Poincaré characteristic on spectral sequences). However the argument used above is applicable to some more general situations. For instance one can show the following

STATEMENT. *Let X be a compact complex space, let $\mathcal{E}$ be a locally free coherent sheaf on X (or more generally, let $\mathcal{E} \in \operatorname{Coh} X$) and let $\mathcal{F} \in \operatorname{Coh} X$. Assume $\mathcal{E}$ generated by its global sections. Then the $\mathbf{C}$-algebra $\mathcal{A}(X, \mathcal{E}) = \bigoplus_{0}^{\infty} \Gamma(X, S^n(\mathcal{E}))$ is finitely generated and for every q, $\bigoplus_{m=0}^{\infty} H^q(X, \mathcal{F} \otimes S^m(\mathcal{E}))$ is an $\mathcal{A}(X, \mathcal{L})$-module of finite type. In particular the functions $m \to \dim H^q(X, \mathcal{F} \otimes S^m(\mathcal{E}))$ are polynomials for $m \gg 0$.* (Here $S^m(\mathcal{E})$ denote the m-th symmetric tensor power of $\mathcal{E}$).

2) By the same type of arguments one gets the following

STATEMENT. *Let X be a complete algebraic variety over an algebraically closed field k. Let $\mathcal{L}$ be an invertible sheaf and $\mathcal{F}$ an algebraic coherent sheaf.*

(i) *Assume that $\mathcal{A}(X, \mathcal{L})$ has no fixed points. Then the k-algebra $\mathcal{A}(X, \mathcal{L}) = \bigoplus_{0}^{\infty} \Gamma(X, \mathcal{L}^m)$ is finitely generated and, for any q, $\bigoplus_{m=0}^{\infty} H^q(X, \mathcal{F}(m))$ is an $\mathcal{A}(X, \mathcal{L})$-module of finite type.*

(ii) *Assume that the elements of $\Gamma(X, \mathcal{L})$ have no common zero. Then for any q the function $m \to \dim_k H^q(X, \mathcal{F}(m))$ is polynomial if $m \gg 0$ of degree $\leqslant \dim \mathcal{F}$, while the function $m \to \chi(X, \mathcal{F}(m))$ is just a polynomial.*

(iii) *Assume that for each pair* $x \neq x'$ *there exists* $s \in \Gamma(X, \mathcal{L})$ *with* $s(x) = 0$ *but* $s(x') \neq 0$. *Then the degree of the polynomial* $m \to \chi(X, \mathcal{F}(m))$ *equals* $\dim \mathcal{F}$.

Indeed the proof is more simple than in the analytic case since the coherence theorem for graded sheaves is easier in the algebraic context ([16], 2.4.1 and 3.3.1). We note the construction of a section s as in the proof of (iii) given above can also be done, by the same arguments, using the theory of singular sets of algebraic coherent sheaves which is carried out in the algebraic case in [27].

From this statement one can derive the results of Zariski mentioned above.

3) In [17] one shows that the function $m \to \chi(X, \mathcal{F}(m))$ is polynomial for every invertible sheaf $\mathcal{L}$ on a complete algebraic variety and for any algebraic coherent sheaf $\mathcal{F}$ (Snapper theorem).

According to [1] if X is a compact complex space (or more generally a pseudoconcave space) and if $\mathcal{L}$ is an invertible sheaf on X the function $m \to \dim \Gamma(X, \mathcal{L}^m)$ is bounded by a polynomial of degree $\leqslant \dim X$. One can ask the following question to which we do not know the answer:

« Let X be a compact complex space, let $\mathcal{L}$ be an invertible sheaf and let $\mathcal{F} \in \operatorname{Coh} X$. When is the function $m \to \Sigma(-1)^q \dim H^q(X, \mathcal{F}(m))$ a polynomial? Are the functions $m \to \dim H^q(X, \mathcal{F}(m))$ bounded by polynomials of degree $\leqslant \dim X$? ».

A very partial answer is given by the following (1)

STATEMENT. *Let* X *be a Moishezon space, let* $\mathcal{L}$ *be an invertible sheaf on* X, *let* $\mathcal{F} \in \operatorname{Coh} X$ *and let* $\mathcal{F}(m)$ *denote the associated twisted sheaves. Then the function* $m \to \Sigma(-1)^q \dim H^q(X, \mathcal{F}(m))$ *is a polynomial* (2).

