ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 6, nº 4 (1979), p. 703-717

<http://www.numdam.org/item?id=ASNSP_1979_4_6_4_703_0>

Diophantine Approximation in Short Intervals.

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1. – Introduction.

A simple consequence of some basic inequalities in the elementary theory of continued fractions is that

$$(1) \qquad \max_{1 \leqslant q \leqslant Q} \frac{1}{\sin^2(\pi q \xi)} \ll Q^2,$$

or that

$$\sum_{q=1}^{Q} \frac{1}{\sin^2(\pi q \xi)} \ll Q^2$$

(see [2]), if and only if ξ is a « badly approximable » irrational number, i.e. one having bounded partial quotients in its continued fraction expansion. Similar bounds frequently occur e.g. in Fourier analysis, and one may need conditions ensuring the existence of such ξ satisfying suitable additional assumptions. Clearly the best possible constant implied by the $\ll$-symbol in (1) will be obtained, for any sufficiently large Q, when ξ is equivalent to

$$\xi' = \frac{\sqrt{5}+1}{2} = 1 + \frac{1}{1+\dots}$$

under a unimodular transformation, i.e.

$$\xi = \frac{p\xi' + r}{q\xi' + s}, \qquad p, q, r, s \in \boldsymbol{Z},\ ps - qr = \pm 1 .$$

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Pervenuto alla Redazione il 29 Settembre 1978 ed in forma definitiva il 22 Maggio 1979.

For any fixed $\alpha \in \boldsymbol{R}$ and any $\delta > 0$, the interval $(\alpha - \delta, \alpha + \delta)$ contains infinitely many such ξ, and it is natural to expect that there will be at least one satisfying

$$\max_{1 \leqslant q \leqslant Q} \frac{1}{\sin^2(\pi q \xi)} \leqslant CQ^2,$$

with the appropriate constant C independent of ξ, as soon as

$$Q \geqslant Q_0 = f(\delta),$$

provided the function $f(\delta)$ of the length of the interval tends rapidly enough to infinity as $\delta \to 0$.

The problem of finding suitable $\alpha \in \boldsymbol{R}$ and functions $f(\delta)$ of suitable growth, such that for every δ in some range $0 < \delta < \Delta$ there exists $\xi = \xi(\delta)$ as above, seems not to have been stated so far, and is the object of the present paper. This arises in a natural way in the formulation of the Dirichlet problem with approximate data for the vibrating string equation ([5]).

2. – Definitions and statement of results.

Let $0 < \Delta \leqslant \infty$ and let $f(\delta) > 0$ be a function in the interval $0 < \delta < \Delta$ such that

$$\lim_{\delta \to 0} f(\delta) = \infty .$$

DEFINITION. *$\mathscr{S}(f, \Delta) = \mathscr{S}(f(\delta), \Delta)$ is the set of real numbers α satisfying the following condition:*

For any δ, $0 < \delta < \Delta$, there exists ξ such that

$$\begin{cases} |\xi - \alpha| < \delta \\ \min\limits_{1 \leqslant q \leqslant Q} \|q\xi\| \geqslant \dfrac{3 - \sqrt{5}}{2} \dfrac{1}{Q} \qquad \textit{for all integers } Q \geqslant f(\delta) . \end{cases}$$

Here $\|x\|$ denotes the distance from the real number x to the nearest integer. Thus

$$\max_{1 \leqslant q \leqslant Q} \big(\sin(\pi q \xi)\big)^{-2} = \big(\sin\big(\pi \min_{1 \leqslant q \leqslant Q} \|q\xi\|\big)\big)^{-2} .$$

From the theory of continued fractions (inequality (4) below) one sees that the constant $(3 - \sqrt{5})/2$ is best possible, and corresponds to a suitable choice of ξ in the equivalence class of $(\sqrt{5} + 1)/2$. Replacing $(3 - \sqrt{5})/2$

by some smaller positive constant in the above definition would allow one to take ξ in other equivalence classes, but would not affect the set $\mathcal{S}(f, \Delta)$.

