ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

D. KINDERLEHRER
L. NIRENBERG
J. SPRUCK

Regularity in elliptic free boundary problems. II. Equations of higher order

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 6, nº 4 (1979), p. 637-683
<http://www.numdam.org/item?id=ASNSP_1979_4_6_4_637_0>

Regularity in Elliptic Free Boundary Problems.

II. Equations of Higher Order.

D. KINDERLEHRER [1] (*) - L. NIRENBERG [1] (**)
J. SPRUCK [1] (***)

dedicated to the memory of Guido Stampacchia

1. – Introduction.

In part I [8] we demonstrated the regularity of free boundaries which occur in some problems involving second order elliptic equations on one or both sides of a « free » hypersurface Γ in $\boldsymbol{R}^n$, subject to overdetermined boundary conditions on Γ. In this paper we take up higher order elliptic equations. As before, our aim is to prove optimal regularity of Γ and of the solution of the equations up to Γ, assuming some initial degree of regularity. Our results are local; we consider real equations for real functions in a small ball B about the origin, which lies on Γ, and we denote by $\Omega^{\pm}$ the two components of B separated by Γ. We always assume Γ to be *at least* C^1 and the unknowns in our equations to have a certain initial regularity in $\Omega^{\pm} \cup \Gamma$. At the origin, the positive x_n axis is taken normal to Γ, pointing into Ω^+.

This paper treats several model problems, for which some of the results have been announced in [4, 6]. It still remains to establish corresponding

([1]) The first author was partially supported by National Science Foundation grant MPS-75-06489. The second was partially supported by an NSF grant MCS 76-07039 and by a grant from the Office of Naval Research, N000014-76-C-0439. The third was partially supported by NSF grant MPS 75-06490 and by a Sloan Foundation Grant to the Courant Institute.

(*) University of Minnesota.
(**) Courant Institute, New York University.
(***) Brooklyn College.

Pervenuto alla Redazione il 19 Settembre 1978.

results for general elliptic systems and we hope to return to this at some later time. To illustrate the kind of problems treated we now describe two results.

THEOREM 1. *Let $u \in C^{2m}(\Omega^+ \cup \Gamma)$ satisfy a nonlinear elliptic equation in Ω^+ of order $2m$*

$$F(x, u, \nabla u, \ldots, \nabla^{2m} u) = 0 ,$$

and the boundary conditions

$$\nabla^j u = 0 \text{ on } \Gamma \quad \text{for } j \leqslant m , \qquad \partial_n^{m+1} u(0) \neq 0 .$$

If F is analytic (C^∞) in all its arguments, then Γ is analytic (C^∞) and so is u in $\Omega^+ \cup \Gamma$ near the origin.

Here we have used the notation $\partial_j = \partial/\partial x_j$, $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$, and also the notation ∇^j to represent all derivatives of order j. Observe that the boundary conditions consist of the usual homogeneous Dirichlet boundary conditions plus one more condition: $\partial_n^m u = 0$ on Γ. This result is a special case of Theorem 4.1 in section 4. For $m = 1$ the result was proved in [5] where it was shown how the problem arises in connection with an obstacle problem for a membrane.

Let us consider the corresponding problem for a plate constrained to lie above an obstacle. Let $w(x)$ and $\psi(x)$ denote the heights of the plate and the obstacle respectively above a point x in a bounded domain G in R^2. Assuming $w = |\text{grad}\, w| = 0$ on ∂G, we seek to minimize energy $I[w]$ under the restraint that $w \geqslant \psi$ in G,

$$I[w] = \int_G w_{x_i x_j}^2 \, dx .$$

Let us suppose that ψ is regular, and that $\psi < 0$ on G and $\psi > 0$ somewhere in D. The existence of a solution has been proved by Stampacchia [13] and it has been shown by Frehse [3] that it is of class $H^3(G)$ i.e. has square integrable third derivatives. Very recently, Caffarelli and Friedman [2] have shown that the solution is of class C^2 when $n = 2$.

Typically two cases arise. The plate may coincide with the obstacle in a region of space whose projection onto G is bounded by a curve Γ. In this case let us denote by Ω^+ a region of G bounded partly by Γ where $w > \psi$. We then have

$$u \underset{\text{def}}{=} w - \psi > 0 \quad \text{and} \quad \Delta^2 u = - \Delta^2 \psi \text{ in } \Omega^+$$

while

$$u = |\text{grad}\, u| = 0 \text{ on } \Gamma ,$$

so *Theorem* 1 applies. On the other hand, in some cases one finds that the plate touches the obstacle just along a curve γ. If Γ denotes the projection of γ in G then (locally) on both sides $\Omega^{\pm}$ of Γ we have

$$u = w - \psi > 0 \quad \text{and} \quad \Delta^2 w = 0 \text{ in } \Omega^{\pm},$$

while

$$u = |\operatorname{grad} u| = 0 \text{ on } \Gamma,$$

and the second derivatives of u are continuous across Γ. Thus

$$\Delta^2 u = -\Delta^2 \psi \text{ in } \Omega^{\pm}. \tag{1}$$

It is natural to ask whether Γ is smooth. Assuming some further regularity, which up to now has not been established, we can answer in the affirmative:

THEOREM 2. *In the preceding situation assume Γ is of class C^1, that $u \in C^2(\Omega^+ \cup \Gamma \cup \Omega^-) \cap C^4(\Omega^+ \cup \Gamma) \cap C^4(\Omega^- \cup \Gamma)$, and that*

$$u = |\operatorname{grad} u| = 0 \text{ on } \Gamma; \quad \partial_n^2 u(0) \neq 0. \tag{2}$$

If u satisfies (1) *in $\Omega^{\pm}$ with ψ analytic (C^∞) in $\Omega^+ \cup \Gamma \cup \Omega^-$ then Γ is analytic (C^∞) and so is u in $\Omega^{\pm} \cup \Gamma$.*

The conditions $u = |\text{grad } u| = 0$ on Γ are just Dirichlet boundary conditions for u in both $\Omega^{\pm}$. The additional boundary condition is the assumption that the second derivatives of u are continuous across Γ. Theorem 2 is a special case of Theorem 6.2.

As in [5] and [8] our results are proved with the aid of suitably constructed mappings which flatten Γ. In the new independent variables we introduce new dependent variables satisfying transformed elliptic equations—to which elliptic boundary regularity may be applied. In section 2 various procedures for attempting this are described but in the end a different procedure is used. In section 3 we transform our given elliptic equations to highly overdetermined elliptic systems (i.e. more equations than unknowns). We are then in a position to use regularity results for overdetermined systems and boundary conditions. Results of this kind concerning C^∞ (and finitely differentiable) regularity are due to Solonnikov [11], [12]. A corresponding analyticity result is needed and it is described in the Appendix here.

The principal results are in sections 4 to 7. In section 4 we treat a rather general nonlinear problem in Ω^+. In section 5 we present a generalization of a result of Lewy [9]; this is based on Lemma 5.2 which is also used in section 6 and is of independent interest. Sections 6 and 7 are concerned with problems for functions defined in both Ω^+ and Ω^-. In section 8, as a first step in studying general elliptic systems, we present a special result for first order systems. Finally, in the Appendix we describe an analyticity result at the boundary, for general nonlinear overdetermined elliptic systems and boundary conditions.

This paper is dedicated to the memory of our close friend Guido Stampacchia. His fundamental work in free boundary problems, and his ideas, continue to have a strong influence on our work in the subject.

2. – Remarks on higher order « hodograph » and « Legendre » transforms.

In part I [8], and in [5], we proved regularity with the aid of suitably constructed mappings $x \mapsto y$ of $\Omega^\pm$ into a region U, in $y_n > 0$, which mapped Γ into the hyperplane $y_n = 0$ on ∂U. The partial differential equations in $\Omega^\pm$ transformed into other equations in U, and the boundary conditions on Γ into conditions on $y_n = 0$. In U we then applied known regularity theorems for elliptic equations up to the boundary to obtain the desired results. The mappings from Ω^+ were of the form $x \mapsto y = (x_1, \ldots, x_{n-1}, y_n)$ where y_n was one of the functions w or $-\partial_n w$, according as w vanished to first or second order on Γ. With the aid of our « zero » of « first » order Legendre transforms any equation satisfied by w transformed into a new equation for the corresponding « partial Legendre transform » of zero or first order.

In dealing with higher order problems we encounter functions w (in Ω^+ say) vanishing to order $p \geqslant 2$ on Γ. For example, in Theorem 1, $p = m + 1$ and $u = w$. Supposing $\partial_n^{p+1} w < 0$ on Γ, we introduce the mapping

$$x \mapsto y = (x_1, \ldots, x_{n-1}, -\partial_n^p w) \tag{2.1}$$

which is locally $1-1$, maps Ω^+ into the region $y_n \geqslant 0$ and Γ into S in $y_n = 0$. This leads to the following:

QUESTION. Is there an analogue of the partial Legendre transform i.e. some function, or functions, of y in terms of which one may express all the x-derivatives of w and such that an elliptic equation for w is transformed to one (or a system) for the Legendre transform?

A reasonable choice is the first order partial Legendre transform of $\partial_n^{p-1} w$:

$$f(y) = \partial_n^{p-1} w - x_n \partial_n^p w\,. \tag{2.2}$$

Then we have (see [5], [8]),

$$(2.3)\qquad \partial_{x_n} = -\,\partial_n^{p+1} w\cdot\partial_{y_n}\,,\qquad \partial_{x_\alpha} = \partial_\alpha - \partial_\alpha \partial_n^p w\cdot\partial_{y_n}\,,\qquad \alpha<n\,,$$

and

$$(2.4)\qquad x_n = f_n\,,\qquad \partial_\alpha\partial_n^{p-1} w = f_\alpha\,,\qquad \alpha<n\,,$$

so

$$(2.5)\qquad \partial_n^{p+1} w = -\frac{1}{f_{nn}}\,,\qquad \partial_\alpha\,\partial_n^p w = \frac{f_{n\alpha}}{f_{nn}}\,,\qquad \alpha<n\,.$$

Here the derivatives of f (denoted via subscripts) are all with respect to the y variables. However, one cannot express all x-derivatives of w in terms of derivatives of f.

Next one might attempt to introduce the following system of functions:

$$g^k(y) = \sum_{j=0}^{p-k}\frac{(-x_n)^j}{j!}\,\partial_n^{j+k} w\,,\qquad k=0,\dots,p-1\,.$$

Note that $g^{p-1}\equiv f$. In terms of the g^k and their y-derivatives one may express all x-derivatives of w; in fact for derivatives of order $\leqslant p$:

$$\partial_n^k w = \sum_{j=0}^{p-k-1}\frac{1}{j!}\,f_n^j g^{j+k} - \frac{1}{(p-k)!}\,y_n f_n^{p-k}\,,\qquad k=0,\dots,p-1\,,$$

and

$$\partial_{\alpha_1}\dots\partial_{\alpha_r}\,\partial_n^k w = \sum_{j=0}^{p-k-1}\frac{f_n^j}{j!}\,\partial_{\alpha_1}\dots\partial_{\alpha_r} g^{j+k}$$

for $r+k\leqslant p$, $r\geqslant 1$, $\alpha_i<n$. All higher derivatives of w can be computed from these with the aid of (2.3)-(2.5). We remark also that

$$g_n^k = -\frac{(-x_n)^{p-k}}{(p-k)!} = -\frac{(-f_n)^{p-k}}{(p-k)!}\,.$$

We shall not prove these identities here since we will make no use of them: we do not see how to transform an elliptic equation for w into an elliptic system for the g^j. Consider for example the following special case of Theorem 1.

$$\Delta^2 w = 0 \ \text{ in } \Omega^+$$

$$w = w_\nu = w_{\nu\nu} = 0\,,\qquad w_{\nu\nu\nu}\neq 0 \ \text{ on } \Gamma\,.$$

Here the subscript ν denotes normal differentiation to Γ (into Ω^+). With $p = 2$ we may perform the mapping

$$x \to y = (x_1, \ldots, x_{n-1}, -\partial_n^2 w)$$

and introduce the functions $g^0 = g$, $g^1 = f$ as above:

$$f = \partial_n w - x_n \partial_n^2 w$$
$$g = w - x_n \partial_n w + \tfrac{1}{2} x_n^2 \partial_n^2 w \,.$$

Recall $g_n = -\frac{1}{2} f_n^2$. If one computes all the derivatives of w in terms of g and f one finds

$$w_{nnnn} = -\frac{f_{nnn}}{f_{nn}^3}$$

$$w_{\alpha\beta\gamma\delta} \equiv g_{\alpha\beta\gamma\delta} + f_n f^{\alpha\beta\gamma\delta} + \text{lower order terms in } g \text{ and } f\,, \qquad \text{for } \alpha, \beta, \gamma, \delta < n\,.$$

We see that the term w_{nnnn} in $\Delta^2 w$ involves only third order derivatives of f while the terms $w_{\alpha\alpha\alpha\alpha}$, $\alpha < n$ involve fourth order derivatives of f. It appears therefore that this equation $\Delta^2 w = 0$ does not transform into an elliptic system for g, f.

We have not found a very satisfactory answer to our request for a suitable analogue of the Legendre transform. Instead, in the domain U in the y-space we introduce (in the next section) in addition to $f(y)$, essentially *all* partial derivatives of w up to order p, as new unknown functions. Corresponding to an elliptic equation for w we construct an enormous overdetermined elliptic system for these functions. Boundary conditions for w then become conditions on $y_n = 0$.