PROOF. Let $\tilde{X} \xrightarrow{\pi} X$ be a proper morphism of complex spaces, let $\mathcal{L} \in \operatorname{Pic} X$, and let $\tilde{\mathcal{L}}$ denote the reciprocal image of $\mathcal{L}$ under π; then for any $\mathcal{G} \in \operatorname{Coh} \tilde{X}$ there exist natural isomorphisms $R^q\pi_*(\mathcal{G}(m)) \simeq (R^q\pi_*(\mathcal{G}))(m)$. In the left side the twisted sheaves are relative to $\tilde{\mathcal{L}}$ but in the right side relative to $\mathcal{L}$. To get them, we consider a covering $\mathcal{U} = (U_i)_i$ of X which trivializes $\mathcal{L}$ and let $\xi = (\xi_{ij})$, $\xi_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$, be the gluing functions. The sheaf $\tilde{\mathcal{L}}$ is obtained gluing together the sheaves $\mathcal{O}_{\tilde{X}}|\pi^{-1}(U_i)$ by means of the func-

(1) E. Selder (München) has proved that the function $m \to \chi(X, \mathcal{F}(m))$ is polynomial when X is of dimension 2 (reduced) and $\mathcal{L}$ is associated to a divisor.

(2) The statement has been proved by C. Horst (München) for a special class of Moishezon spaces.

tions $\pi^*\xi_{ij}$. One verifies easily that both sheaves $R^q\pi_*(\mathcal{G}(m))$, $(R^q\pi_*\mathcal{G})(m)$ are equal to $R^q\pi_*(\mathcal{F})$ on each U_i and moreover on each $U_i \cap U_j$ the gluing functions are for both the multiplication by ξ_{ij}.

We now go back to the proof of the statement. We proceed inductively on $\dim X$. If $\dim X = 0$ the conclusion is obvious. Let suppose the assertion already proved for Moishezon spaces of dimension $< \dim X$ and let prove it for X. First note the following: if $\mathcal{F} \in \operatorname{Coh} X$ and $\mathcal{F} = 0$ outside a closed analytic set Y of dimension $< \dim X$, then the conclusion holds for $\mathcal{F}$. Indeed, if $\mathfrak{J} = \mathfrak{J}(Y)$ is the maximal ideal sheaf assigned to Y, then $\mathfrak{J}^k\mathcal{F} = 0$ for k sufficiently large and the required statement derives by additivity from the exact sequences

$$0 \to \mathfrak{J}^k\mathcal{F}/\mathfrak{J}^{k+1}\mathcal{F} \to \mathcal{F}/\mathfrak{J}^{k+1}\mathcal{F} \to \mathcal{F}/\mathfrak{J}^k\mathcal{F} \to 0$$

using induction on k and the fact that $(Y, \mathcal{O}_X/\mathfrak{J}|_Y)$ is Moishezon.

Now X is bimeromorphically equivalent to an algebraic projective variety [18] hence there exists an algebraic projective variety $\tilde{X}$ and a proper surjective morphism $\pi\colon \tilde{X} \to X$ biholomorphic outside an analytic set Y of dimension smaller than $\dim X$. Let $\tilde{\mathfrak{L}} = \pi^*(\mathfrak{L})$ and $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$. There exists a natural morphism $\mathcal{F} \to \pi_*(\tilde{\mathcal{F}})$ which is an isomorphism outside Y. In virtue of the former remark the statement holds for ker and coker of that morphism, thus to prove that the function $m \to \chi(X, \mathcal{F}(m))$ is polynomial we need only to show that this happens for the sheaf $\pi_*(\tilde{\mathcal{F}})$.

For each m there exists a spectral sequence of term $E_2^{p,q}(m) = H^p\big(X, R^q\pi_*(\tilde{\mathcal{F}}(m))\big)$ which converges to $H^{p+q}(\tilde{X}, \tilde{\mathcal{F}}(m))$. The remark at the begining of the proof gives us that $E_2^{p,q}(m) = H^p\big(X, (R^q\pi_*(\tilde{\mathcal{F}}))(m)\big)$. By the invariance of the Euler-Poincaré characteristic in a spectral sequence we get $\chi(\tilde{X}, \tilde{\mathcal{F}}(m)) = \chi(X, \pi_*(\tilde{\mathcal{F}})(m)) - \chi(X, R^1\pi_*(\tilde{\mathcal{F}})(m)) + \ldots$; Moreover $R^q\pi_*(\tilde{\mathcal{F}})$ are coherent sheaves on X which are zero on $X\setminus Y$ for $q \geqslant 1$, thus the functions $m \to \chi(X, R^q\pi_*(\tilde{\mathcal{F}})(m))$ are polynomial when $q \geqslant 1$. Similarly behaves the function $m \to \chi(\tilde{X}, \tilde{\mathcal{F}}(m))$ by Snapper theorem (and by GAGA) and the proof is finished.

It is easy to see that the degree of the polynomial which appears in the statement is smaller than $\dim \mathcal{F}$.