If $\Delta_1 \leqslant \Delta_2$, then obviously

$$\mathcal{S}(f, \Delta_1) \supset \mathcal{S}(f, \Delta_2) .$$

Similarly, if $f(\delta) \leqslant g(\delta)$ then

$$\mathcal{S}(f, \Delta) \subset \mathcal{S}(g, \Delta) .$$

The set $\mathcal{S}(f, \Delta)$ is clearly defined modulo 1. In other words,

$$\mathcal{S}(f, \Delta) = \mathcal{S}(f, \Delta) + m \tag{2}$$

for every integer m.

Our theorem 1 is:

$$\mathcal{S}\left(\frac{1}{\delta}, \infty\right) = \boldsymbol{R} .$$

We must therefore analyse the structure of $\mathcal{S}(f, \Delta)$ when $f(\delta) = o(1/\delta)$ $(\delta \to 0)$. Theorem 1 is in fact essentially best possible if one requires $\mathcal{S}(f, \Delta) = \boldsymbol{R}$, for we prove (corollary 1) that

For any $\Delta > 0$ and any function $f(\delta) = o(1/\delta)$ $(\delta \to 0)$, every number $\alpha \in \mathcal{S}(f, \Delta)$ is irrational.

This result is obtained as a corollary of a fairly general statement (theorem 2), from which we also derive (corollary 2):

Let $\alpha \notin \boldsymbol{Q}$ and let $\omega > 1$. If there exist infinitely many $p/q \in \boldsymbol{Q}$ such that

$$\left|\alpha - \frac{p}{q}\right| \ll \frac{1}{q^{1+\omega}} ,$$

then, for any $\Delta > 0$ and any function $f(\delta) = o(\delta^{-\omega/(1+\omega)})$ $(\delta \to 0)$, we have

$$\alpha \notin \mathcal{S}(f, \Delta) .$$

In the opposite direction we prove (theorem 3):

If

$$\left|\alpha - \frac{p}{q}\right| \geqslant \frac{1}{(Hq)^{1+\omega}}$$

for some constants $H > 0$, $\omega \geqslant 1$ *and for all* $p/q \in \mathbf{Q}$, *then*

$$\alpha \in \mathscr{S}\left(H\left(\frac{2}{\delta}\right)^{\omega/(1+\omega)}, \infty\right).$$

For convenience we introduce the following

DEFINITION. *Let* $\alpha \notin \mathbf{Q}$. *The type of* α *is the upper bound* Ω *of the set of* ω *satisfying*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{1+\omega}}$$

for infinitely many $p/q \in \mathbf{Q}$.

If α has type Ω, then $1 \leqslant \Omega \leqslant \infty$. By Roth's theorem, every algebraic irrational number has type 1. Also, by well known results in the metric theory of continued fractions ([4]), or as a simple consequence of corollary 3 and theorem 4 below, we have that almost all numbers (i.e. all except a set of measure zero) have type 1.

From the above results we immediately obtain (corollaries 3 and 6):

If α *has type* $\Omega > 1$, *then for any* $\theta < \Omega/(1+\Omega)$ *and any* K, $\Delta > 0$ *we have*

$$\alpha \notin \mathscr{S}(K\delta^{-\theta}, \Delta).$$

If α *has type* $\Omega < \infty$, *then for any* $\theta > \Omega/(1+\Omega)$ *there exists* $K = K(\theta, \alpha) > 0$ *such that*

$$\alpha \in \mathscr{S}(K\delta^{-\theta}, \infty).$$

The latter statement shows that the sets $\mathscr{S}(K\delta^{-\theta}, \infty)$, where $\frac{1}{2} < \theta < 1$, are « large ». However, since the dependence of the constant $K(\theta, \alpha)$ on α is highly erratic, it is desirable to evaluate the measure of $\mathscr{S}(K\delta^{-\theta}, \infty)$ for fixed θ, K. We have (theorem 4):

For any θ *such that* $\frac{1}{2} < \theta < 1$ *and any* $K > 0$, *the Lebesgue measure of the set*

$$\mathscr{S}(K\delta^{-\theta}, \infty) \cap [0, 1)$$

exceeds

$$1 - \frac{\theta}{2\theta - 1}\left(\frac{2}{K}\right)^{1/(1-\theta)}.$$

In particular, for any fixed θ, $\frac{1}{2} < \theta < 1$, and any $\varepsilon > 0$, the above measure exceeds $1 - \varepsilon$ provided $K \geqslant C(\theta)(1/\varepsilon)^{1-\theta}$.