Before proceeding, we would like to describe an alternative approach to the problems considered here of regularity of free boundaries. This involves a different way of mapping Ω^+ to U and avoids consideration of overdetermined systems. For definiteness let us return to the situation of Theorem 1 and suppose $\partial_n^{m+1} w(0) > 0$. Consider the mapping of Ω^+ to $y_n > 0$:

$$x \mapsto y = \big(x_1, \ldots, x_{n-1}, (m+1) w^{1/(m+1)}\big)\,.$$

Then

$$\partial_{x_n} = w^{-m/(m+1)} w_n \partial_{y_n}\,, \qquad \partial_{x_\alpha} = \partial_{y_\alpha} + w^{-m/(m+1)} w_\alpha \partial_{y_n}\,, \qquad \alpha < n\,.$$

Introduce as an unknown function in U the zero order Legendre transform ([5], [8]),

$$x_n = \psi(y) .$$

Denoting all y differentiations of ψ by subscripts we find

$$\psi_n = w^{m/(m+1)} w_n^{-1} = \left(\frac{y_n}{m+1}\right)^m w_n^{-1}, \qquad \psi_\alpha = -\frac{w_\alpha}{w_n}, \qquad \alpha < n .$$

Thus

$$w_n = \left(\frac{y_n}{m+1}\right)^m \psi_n^{-1}, \qquad w_\alpha = -\left(\frac{y_n}{m+1}\right)^m \frac{\psi_\alpha}{\psi_n}, \qquad \alpha < n ,$$

and hence

$$\partial_{x_n} = \psi_n^{-1}\partial_{y_n}, \qquad \partial_{x_\alpha} = \partial_{y_\alpha} - \frac{\psi_\alpha}{\psi_n}\partial_{y_n} .$$

We see that we can compute any x derivative of w in terms of derivatives of $\psi(y)$; therefore any differential equation satisfied by w transforms into a differential equation for ψ—but a singular one as we approach $y_n = 0$.

For example, if we take $m = 1$ and suppose that w satisfies

$$\Delta w = a(x)$$

then ψ satisfies

$$-\frac{y_n}{2}\left[\frac{\psi_{nn}}{\psi_n^3}\Big(1 + \sum_{\alpha<n}\psi_\alpha^2\Big) - \frac{2}{w_n^2}\sum_{\alpha<n}\psi_\alpha\psi_{\alpha n} + \frac{1}{\psi_n}\sum_{\alpha<n}\psi_{\alpha\alpha}\right] + \\ + \frac{1}{2}\psi_n^{-2}\Big(1 + \sum_{\alpha<n}\psi_\alpha^2\Big) = a(y_1, \dots, y_{n-1}, \psi) \text{ in } U.$$

What about a boundary condition on $y_n = 0$? It is, simply, that w is somewhat regular as $y_n \to 0$ and $w_n \neq 0$. Under these conditions if a is analytic one should be able to conclude that ψ is analytic in $y_n \geqslant 0$. However no regularity result of this kind seems to be known for nonlinear degenerate elliptic equations. (We note that if $m = 2$ and w satisfies

$$\Delta^2 w = 0$$

then, in U, ψ satisfies an elliptic quasilinear equation whose leading terms have a factor of y_n^2, and so on for higher order).

We have described the mapping (2.1) of Ω^+ into U. In case Ω^- is involved we use a reflection as in Part 1 mapping $U \cup S$ to $\Omega^- \cup \Gamma$ given by

$$y \mapsto x = (y_1, \dots, y_{n-1}, f_n - Cy_n) , \tag{2.6}$$

where C is a constant greater than f_{nn}. A simple computation yields the transformation laws:

$$(2.7)\qquad \partial_{x_n} = \frac{1}{f_{nn} - C}\,\partial_{y_n} \qquad \partial_{x_\alpha} = \partial_{y_\alpha} - \frac{f_{n\alpha}}{f_{nn} - C}\,\partial_{y_\alpha}, \qquad \alpha < n .$$

In section 6, example 6.1 we illustrate the use of this reflection in a problem analogous to Example 3.2 of Part. I [8]. This is somewhat complicated: if v satisfies an elliptic equation of order $2m$ in Ω^-, it is necessary to introduce all derivatives of v of order $\leqslant p$ as new unknowns via (2.6) and then use the transformation laws (2.7).

3. – Construction of overdetermined elliptic systems.

We consider a real function $u(x) \in C^{2m}(\Omega^+ \cup \Gamma)$ satisfying a nonlinear elliptic equation

$$(3.1)\qquad F(x, u, \ldots, \nabla^{2m} u) = 0 ,$$

of order $2m$, and the boundary conditions on Γ:

$$(3.2)\qquad \nabla^j u = 0 , \quad j = 0, \ldots, p , \quad \partial_n^{p+1} u(0) < 0 ,$$

where, we recall, the positive x_n axis is normal to Γ at the origin and points into Ω^+. We have denoted differentiations of order j by ∇^j. As described in § 2 we map a neighborhood of the origin in $\Omega^+ \cup \Gamma$ into $y_n \geqslant 0$ by the mapping

$$(3.3)\qquad x \to y = (x_1, \ldots, x_{n-1}, -\partial_n^p u) .$$

The image of $\Omega^+(\Gamma)$ is denoted by $U(S)$. Our purpose is to associate with the eq. (3.1) for u an overdetermined system of equations in U for derivatives of u up to order p. (In section 7 of [4] we treated a special case in which we were able to construct an associated *determined* elliptic system.)

First, let us review the definition of ellipticity for a general overdetermined system defined in a neighbourhood U of 0 in a half space $\{y = (y_1, \ldots, y_n)\ y_n \geqslant 0\}$. We use the notation $\xi = (\xi_1, \ldots, \xi_n)$ for the dual variables, and we employ summation convention.

DEFINITION 3.1. A system of linear equations

$$(3.2)\qquad L_{rj}\left(y, \frac{1}{i}\partial\right) u^j(y) = f_r(y) , \qquad y \in U,\ 1 \leqslant r \leqslant M$$

in the unknowns $u^1, \ldots, u^N$, $M \geqslant N$, where the $L_{rj}(y, (1/i)\partial)$ are linear differential operators with complex valued coefficients, is said to be *elliptic* with weights $s_1, \ldots, s_M$, $\max s_k = 0$ and $t_1, \ldots, t_N$, integers, provided

$$\text{order } L_{rj} \leqslant s_r + t_j\,, \qquad 1 \leqslant r \leqslant M\,, \quad 1 \leqslant j \leqslant N\,,$$

and the principal symbol matrix $(L'_{rj}(y, \xi))$ has rank N for $0 \neq \xi \in R^n$ and $y \in U$. Here $L'_{rj}(y, \xi)$ consists of terms which are precisely of order $s_r + t_j$. It is understood that $L_{rj} = 0$ if $s_r + t_j < 0$.

A *nonlinear* system in the $u^1, \ldots, u^N$ of the form

$$\varphi_r\left(y, \left(\frac{1}{i}\,\partial\right)^{\alpha} u\right) = 0\,, \qquad r = 1, \ldots, M, \tag{3.4}$$

is *elliptic* (at the solution) if φ_r is at most of order $s_r + t_j$ with respect to the u^j, $r = 1, \ldots, M$; $j = 1, \ldots, N$, and

$$L_{rj}\left(y, \left(\frac{1}{i}\,\partial\right)^{\alpha} u\right)\bar{u}^j \equiv \frac{d}{dt}\,\varphi_r\left(y, \left(\frac{1}{i}\,\partial\right)^{\alpha}(u + t\bar{u})\right)\bigg|_{t=0} = 0 \tag{3.5}$$

is elliptic in the sense just described.

The condition of ellipticity is an open condition; when verifying it locally, it suffices to do so only at the origin.

We now return to (3.1) and the transformation (3.3). We will restrict ourselves to p in the range $2 \leqslant p \leqslant 2m - 1$. The cases $p = 0$ or 1 are simpler, and were treated in [8] using transformations of order zero or one respectively. In each of those cases the appropriate Legendre transform satisfies a single nonlinear elliptic equation of order $2m$. As we shall see in the next section, the cases $p \leqslant m$ are the most interesting.

In $U \cup S$ we now introduce new unknown functions of y in addition to the function of (2.2),

$$f(y) = \partial_n^{p-1} u - x_n \partial_n^p u\,, \tag{3.6}$$

almost all the derivatives of u of order $\leqslant p$. For $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\alpha_i < n$, set

$$\begin{cases} w^{\alpha,k} \underset{\text{def}}{=} \partial_{\alpha_1} \ldots \partial_{\alpha_r} \partial_n^k u(x)\,, & k \leqslant p - 1\,,\ r + k \leqslant p \\ w^{\alpha,p-r} \underset{\text{def}}{=} \partial_{\alpha_1} \ldots \partial_{\alpha_r} \partial_n^{p-r} u(x)\,, & 2 \leqslant r\,. \end{cases} \tag{3.7}$$

The length of α is $|\alpha| = r$. We must point out that the use of the multi-index α differs from the usual one and we will employ the notation

$$\partial^\alpha = \partial_{\alpha_1} \cdot \partial_{\alpha_2} \cdots \partial_{\alpha_r}$$

in either x or y variables and $\xi^\alpha = \xi_{\alpha_1} \cdot \xi_{\alpha_2} \dots \xi_{\alpha_r}$ for $\xi \in R^n$. This differs from our earlier notation. We will use the symbol 0 for the empty multi-index.

We consider the $w^{\alpha,k}$ as functions of y, and symmetric in the indices $\alpha_1, \dots, \alpha_r$. Note that we have not introduced as new unknowns in U those derivatives of order p of the form $\partial_n^p u$, $\partial_\alpha \partial_n^{p-1} u$, $\alpha < n$; the first is simply $-y_n$ and, by (2.4), the remainder are f_α. For convenience in presentation we will sometimes use the representations

$$(3.8) \qquad w^{0,p} \text{ for } -y_n \text{ and } w^{\alpha,p-1} \text{ for } f_\alpha , \qquad |\alpha| = 1 ,$$

for example in (3.11) below.

Note that we always have $k \leqslant p-1$ in $w^{\alpha,k}$ $\alpha \neq 0$. In the following, Greek letters σ, τ vary from 1 to $n-1$.

Being x-derivatives of u, the $w^{\alpha,k}$ satisfy the following compatibility relations—obtained with the aid of (2.3)-(2.5). First, those for derivatives of order p:

$$(3.9) \qquad \left(\partial_\tau - \frac{f_{n\tau}}{f_{nn}} \partial_n\right) w^{\alpha,k} = \frac{1}{f_{nn}} \partial_n w^{(\alpha,\tau),k-1} \qquad \text{if } k > 0,\ |\alpha| + k = p .$$

Here if $\alpha = (\alpha_1, \dots, \alpha_r)$ then (α, τ) is obviously $(\alpha_1, \dots, \alpha_r, \tau)$.

$$(3.10) \qquad \left(\partial_\tau - \frac{f_{n\tau}}{f_{nn}} \partial_n\right) w^{(\alpha,\sigma),0} = \left(\partial_\sigma - \frac{f_{n\sigma}}{f_{nn}} \partial_n\right) w^{(\alpha,\tau),0} \qquad \text{if } 1 + |\alpha| = p .$$

For lower order derivatives, we have

$$(3.11) \qquad \begin{cases} \dfrac{1}{f_{nn}} \partial_n w^{\alpha,k} = w^{\alpha,k+1} , \\ \left(\partial_\tau - \dfrac{f_{n\tau}}{f_{nn}} \partial_n\right) w^{\alpha,k} = w^{(\alpha,\tau),k}, \qquad |\alpha| + k < p . \end{cases}$$

In addition to these compatibility conditions we must represent our original equation (3.1). In F we replace x by (y', f_n), here $y' = (y_1, \dots, y_{n-1})$, and the derivatives of u of order $\leqslant p$ by the corresponding $w^{\alpha,k}$. Consider $\partial^\alpha \partial_n^k u$ with $|\alpha| + k > p$. If $k \geqslant p-1$ then, according to (3.8) and (2.3)-(2.5) we may represent this derivative in terms of derivatives of f, and we replace it by this expression in F. If $k < p-1$ we write $\partial^\alpha \partial_n^k u = \partial^\beta (\partial^\gamma \partial_n^k u)$ where $|\gamma|$ is as large as possible consistent with the restriction $|\gamma| + k = p$. We may then express this in terms of $w^{\gamma,k}$ and f. This choice is certainly not unique but this will not be important. Equation (3.1) now takes the form

$$(3.12)\qquad F(y', f_n, \ldots, \nabla^{2m-p+1} f, \ldots, \nabla^{2m-p} w^{\alpha,k}, \ldots, w^{\beta,k}) = 0$$

where f appears with derivatives to order $2m-p+1$, the $w^{\alpha,k}$, $|\alpha|+k=p$ appear to order $2m-p$, and the $w^{\beta,k}$ to *zeroth* order for $|\beta|+k<p$.

To this huge system (3.9)-(3.12) we now assign the following weights:

$$(3.13)\qquad \begin{cases} t_f = 2m-p+1\,, & \\ t_{w^{\alpha,k}} = 2m-p & \text{if } |\alpha|+k=p\,, \\ t_{w^{\beta,k}} = 1 & \text{if } |\beta|+k<p \\ s = 1+p-2m & \text{for equations (3.9), (3.10)} \\ s = 0 & \text{for equations (3.11), (3.12).} \end{cases}$$

Observe that the weights are consistent and have the crucial property that the $w^{\beta,k}$ for $|\beta|+k<p$ are « invisible » in eq. (3.12) in the sense that they will not appear in the principal part of the system.

We can now prove the main result of the section.

THEOREM 3.1. *The system* (3.9)-(3.12) *is elliptic with the choice of weights* (3.13).