4. Another open question is the following

« Let X be a strongly pseudoconcave space, $\mathfrak{L}$ and invertible sheaf on X such that the elements of some power $\mathfrak{L}^r$ $(r > 0)$ have no common zeros. Let $\mathcal{F}$ be a coherent sheaf and torsion-free. Is $\mathcal{A}(X, \mathfrak{L})$ finitely generated? Is $\bigoplus_0^\infty \Gamma(X, \mathcal{F}(m))$ an $\mathcal{A}(X, \mathfrak{L})$-module of finite type? »

Particular answers could be given in the case the pseudoconcave space is compactificable and the data is extendible on the compactification.

A still very partial answer to this type of questions is given by the

STATEMENT. *Let X be a strongly pseudoconcave space, let $\mathfrak{L}$ be an invertible sheaf on X and let $\mathcal{F}$ be a locally free coherent sheaf. Assume that there exist* depth $X - 1$ *global sections of some power $\mathfrak{L}^r$, $r > 0$, without common zeros. Then $\mathcal{A}(X, \mathfrak{L})$ is finitely generated and $\bigoplus_0^\infty \Gamma(X, \mathcal{F}(m))$ is an $\mathcal{A}(X, \mathfrak{L})$-module of finite type.*

PROOF. As in the proof of part (i) of the previous theorem it is sufficient to prove the assertion when $r = 1$. Let $s_0, \ldots, s_n$ be global sections of $\mathfrak{L}$ without common zeros, $n = \operatorname{depth} \mathcal{O}_X - 2$. Consider the morphism $X \xrightarrow{\pi} \boldsymbol{P}^n$ given by those sections. The sheaf $\mathfrak{L}$ is isomorphic to $\pi^*(\mathcal{O}_{\boldsymbol{P}^n}(1))$.

One gets isomorphisms $\pi_*(\mathcal{F} \otimes \mathfrak{L}^m) \simeq \pi_*(\mathcal{F}) \otimes \mathcal{O}_{\boldsymbol{P}^n}(1)^m$ which agree with the graduation. Consequently,

$$\mathcal{A}(X, \mathfrak{L}) \simeq \bigoplus \Gamma(\boldsymbol{P}^n, \pi_*(\mathcal{O}_X)(m)) \quad \text{and} \quad \bigoplus \Gamma(X, \mathcal{F}(m)) = \bigoplus \Gamma(\boldsymbol{P}^n, \pi_*(\mathcal{F})(m))$$

where on $\boldsymbol{P}^n$ the twisted sheaves are considered relative to the tautological line bundle $\mathcal{O}_{\boldsymbol{P}^n}(1)$. On the other hand as π is a strongly pseudoconcave map and $\operatorname{depth} \mathcal{O}_X = n + 2 = \dim \boldsymbol{P}^n + 2$, $\operatorname{depth} \mathcal{F} = \dim \boldsymbol{P}^n + 2$ by ([21], theorem 3) both $\pi_*(\mathcal{O}_X)$ and $\pi_*(\mathcal{F})$ are coherent sheaves on $\boldsymbol{P}^n$. By [24] it follows that $\mathcal{A}(X, \mathfrak{L})$ and $\bigoplus \Gamma(X, \mathcal{F}(m))$ are $\boldsymbol{C}[t_0, \ldots, t_n]$-modules of finite type. The structure of module is given by setting $t_i \to s_i$ and the proof is over.

5. STATEMENT. *Let X be a complex space, let $\mathfrak{J} \subset \mathcal{O}_X$ be a coherent ideal sheaf such that $Z = \operatorname{supp}(\mathcal{O}_X/\mathfrak{J})$ is compact and let $\mathcal{F} \in \operatorname{Coh} X$. Assume that $\Gamma(X, \mathfrak{J})$ generates the fibres $\mathfrak{J}_z$ for all $z \in Z$. Then the function $m \to \to \Sigma(-1)^q \dim H^q(X, \mathcal{F}/\mathfrak{J}^m\mathcal{F})$ is polynomial for $m \gg 0$ and for any q the function $m \to \dim H^q(X, \mathcal{F}/\mathfrak{J}^m\mathcal{F})$ is bounded by a polynomial.*