It is plain from the above results that $\theta = \frac{1}{2}$ is a critical value for the sets $\mathscr{S}(K\delta^{-\theta}, \Delta)$. In this case we prove (corollaries 5 and 7):

If α has unbounded partial quotients in its continued fraction expansion then, for any $K, \Delta > 0$,

$$\alpha \notin \mathscr{S}(K\delta^{-\frac{1}{2}}, \Delta)\,.$$

This implies in particular that $\mathscr{S}(K\delta^{-\frac{1}{2}}, \Delta)$ is a set of measure zero for every $K, \Delta > 0$ (see [4]).

If $\alpha \notin \boldsymbol{Q}$ has bounded partial quotients $a_i \leqslant A$, then

$$\alpha \in \mathscr{S}(K\delta^{-\frac{1}{2}}, \infty)$$

with $K = \sqrt{2A+4}$.

On combining these results we have (corollary 8):

$\bigcup_{K>0} \mathscr{S}(K\delta^{-\frac{1}{2}}, \infty)$ is the set of irrational numbers with bounded partial quotients.

The method of theorems 2 and 3 applies also to numbers α of infinite type. In this case, the definition of type can be naturally modified by introducing any positive, continuous and decreasing function $\eta(x)$ of the real variable $x \geqslant 1$ satisfying $\|q\alpha\| \geqslant \eta(q)$ for all integers $q \geqslant 1$, and $\|q_n\alpha\| \ll \ll \eta(q_n)$ for infinitely many integers $q_n \geqslant 1$. We let $\psi(x) = \eta(x)/x$ and denote by ψ^{-1} the inverse function.

The following is again an immediate consequence of theorem 2:

Let $\alpha \notin \boldsymbol{Q}$, let $\psi(x) > 0$ be a decreasing continuous function of $x \geqslant 1$, and let $p_n/q_n \in \boldsymbol{Q}$ be a sequence satisfying

$$\left|\alpha - \frac{p_n}{q_n}\right| \ll \psi(q_n) = o\left(\frac{1}{q_n^2}\right).$$

Then, for any $\Delta > 0$ and any function $f(\delta) = o\big(1/\delta\psi^{-1}(\delta)\big)$ $(\delta \to 0)$, we have

$$\alpha \notin \mathscr{S}(f, \Delta)\,.$$

On the other hand, an argument similar to the proof of theorem 3 yields (theorem 5):

Let $\eta(x) > 0$ be a decreasing continuous function of $x \geqslant 1$ satisfying

$$\|q\alpha\| \geqslant \eta(q)$$

for every integer $q \geqslant 1$, *and let* $\psi(x) = \eta(x)/x$. *Then*

$$\alpha \in \mathscr{S}\left(\frac{2}{\delta\psi^{-1}(\delta)}, \psi(1)\right).$$

This gives (corollary 9):

For every $\alpha \notin \boldsymbol{Q}$ *there exist* $\Delta > 0$ *and* $f(\delta) = o(1/\delta)$ $(\delta \to 0)$ *such that*

$$\alpha \in \mathscr{S}(f, \Delta).$$

3. – Notation.

Vinogradov's symbol $\ll$ indicates inequality containing an unspecified positive constant factor.

As previously stated, $\|x\|$ is the distance from x to the nearest integer.