PROOF. In checking ellipticity it suffices to consider the principal part of the system linearized at the origin. This calculation is greatly simplified by observing that $w^{\alpha,k}$, $\partial_n w^{\alpha,k}$ for $|\alpha|+k=p$, and $f_{n\tau}$, $\tau<n$ all vanish at the origin. We thus obtain the linearized system:

$$(3.14)\qquad \partial_\tau \overline{w}^{\alpha,k} = \frac{1}{f_{nn}(0)}\,\partial_n \overline{w}^{(\alpha,\tau),k-1}\,, \qquad k>0,\ |\alpha|+k=p,$$

$$(3.15)\qquad \partial_\tau \overline{w}^{(\alpha,\sigma),0} = \partial_\sigma \overline{w}^{(\alpha,\tau),0} \quad \text{if } 1+|\alpha|=p\,.$$

$$(3.16)\qquad \frac{1}{f_{nn}(0)}\,\partial_n \overline{w}^{\alpha,k} = 0 \qquad \partial_\tau \overline{w}^{\alpha,k} = 0\,, \qquad |\alpha|+k<p$$

$$(3.17)\qquad a_{(0),2m}\left(\frac{1}{f_{nn}(0)}\,\partial_n\right)^{2m-p+1}\overline{f} + \sum_{\substack{|\beta|=2m-p\\ |\alpha|+k=p\\ k<p}} a_{(\beta,\alpha),k}\,\partial^\beta \overline{w}^{\alpha,k} + \ldots = 0$$

where

$$a_{(\gamma),k} = \frac{\partial F}{\partial(\partial^\gamma \partial_n^k u)}\,(0, \ldots, \nabla^{2m} u(0))\,,$$

and the dots represent the remaining terms $\partial^\gamma \partial_n^k u$ with $k \geqslant p$ and $|\gamma| + k = 2m$ as in our discussion above.

The ellipticity of the above system is checked by passing to the symbol level (that is replacing ∂ by $i\xi$) and showing that the corresponding *algebraic* equations have only the trivial solution for $\xi \neq 0$ in R^n. For simplicity we may assume $f_{nn}(0) = 1$. Using the same symbols to represent algebraic quantities we must study the following equations:

$$\xi_\tau \overline{w}^{\alpha,k} = \xi_n \overline{w}^{(\alpha,\tau),k-1}, \quad k > 0, \ |\alpha| + k = p. \tag{3.18}$$

Recall

$$\overline{w}^{\alpha,p-1} = i\xi^\alpha \bar{f} \quad \text{for } |\alpha| = 1.$$

$$\xi_\tau \overline{w}^{(\alpha,\sigma),0} = \xi_\sigma \overline{w}^{(\alpha,\tau),0}, \quad 1 + |\alpha| = p \tag{3.19}$$

$$\xi_n \overline{w}^{\alpha,k} = 0 \tag{3.20}$$

$$\xi_\tau \overline{w}^{\alpha,k} = 0 \tag{3.21}$$

(3.20), (3.21): $|\alpha| + k < p$

$$i a_{(0),2m} \xi_n^{2m-p+1} \bar{f} + \sum_{\substack{|\beta| = 2m-p \\ |\alpha|+k=p \\ k<p}} a_{(\beta,\alpha),k} \xi^\beta \overline{w}^{\alpha,k} + \ldots = 0. \tag{3.22}$$

We can conclude immediately from (3.20), (3.21) that all $w^{\alpha,k} = 0$ for $|\alpha| + k < p$. The ellipticity of our system will follow easily from

LEMMA 3.1. *For $|\alpha| + k = p$ let $\overline{w}^{\alpha,k}$ be symmetric in α and satisfy* (3.18), (3.19). *Then $\overline{w}^{\alpha,k}$ is of the form*

$$\overline{w}^{\alpha,k} = \xi^\alpha \hat{w}^k, \quad k = 0, \ldots, p-1 \quad \text{if } \xi' = (\xi_1, \ldots, \xi_{n-1}, 0) \neq 0 \tag{3.23}$$

for some numbers $\hat{w}^k$, $k = 0, \ldots, p-1$.

PROOF. Observe that (3.18), (3.19) imply

$$\xi_\tau \overline{w}^{(\alpha,\sigma),k} = \xi_\sigma \overline{w}^{(\alpha,\tau),k}, \quad 0 \leqslant k \leqslant p-2, \ |\alpha| + k = p-1, \tag{3.24}$$

and

$$\overline{w}^{\alpha,p-1} = i\xi^\alpha \bar{f}, \quad |\alpha| = 1,$$

so that (3.23) holds for $|\alpha| = p - k = 1$. We prove (3.23) by induction on $|\alpha|$. More precisely, assume the desired result for $|\alpha| = r$, i.e. that symmetric tensors $\overline{w}^{\alpha_1,\ldots,\alpha_r}$ satisfying the identities

$$\xi_\tau \overline{w}^{\alpha_1,\ldots,\alpha_r,\sigma} = \xi_\sigma \overline{w}^{\alpha_1,\ldots,\alpha_r,\tau} \tag{3.25}$$

necessarily satisfy $\overline{w}^{\alpha_1 \dots \alpha_r} = \xi_{\alpha_1} \dots \xi_{\alpha_r} \hat{w}$. This clearly holds for $r = 1$. We wish to establish it for $|\alpha| = r + 1$. So suppose a symmetric tensor $\overline{w}^{\alpha_1, \dots, \alpha_{r+1}}$ of rank $r + 1$ satisfies

$$\xi_\tau \overline{w}^{\alpha_2, \dots, \alpha_{r+1}, \sigma} = \xi_\sigma \overline{w}^{\alpha_2, \dots, \alpha_{r+1}, \tau} .$$

Multiplying by ξ_τ and summing on τ we find

$$|\xi'|^2 \overline{w}^{\alpha_2, \dots, \alpha_{r+1}, \sigma} = \xi_\sigma \sum_\tau \xi_\tau \overline{w}^{\alpha_2, \dots, \alpha_{r+1}, \tau} . \tag{3.26}$$

But

$$\overline{w}^{\alpha_2, \dots, \alpha_{r+1}} \underset{\text{def}}{=} \sum_\tau \xi_\tau \, \overline{w}^{\alpha_2, \dots, \alpha_{r+1}, \tau} ,$$

the contraction of the tensor $\overline{w}^{\alpha_1, \dots, \alpha_{r+1}}$, is a tensor of rank r which clearly satisfies (3.25). Hence, by induction, $\overline{w}^{\alpha_2, \dots, \alpha_{r+1}} = \xi_{\alpha_2} \dots \xi_{\alpha_{r+1}} \hat{w}$ so that by (3.26),

$$\overline{w}^{\alpha_2, \dots, \alpha_{r+1}, \sigma} = \xi_{\alpha_2} \dots \xi_{\alpha_{r+1}} \xi_\sigma \frac{\hat{w}}{|\xi'|^2}$$

and the lemma is proved.

Returning to the proof of Theorem 3.1 we distinguish two cases.

CASE 1. $\xi_n = 0$, $\xi' \neq 0$. Then (3.18), (3.19) imply $\bar{f} = w^{\alpha,k} = 0$ for $k > 0$, $|\alpha| + k = p$, while $w^{\alpha,0} = \xi^\alpha \hat{w}^0$, $|\alpha| = p$, by Lemma 3.1. Substitution into (3.22) gives

$$\Big(\sum_{\substack{|\beta| = 2m - p \\ |\alpha| = p}} a_{(\beta,\alpha),0} \xi^\beta \xi^\alpha \Big) \hat{w}^0 = 0 .$$

By ellipticity of F (i.e. the nonvanishing of the symbol at ξ' of the linearization), the coefficient of $\hat{w}^0$ does not vanish. Hence $\hat{w}^0 = \hat{w}^{\alpha,0} = 0$.

CASE 2. $\xi_n \neq 0$. Then (3.18), (3.19) imply

$$\xi_n^{p-k-1} \overline{w}^{\alpha,k} = i \xi^\alpha \bar{f} , \qquad k = 0, \dots, p-1 , \ |\alpha| + k = p .$$

Substitution into (3.22) yields

$$i a_{(0),2m} \xi_n^{2m-p+1} \bar{f} + \sum_{\substack{|\beta| = 2m - p \\ |\alpha| + k = p}} a_{(\beta,\alpha),k} \xi^\beta \frac{i \xi^\alpha}{\xi_n^{p-k-1}} \bar{f} + \dots = 0$$

or

$$\Big(a_{(0),2m} \xi_n^{2m} + \sum_{\substack{|\beta| = 2m - p \\ |\alpha| + k = p}} a_{(\beta,\alpha),k} \xi^\beta \xi^\alpha \xi_n^k + \dots \Big) \bar{f} = 0 . \tag{3.27}$$

The coefficient of $\bar{f}$ in (3.27) is precisely the full symbol of the linearization of the equation $F = 0$. Hence by ellipticity of this symbol, $\bar{f} = \bar{w}^{\alpha,k} = 0$ and the theorem is proved.

4. – Free boundary regularity for a single higher order elliptic equation.

In the previous section we associated with a single nonlinear elliptic equation of order $2m$ in $\Omega^+ \cup \Gamma$ an overdetermined elliptic system locally in a half space $U \cup S$. We now investigate the problem of determining boundary conditions on u in addition to (3.2) that lead to the regularity of Γ. This depends on knowing which boundary conditions for our overdetermined system lead to regularity. The relevant notion is given in the following

DEFINITION 4.1. A set of linear boundary conditions $B_{qj}(y, (1/i)\partial)\, u^j = \varphi_q$ in $y_n = 0$, $q = 1, \dots, \mu$ is said to be *covering* for the overdetermined system of differential equations (3.2) in $y_n \geqslant 0$ provided

(i) the system (2.1) is elliptic,

(ii) there exist integers r_q, $q = 1, \dots, \mu$ such that the order of $B_{qj}(y, (1/i)\partial)$ is at most $r_q + t_j$ and if $B'_{qj}(y^0, (1/i)\partial)$ denotes the homogeneous part of order $r_q + t_j$ (with coefficients frozen at $y^0 = (y_1^0, \dots, y_{n-1}^0, 0)$) in B_{qj}, the homogeneous boundary problem

$$L'_{rj}\left(y^0, \frac{1}{i}\,\partial\right) u^j = 0 \qquad \text{in } y_n \geqslant 0,\ r = 1, \dots, M$$

$$B'_{qj}\left(y^0, \frac{1}{i}\,\partial\right) u^j = 0 \qquad \text{on } y_n = 0,\ q = 1, \dots, \mu$$

has no nontrivial exponential solution of the form $u^j = \exp\,[iy' \cdot \xi']\, u^j(y_n)$, $\xi' \neq 0$ which decays as $y_n \to \infty$.

Similarly, a set of nonlinear boundary conditions $\chi_q(y, (-i\,\partial)^\alpha u) = 0$, $q = 1, \dots, \mu$ is covering for the system (3.3) if χ_q is at most of order $r_q + t_j$ in u^j and the system of linearized boundary conditions

$$B_{qj}\left(y, \frac{1}{i}\,\partial\right) \bar{u}^j = \frac{d}{dt}\, \chi_q\left(y, \left(\frac{1}{i}\,\partial\right)^\alpha (u + t\bar{u})\right)\Bigg|_{t=0} = 0$$

is covering for the linearized system (3.5).

The covering condition, like the ellipticity condition is an *open* condition and we need only verify it at the origin. Our basic references for regularity results for overdetermined elliptic systems are two papers of Solonnikov [11], [12] where much of the linear theory analogous to that of [1], [10] is worked out. However this is not quite sufficient to treat our problems, and in the Appendix we describe a regularity theorem for non-linear systems of sufficient generality for our purposes.

We now return to the question of determining boundary conditions for solutions of

$$F(x, u, \dots, \nabla^{2m} u) = 0 \quad \text{in } \Omega^+ \tag{4.1}$$

which together with (3.2):

$$\partial^j u = 0\,, \quad j = 0, \dots, p\,, \quad \partial_n^{p+1} u < 0 \text{ on } \Gamma, \quad p \leqslant 2m-1\,, \tag{4.2}$$

are covering boundary conditions for our overdetermined elliptic system (3.9)-(3.12). If we look for decaying exponential solutions of this system, we are led to the following system of ordinary differential equations in $y_n \geqslant 0$ corresponding to equations (3.14)-(3.17) (here σ, τ range from 1 to $n-1$ and we have set $y_n = t$),

$$\xi_\tau \overline{w}^{\alpha,k} = \frac{1}{f_{nn}(0)} \frac{1}{i} \frac{d}{dt} \overline{w}^{(\alpha,\tau),k-1}\,, \qquad k > 0,\ |\alpha| + k = p\,, \tag{4.3}$$

recall $\overline{w}^{\alpha,p-1} \equiv i\xi^\alpha \bar{f}$, $|\alpha| = 1$,

$$\xi_\tau \overline{w}^{(\alpha,\sigma),0} = \xi_\sigma \overline{w}^{(\alpha,\tau),0}\,, \qquad 1 + |\alpha| = p\,. \tag{4.4}$$

$$\partial_n \overline{w}^{\alpha,k} = 0 \tag{4.5}$$

$$\xi_\tau \overline{w}^{\alpha,k} = 0 \qquad |\alpha| + k < p \tag{4.6}$$

$$i^{p-2m} a_{(0),2m} \left(\frac{1}{f_{nn}(0)} \frac{d}{dt}\right)^{2m-p+1} \bar{f} + \sum_{\substack{|\beta| = 2m-p \\ |\alpha|+k=p \\ k<p}} a_{(\beta,\alpha),k} \xi^\beta \overline{w}^{\alpha,k} + \dots = 0\,. \tag{4.7}$$

Observe that (4.6) immediately implies $\overline{w}^{\alpha,k} = 0$, $|\alpha| + k < p$.