PROOF. In order to prove the first assertion it suffices to show that the difference function is polynomial when $m \gg 0$. Making use of the exact sequences

$$(*) \qquad 0 \to \mathfrak{J}^m\mathcal{F}/\mathfrak{J}^{m+1}\mathcal{F} \to \mathcal{F}/\mathfrak{J}^{m+1}\mathcal{F} \to \mathcal{F}/\mathfrak{J}^m\mathcal{F} \to 0$$

it is only necessary to prove that for every q the function

$$m \to \dim H^q(X, \mathfrak{J}^m\mathcal{F}/\mathfrak{J}^{m+1}\mathcal{F})$$

is polynomial for $m \gg 0$. By the assumption there exist sections $s_1, \ldots, s_N$ in $\Gamma(X, \mathfrak{J})$ which span the fibre $\mathfrak{J}_x$ for every $x \in Z$. The direct summ $\mathcal{F}/\mathfrak{J}\mathcal{F} \oplus \oplus \mathfrak{J}\mathcal{F}/\mathfrak{J}^2\mathcal{F} \oplus \ldots$ has a natural structure of graded $\mathcal{O}_Z[T_1, \ldots, T_N]$-module $(\mathcal{O}_Z = \mathcal{O}_X/\mathfrak{J}|_Z)$ by the mapping $T_i \to$ class of s_i mod $\mathfrak{J}^2$; moreover its coherence over $\mathcal{O}_Z[T_1, \ldots, T_N]$ is easily established. By the theorem of finiteness we get that $\oplus H^q(Z, \mathfrak{J}^m\mathcal{F}/\mathfrak{J}^{m+1}\mathcal{F})$ is a $C[T_1, \ldots, T_N]$-module of finite type and the first conclusion follows.

For the second assertion it suffices to show that the difference function $m \to \dim H^q(X, \mathcal{F}/\mathfrak{J}^{m+1}\mathcal{F}) - \dim H^q(X, \mathcal{F}/\mathfrak{J}^m\mathcal{F})$ is bounded by a polynomial. From the exact sequence

$$H^q(X, \mathfrak{J}^m\mathcal{F}/\mathfrak{J}^{m+1}\mathcal{F}) \to H^q(X, \mathcal{F}/\mathfrak{J}^{m+1}\mathcal{F}) \to H^q(X, \mathcal{F}/\mathfrak{J}^m\mathcal{F})$$

we derive that this difference is bounded by $\dim H^q(X, \mathfrak{J}^m\mathcal{F}/\mathfrak{J}^{m+1}\mathcal{F})$ and by this the proof is over.

We do not know if the assumption on $\Gamma(X, \mathfrak{J})$ can be weakened (if X is Moishezon, then the function $m \to \chi(X, \mathcal{F}/\mathfrak{J}^m\mathcal{F})$ is polynomial without any additional hypothesis on $\mathfrak{J}$; this derives from the statement given in section 3., using an argument due to Ramanujam [20] which reduces this problem to a problem concerning invertible sheaves via a blowing-up along $\mathfrak{J}$).

The required condition on $\mathfrak{J}$ is fulfilled when Z is a compact analytic set which has in X a strongly pseudoconvex neighbourhood and when $\mathfrak{J} = \mathfrak{J}(Z)$ is the maximal ideal sheaf of Z (use Remmert reduction).

We may also note the following case. Let $X \xrightarrow{f} Y$ be a proper morphism, let $y \in Y$ and $\mathfrak{J} = \widehat{\mathfrak{m}}_y\mathcal{O}_X$ the ideal sheaf of X generated by the maximal ideal of $\mathcal{O}_{Y,y}$. Then Z is the analytic fibre $X_y = f^{-1}(y)$ and we get that the function

$$m \to \sum_q (-1)^q \dim H^q(X, \mathcal{F}/\widehat{\mathfrak{m}}_y^m\mathcal{F})$$

is polynomial (see also [9]) and that the functions

$$m \to \dim H^q(X, \mathcal{F}/\widehat{\mathfrak{m}}_y^m\mathcal{F})$$

are bounded by polynomials in this case of degrees $\leqslant \dim_y Y$.

In [9] it is also proved that the functions $m \to \dim H^q(X, \mathcal{F}/\widehat{\mathfrak{m}}_y^m\mathcal{F})$ are polynomial for $m \gg 0$ provided that $\mathcal{F}$ is flat over Y.

3. – Twisted sheaves on projective spaces.

In this paragraph we deal with twisted sheaves relative to the tautological line bundle of the projective space. We are interested of the algebraic geometry over an arbitrary algebraically closed ground field k and on algebraic sheaves. When $k = \boldsymbol{C}$ one recovers results concerning coherent analytic sheaves making use of GAGA.

a) STATEMENT. *Let $\boldsymbol{P}^n$ be the n-dimensional projective space over the field k, let $\mathcal{F}$ be an algebraic coherent sheaf on $\boldsymbol{P}^n$ and let q be an integer. Then*

(i) depth $\mathcal{F} \geqslant q$ *if and only if* $H^r(\boldsymbol{P}^n, \mathcal{F}(-m)) = 0$ *for* $r < 0$ *and* $m \gg 0$;

(ii) $\dim \mathcal{F} \leqslant q$ *if and only if* $H^r(\boldsymbol{P}^n, \mathcal{F}(-m)) = 0$ *for* $r > q$ *and* $m \gg 0$;

(iii) *whichever is* $m \gg 0$ *and* $x \in \boldsymbol{P}^n$ *such that* depth $\mathcal{F}_x = q$ *or* $\dim \mathcal{F}_x = q$, *there exists a class* $\xi \in H^q(\boldsymbol{P}^n, \mathcal{F}(-m))$ *such that* supp $\xi = \{x\}$.