We denote by $[a, b]$ the closed interval $a \leqslant x \leqslant b$ or $b \leqslant x \leqslant a$, by (a, b) the corresponding open interval, and similarly for $[a, b)$. As usual, the simple continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}},$$

where the partial quotients $a_0, a_1, a_2, \ldots$ are integers satisfying $a_i \geqslant 1$ for $i \geqslant 1$, is denoted by

$$[a_0, a_1, a_2, \ldots].$$

We follow Cassels' tract [1] for the notation of convergents: if

$$\xi = [a_0, a_1, a_2, \ldots] \tag{3}$$

we let

$$\begin{cases} p_0 = 1, & p_1 = a_0 \\ q_0 = 0, & q_1 = 1 \end{cases}$$

and

$$\begin{cases} p_{i+1} = a_i p_i + p_{i-1} \\ q_{i+1} = a_i q_i + q_{i-1} \end{cases} \qquad (i \geqslant 1),$$

so that the convergents to ξ are

$$\frac{p_{i+1}}{q_{i+1}} = [a_0, a_1, \ldots, a_i] \qquad (i \geqslant 0).$$

Also, we denote by $\xi_i = [a_i, a_{i+1}, \ldots]$ the complete quotient of the continued fraction (3) corresponding to the partial quotient a_i.

We shall use in the sequel several elementary results about continued fractions and Farey sequences, without quoting them explicitly. They can all be found e.g. in [3], [4].

4. – Proofs.

LEMMA 1. *Let p/q, $r/s \in [0, 1]$ be fractions satisfying $ps - qr = \pm 1$. Then there exists $\xi \in (p/q, r/s)$ such that*

$$\min_{1\leqslant k\leqslant Q} \|k\xi\| \geqslant \frac{3-\sqrt{5}}{2}\frac{1}{Q}$$

for all integers $Q \geqslant \max\{q, s\}$.

PROOF. If $q = s = 1$, so that $[p/q, r/s] = [0, 1]$, we take

$$\xi = [0, 1, 1, 1, \ldots] = \frac{\sqrt{5}-1}{2}.$$

The sequence (q_i) of the denominators of the convergents to ξ is the Fibonacci sequence $q_1 = q_2 = 1, q_3 = 2, \ldots$. For any integer $Q \geqslant \max\{q, s\} = 1$ there exists $i \geqslant 2$ such that $q_i \leqslant Q < q_{i+1}$. Then

$$\min_{1\leqslant k\leqslant Q} \|k\xi\| = \|q_i\xi\|, \qquad \text{and} \qquad \xi_i = [1, 1, 1, \ldots] = \frac{\sqrt{5}+1}{2}.$$

Hence

$$Q \min_{1\leqslant k\leqslant Q} \|k\xi\| \geqslant q_i \|q_i\xi\| = \frac{1}{\xi_i + q_{i-1}/q_i} \geqslant \frac{1}{(\sqrt{5}+1)/2+1} = \frac{3-\sqrt{5}}{2}. \tag{4}$$

If $q \neq s$, assume e.g. $q > s$ and let

$$\xi = \frac{p\xi' + r}{q\xi' + s},$$

where

$$\xi' = [1, 1, 1, \ldots] = \frac{\sqrt{5}+1}{2}.$$

(If $q < s$, we should define $\xi = (p + r\xi')/(q + s\xi')$, and proceed similarly).

Then r/s and p/q are two consecutive convergents to ξ, whence $\xi \in (p/q, r/s)$, and, for a suitable $m \geqslant 2$,

$$\xi = [0, a_1, \ldots, a_{m-1}, 1, 1, 1, \ldots]$$

where

$$\frac{r}{s} = [0, a_1, \ldots, a_{m-2}]\,, \qquad \frac{p}{q} = [0, a_1, \ldots, a_{m-1}]\,.$$

The sequence (q_i) for ξ is now such that $q_{m-1} = s$, $q_m = q$. Hence for any integer $Q \geqslant \max\{q, s\} = q = q_m$ there exists $i \geqslant m$ such that $q_i \leqslant Q < q_{i+1}$. Again

$$\min_{1\leqslant k\leqslant Q} \|k\xi\| = \|q_i\xi\|\,, \qquad \xi_i = [1, 1, 1, \ldots] = \frac{\sqrt{5}+1}{2}\,,$$

and (4) follows.