Using (4.3), (4.4) and Lemma 3.1 we obtain the relations

$$\tag{4.8} \left\{ \begin{aligned} &\overline{w}^{\alpha,k} = \xi_\alpha \hat{w}^k\,, \qquad |\alpha| + k = p \\ &\hat{w}^k = \frac{1}{i} \frac{1}{f_{nn}(0)} \frac{d}{dt} \hat{w}^{k-1} = \left(\frac{1}{i} \frac{1}{f_{nn}(0)} \frac{d}{dt}\right)^k \hat{w}^0\,, \quad k > 0 \\ &\hat{w}^{p-1} = i\bar{f} = \left(\frac{1}{i} \frac{1}{f_{nn}(0)} \frac{d}{dt}\right)^{p-1} \hat{w}^0\,. \end{aligned} \right.$$

Substitution of these into (4.7) then yields the equation

$$\left(a_{(0),2m}\left(\frac{1}{if_{nn}(0)}\frac{d}{dt}\right)^{2m}+\sum_{\substack{|\beta|=2m-p\\|\alpha|+k=p\\k=0,\dots,p-1}}a_{\beta,\alpha,k}\xi^{\beta}\xi^{\alpha}\left(\frac{1}{if_{nn}(0)}\frac{d}{dt}\right)^{k}+\dots\right)\hat{w}^{0}=0\,. \tag{4.9}$$

That is, $\hat{w}^0$ satisfies the equation

$$L'\left(i\xi',\frac{1}{f_{nn}(0)}\frac{d}{dt}\right)\hat{w}_0=0 \tag{4.10}$$

where L' is the principal part of the linearization of (4.1) at the origin. Moreover since $\left((1/if_{nn}(0)\left((d/dt)\right)^k\hat{w}^0=\hat{w}^k,\ k=0,\dots,p-1\right.$ and u satisfies (4.2), the $w^{\alpha,k}$ which represent derivatives of u of order p vanish on $y_n=0$, and thus so does $\hat{w}^k$; therefore

$$\left(\frac{d}{dt}\right)^{k}\hat{w}_0=0 \qquad \text{at } y_n=0,\ k=0,\dots,p-1. \tag{4.11}$$

This gives p boundary conditions of Dirichlet type for $\hat{w}^0$ and provides a significant reduction of our problem, which we state for future reference in the following way:

PROPOSITION 4.1. *Let u satisfy* (4.1), (4.2). *Then $\hat{w}^0$ (which is related via* (4.8) *to the linearized system of ordinary differential equations* (4.3)-(4.7) *for the elliptic system* (3.9)-(3.12)) *satisfies*

$$L'\left(i\xi',\frac{1}{f_{nn}(0)}\frac{d}{dt}\right)\hat{w}^0=0\,,\qquad y_n\geqslant 0$$

and (4.11), *where L' is the principal part of the linearization of $F=0$ at the origin. Moreover, a set of boundary conditions for the $w^{\alpha,k}$ and f are covering if and only if they imply $\hat{w}^0\equiv 0$ for each $\xi'\in R^{n-1}\setminus 0$.*

Note that $\hat{w}^0$ is simply determined by the conditions

$$\left(\frac{-i}{f_{nn}(0)}\frac{d}{dt}\right)^{p-1}\hat{w}^0(t)=i\bar{f}(t) \qquad \text{for } t>0, \text{ and (4.11)},$$

where $\bar{f}(t)$ (or $\exp[i\xi'\cdot y']\bar{f}(t)$) is the linearization of f in (2.2).

In case $p\geqslant m$ the conditions (4.2) are already sufficient and we need no additional boundary conditions:

THEOREM 4.1. *Let* $u \in C^{2m}(\Omega^+ \cup \Gamma) \cap C^{p+1}(\Omega^+ \cup \Gamma)$ *satisfy the nonlinear elliptic equation* $F(x, u, \ldots, \partial^{2m} u) = 0$ *in* Ω^+ *and the boundary conditions*

$$(4.2)' \qquad \partial^j u = 0\,, \quad j = 0, \ldots, p\,, \qquad \partial_n^{p+1} u \neq 0 \quad \text{on } \Gamma$$

where $p \geq m$. *If* F *is analytic* (C^∞) *in all its arguments, then* Γ *is analytic* (C^∞).

PROOF. Assume first that $m \leq p \leq 2m-1$. Then our construction of the associated overdetermined system (3.9)-(3.12) is valid and we take as boundary conditions $f = w^{\alpha,k} = 0$ with the obvious choice of weights. Then these boundary conditions are covering, for according to Proposition 4.1 $\hat{w}_0$ satisfies (4.10) with the boundary conditions $(d/dt)^j \hat{w}^0 = 0$, $j = 0, \ldots, m-1$ which are just Dirichlet boundary conditions, and imply $\hat{w}^0 \equiv 0$. Thus Theorem A of the appendix can be applied and we conclude that f is analytic (C^∞) in $U \cup S$. Since $x_n = f_n(y', 0)$ parametrizes Γ, and $f_{nn} \neq 0$, Γ is analytic (C^∞).

Suppose now that $p \geq 2m$. Then (4.1), (4.2) imply that $\partial_n^{p-2m} F(x, 0, \ldots, 0) = 0$ and $\partial_n^{p-2m+1} F(x, 0, \ldots, 0) \neq 0$ on Γ which follows easily by differentiating (4.1) with respect to x_n and using (4.2)$'$. Applying the ordinary implicit function theorem gives the desired regularity.

Theorem 4.1 is a generalization of Theorem 1$'$ of [5] which corresponds to the case $p = m = 1$. Before taking up a more general result, it is worthwhile to consider another special case which corresponds to Theorem 2 of [5]. We will assume $m \geq 2$.

THEOREM 4.2. *Let* $u \in C^{2m}(\Omega^+ \cup \Gamma)$ *satisfy the nonlinear elliptic equation* $F(x, \ldots, \nabla^{2m} u) = 0$ *in* Ω^+ *and the boundary conditions*

$$(4.12) \quad \partial^j u = 0\,, \quad j = 0, \ldots, m-1\,, \qquad \partial_n^m u \neq 0\,, \qquad g(x, \nabla^m u) = 0 \text{ on } \Gamma$$

where Γ *is noncharacteristic for* g *at* 0 $\big($*i.e.* $\partial g / \partial(\partial_n^m u) \neq 0$ *at* $x = 0\big)$. *If* F *and* g *are analytic* (C^∞) *in all arguments, then* Γ *is analytic* (C^∞).

PROOF. We apply our construction of the overdetermined elliptic system (3.9)-(3.12) with $p = m-1$ $\big($assuming for convenience that $f_{nn}(0) = 1\big)$. It must be shown that (4.12) corresponds to covering boundary conditions for this system. The boundary conditions on $y_n = 0$ corresponding to (4.12) are

$$(4.13) \qquad f = w^{\alpha,k} = 0, \quad |\alpha| + k = m-1$$

and

$$(4.14) \qquad g\left(y', f_n, -\frac{1}{f_{nn}}, \frac{f_{n\alpha}}{f_{nn}}, \left(\partial_\beta - \frac{f_{n\beta}}{f_{nn}}\, \partial_n\right) w^{\alpha,k}\right) = 0$$

where in the last arguments, $|\alpha| + k = m - 1$. This last condition is obtained from the condition $g(x, \nabla^m u) = 0$ by observing that the m-th derivatives of u may be written in the form

$$\partial_n^m u = -\frac{1}{f_{nn}}, \qquad \partial_\alpha \partial_n^{m-1} u = \frac{f_{n\alpha}}{f_{nn}}, \qquad \partial_\beta \partial^\alpha \partial_n^k u = \left(\partial_\beta - \frac{f_{n\beta}}{f_{nn}} \partial_n\right) w^{\alpha,k},$$
$$0 \leqslant k \leqslant m-2, \; |\alpha| + k = m - 1.$$

The linearized boundary conditions corresponding to (4.13), (4.14) are: on $y_n = 0$,

$$\bar{f} = \overline{w}^{\alpha,k} = 0, \quad |\alpha| + k = m - 1, \tag{4.15}$$

$$a_n \bar{f}_{nn} + \sum_{\alpha<n} a_\alpha \bar{f}_{n\alpha} + \sum_{\beta,\alpha;k} b_{\beta,\alpha,k} \partial_\beta \overline{w}^{\alpha,k} = 0, \tag{4.16}$$

where

$$a_i = \frac{\partial g}{\partial(\partial_i \partial_n^{m-1} u)}\bigg|_{x=0}, \qquad i = 1, \dots, n, \qquad b_{(\beta,\alpha),k} = \frac{\partial g}{\partial(\partial_\beta \partial^\alpha \partial_n^k u)}\bigg|_{x=0}$$

and the a_i, $b_{(\beta,\alpha),k}$ are real and $a_n \neq 0$. Since $\overline{w}^{\alpha,k} = 0$ on $y_n = 0$ for $|\alpha| + k = m - 1$ and $\beta < n$, we may replace (4.16) by

$$a_n \bar{f}_{nn} + \sum_{\alpha<n} a_\alpha \bar{f}_{n\alpha} = 0, \quad \text{on } y_n = 0. \tag{4.17}$$

According to Proposition 4.1 the covering property of the boundary conditions (4.13), (4.14) is equivalent to the assertion that any solution of the problem

$$L'\left(i\xi', \frac{d}{dy_n}\right)\hat{w}^0 = 0, \qquad y_n \geqslant 0, \tag{4.18}$$

$$\frac{d^j}{dy_n^j}\hat{w}^0 = 0, \quad j = 0, \dots, m-2 \text{ on } y_n = 0 \tag{4.19}$$

$$a_n \frac{d^m}{dy_n^m}\hat{w}^0 + i \sum_{\alpha<n} a_\alpha \xi_\alpha \frac{d^{m-1}}{dy_n^{m-1}}\hat{w}^0 = 0 \qquad \text{on } y_n = 0 \tag{4.20}$$

which decays as $y_n \to +\infty$ is trivial. Since L' is elliptic, and has real coefficients, the characteristic polynomial of L has exactly m complex roots $\tau_1, \dots, \tau_m$ with $\operatorname{Im} \tau_i > 0$ and $\hat{w}^0$ is a solution of the m-th order equation

$$\left(\frac{1}{i}\partial_n - \tau_1\right) \cdots \left(\frac{1}{i}\partial_n - \tau_m\right)\hat{w}^0 = 0 \qquad \text{in } y_n \geqslant 0. \tag{4.21}$$

Combining (4.19) and (4.21) this expression simplifies to

$$\partial_n^m \hat{w}^0 - i \sum_{k=1}^{m} \tau_k \partial_n^{m-1} \hat{w}^0 = 0 \quad \text{on } y_n = 0 . \tag{4.22}$$

Since $i \sum_{\alpha<n} a_\alpha \xi_\alpha$ is purely imaginary and $-i \sum \tau_k$ has strictly positive real part we conclude from (4.20), (4.22) that

$$\partial_n^m \hat{w}^0 = \partial_n^{m-1} \hat{w}^0 = 0 \quad \text{on } y_n = 0 .$$

These relations together with (4.19) obviously imply $\hat{w}^0 \equiv 0$ so our conditions are covering. Appealing to Theorem A of the Appendix, we conclude that f is analytic (C^∞) and therefore Γ is analytic (C^∞).

In order to prove a more general result we first clarify what we mean by general boundary conditions for a free boundary problem since, in the natural Cartesian coordinates x, normal and tangential boundary operators implicitly involve still higher order differentiation of the unknown u (the relations $\partial_n^p u = 0$, $\partial_n^{p+1} u \neq 0$ on Γ, determine Γ).

We will use natural tangential and normal boundary operators on $\Gamma, \Lambda_\alpha, \partial_\nu$, associated with the parametrization of Γ as a graph: $x_n = \sigma(x')$ —in fact, $\sigma = f_n(x', 0)$. Here $x' = (x_1 ,\dots, x_{n-1})$. In terms of this representation the unit normal to Γ pointing into Ω^+ is

$$\nu = W^{-1}(-\sigma_1(x'), \dots, -\sigma_{n-1}(x'), 1) , \qquad W = \sqrt{1 + \sum_{\beta<n} \sigma_\beta^2(x')}$$

and

$$T_\alpha = \big(0, \dots, \underset{\alpha^{th}}{1}, \dots, 0, \sigma_\alpha(x')\big) , \qquad 1 \leq \alpha \leq n-1$$

is a basis for the tangent space to Γ. We set

$$\begin{cases} \partial_\nu = W^{-1}\big(\partial_n - \sum_{\beta<n} \sigma_\beta(x')\partial_\beta\big) \\ \Lambda_\alpha = \partial_\alpha + \sigma_\alpha(x')\partial_n . \end{cases} \tag{4.23}$$

In the following we will consider a class of first order operators of the form

$$X = \sum_{i=1}^{n} \gamma_i(x) \frac{\partial}{\partial x_i} + \sum_{\alpha<n} a_\alpha(x, \nabla\sigma) \Lambda_\alpha + a_n(x)\, \partial_\nu . \tag{4.24}$$

Note that we allow the a_α to depend on the tangent plane to Γ but the coefficient a_n of ∂_ν is permitted to depend only on position. For this reason

$\sum_{i=1}^{n} \gamma_i(x)(\partial/\partial x_i)$ is independent of the last two terms in (4.24) and cannot be expressed in their form. More precisely,

$$\sum \gamma_i(x) \frac{\partial}{\partial x_i} = \sum_{\alpha<n} a_\alpha \Lambda_\alpha + a_n \partial_\nu$$

where

$$a_\alpha = \sum_{\beta<n} \left(\delta_{\alpha\beta} - \frac{\sigma_\alpha \sigma_\beta}{W^2}\right)\gamma_\beta + \frac{\sigma_\alpha}{W^2}\gamma_n, \qquad \alpha<n \qquad a_n = \frac{1}{W}\left(\gamma_n - \sum_{\beta<n} \sigma_\beta \gamma_\beta\right)$$

so that a_n clearly depends on the tangent plane.

DEFINITION 4.2. (i) Let u satisfy (4.1), (4.2). A first order analytic (C^∞) free boundary operator is an expression of the form (4.24) where the coefficients $\gamma_i(x)$, $a_\alpha(x, \nabla\sigma)$, $a_n(x)$ are analytic (C^∞) in all arguments. Similarly an l-th order analytic (C^∞) *free* boundary operator is a finite linear combination of products $X = X^1 \cdot \ldots \cdot X^l$ with X^i of the form (4.24). By an analytic (C^∞) free boundary condition $\chi(u) = 0$ we mean a boundary condition where χ is an analytic (C^∞) (possibly nonlinear) function of a finite number of such products $X = X^1 \ldots X^j$, $j \leqslant l$; then χ is said to be of order l.