Assertion (i) was proved in ([24], Ch. III, § 5, th. 2 and also [15], Exposé XII). The assertion (ii) can be proved in the same way using theorem 1 of Ch. III, § 5 of [24] and a convenient characterization of the dimension over regular local rings. For the assertion (iii) we reconsider the argument given in theorems 3 and 4. Precisely we will show that the sheaves $\mathcal{F}(-m)$ verify the dual theorem A when $m \gg 0$, i.e. the maps

$$H^r_x(\boldsymbol{P}^n, \mathcal{F}(-m)) \to H^r(\boldsymbol{P}^n, \mathcal{F}(-m))$$

are injective on the socle $\gamma\big(H^r_x(\boldsymbol{P}^n, \mathcal{F}(-m))\big)$.

For a coherent sheaf $\mathcal{G}$ the $\mathcal{O}_x$-module $H^r_x(\boldsymbol{P}^n, \mathcal{G})$ is null iff its socle is null (indeed, in the algebraic case any element of $H^r_x(\boldsymbol{P}^n, \mathcal{G})$ is annulated by some power of the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_x$). From this remark the assertion (iii) can be derived as $H^q_x(\boldsymbol{P}^n, \mathcal{F}(-m)) \simeq H^q_x(\boldsymbol{P}^n, \mathcal{F}) \simeq H^q_{\mathfrak{m}_x}(\operatorname{Spec} \mathcal{O}_x, \tilde{\mathcal{F}}_x)$ is $\neq 0$ for $q = $ depth $\mathcal{F}_x$ or $q = \dim \mathcal{F}_x$ by [15]. Now $H^r(\boldsymbol{P}^n, \mathcal{F}(-m))$ is finite-dimensional over k and its dual is isomorphic to $\operatorname{Ext}^{n-r}(\boldsymbol{P}^n; \mathcal{F}(-m), \Omega_{\boldsymbol{P}^n})$. Use the spectral sequences which tie the global with the local $\mathcal{E}xt$'s, use the isomorphism

$$\mathcal{E}xt^{\bullet}(\mathcal{F}(-m), \Omega) \simeq \mathcal{E}xt^{\bullet}(\mathcal{F}, \Omega)(m)$$

and the theorem B of Serre. Then we get

$$\operatorname{Ext}^{n-r}(\boldsymbol{P}^n; \mathcal{F}(-m), \Omega) \simeq \Gamma\big(X, \mathcal{E}xt^{n-r}(\mathcal{F}(-m), \Omega)\big) \quad \text{if } m \gg 0\,.$$

On the other hand, $\mathcal{E}xt^{n-r}(\mathcal{F}(-m), \Omega) \simeq \mathcal{E}xt^{n-r}(\mathcal{F}, \Omega)(m)$ is spanned by global sections when $m \gg 0$, i.e. for any x the map

$$(*) \qquad \Gamma\big(X, \mathcal{E}xt^{n-r}(\mathcal{F}(-m), \Omega)\big) \to$$
$$\to \operatorname{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}(-m)_x, \Omega_x)/\mathfrak{m}_x \operatorname{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}(-m)_x, \Omega_x)$$

is surjective. By local duality over a regular local ring [15], considering the $\mathfrak{m}_x$-adic topology on the $\mathcal{O}_x$-module of finite type $\mathrm{Ext}^{n-r}_{\mathcal{O}_x}(\mathcal{F}(-m)_x, \Omega_x)$, its topological dual is isomorphic to

$$H^r_{\mathfrak{m}_x}(\mathrm{Spec}\ \mathcal{O}_x, \tilde{\tilde{\mathcal{F}}}(-m)_x) \simeq H^r_x(\boldsymbol{P}^n, \mathcal{F}(-m)) .$$

The target space of the map (*) is finite dimensional over k and its dual is just the socle of $H^r_x(\boldsymbol{P}^n, \mathcal{F}(-m))$. Taking thus the dual of (*) we obtain our contention.

b) Let us recall that $\mathcal{F}$ is called *m-regular* if $H^r(\boldsymbol{P}^n, \mathcal{F}(m-r)) = 0$ for $r > 0$ ([19], lecture 14). We will also say that $\mathcal{F}$ is *m-coregular* if $H^r(\boldsymbol{P}^n, \mathcal{F}(m-r)) = 0$ for $r < \mathrm{depth}\, \mathcal{F}$. In particular if $\mathcal{F}$ is locally free this last condition becomes

$$H^r(\boldsymbol{P}^n, \mathcal{F}(m-r)) = 0 \quad \text{for } r < n .$$

Analogously to Castelnuovo Lemma ([19], lecture 14) we have the following

LEMMA. *Let* $\mathcal{F} \in \mathrm{Coh}\, \boldsymbol{P}^n$ *be m-coregular and let* $q_0 = \mathrm{depth}\, \mathcal{F}$.