LEMMA 2. *Let $\alpha \in [0, 1)$. For $0 < \delta < \Delta$ let $n = n(\delta)$ be the least integer $\geqslant f(\delta)$, $\mathcal{F}_n$ the Farey sequence of order n, and let p/q and r/s be two successive terms of $\mathcal{F}_n$ such that*

$$\alpha \in \left[\frac{p}{q}, \frac{r}{s}\right].$$

If

$$qs \geqslant \frac{1}{\delta}$$

for every $0 < \delta < \Delta$, then

$$\alpha \in \mathcal{S}(f, \Delta)\,.$$

PROOF. We have $ps - qr = \pm 1$ and

$$\left|\frac{p}{q} - \frac{r}{s}\right| = \frac{1}{qs} \leqslant \delta\,.$$

If ξ is as in lemma 1,

$$|\xi - \alpha| < \left|\frac{p}{q} - \frac{r}{s}\right| \leqslant \delta$$

and for any integer $Q \geqslant f(\delta)$ we have $Q \geqslant n \geqslant \max\{q, s\}$, whence, by lemma 1,

$$\min_{1\leqslant k\leqslant Q} \|k\xi\| \geqslant \frac{3-\sqrt{5}}{2}\,\frac{1}{Q}\,.$$

THEOREM 1:

$$\mathscr{S}\left(\frac{1}{\delta}, \infty\right) = \boldsymbol{R}.$$

PROOF. By (2) it suffices to prove that every $\alpha \in [0, 1)$ belongs to $\mathscr{S}(1/\delta, \infty)$. We apply lemma 2 with $\Delta = \infty$, $f(\delta) = 1/\delta$. If n, p/q and r/s are as in lemma 2, we have

$$qs \geqslant n \geqslant f(\delta) = \frac{1}{\delta},$$

for if $q = 1$ then $s = n$, if $s = 1$ then $q = n$, and otherwise $q \geqslant 2$, $s \geqslant 2$, $qs \geqslant q + s > n$. The theorem follows from lemma 2.

THEOREM 2. *Let $\alpha \in \boldsymbol{R}$ and $f(\delta)$ be such that there exist sequences $\delta_n \to 0$ and $p_n/q_n \in \boldsymbol{Q}$ satisfying*

i) $\left|\alpha - \frac{p_n}{q_n}\right| \ll \delta_n = o\left(\frac{1}{q_n^2}\right);$

ii) $f(\delta_n) = o\left(\frac{1}{q_n \delta_n}\right).$

Then, for any $\Delta > 0$,

$$\alpha \notin \mathscr{S}(f, \Delta).$$

PROOF. We will show that for any constant $C > 0$ and any sufficiently large n, if

$$|\xi - \alpha| < \delta_n, \tag{5}$$

then $Q \min_{1 \leqslant k \leqslant Q} \|k\xi\| < C$ for some $Q \geqslant f(\delta_n)$. By (5) and i) we obtain

$$\left|\xi - \frac{p_n}{q_n}\right| \ll \delta_n. \tag{6}$$

If we let

$$\varrho_n = \max\{q_n, f(\delta_n)\}$$

then, by i) and ii),

$$\varrho_n = o\left(\frac{1}{q_n \delta_n}\right).$$

Hence, for any integer Q satisfying

$$\varrho_n \leqslant Q \leqslant \varrho_n + o\left(\frac{1}{q_n \delta_n}\right) = o\left(\frac{1}{q_n \delta_n}\right),$$

we have by (6)

$$Q \min_{1\leqslant k\leqslant Q} \|k\xi\| \leqslant Q\|q_n\xi\| \leqslant Q|q_n\xi - p_n| \ll Qq_n\delta_n = o(1) \qquad (n\to\infty)\,.$$

COROLLARY 1. *For any* $\Delta > 0$ *and any function* f *satisfying* $f(\delta) = o(1/\delta)$ $(\delta\to 0)$, *we have*

$$\mathscr{S}(f, \Delta) \cap \boldsymbol{Q} = \varnothing\,.$$

PROOF. We take $\alpha = p/q \in \boldsymbol{Q}$ in theorem 2, and choose

$$\frac{p_n}{q_n} = \frac{p}{q} = \alpha$$

for all n. The assumptions of theorem 2 are then verified for any sequence $\delta_n \to 0$. Hence $p/q \notin \mathscr{S}(f, \Delta)$.