(ii) A set of free boundary conditions $\chi_q(u) = 0$, $q = 1, \ldots, \mu$, is *covering* for equation (4.1) if, assuming Γ known, that is $\sigma = f_n(x', 0)$ given, they are covering in the ordinary sense of Definition 4.1. The boundary condition is to be interpreted in the following way. After differentiating out all the terms $X = X^1 \ldots X^j u$, $j \leqslant l$ the terms having as factors derivatives of u of order $\leqslant p$ are to be omitted.

Now that we have given a meaning to our free boundary conditions we can state the main theorem.

THEOREM 4.3. *Let $u \in C^N(\Omega^+ \cup \Gamma)$ satisfy the nonlinear elliptic equation $F(x, \ldots, \nabla^{2m} u) = 0$ in Ω^+ and the boundary conditions*

$$\partial^j u = 0, \quad j = 0, \ldots, p-1 \ (u = 0 \text{ if } p = 0) \tag{4.25}$$

$$\partial_n^p u = 0, \quad \partial_n^{p+1} u \neq 0 \text{ on } \Gamma \tag{4.26}$$

$$\chi_q(u) = 0, \quad q = 1, \ldots, \mu \tag{4.27}$$

where the χ_q are analytic (C^∞) free boundary conditions, F is analytic (C^∞) in all arguments and N is the maximum of $2m$ and the orders of the χ_q. If the boundary conditions (4.25), (4.27) *are covering, Γ is analytic (C^∞).*

PROOF. We need only consider the cases $2 \leqslant p < m$; the case $p \geqslant m$ follows from Theorem 4.1 while the cases $m = 2$, or $p = 0, 1$ are elementary —one may use the simple transformations of order zero or one as in [8]. To prove the theorem we construct our overdetermined elliptic system as in Theorem 3.1 with boundary conditions $f = w^{\alpha,k} = 0$, $|\alpha| + k = p$ (weights $p - 2m - 1$, $p - 2m$ respectively) corresponding to (4.25), (4.26) and with « transformed » boundary conditions corresponding to the conditions (4.27). That is, using relations (4.23) our boundary conditions can be expressed in the original x variables and then represented in terms of f and the $w^{\alpha,k}$ just as we did in the construction of our system in section 3. Without loss of generality we may assume in (4.27) that the order of χ_a is at least $p + 1$ for otherwise by (4.25), (4.26) it is vacuous. If χ_a is of order $l \geqslant p + 1$ we assign its corresponding transformed boundary condition $\tilde{\chi}_a(u) = 0$ the weight $l - 2m$. We will show that these weights are consistent and that if (4.25), (4.27) are covering for $F = 0$ the transformed boundary conditions are covering for the overdetermined system (3.9)-(3.12). Consider one term of an l-th order boundary operator applied to u

$$(4.28) \qquad Xu = \prod_{i=1}^{l} \left\{ \gamma_j^i(x) \frac{\partial}{\partial x_j} + \sum_{\alpha<n} a_\alpha^i(x, \nabla\sigma(x'))\, \Lambda_\alpha + a_n^i(x)\, \partial_\nu \right\} u$$

which might appear in $\chi_a(u)$. The coefficients a_α^i implicitly contain terms $f_{n\beta}(x', 0)$ $\big(\text{since } \sigma = f_n(x', 0)\big)$ which might lead to difficulties if these coefficients are differentiated too often. However since all derivatives of u of order $\leqslant p$ vanish on Γ, at most $l - p - 1$ derivatives can act on a_α^i and this transforms to at most $l - p + 1$ derivatives of f. Since ∂_ν and Λ_α are given by (4.23) a similar argument applies to the terms $f_{n\beta}(x', 0)$ contained in these operators. Finally using the procedure of section 3 to transform derivatives of u, it is easily seen that derivatives of u of order $l \geqslant p + 1$ transform to derivatives of $w^{\alpha,k}$, $|\alpha| + k = p$, of order at most $l - p$ and to derivatives of f of order at most $l - p + 1$. Therefore our choice of weight $l - 2m$ for $\tilde{\chi}_a(u) = 0$ is consistent.

Next we consider those terms in $\chi_a(u)$ which may contribute to the *linearization at the origin* of the principal part of $\chi_a(u) = 0$. We first observe that terms where the a_α^i are differentiated $l - p - 1$ times do not contribute, for a typical such term is of the form $\big(\partial^{l-p-1} a_\alpha^i(x, \nabla\sigma(x'))\big)\cdot\ \partial^p \Lambda_\alpha u$ vanishes at the origin (since only the pure derivative $\partial_n^{p+1} u$ is nonzero at the origin). Similarly terms of top order arising from the coefficients $f_{n\beta}(x', 0)$ in the operators Λ_α and ∂_ν also do not contribute to the linearization. Hence for the purpose of considering the linearization at the origin we may consider $\Lambda_\alpha, \partial_\nu$ equivalent to $\partial_\alpha, \partial_n$ respectively and consider only terms in which all derivatives fall on u. Consider a typical term (ignoring the coefficient) $\partial^\alpha \partial_n^k u$, $|\alpha| + k = l$. Arguing as in section 3 the transform of $\partial^\alpha \partial_n^k u$ can be written

$$(4.29) \qquad \partial^\alpha \partial_n^k u = \begin{cases} \partial^\alpha \dfrac{\partial_n^{k-p+1} f}{f_{nn}^{k-p+1}} + \dots, & k \geqslant p-1 \\ \partial^\beta w^{\gamma,k} + \dots, & k < p-1, \ \alpha = (\beta, \gamma) \ \ |\gamma| + k = p \end{cases}$$

where the dots represent inessential terms which do not contribute to the linearization at the origin. In order to check the covering property we apply Proposition 4.1. Using relations (4.8) we see that the linearization of $\widetilde{\partial^\alpha \partial_n^k u}$ corresponds to the term

$$\frac{1}{i^p} (i^{|\alpha|} \xi_\alpha) \left(\frac{\partial_n}{f_{nn}(0)}\right)^k \hat{w}^0 .$$

This shows (since $i^{|\alpha|}\xi_\alpha$ corresponds to ∂^α) that our transformed boundary conditions are covering if the original boundary conditions (4.25), (4.27) are covering for $F(x, \dots, \partial^{2m} u) = 0$. Appealing to Theorem A of the appendix we conclude that f is analytic (C^∞) and therefore Γ is analytic (C^∞).

REMARK. The form of the boundary operator (4.24) is sufficiently general to handle all applications that we have encountered. More general boundary conditions can be studied in the same way but have the unpleasant feature that the equivalence of the covering property for the original and transformed system is destroyed. Hence each case must be treated in an ad hoc way.

The virtue of the class of boundary conditions we have used is that it gives an algebraic criterion in the original variables for free boundary regularity; thus complicated calculations and transformations are unnecessary.

5. – An extension of a Theorem of Lewy.

We defer our study of free boundary problems for another look at the theorem of Lewy in [9] regarding systems. In Theorem 4.1 of part I we gave a simple proof of this theorem for elliptic equations of arbitrary order. In this section we will extend that result to other boundary conditions. As always, the result is local.

Let L be an elliptic operator of order $2m$ with analytic (C^∞) coefficients in a neighborhood of the origin in R^n. Let $u, v \in C^{2m}$ in $x_n \geqslant 0$ (near the origin) satisfy

$$Lu = f_1\,, \qquad Lv + \lambda(x)v = f_2 \tag{5.1}$$

where λ, f_1, f_2 are analytic (C^∞) in a full neighborhood of the origin and $\lambda(0) \neq 0$.

THEOREM 5.1. *Under the conditions above, if u and v satisfy the boundary conditions for some l, $0 \leqslant l \leqslant m$:*

$$\left.\begin{array}{lll} (5.2) & \partial_n^j u = \partial_n^j v = 0\,, & j \leqslant l-1\,, \\ (5.3) & \partial_n^j (u - v) = 0\,, & j = l, \ldots, 2m-l-1\,, \end{array}\right\} \quad \text{on } x_n = 0$$

then u and v are analytic (C^∞) in $x_n \geqslant 0$ (near the origin).

Observe that (5.2), (5.3) constitute $2m$ conditions. For $l = 0$, in which case (5.2) is vacuous, this corresponds to Theorem 4.1 of part I. The idea of the proof is the same as that of the earlier result: by rewriting the system in terms of u and $w = v - u$ we show that, with suitable weights, (5.1) and the boundary conditions (5.2), (5.3) form a coercive system. On the face of it this seems far from obvious and in fact the proof of Theorem 5.1 involves considerably more work than that of the earlier result.

PROOF. For u and w the system takes the form

$$\left.\begin{array}{ll} (5.4) & Lu = f_1\,, \\ (5.5) & Lw + \lambda(w+u) = f_2\,, \end{array}\right\} \quad \text{in } x_n > 0$$

while on $x_n = 0$ we have

$$\partial_n^j u = 0\,, \quad j \leqslant l-1\,, \quad \partial_n^k w = 0, \quad k \leqslant 2m-l-1\,. \tag{5.6}$$

We assign the weights $t_u = 2m$, $t_w = 4m$, and $s = 0$, $s = -2m$ respectively to equations (5.4), (5.5). The boundary conditions are then assigned the obvious weights. If M is the principal part of L then the principal part of this system is

$$Mu\,, \quad Mw + \lambda u\,.$$

Since $\lambda \neq 0$ this is elliptic. The main point to be verified is that the boundary conditions are coercive. To show this at the origin we have to study the following system of ordinary differential equations; here

$\xi' = (\xi_1, \dots, \xi_{n-1}) \in R^{n-1} \setminus 0$.

$$M\left(\xi', \frac{1}{i}\,\partial_n\right)\bar{u} = 0 \qquad M\left(\xi', \frac{1}{i}\,\partial_n\right)\bar{w} + \lambda(0)\bar{u} = 0 \qquad \text{in } x_n \geqslant 0$$

with boundary conditions

$$\partial_n^j \bar{u}(0) = 0\,, \quad j \leqslant l-1\,, \quad \partial_n^k \bar{w}(0) = 0\,, \quad k \leqslant 2m-l-1\,.$$

Writing $x_n = t$, we have to prove that $\bar{u} = \bar{w} = 0$ is the only solution of this system in $t \geqslant 0$ decaying as $t \to \infty$. Fixing ξ' we simply write the equations for $\bar{u}, \bar{w}$ as

$$M\left(\frac{1}{i}\,\partial_t\right)\bar{u} = 0 \tag{5.7}$$

$$t > 0$$

$$M\left(\frac{1}{i}\,\partial_t\right)\bar{w} + \lambda(0)\,\bar{u} = 0\,. \tag{5.8}$$

After dividing by a constant we may suppose that $M(z)$ is a monic polynomial, and, since the coefficients of L are real, we have

$$M(z) = \prod_1^m (z - z_j) \prod_1^m (z - \bar{z}_j) = M_+(z)\, M_-(z)$$

where $z_1, \dots, z_m$ lie in the complex upper half plane. The boundary conditions are still

$$\partial_t^j \bar{u}(0) = 0\,, \quad j \leqslant l-1\,, \quad \partial_t^k \bar{w}(0) = 0\,, \quad k \leqslant 2m-l-1\,. \tag{5.9}$$

Since $\bar{u}, \bar{w}$ decay as $t \to \infty$ we see that they satisfy

$$M_+\left(\frac{1}{i}\,\partial_t\right)\bar{u} = 0 \tag{5.10}$$

and hence

$$M\left(\frac{1}{i}\,\partial_t\right) M_+\left(\frac{1}{i}\,\partial_t\right)\bar{w} = 0\,,$$

and consequently

$$M_+^2\left(\frac{1}{i}\,\partial_t\right)\bar{w} = 0\,. \tag{5.11}$$

In addition, since $\lambda(0) \neq 0$, $\overline{w}$ satisfies the boundary conditions:

$$\partial_t^j M\left(\frac{1}{i}\,\partial_t\right)\overline{w} = 0\,, \qquad j \leqslant l-1\,, \qquad \partial_j^k \overline{w} = 0\,, \qquad k \leqslant 2m-l-1 \quad \text{at } t=0\,. \tag{5.12}$$

We will prove that $\overline{w} = 0$; from (5.8) it follows that $\overline{u} = 0$ and the proof will be finished. For convenience in the remainder of this section we will drop the bars over w.

LEMMA 5.1. *The only function $w(t)$ satisfying* (5.11) *in $t > 0$ and the boundary conditions* (5.12) *is $w \equiv 0$.*

Lemma 5.1 is based on the following algebraic lemma which is proved later in the section.

LEMMA 5.2. *Let $P_1(z), P_2(z)$ be monic polynomials of the degree $m+k$, $0 \leqslant k < m$, such that $z_1, \ldots, z_m$ in the upper half plane are roots of P_1 while $\bar{z}_1, \ldots, \bar{z}_m$ are roots of P_2. Then $P_1 - P_2$ has degree $> m-k-2$.*

PROOF OF LEMMA 5.1. The general solution of (5.11) has the form

$$w(t) = \oint \exp[izt]\,\frac{A(z)}{M_+^2(z)}\,dz \tag{5.13}$$

where A is an arbitrary polynomial of degree $2m-1$ and the contour encloses all the roots of M_+. If A has degree $\leqslant l-1$ then the function w satisfies the boundary conditions

$$\partial_t^k w(0) = 0\,, \qquad k \leqslant 2m-l-1\,, \tag{5.14}$$

for

$$\left(\frac{1}{i}\,\partial_t\right)^k w(0) = \int \frac{z^k A(z)}{M_+^2(z)}\,dz$$

and, if we integrate over a circle $|z| = R$ and let $R \to \infty$ we see that the integral tends to zero. Conversely if $w(t)$ in (5.13) satisfies (5.14) then $\deg A \leqslant l-1$. For we have

$$\oint \frac{z^k A(z)}{M_+^2(z)}\,dz = 0 \qquad k \leqslant 2m-l-1\,. \tag{5.15}$$

Suppose $A(z) = a_r z^r +$ lower order terms, with $r > l-1$ and $a_r \neq 0$. We may then choose $k = 2m-r-1$ in (5.15). Integrating once more over $|z| = R$

and letting $R \to \infty$ we see that the integral in (5.15) tends to $2\pi i \cdot a_r$ — — contradiction.