(i) *then* $\mathcal{F}$ *is* m'*-coregular for every* $m' \leqslant m$, *i.e.*

$$H^r(\boldsymbol{P}^n, \mathcal{F}(k)) = 0 \quad \textit{for } r < \mathrm{depth}\, \mathcal{F} \textit{ and } r + k \leqslant m ;$$

(ii) *if* $k + q_0 < m$ *and* $\xi \in H^{q_0}(\boldsymbol{P}^n, \mathcal{F}(k))$ *is such that* $T\xi = 0$ *for any linear form* T, *then* $\xi = 0$;

(iii) *if* $k + q_0 \leqslant m$ *and if* $x \in \boldsymbol{P}^n$ *is such that* $\mathrm{depth}\, \mathcal{F}_x = q_0$, *then there exists* $\xi \in H^{q_0}(\boldsymbol{P}^n, \mathcal{F}(k))$ *with* $\mathrm{supp}\, \xi = \{x\}$.

PROOF. (i) When $\mathrm{depth}\, \mathcal{F} = \infty$ (i.e. $\mathcal{F} = 0$) the statement is trivial. Let us suppose $\mathcal{F} \neq 0$ and use induction on $\mathrm{depth}\, \mathcal{F}$. If $\mathrm{depth}\, \mathcal{F} = 0$ there is nothing to prove. We can thus suppose $\mathrm{depth}\, \mathcal{F} \geqslant 1$. By ([24], Ch. III, 8) we can find a linear form T such that the multiplication morphism $\mathcal{F}(-1) \to \mathcal{F}$ is injective and such that the hyperplane associated to T passes through a point x in which $\mathrm{depth}\, \mathcal{F}_x = \mathrm{depth}\, \mathcal{F}$ (one could give an argument for the existence of T as in the proof of (iii) of theorem 7). One has the exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$$

and $\mathrm{depth}\, \mathcal{G} = \mathrm{depth}\, \mathcal{F} - 1$. For any k one has the exact sequence

$$0 \to \mathcal{F}(k-1) \to \mathcal{F}(k) \to \mathcal{G}(k) \to 0 .$$

Considering the associated exact sequence we derive that $\mathcal{G}$ is m-coregular. By the inductive hypothesis then $\mathcal{G}$ is m'-coregular for $m' \leqslant m$. By induction on $k \geqslant 0$ one deduces then that $\mathcal{F}$ is $(m-k)$-coregular.

(ii) When depth $\mathcal{F} = \infty$ there is nothing to prove. Assume that depth $\mathcal{F} = 0$ and choose $\xi \in H^0(\boldsymbol{P}^n, \mathcal{F}(k))$ such that $T\xi = 0$ for any linear form T. For every $x \in \boldsymbol{P}^n$ take a form such that $T(x) \neq 0$, then it follows that $\xi_x = 0$ for every $x \in \boldsymbol{P}^n$, thus $\xi = 0$. Assume now depth $\mathcal{F} > 0$ and proceed by induction on depth $\mathcal{F}$. With the same notations as above we have the exact sequence

$$H^{q_0-1}(\boldsymbol{P}^n, \mathcal{G}(k+1)) \to H^{q_0}(\boldsymbol{P}^n, \mathcal{F}(k)) \to H^{q_0}(\boldsymbol{P}^n, \mathcal{F}(k+1)) .$$

Let $\xi \in H^{q_0}(\boldsymbol{P}^n, \mathcal{F}(k))$ as in (ii). In particular for the linear form T used for the exact sequence we have $T\xi = 0$, i.e. the image of ξ through the second arrow is zero. There exists thus an element $\eta \in H^{q_0-1}(\boldsymbol{P}^n, \mathcal{G}(k+1))$ whose image in $H^{q_0}(\boldsymbol{P}^n, \mathcal{F}(k))$ is ξ. It is enough to show that for any linear form S, $S\eta = 0$, because then by induction $\eta = 0$ and thus $\xi = 0$.

Now $H^{q_0-1}(\boldsymbol{P}^n, \mathcal{F}(k+2)) = 0$ by (i) so that the map

$$H^{q_0-1}(\boldsymbol{P}^n, \mathcal{G}(k+2)) \to H^{q_0}(\boldsymbol{P}^n, \mathcal{F}(k+1))$$

is injective. The image of $S\eta$ under this map is zero because equal to $S\xi$.