COROLLARY 2. *Let* $\alpha \notin \boldsymbol{Q}$ *and let* $\omega > 1$. *If there exist infinitely many* $p/q \in \boldsymbol{Q}$ *such that*

$$\left|\alpha - \frac{p}{q}\right| \ll \frac{1}{q^{1+\omega}}\,,$$

then, for any $\Delta > 0$ *and any function* $f(\delta) = o(\delta^{-\omega/(1+\omega)})$ $(\delta\to 0)$, *we have*

$$\alpha \notin \mathscr{S}(f, \Delta)\,.$$

PROOF. Let $p_n/q_n \in \boldsymbol{Q}$ be a sequence satisfying $|\alpha - p_n/q_n| \ll 1/q_n^{1+\omega}$. We may apply theorem 2 with the choice $\delta_n = 1/q_n^{1+\omega}$, since $1+\omega > 2$ and

$$\delta_n^{-\omega/(1+\omega)} = q_n^{\omega} = \frac{1}{q_n\delta_n}\,.$$

COROLLARY 3. *If* α *has type* $\Omega > 1$, *then for any* $\theta < \Omega/(1+\Omega)$ *and any constants* $K, \Delta > 0$ *we have*

$$\alpha \notin \mathscr{S}(K\delta^{-\theta}, \Delta)\,.$$

PROOF. By corollary 2 with $1 < \omega < \Omega$ such that $\theta < \omega/(1+\omega)$.

COROLLARY 4. *Let*

$$\alpha = [a_0, a_1, a_2, \ldots] \notin \boldsymbol{Q}$$

have unbounded partial quotients a_i, let (a_{i_n}) be a subsequence of (a_i) such that

$$\lim_{n\to\infty} a_{i_n} = \infty\,,$$

and let (q_i) denote the sequence of the denominators of the convergents to α. If $f(\delta)$ satisfies

$$f\left(\frac{1}{q_{i_n} q_{i_n+1}}\right) = o(q_{i_n+1})\,,$$

then, for any $\Delta > 0$,

$$\alpha \notin \mathscr{S}(f, \Delta)\,.$$

PROOF. We have $q_{i_n+1}/q_{i_n} > a_{i_n} \to \infty$, whence

$$q_{i_n} = o(q_{i_n+1})\,. \tag{7}$$

Since $|\alpha - p_i/q_i| < 1/q_i q_{i+1}$ for all i, we may apply theorem 2 to the sequences $\delta_{i_n} = 1/q_{i_n} q_{i_n+1}$ and p_{i_n}/q_{i_n}.

COROLLARY 5. *If $\alpha \notin \boldsymbol{Q}$ has unbounded partial quotients then, for any constants K, $\Delta > 0$,*

$$\alpha \notin \mathscr{S}(K\delta^{-\frac{1}{2}}, \Delta)\,.$$

PROOF. We apply corollary 4 to the function $f(\delta) = K\delta^{-\frac{1}{2}}$. The above condition $f(1/q_{i_n} q_{i_n+1}) = o(q_{i_n+1})$ becomes

$$K q_{i_n}^{\frac{1}{2}} q_{i_n+1}^{\frac{1}{2}} = o(q_{i_n+1})$$

and follows immediately from (7).

THEOREM 3. *Let $\alpha \notin \boldsymbol{Q}$. If*

$$\left|\alpha - \frac{p}{q}\right| \geqslant \frac{1}{(Hq)^{1+\omega}}$$

for some constants $H > 0$, $\omega \geqslant 1$ and for all $p/q \in \boldsymbol{Q}$, then

$$\alpha \in \mathscr{S}\left(H\left(\frac{2}{\delta}\right)^{\omega/(1+\omega)}, \infty\right)\,.$$