Thus we know that our w has the form (5.13) with $\deg A \leqslant l-1$. The remaining boundary conditions in (5.12) assert that

$$\oint \frac{z^j M(z) A(z)}{M_+^2(z)} dz = 0 , \qquad j \leqslant l-1 ,$$

i.e.

$$\oint \frac{M_-}{M_+} A(z) z^j dz = 0 , \qquad j \leqslant l-1 .$$

We may write

$$M_- A = Q M_+ + R$$

where Q is a polynomial and R is a polynomial of degree $m-1$. The preceding condition then means

$$\oint z^j \frac{R}{M_+} dz = 0 \qquad \text{for } j \leqslant l-1 .$$

By the preceding argument we see that R is then necessarily of degree $\leqslant m-l-1$. We then have

$$M_- A - Q M_+ = R$$

has degree $\leqslant m-l-1$. If $A \not\equiv 0$ we may suppose it is a monic polynomial of degree $k \leqslant l-1$. Then Q is necessarily also a monic polynomial of the same degree, while $\deg R \leqslant m-l-1 \leqslant m-k-2$. But this contradicts Lemma 5.2. Hence $A \equiv 0$ and $w \equiv 0$. Lemma 5.1 is proved.

PROOF OF LEMMA 5.2. For any polynomial $P(z)$ we shall denote by $\hat{P}(z)$ the polynomial with its coefficients replaced by their complex conjugates.

Suppose the result is false for some m and k, i.e. for some monic polynomials A, B of degree k the polynomial

$$\prod_1^m (z - z_j) A(z) - \prod_1^m (z - \bar{z}_j) B(z) = R \tag{5.16}$$

has degree $\leqslant m-k-2$. Taking complex conjugate of (5.16), with z real, we obtain the following—which then automatically holds for all complex z—

$$\prod (z - \bar{z}_j) \hat{A} - \prod (z - z_j) \hat{B} = \hat{R} .$$

Taking half of the difference of this from the preceding we find

$$\prod (z - z_j)\frac{A + \hat{B}}{2} - \prod (z - \bar{z}_j)\frac{\hat{A} + B}{2} = \frac{R - \hat{R}}{2}$$

having degree $\leqslant m - k - 2$. We may write this in the form (with a new R):

$$\prod_1^m (z - z_j)\prod_1^k (z - \tau_i) - \prod_1^m (z - \bar{z}_j)\prod_1^k (z - \bar{\tau}_i) = R$$

having degree $\leqslant m - k - 2$. If some of the τ_i are real let us divide by the corresponding $(z - \tau_i)$—which must factor R. We then obtain, again with a new R, and some $r \leqslant k$

$$\prod_1^m (z - z_j)\prod_1^r (z - \tau_i) - \prod_1^m (z - \bar{z}_j)\prod_1^r (z - \bar{\tau}_i) = R$$

having degree $\leqslant m - r - 2$, and where no τ_i is real. Thus the product

$$f(z) = \prod_1^m \frac{z - z_j}{z - \bar{z}_j}\cdot\prod_1^r \frac{z - \tau_i}{z - \bar{\tau}_i}$$

satisfies

$$f(z) - 1 = \frac{R}{\prod (z - \bar{z}_j)\prod (z - \bar{\tau}_i)}\,.$$

For z real the absolute value of $f(z)$ is 1 and $f(\infty) = 1$. As z goes from $-\infty$ to $+\infty$ on the real axis $f(z)$ moves on the unit circle and its winding number around the origin is exactly equal to the number of roots of f in the upper half plane minus the number of poles. This is at least $m - r$. It follows that $f(z) - 1$ has at least $m - r - 1$ roots on the real axis $-\infty < z < \infty$. However R has degree $< m - r - 1$ so this is impossible.

The proofs of the lemma, and of Theorem 5.1 are complete.

Remark. Lemma 5.2 need not hold if P_1 and P_2 are merely required to have at least m roots in the upper and lower half planes respectively.

It is natural to ask what happens in Theorem 5.1 if the functions u, v, and the coefficients in the equations (5.1) are allowed to be complex? (It is clear that the result still holds if the coefficients of the highest order terms are real.) Let us assume that the corresponding polynomial $M(\xi', \xi_n)$, $\forall \xi' \in R^{n-1}\setminus 0$ has m roots ξ_n in the upper and m in the lower half planes (this is automatic if $n > 2$). For general $l > 0$, we do not know if the result still holds. However for special values of l it does.

THEOREM 5.1′. *Theorem* 5.1 *holds in the complex case* (under the proviso above) *if* $l = 1$ *or* m.

PROOF. The case $l = m$ is obvious for in this case the conditions (5.2) are simply the Dirichlet boundary conditions which are coercive for either equation in (5.1). So suppose $l = 1$. Following the proof of Theorem 5.1 we are led to the system (5.7), (5.8) and the boundary conditions (5.9). For convenience we drop the bars over u and w. Factor

$$M(z) = \prod_1^m (z - z_j) \prod_1^m (z - \zeta_k) = M_+(z)\, M_-(z)$$

where the z_i and ζ_i are in the upper and lower half planes respectively. Since u, w decay as $t \to \infty$ they satisfy as before

$$M_+\left(\frac{1}{i}\,\partial_t\right) u = 0 \qquad M_+^2\left(\frac{1}{i}\,\partial_t\right) w = 0\,, \tag{5.17}$$

and w satisfies the boundary conditions (5.12) with $l = 1$:

$$M\left(\frac{1}{i}\,\partial_t\right) w = 0\,, \qquad \partial_t^k w = 0\,, \qquad k \leqslant 2m - 2 \qquad \text{at } t = 0\,. \tag{5.18}$$

Thus the Cauchy data of w for equation (5.17) is zero at the origin except possibly for $\partial_t^{2m-1} w(0)$. To show that $w \equiv 0$ we have only to show that this derivative is also zero at the origin. This follows easily with the aid of the remaining condition in (5.18). Using it and the equations (5.17) we see that

$$\left[M\left(\frac{1}{i}\,\partial_t\right) - M_+^2\left(\frac{1}{i}\,\partial_t\right)\right] w(0) = 0\,.$$

It follows that

$$a\,\partial_t^{2m-1} w(0) = 0$$

where a is the coefficient of z^{2m-1} in $M(z) - M_+^2(z) = M_+(z)\big(M_-(z) - M^+(z)\big)$, i.e.

$$a = \sum (z_j - \zeta_j)\,.$$

This has positive imaginary part and hence $\partial_t^{2m-1} w(0) = 0$. Consequently $w = 0$, and also $u = 0$, and the proof is complete.

REMARK. Lemma 5.2 has as corollary a simple generalization of the classical fact that a monic polynomial of degree n with real coefficients has at most $[n/2]$ roots on either side of the real axis. Namely we can assert

COROLLARY. *Let P be a monic polynomial of degree n. Suppose that $P - \hat{P}$ has degree d. If P has m roots on one side of the real axis, then necessarily*

$$m < 1 + \frac{n+d}{2} .$$

6. – Examples.

To illustrate the applicability of our methods we present several examples of free boundary problems and regularity results.

EXAMPLE 6.1. Let $u, v \in C^{2+\alpha}(\Omega^+ \cup \Gamma)$, $\Gamma \in C^{2+\alpha}$ satisfy

$$\Delta u = b(x, u, v) \quad \Delta v = c(x, u, v) \qquad \text{in } \Omega^+ \tag{6.1}$$

$$u = v = 0 , \quad u_\nu = v_\nu , \quad \text{on } \Gamma , \tag{6.2}$$

where b, c are analytic (C^∞) in a neighborhood of Γ.

Case (1). $(u - v)_{\nu\nu} \neq 0$ on Γ,

Case (2). $(u - v)_{\nu\nu} \equiv 0$ on Γ and

$$(u - v)_{\nu\nu\nu} \neq \text{on } \Gamma . \tag{6.3}$$

REMARK. The special case $b \equiv 0$, $c = \lambda(x) v$, $\lambda \neq 0$ was given as an example in Part I (Example 4.1). Notice that in this case the conditions of Case (2) follow from the conditions $u = v = 0$, $u_\nu = v_\nu \neq 0$ on Γ.

For Case (1) the proof of regularity of Γ follows as in Theorem 4.2 of [8] so we shall only treat Case (2). As in Part I, we set $w = v - u \in C^{2+\alpha}(\Omega^+ \cup \Gamma)$ and rewrite the system as

$$\Delta u = b(x, u, u + w) \quad \text{in } \Omega^+ \tag{6.4}$$

$$\Delta w = c(x, u, u + w) - b(x, u, u + w) \qquad \text{in } \Omega^+ . \tag{6.5}$$

$$u = w = w_\nu = w_{\nu\nu} = 0 \quad \text{on } \Gamma \tag{6.6}$$

$$w_{\nu\nu\nu} \neq 0 \qquad \text{on } \bar{\Gamma} . \tag{6.7}$$

We claim first that $w \in C^{4+\alpha}(\Omega^+ \cup \Gamma)$, which justifies our writing condition (6.3). To see this observe that any first derivative w_i of w is a $C^{1+\alpha}$ weak solution of an equation of the form $\Delta w_i = h \in C^{1+\alpha}$ and $w_i = 0$ on $\Gamma \in C^{2+\alpha}$. It follows from elliptic regularity theory [1], [10] that

$w_i \in C^{2+\alpha}(\Omega^+ \cup \Gamma)$, hence $w \in C^{3+\alpha}(\Omega^+ \cup \Gamma)$. Repeating the argument once again with any second derivative w_{ij} gives $w \in C^{4+\alpha}(\Omega^+ \cup \Gamma)$. With this regularity we can apply the Δ operator to both sides of (6.5); using (6.4), (6.5) we find

$$\Delta^2 w = h(x, u, \partial u, w, \partial w) \quad \text{in } \Omega^+ \tag{6.8}$$

where h is analytic (C^∞) in all arguments. We now use (6.7) to introduce new independent variables

$$y = (x', -w_{nn}) ,$$

and transform functions:

$$\begin{cases} f(y) = w_n - x_n w_{nn} \\ w^{\alpha\beta}(y) = w_{\alpha\beta} \quad 1 \leqslant \alpha, \beta \leqslant n-1 \\ w^i(y) = w_i \quad 1 \leqslant i \leqslant n \\ w^0(y) = w \\ \varphi(y) = u(x) . \end{cases} \tag{6.9}$$

The system (6.4), (6.8), (6.6) transforms to the following overdetermined system for the unknowns (6.9):

$$\frac{1}{f_{nn}}\left(\frac{\varphi_n}{f_{nn}}\right)_n + \sum_{\alpha<n}\left(\varphi_\alpha - \frac{f_{n\alpha}}{f_{nn}}\varphi_n\right)_\alpha - \frac{f_{\alpha n}}{f_{nn}}\left(\varphi_\alpha - \frac{f_{n\alpha}}{f_{nn}}\varphi_n\right)_n = b(y', f_n, \varphi, \varphi + w^0) , \tag{6.10}$$

$$\frac{f_{nnn}}{f_{nn}^3} + 2\sum_{\alpha,\beta<n}\left[\left(w^{\alpha\alpha}_\beta - \frac{f_{n\beta}}{f_{nn}} w^{\alpha\alpha}_n\right)_\beta - \frac{f_{n\beta}}{f_{nn}}\left(w^{\alpha\alpha}_\beta - \frac{f_{n\beta}}{f_{nn}} w^{\alpha\alpha}_n\right)_n\right] + 2\sum_{\alpha<n}\left[\left(\frac{f_{n\alpha}}{f_{nn}}\right)_\alpha - \frac{f_{n\alpha}}{f_{nn}}\left(\frac{f_{n\alpha}}{f_{nn}}\right)_n\right] = h\left(y', f_n, \varphi, \varphi_\alpha - \frac{f_{n\alpha}}{f_{nn}}\varphi_n, \frac{\varphi_n}{f_{nn}}, w, w^i\right) \tag{6.11}$$

and, with $1 \leqslant \alpha, \beta, \tau \leqslant n-1$:

$$\left(\partial_\tau - \frac{f_{n\tau}}{f_{nn}}\partial_n\right) f_\alpha = \frac{1}{f_{nn}} w^{\alpha\tau}_n \tag{6.12}$$

$$\left(\partial_\tau - \frac{f_{n\tau}}{f_{nn}}\partial_n\right) w^{\alpha\beta} = \left(\partial_\beta - \frac{f_{n\beta}}{f_{nn}}\partial_n\right) w^{\alpha\tau} \tag{6.13}$$

$$(6.14)\qquad \begin{cases} \dfrac{1}{f_{nn}}\, w^{n} = f_{\tau}, & \dfrac{1}{f_{nn}}\, w^{n}_{n} = -\, y_n \\ \left(\partial_\tau - \dfrac{f_{n\tau}}{f_{nn}}\,\partial_n\right) w^{\alpha} = w^{\alpha\tau}\,, & \left(\partial_\tau - \dfrac{f_{n\tau}}{f_{nn}}\,\partial_n\right) w^{n} = f_\tau \end{cases}$$

$$(6.15)\qquad \frac{1}{f_{nn}}\, w^0_n = w^n\,, \qquad \left(\partial_\tau - \frac{f_{n\tau}}{f_{nn}}\,\partial_n\right) w^0 = w^\tau\,;$$

with boundary conditions

$$(6.16)\qquad \varphi = f = w^{\alpha\beta} = w^i = w^0 = 0 \quad \text{on } y_n = 0\,.$$

To the system we assign the weights

$$(6.17)\qquad \begin{cases} t_f = 3\,, \quad t_{w^{\alpha\beta}} = t_\varphi = 2\,, \quad t_{w^i} = t_{w^0} = 1 \\ s = 0 \quad \text{to equations (6.10), (6.11), (6.14), (6.15)} \\ s = -1 \quad \text{to equations (6.12), (6.13)}\,. \end{cases}$$

To check ellipticity of (6.10)-(6.15) we linearize at the origin and observe that equations (6.11)-(6.15) yield a linear system of the type discussed in section 3 (in which φ does not appear) corresponding to the equation $\Delta^2 w = 0$. Thus we obtain on linearization (assuming as usual $f_{nn}(0) = 1$):

$$(6.18)\qquad \Delta\big(\bar{\varphi} - \varphi_n(0)\,\bar{f}_n\big) = 0\,,$$

(6.19) The elliptic system (3.14)-(3.17) in $\bar{f}, \overline{w}^{\alpha\beta}, \overline{w}^i, \overline{w}^0$ corresponding to Δ^2 and $p = 2$.