(iii) We proceed by induction on k, $q_0+k \leqslant m$. For k small enough there exists $\xi \in H^{q_0}(\boldsymbol{P}^n, \mathcal{F}(k))$ such that $\operatorname{supp} \xi = \{x\}$. This follows from point (iii) of the statement given before. In accordance with (ii) if $q_0+k < m$ there exist a linear form T such that $T\xi \in H^{q_0}(\boldsymbol{P}^n, \mathcal{F}(k+1))$ is not zero. As $\operatorname{supp}(T\xi) \subset \operatorname{supp} \xi$ one gets $\operatorname{supp} T\xi = \{x\}$. This shows that we can proceed with the induction.

c) THEOREM 8. *Let $\mathcal{F}$ be an algebraic coherent sheaf on the projective space $\boldsymbol{P}^n$.*

(i) *Assume that the Hilbert polynomial $m \to \chi(\boldsymbol{P}^n, \mathcal{F}(m))$ equals the Hilbert polynomial of a free sheaf. Then $\mathcal{F}$ is also free if and only if* 0*-regular; if in addition $\mathcal{F}$ is locally free this last condition is also equivalent with the condition for $\mathcal{F}$ to be* (-1)*-coregular.*

(ii) *Assume $\mathcal{F}$ locally free; then $\mathcal{F}$ is a direct factor of a free sheaf (of finite rank) if and only if it is* 0*-regular and* (-1)*-coregular.*

PROOF. The « only if » implications follow from the formulae which compute $\dim H^r(\boldsymbol{P}^n, \mathcal{O}(m))$ of [24].

(i) The first assertion follows directly from Castelnuovo Lemma. One can also argue by induction on n. If $n = 0$ there is nothing to prove. Assume $n > 0$. We take a linear form T such that the multiplication gives rise to an injective morphism $\mathcal{F}(-1) \to \mathcal{F}$. Let denote by $H = \boldsymbol{P}^{n-1}$ the corresponding hyperplane. We have exact sequences

$$0 \to \mathcal{O}_{\boldsymbol{P}}(-1) \to \mathcal{O}_{\boldsymbol{P}} \to \mathcal{O}_H \to 0\,, \qquad 0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0$$

where $\mathcal{F}_H = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}_H$. Assume that the polynomial $m \to \chi(\boldsymbol{P}^n, \mathcal{F}(m))$ equals the polynomial $m \to \chi(\boldsymbol{P}^n, \mathcal{O}^p_{\boldsymbol{P}^n}(m))$. Since

$$\chi(\boldsymbol{P}^{n-1}, \mathcal{F}_H(m)) = \chi(\boldsymbol{P}^n, \mathcal{F}(m)) - \chi(\boldsymbol{P}^n, \mathcal{F}(m-1))$$

the Hilbert polynomial associated to $\mathcal{F}_H$ equals the Hilbert polynomial associated to $\mathcal{O}^p_H$.

On the other hand, $\mathcal{F}_H$ must be 0-*regular*. By the inductive hypothesis $\mathcal{F}_H$ is free and in fact isomorphic to $\mathcal{O}^p_H$. Let us choose $s_1, \dots, s_p \in H^0(\boldsymbol{P}^n, \mathcal{F}_H)$ giving rise to the isomorphism $\mathcal{O}^p_H \simeq \mathcal{F}_H$. The map $H^0(\boldsymbol{P}^n, \mathcal{F}) \to H^0(\boldsymbol{P}^n, \mathcal{F}_H)$ is surjective as the obstruction lies in $H^1(\boldsymbol{P}^n, \mathcal{F}(-1))$ which is zero by the hypothesis of 0-regularity. Therefore there exist $t_1, \dots, t_p \in H^0(\boldsymbol{P}^n, \mathcal{F})$ whose images are just $s_1, \dots, s_p$. The corresponding morphism $\mathcal{O}^p \xrightarrow{\theta} \mathcal{F}$ induces the isomorphism $\mathcal{O}^p_H \simeq \mathcal{F}_H$. By Nakayama lemma θ is surjective at all points of H, thus $\operatorname{supp}(\operatorname{Coker}\theta)$ has dimension $< n$. One has

$$\chi(\boldsymbol{P}^n, \ker\theta(m)) - \chi(\boldsymbol{P}^n, \mathcal{O}^p(m)) + \chi(\boldsymbol{P}^n, \mathcal{F}(m)) - \chi(\boldsymbol{P}^n, (\operatorname{Coker}\theta)(m)) = 0\,.$$