PROOF. As in the proof of theorem 1 we may assume $\alpha \in (0, 1)$. We now apply lemma 2 with $\Delta = \infty$, $f(\delta) = H(2/\delta)^{\omega/(1+\omega)}$. If n, p/q and r/s

are as in lemma 2, we have $q + s > n$ and

$$\frac{1}{qs} = \left|\frac{p}{q} - \frac{r}{s}\right| = \left|\alpha - \frac{p}{q}\right| + \left|\alpha - \frac{r}{s}\right| \geqslant \frac{1}{(Hq)^{1+\omega}} + \frac{1}{(Hs)^{1+\omega}}.$$

On multiplying by $(Hqs)^{1+\omega}$ we obtain

$$H^{1+\omega}(qs)^{\omega} \geqslant q^{1+\omega} + s^{1+\omega}.$$

By Hölder's inequality,

$$(q + s)^{1+\omega} \leqslant 2^{\omega}(q^{1+\omega} + s^{1+\omega}) .$$

Therefore

$$H^{1+\omega}(qs)^{\omega} \geqslant 2^{-\omega}(q + s)^{1+\omega} > 2^{-\omega} n^{1+\omega} \geqslant 2^{-\omega} f(\delta)^{1+\omega} = \frac{H^{1+\omega}}{\delta^{\omega}} ,$$

whence

$$qs > \frac{1}{\delta} ,$$

and the theorem follows from lemma 2.

COROLLARY 6. *If α has type $\Omega < \infty$, then for any $\theta > \Omega/(1 + \Omega)$ there exists $K = K(\theta, \alpha) > 0$ such that*

$$\alpha \in \mathscr{S}(K\delta^{-\theta}, \infty) .$$

PROOF. By theorem 3 with $\theta = \omega/(1 + \omega)$, so that $\omega > \Omega$, and $K = 2^{\theta}H$.

COROLLARY 7. *If*

$$\alpha = [a_0, a_1, a_2, \ldots] \notin \boldsymbol{Q}$$

has bounded partial quotients, $a_i \leqslant A$, then

$$\alpha \in \mathscr{S}(K\delta^{-\frac{1}{2}}, \infty)$$

with $K = \sqrt{2A + 4}$.

PROOF. For $q_i \leqslant q < q_{i+1}$ we have

$$q\|q\alpha\| \geqslant q_i\|q_i\alpha\| > \frac{1}{a_i + 2} \geqslant \frac{1}{A + 2}.$$

We may therefore apply theorem 3 with $\omega = 1$ and $H = \sqrt{A+2}$. Hence $\alpha \in \mathscr{S}(K\delta^{-\frac{1}{2}}, \infty)$ with $K = \sqrt{2}H$.

COROLLARY 8. $\bigcup_{K>0} \mathscr{S}(K\delta^{-\frac{1}{2}}, \infty)$ *is the set of irrational numbers with bounded partial quotients.*

PROOF. By corollaries 1, 5 and 7.

THEOREM 4. *For any* θ *such that* $\frac{1}{2} < \theta < 1$ *and for any* $K > 0$,

$$\mu\{\mathscr{S}(K\delta^{-\theta}, \infty) \cap [0,1)\} > 1 - \frac{\theta}{2\theta - 1}\left(\frac{2}{K}\right)^{1/(1-\theta)},$$

where μ *denotes the Lebesgue measure.*

REMARK. It is clear from the following proof that the constant $\theta/(2\theta - 1)$ occurring above could be replaced by

$$\zeta\left(\frac{\theta}{1-\theta}\right)\Big/\zeta\left(\frac{1}{1-\theta}\right),$$

where ζ is the Riemann zeta-function.

PROOF. Let D be the set of those $\delta > 0$ that are either rational or such that $f(\delta) = K\delta^{-\theta}$ is an integer. D is countable and dense in $(0, \infty)$. For any $\delta > 0$, let Ξ_δ be the set of irrational ξ satisfying

$$\min_{1 \leqslant k \leqslant Q} \|k\xi\| \geqslant \frac{3 - \sqrt{5}}{2}\,\frac{1}{Q}$$

for all integers $Q \geqslant f(\delta)$. Clearly

$$\mathscr{S}(f, \infty) = \bigcap_{\delta \in D} \bigcup_{\xi \in \Xi_\delta} (\xi - \delta, \xi + \delta)\,.$$

Therefore $\mathscr{S}(f, \infty) \cap [0,1)$ is a Borel set and hence is Lebesgue-measurable.