This is clearly elliptic. We claim the boundary conditions (6.16) (with the obvious weights) are covering for this system. This is clear since the conditions $\bar{f} = \overline{w}^{\alpha\beta} = \overline{w}^i = \overline{w}^0 = 0$ cover the system (6.19) by Proposition 4.1, and then $\bar{\varphi} = 0$ covers (6.18) (since $\bar{f} \equiv 0$ at this point). Therefore we have constructed an overdetermined elliptic system with covering boundary conditions. By Theorem A of the appendix f is analytic (C^∞). Hence Γ is analytic (C^∞).

It is clear from the proof that we may replace the Δ operator by any second order linear elliptic operator L and we see therefore that the following more general result holds:

THEOREM 6.1. *Let* $u, v \in C^{2+\alpha}(\Omega^+ \cup \Gamma)$, $\Gamma \in C^{2+\alpha}$ *satisfy the system*

$$Lu = b(x, u, v) \quad \text{in } \Omega^+$$

$$Lv = c(x, u, v)\,, \quad \text{in } \Omega^+$$

$$u = v = 0\,, \quad u_\nu = v_\nu\,, \quad u_{\nu\nu} = v_{\nu\nu} \qquad \text{on } \Gamma\,,$$

where $L = \sum a_{ij}\,\partial_i\,\partial_j + \sum b_i\,\partial_i$ *is elliptic,* a_{ij}, b_i, b, c *analytic* (C^∞) *in all arguments in a neighborhood of* Γ. *If* $(u - v)_{\nu\nu\nu} \neq 0$ *on* Γ, Γ *is analytic* (C^∞).

Here is a related result for a triple of functions; we confine ourselves to a very simple (but curious) example.

THEOREM 6.1′. *Let* $u, v, w \in C^{2+\alpha}(\Omega^+ \cup \Gamma)$, $\Gamma \in C^1$, *satisfy*

$$\Delta u = 0$$

$$\Delta v + v = 0 \qquad\qquad \text{in } \Omega^+$$

$$\Delta w + 2w \equiv a(x)\,, \qquad a(0) \neq 0$$

and

$$u = v = w\,, \qquad \operatorname{grad} u = \operatorname{grad} v = \operatorname{grad} w \qquad \text{on } \Gamma\,.$$

If a is analytic (C^∞) *in a full neighborhood of the origin then* Γ *is analytic* (C^∞) *near the origin.*

Proof. Setting $v = u + \beta$, $w = u + \gamma$ the system takes the form in Ω^+:

$$\Delta u = 0$$

$$\Delta\beta + \beta + u = 0$$

$$\Delta\gamma + 2\gamma + 2u = a$$

hence

$$\Delta(\gamma - 2\beta) + 2\gamma - 2\beta = a\,, \tag{6.1$'$}$$

$$\Delta(\Delta + 1)\beta = 0\,, \tag{6.1$''$}$$

and on Γ:

$$\beta = \gamma = 0\,, \qquad \operatorname{grad}\beta = \operatorname{grad}\gamma = 0\,.$$

Following the analysis in Example 6.1 one establishes first that $\beta, \gamma \in C^{3+\alpha}(\Omega^+ \cup \Gamma)$. Since $a(0) \neq 0$ we may apply an earlier regularity result for free boundaries, Theorem 1 [5] using (6.1)′ and infer that $\Gamma \in C^{4+\alpha}$.

Repeating the earlier analysis one finds that $\beta, \gamma \in C^{4+\alpha}$ in $\Omega^+ \cup \Gamma$ near the origin. Applying Theorem 1 of [5] once more we conclude that $\Gamma \in C^{5+\alpha}$. Differentiating $(6.1)'$ one finds that the first derivatives of $\gamma - 2\beta$ are in $C^{5+\alpha}$. Hence $\gamma - 2\beta \in C^{6+\alpha}(\Omega^+ \cup \Gamma)$. Since β has zero Dirichlet data on Γ for the equation $(6.1)''$ we see also that $\beta \in C^{5+\alpha}$. Thus β and $\gamma \in C^{5+\alpha}$. Once more by Theorem 1 of [5] we infer that $\Gamma \in C^{6+\alpha}$, and so on. Thus one may conclude that $\Gamma \in C^\infty$ and $\beta, \gamma \in C^\infty(\Omega^+ \cup \Gamma)$. We still have to prove analyticity in case a is analytic.

Setting $\gamma - 2\beta = \delta$ we have from $(6.1)'$,

$$(\Delta + 2)\delta + 2\beta = a \quad \text{in } \Omega^+$$
$$\Delta(\Delta + 1)\beta = 0\,, \quad \text{in } \Omega^+$$

and on Γ:

$$\delta = \beta = 0\,, \quad \operatorname{grad} \delta = \operatorname{grad} \beta = 0\,.$$

Eliminating β we find

$$\Delta(\Delta + 1)(\Delta + 2)\delta = (\Delta + 1)a \quad \text{in } \Omega^+$$
$$\delta = \delta_n = 0\,, \quad (\Delta + 2)\delta = a\,, \quad \frac{\partial}{\partial x_n}(\Delta + 2)\delta = a_n$$

on Γ. The desired result then follows with the aid of Theorem 4.3 or Proposition 4.1. Indeed, in the case at hand, the equation transforms with $p = 1$ to a single equation for f of sixth order with f subject to three boundary conditions which are derived from the first, third, and fourth conditions on δ.

PROBLEM. Can one prove a similar result for such systems with variable coefficients? or for nonlinear systems?

EXAMPLE 6.2. Consider $u \in C^{2m-p-1}(\Omega^+ \cup \Gamma \cup \Omega^-)$ satisfying elliptic equations

$$F(x, \ldots, \partial^{2m} u) = 0 \quad \text{in } \Omega^+$$
$$G(x, \ldots, \partial^{2m} u) = 0 \quad \text{in } \Omega^-$$
$$\partial^j u = 0\,, \quad j = 0, \ldots, p\,, \quad \partial_n^{p+1} u \neq 0 \quad \text{on } \Gamma$$

and assume

$$u \in C^{2m}(\Omega^+ \cup \Gamma) \cap C^{2m}(\Omega^- \cup \Gamma)\,.$$

THEOREM 6.2. *Let u satisfy the hypotheses of example 6.2 with F, G analytic (C^∞) in all their arguments.* (i) *If $p = 0$ or $p \geq m - 1$, then Γ is* *analytic* (C^∞). (ii) *If F and G are semilinear, i.e., the leading parts are linear, and if these linear operators are the same, then, for any $p \geq 0$, Γ is analytic* (C^∞).

PROOF. We may suppose $\partial_n^{p+1} u(0) = -1$. If $p \geq m$ the result is already contained in Theorem 4.1. So we shall suppose $p \leq m-1$. Introduce the change of variables $y = (x', -\partial_n^p u)$ which maps Ω^+ into $U^+ \subset \{y_n \geq 0\}$, Γ into a flat boundary $S \subset \{y_n = 0\}$, and Ω^- into $U^- \subset \{y_n \leq 0\}$ in a locally 1-1 way. We apply the construction of section 3 to both equations $F = 0$ and $G = 0$ in U^+, U^- respectively to obtain systems of the form (3.7)-(3.12) in the unknowns f^+, $w^{+\alpha,k}$ in U^+ and f^-, $w^{-\alpha,k}$ in U^- with boundary conditions $f^+ = w^{+\alpha,k} = f^- = w^{-\alpha,k} = 0$ on S. We then «reflect» the system defined in U^- across $\{y_n = 0\}$ to a system defined in U^+ by the usual rule

$$f(y', y_n) = f^-(y', -y_n) \qquad y_n \geq 0$$

$$w^{\alpha,k}(y', y_n) = w^{-\alpha,k}(y', -y_n)\,.$$

In this way we obtain two uncoupled elliptic systems in U^+ in the unknowns f^+, $w^{+\alpha,k}$, f, $w^{\alpha,k}$ and these unknowns vanish on S. This system is clearly elliptic. We obtain additional boundary conditions from our assumption $u \in C^{2m-p-1}$ across Γ. Namely

$$\partial_n^k f^+ = \partial_n^k f^-\,, \qquad k = 1, \dots, 2m-2p \qquad \text{on } y_n = 0\,.$$

Reflecting these conditions we obtain the boundary conditions

$$\partial_n^k f^+ = (-1)^k \partial_n^k f\,, \qquad k = 1, \dots, 2m-2p\,.$$

To check the covering property of our boundary conditions is equivalent by Proposition 4.1 to showing that the system of ordinary differential equations

$$\left.\begin{aligned} L'\left(\xi', \frac{1}{i}\frac{d}{dy_n}\right) w^+ &= 0\,, \\ P'\left(\xi', -\frac{1}{i}\frac{d}{dy_n}\right) w &= 0\,, \end{aligned}\right\} \quad \text{in } y_n \geq 0$$

$$\left.\begin{aligned} &\partial_n^j w^+ = \partial_n^j w = 0\,, && j = 0, \dots, p-1 && (6.20) \\ &\partial_n^k w^+ = (-1)^k \partial_n^k w\,, && k = p, \dots, 2m-p-1 && (6.21) \end{aligned}\right\} \quad y_n = 0$$

has no nontrivial solutions which decay as $y_n \to \infty$. Here L', P', are the principal parts of the linearizations at the origin of F, G respectively.

We shall argue as in section 5. Fixing ξ', and dividing by the coefficients of ∂_n^{2m}, we may write

$$L'\left(\xi', \frac{1}{i}\partial_n\right) = L'\left(\frac{1}{i}\partial_n\right) = L_+\left(\frac{1}{i}\partial_n\right)L_-\left(\frac{1}{i}\partial_n\right)$$

$$P'\left(\xi', -\frac{1}{i}\partial_n\right) = P'\left(-\frac{1}{i}\partial_n\right) = P_+\left(-\frac{1}{i}\partial_n\right)P_-\left(-\frac{1}{i}\partial_n\right)$$

where

$$L_+(z) = \prod_1^m (z - z_j)\,, \quad L_-(z) = \prod_1^m (z - \bar{z}_j)\,, \quad P_+ = \prod_1^m (z - \zeta_j)\,, \quad P_- = \prod_1^m (z - \bar{\zeta}_j)$$

where the z_i and ζ_i lie in the upper half plane. Setting $y_n = t$ we see that w^+, w are solutions of

$$(6.22)\qquad \begin{cases} L_+\left(\frac{1}{i}\partial_t\right)w^+ = \prod_1^m\left(\frac{1}{i}\partial_t - z_j\right)w^+ = 0 \\ (-1)^m P_-\left(-\frac{1}{i}\partial_t\right)w = \prod_1^m\left(\frac{1}{i}\partial_t + \bar{\zeta}_j\right)w = 0\,. \end{cases} \qquad \text{for } t \geqslant 0$$

Suppose $p = m - 1$. According to (6.20), (6.21) we have at $t = 0$:

$$(6.23)\qquad \partial_t^j w^+ = \partial_t^j w = 0\,, \quad j \leqslant m - 2\,,$$

$$(6.24)\qquad \begin{cases} \partial_t^{m-1} w^+ = (-1)^{m-1}\partial_t^{m-1} w \\ \partial_t^m w^+ = (-1)^m \partial_t^m w\,. \end{cases}$$

Inserting (6.23) in (6.22) we find at $t = 0$:

$$\left[\left(\frac{1}{i}\partial_t\right)^m - \sum z_j\left(\frac{1}{i}\partial_t\right)^{m-1}\right]w^+ = 0$$

$$\left[\left(\frac{1}{i}\partial_t\right)^m + \sum \bar{\zeta}_j\left(\frac{1}{i}\partial_t\right)^{m-1}\right]w = 0$$

or, from (6.24)

$$\left[\left(\frac{1}{i}\partial_t\right)^m + \sum z_j\left(\frac{1}{i}\partial_t\right)^{m-1}\right]w = 0\,.$$

Subtracting these last two equations we find since $\operatorname{Im}\sum(\bar{z}_j - \bar{\zeta}_j) > 0$,

$$\partial_t^{m-1} w(0) = \partial_t^{m-1} w^+(0) = 0\,.$$

Thus for equations (6.22) w^+ and w have zero Cauchy data at the origin and therefore both vanish identically.