It follows that the Hilbert polynomial of $\ker\theta$ equals the one of $\operatorname{Coker}\theta$. Hence $\dim\operatorname{supp}(\operatorname{Ker}\theta) < n$. But this is possible only if $\operatorname{Ker}\theta = 0$ as $\operatorname{Ker}\theta \subset \mathcal{O}^p$. Consequently $\operatorname{Coker}\theta = 0$, that is $\mathcal{F} \simeq \mathcal{O}^p$. Now assume $\mathcal{F}$ locally free and (-1)-coregular. It is sufficiently to show that the dual sheaf $\check{\mathcal{F}}$ is free. The Hilbert polynomial of $\check{\mathcal{F}}$ is given by $m \to \Sigma(-1)^q \cdot \dim H^q(\boldsymbol{P}^n, \check{\mathcal{F}}(m))$. But $H^q(\boldsymbol{P}^n, \check{\mathcal{F}}(m))$ is isomorphic to the dual of $H^{n-q}(\boldsymbol{P}^n, \mathcal{F}(-m)\otimes\Omega) = H^{n-q}(\boldsymbol{P}^n, \mathcal{F}(-m-n-1))$ as $\Omega \simeq \mathcal{O}(-n-1)$. It follows then easily that the Hilbert polynomial of $\check{\mathcal{F}}$ equals the one of a free sheaf; as the condition « $\check{\mathcal{F}}$ is 0-regular » reduces by duality to the condition « $\mathcal{F}$ is (-1)-coregular », it follows that $\check{\mathcal{F}}$ is free.

(ii) For this point we apply Castelnuovo Lemma. In virtue of it $\mathcal{F}$ is generated by its global sections. Let $\mathcal{O}^p \to \mathcal{F}$ be an epimorphism and let $\mathcal{G}$ be its Kernel. From the exact sequence

$$H^{r-1}(\boldsymbol{P}^n, \mathcal{F}(-r)) \to H^r(\boldsymbol{P}^n, \mathcal{G}(-r)) \to H^r(\boldsymbol{P}^n, \mathcal{O}^p(-r))$$

and from the fact that $H^r(\boldsymbol{P}^n, \mathcal{O}^p(-r)) = 0$ for $r \geqslant 1$ we deduce that $\mathcal{G}$ is 0-regular.

Now the coherent sheaf $\mathcal{G} \oplus \mathcal{F}$ has Hilbert polynomial equal with that of $\mathcal{O}^p$ and it is 0-regular. By (i) this sheaf must be free and the proof is over.

COROLLARY. *Let $\mathcal{F}$ be a locally free sheaf of finite rank on the projective line $\boldsymbol{P}^1$.*

(i) *Assume $\chi(\boldsymbol{P}^1, \mathcal{F}(m)) = p(m+1)$ for at least an integer m, where $p \geqslant 0$ is an integer. Then $\mathcal{F}$ is free if and only if $H^1(\boldsymbol{P}^1, \mathcal{F}(-1)) = 0$ and if and only if $H^0(\boldsymbol{P}^1, \mathcal{F}(-1)) = 0$.*

(ii) *$\mathcal{F}$ is a direct factor of a free sheaf of finite rank if and only if $H^0(\boldsymbol{P}^1, \mathcal{F}(-1)) = 0$ and $H^1(\boldsymbol{P}^1, \mathcal{F}(-1)) = 0$.*

These facts can also be proved using Grothendieck's structure theorem for vector bundles on the projective line.

STATEMENT. *Let X be a non-singular projective variety and let $\mathcal{F}$ be an algebraic coherent sheaf on X which is 0-regular relative to the hyperplane section and having a Hilbert polynomial equal to the one of a free sheaf. Then $\mathcal{F}$ is free.*

The argument is the same as in the theorem since for a generic hyperplane H, $H \cap X$ is non-singular and the corresponding morphisms $\mathcal{O}_X(-1) \to \mathcal{O}_X$, $\mathcal{F}(-1) \to \mathcal{F}$ are injective.

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Added in proofs.

(1) The implication i $\Rightarrow$ ii of the statement on page 10 was also noted by M. SCHNEIDER and A. SILVA;

(2) D. LEISTNER (Regensburg) has proved that the functions $m \to \dim H^q(X, \mathcal{F} \otimes \mathfrak{L}^m)$ are bounded by polynomials (X compact complex space);

(3) K. UENO and the second author have proved that the function $(n, m) \to \to \chi(X, \mathcal{F} \otimes \mathfrak{L}^n / \mathfrak{J}^m(\mathcal{F} \otimes \mathfrak{L}^n))$ is polynomial for any n and $m \gg 0$ (the only assumption is that $\operatorname{Supp}(\mathcal{O}_X/\mathfrak{J})$ is compact).