On writing

$$K = 2^\theta H, \qquad \theta = \frac{\omega}{1+\omega}$$

we have to show that, for any $\omega > 1$ and any $H > 0$,

$$\mu\left\{\mathscr{S}\left(H\left(\frac{2}{\delta}\right)^{\omega/(1+\omega)}, \infty\right) \cap [0,1)\right\} > 1 - \frac{2\omega}{H^{1+\omega}(\omega - 1)}\,. \tag{8}$$

By theorem 3, if $\alpha \in [0,1)$ and $\alpha \notin \mathcal{S} = \mathcal{S}(H(2/\delta)^{\omega/(1+\omega)}, \infty)$, there exists $p/q \in [0,1]$ such that

$$\alpha \in I_{p,q} = \left(\frac{p}{q} - \frac{1}{(Hq)^{1+\omega}}, \frac{p}{q} + \frac{1}{(Hq)^{1+\omega}}\right).$$

Hence

$$\complement_{[0,1)} \mathcal{S} \subset \bigcup_{p/q \in [0,1]} (I_{p,q} \cap [0,1)),$$

$$\mu\left(\complement_{[0,1)}\mathcal{S}\right) < \sum_{p/q\in[0,1]} \mu(I_{p,q} \cap [0,1)) \leqslant \frac{2}{H^{1+\omega}} + \sum_{p/q\in(0,1)} \mu(I_{p,q})$$
$$= \frac{2}{H^{1+\omega}}\left(1 + \sum_{q=2}^{\infty}\frac{\varphi(q)}{q^{1+\omega}}\right) = \frac{2\zeta(\omega)}{H^{1+\omega}\zeta(1+\omega)}.$$

Trivially

$$\frac{\zeta(\omega)}{\zeta(1+\omega)} < \zeta(\omega) = \sum_{q=1}^{\infty}\frac{1}{q^{\omega}} < 1 + \int_1^{\infty}\frac{dx}{x^{\omega}} = \frac{\omega}{\omega-1},$$

and (8) follows.

THEOREM 5. *Let $\alpha \notin \mathbf{Q}$, let $\eta(x) > 0$ be a decreasing continuous function of $x \geqslant 1$ satisfying*

$$\|q\alpha\| \geqslant \eta(q)$$

for every integer $q \geqslant 1$, and let $\psi(x) = \eta(x)/x$. Then

$$\alpha \in \mathcal{S}\left(\frac{2}{\delta\psi^{-1}(\delta)}, \psi(1)\right).$$

PROOF (*). We may assume $\alpha \in (0,1)$. We apply lemma 2 with $\Delta = \psi(1)$, $f(\delta) = 2/\delta\psi^{-1}(\delta)$. If n, p/q and r/s are as in lemma 2, with $q \geqslant s$, then $q + s > n$ and $q > n/2$, whence

$$\frac{1}{q} < \frac{2}{n} \leqslant \frac{2}{f(\delta)} = \delta\psi^{-1}(\delta).$$

We have

$$\frac{1}{qs} = \left|\frac{p}{q} - \frac{r}{s}\right| > \left|\alpha - \frac{r}{s}\right| \geqslant \frac{\eta(s)}{s},$$

(*) I am indebted to D. Zagier for some remarks which led to the present proof of theorem 5.

and therefore

$$\eta(s) < \frac{1}{q} < \delta\psi^{-1}(\delta) = \eta(\psi^{-1}(\delta)) .$$

Since η is decreasing, this yields

$$s > \psi^{-1}(\delta) > \frac{1}{q\delta} ,$$

whence

$$qs > \frac{1}{\delta} ,$$

and again the theorem follows from lemma 2.

COROLLARY 9. *For every* $\alpha \notin \boldsymbol{Q}$ *there exist* $\Delta > 0$ *and* $f(\delta) = o(1/\delta)$ $(\delta \to 0)$ *such that*

$$\alpha \in \mathscr{S}(f, \Delta) .$$

PROOF. By theorem 5.

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