Consider $p = 0$. On $t < 0$ define

$$w^-(t) = w(-t)\,.$$

Then $w^-(t)$ satisfies

$$P_-\left(\frac{1}{i}\,\partial_t\right)w^- = 0$$

and the boundary conditions at $t = 0$

$$\partial_t^k w^+ = \partial_t^k w^-\,, \qquad k = 0, \ldots, 2m-1\,.$$

Thus w^- and w^+ together form a function $\hat{w}$ of class C^{2m-1} on the real axis decaying as $|t| \to \infty$. On the other hand $\hat{w}(t)$ is a solution of the equation of order $2m$

$$P_-\left(\frac{1}{i}\,\partial_t\right)L_+\left(\frac{1}{i}\,\partial_t\right)\hat{w} = 0\,, \qquad t \neq 0\,.$$

From this equation and the fact that $\hat{w} \in C^{2m-1}$ it follows that $\hat{w} \in C^{2m}$ on the real axis and satisfies the equation everywhere. Since it decays as $t \to \pm\infty$ it can only be zero. Thus Theorem 6.2 (i) is proved.

Turning to (ii) for $0 < p < m-1$, we now have $\zeta_j = z_j$. As in the pıoof of Lemma 5.1 we see that

$$w^+ = \oint \exp\,[izt]\,\frac{A(z)}{L_+(z)}\,dz\,, \qquad w^- = \oint \exp\,[-izt]\,\frac{B(z)}{L_-(z)}\,dz$$

where the contour encloses all the z_j and $\bar{z}_j$ and A, B are plynomials of degrees $\leqslant m-p-1$. According to (6.21) we have

$$\oint z^k\,\frac{AL_- - BL_+}{L_+L_-}\,dz = 0\,, \qquad k = p, \ldots, 2m-p-1\,.$$

As in the proof of Lemma 5.1 this implies that the polynomial $R = AL_- - BL_+$ whose degree is at most $2m-p-1$ has in fact,

$$\deg\,(AL_- - BL_+) \leqslant p-1\,. \tag{6.25}$$

Namely, if the degree of R is actually $r \geqslant p$ we set $k = 2m-r-1$ in the last integral identity and integrate over a large circle about the origin. Letting the radius of the circle tend to ∞ we obtain a contradiction. On the other hand, by Lemma 5.2, (6.25) is possible only if $A \equiv B \equiv 0$, and the theorem is proved.

7. – Another example.

THEOREM 7.1. *Let* $u \in C^{2m}(\Omega^+ \cup \Gamma) \cap C^{2m}(\Omega^- \cup \Gamma) \cap C^{m-1}(\Omega^+ \cup \Gamma \cup \Omega^-)$ *satisfy*

$$\Delta^m u^+ = 0 \quad \text{in } \Omega^+$$

$$\Delta^m u^- = 0 \quad \text{in } \Omega^-;$$

$$\left.\begin{aligned} &\partial_\nu^j u^\pm = 0\,, \quad j = 0, \dots, m-1\,,\\ &g(x, \partial_\nu^m u^+, \partial_\nu^m u^-) = 0\\ &|\partial_\nu^m u^+| + |\partial_\nu^m u^-| \neq 0 \end{aligned}\right\} \text{ on } \Gamma$$

where $u^\pm = u|_{\Omega^\pm}$, *and* g *is analytic* (C^∞) *in all arguments. If*

$$\frac{\partial g}{\partial(\partial_\nu^m u^+)}\,\partial_\nu^m u^+ \neq \frac{\partial g}{\partial(\partial_\nu^m u^-)}\,\partial_\nu^m u^- \tag{7.1}$$

at the origin then Γ *is analytic* (C^∞).

It will be seen from the proof that the theorem holds if the equations $\Delta^m u^\pm = 0$ are replaced by a general nonlinear elliptic operator $F(x, u^\pm, \dots, \partial^m u^\pm) = 0$ in $\Omega^\pm$. Since the proof is long we describe it for the simpler case, omitting some details which are similar to some of sections 3 and 4.

We may suppose $\partial_n^m u^+(0) < 0$ and we introduce the change of variable

$$x \mapsto y = (x', -\partial_n^{m-1} u^+)\colon \Omega^+ \to U \subset \{y_n \geq 0\}$$

and also the elliptic system defined in U constructed in section 3 ($p = m-1$) corresponding to the equation $\Delta^m u^+ = 0$ with the weights (3.13). We will also introduce a system corresponding to the equation $\Delta^m u^- = 0$ via the reflection of section 2. More precisely, define unknowns $v^{\alpha,k}(y) = \partial^\alpha \partial_n^k u^-(x)$, $|\alpha| + k \leq m-1$ where y and x are related by (2.6). Using (2.7) and imitating the construction of section 3, the $v^{\alpha,k}$ satisfy a system of the form (σ, τ

vary from 1 to $n-1$):

$$(7.2)\qquad \left(\partial_\tau - \frac{f_{n\tau}}{f_{nn}-C}\,\partial_n\right) v^{\alpha,k} = \frac{1}{f_{nn}-C}\,\partial_n v^{(\alpha,\tau),k-1)}\,, \qquad |\alpha| + k = m-1$$

$$(7.3)\qquad \left(\partial_\tau - \frac{f_{n\tau}}{f_{nn}-C}\,\partial_n\right) v^{(\alpha,\sigma),0} = \left(\partial_\sigma - \frac{f_{n\sigma}}{f_{nn}-C}\,\partial_n\right) v^{(\alpha,\tau),0}\,, \qquad |\alpha| = m-2$$

$$(7.4)\qquad \begin{cases} \dfrac{1}{f_{nn}-C}\,\partial_n v^{\alpha,k} = v^{\alpha,k+1} \\ \left(\partial_\tau - \dfrac{f_{n\tau}}{f_{nn}-C}\,\partial_n\right) v^{\alpha,k} = v^{(\alpha,\tau),k} \end{cases} \qquad |\alpha| + k < m-1$$

$$(7.5)\qquad \left(\frac{1}{f_{nn}-C}\,\partial_n\right)^{m+1} v^{0,m-1} + \ldots = 0\,.$$

The last equation corresponds to $\Delta^m u^- = 0$. As in section 3 we have chosen some way to express the derivatives of u^- of order $\geqslant m-1$ in terms of derivatives of $v^{\alpha,k}$, $|\alpha| + k = m-1$ of order at most $m+1$ while the derivatives of u^- of order $\leqslant m-2$ are the corresponding $v^{\alpha,k}$, $|\alpha| + k \leqslant m-2$. These are assigned the weights:

$t = m+1$ to $v^{\alpha,k}$ for $|\alpha| + k = m-1$; $t = 1$ to $v^{\alpha,k}$ for $|\alpha| + k < m-1$; $s = -m$ to equations (7.2), (7.3), $s = 0$ to equations (7.4), (7.5).

We combine this system with the system in f and $w^{\alpha,k}$; observe that these two systems are linked through f and that the weights we have given ($t_f = m+2$) are consistent. The linearization of equations (7.2)-(7.5) at the origin (principal part only) takes the following form:

$$(7.6)\qquad \begin{cases} \partial_\tau \bar v^{\alpha,k} = \dfrac{1}{f_{nn}(0)-C}\,\partial_n \bar v^{(\alpha,\tau),k-1} & |\alpha| + k = m-1\,, \\ \partial_\tau \bar v^{0,m-1} - \dfrac{\partial_\nu^m u^-(0)}{f_{nn}(0)-C}\,\bar f_{n\tau} = \dfrac{1}{f_{nn}(0)-C}\,\bar v_n^{\tau,m-2} & k = 1, \ldots, m-2 \end{cases}$$

$$(7.7)\qquad \partial_\tau \bar v^{(\alpha,\sigma),0} = \partial_\sigma \bar v^{(\alpha,\tau),0} \qquad |\alpha| = m-2$$

$$(7.8)\qquad \begin{cases} \dfrac{1}{f_{nn}(0)-C}\,\partial_n \bar v^{\alpha,k} = \bar v^{\alpha,k+1} \\ \partial_\tau \bar v^{\alpha,k} = \bar v^{(\alpha,\tau),k} \end{cases} \qquad |\alpha| + k < m-1$$

$$(7.9)\qquad \left(\frac{1}{f_{nn}(0)-C}\,\partial_n\right)^{m+1} (\bar v^{0,m-1} - \partial_\nu^m u^-(0)\,\partial_n \bar f) + \ldots = 0\,.$$

A word of explanation is in order about equation (7.9). It is easy to see from (7.5) that the terms in $\bar{f}$ arise with a coefficient $v_n^{\alpha,k}(0)$ and these are all zero except possibly for $v_n^{0,m-1}(0)$. It is then easy to check that if $L\bar{v}^{0,m-1}$ is the part of the linearization of (7.5) corresponding to $v^{0,m-1}$, then $-\partial_\nu^m u^-(0) L\bar{f}_n$ is the part of the linearization of (7.5) corresponding to f. These combine to give $L(\bar{v}^{0,m-1} - \partial_\nu^m u^-(0)\bar{f}_n)$.

Our big system is easily seen to be elliptic. Indeed the subsystem in f and the $w^{\alpha,k}$ is elliptic, which means that it is sufficient to show that the subsystem (7.6)-(7.9) with $\bar{f} \equiv 0$ elliptic for the $\bar{v}^{\alpha,k}$. But this follows as in the proof of Theorem 3.1.

Boundary conditions for the unknowns $f, w^{\alpha,k}, v^{\alpha,k}$ are obtained by transforming the boundary conditions on $u^\pm$ as in section 4. Following section 4, in particular proposition 4.1 and Lemma 4.1 we reduce the covering property of these boundary conditions to a corresponding property for a system of ordinary differential equations in the unknowns $\hat{w}^0$ (corresponding to $\Delta^m u^+ = 0$) and $\hat{v}^0$ (corresponding to $\Delta^m u^- = 0$). The function $\hat{w}^0$ is subject to the conditions of proposition 4.1 while (by the same analysis based on equations (7.6)-(7.9)) $\hat{v}^0$ and the $\bar{v}^{\alpha,k}$, $|\alpha| + k = m-1$ are related by

$$(7.10)\qquad \begin{cases} \bar{v}^{\alpha,k} = \xi_\alpha \hat{v}^k \\ \hat{v}^k = \left(\dfrac{-i\partial_n}{f_{nn}(0) - C}\right)^k \hat{v}^0. \end{cases} \qquad |\alpha| + k = m-1,\ k = 0, \ldots, m-2$$

$$(7.11)\qquad \bar{v}^{0,m-1} - \partial_\nu^m u^-(0)\,\partial_n \bar{f} = \left(\frac{-i\partial_n}{f_{nn}(0) - C}\right)^{m-1} \hat{v}^0 .$$

Hence the covering condition is equivalent to the assertion: $\hat{w}^0 \equiv \hat{v}^0 \equiv 0$ is the only decaying solution as $y_n \to +\infty$ of the system of ordinary differential equations in $y_n \geqslant 0$

$$(7.12)\qquad \begin{cases} \left(\left(\dfrac{1}{f_{nn}(0)}\,\partial_n\right)^2 - |\xi'|^2\right)^m \hat{w}^0 = 0 \\ \left(\left(\dfrac{1}{f_{nn}(0) - C}\,\partial_n\right)^2 - |\xi'|^2\right)^m \hat{v}^0 = 0 \end{cases}$$

satisfying at $y_n = 0$:

$$(7.13)\qquad \partial_n^j \hat{w}_0 = 0\,, \quad \partial_n^j \hat{v}^0 = 0\,, \quad j = 0, \ldots, m-2$$

$$(7.14)\qquad f_{nn}(0)\,\partial_\nu^m u^-(0)\left(\frac{1}{f_{nn}(0)}\,\partial_n\right)^{m-1} \hat{w}^0 + \left(\frac{\partial_n}{f_{nn}(0) - C}\right)^{m-1} \hat{v}^0 = 0\,.$$

$$(7.15)\qquad \left(\frac{\partial g}{\partial(\partial_\nu^m u^+)}\right)\left(\frac{\partial_n}{f_{nn}(0)}\right)^m \hat{w}^0 + \frac{\partial g}{\partial(\partial_\nu^m u^-)}\left(\frac{\partial_n}{f_{nn}(0) - C}\right)^m \hat{v}^0 = 0\,.$$

Formula (7.14) is obtained from the condition $\bar{v}^{0,m-1} = 0$ at $y_n = 0$ with the aid of (4.8) and (7.11). Formula (7.15) in turn is obtained from the linearization of the relation $g(x, \partial_\nu^m u^+, \partial_\nu^m u^-) = 0$:

$$\frac{\partial g}{\partial(\partial_\nu^m u^+)} \frac{\bar{f}_{nn}}{f_{nn}^2(0)} + \frac{\partial g}{\partial(\partial_\nu^m u^-)} \frac{\partial_n}{f_{nn}(0) - C} \left(\bar{v}^{0,m-1} - \partial_\nu^m u^-(0)\, \partial_n \bar{f}\right) = 0 .$$

Using (4.8) and (7.11) we obtain (7.15).

Since $\hat{w}^0$ and $\hat{v}^0$ decay as $y_n \to \infty$ it follows from (7.12) that they satisfy (for convenience we set $|\xi'| = 1$) the differential equations

$$\left(\frac{1}{f_{nn}(0)}\, \partial_n + 1\right)^m \hat{w}^0 = 0$$

$$\left(\frac{1}{C - f_{nn}(0)}\, \partial_n + 1\right)^m \hat{v}^0 = 0 . \qquad y_n \geqslant 0$$

From the boundary conditions (7.13) it follows that at $y_n = 0$,

$$\begin{cases} \partial_n^m \hat{w}^0 + f_{nn}(0)\, \partial_n^{m-1} \hat{w}_0 = 0 \\ \partial_n^m \hat{v}^0 - \left(f_{nn}(0) - C\right) \partial_n^{m-1} \hat{v}^0 = 0 . \end{cases} \tag{7.16}$$

Combining these with (7.14) we obtain

$$f_{nn}(0)\, \partial_\nu^m u^-(0) \left(\frac{\partial_n}{f_{nn}(0)}\right)^m \hat{w}^0 - \left(\frac{\partial_n}{f_{nn}(0) - C}\right)^m \hat{v}^0 = 0 \tag{7.17}$$

at $y_n = 0$. From this and (7.15) we see that our boundary conditions are covering if and only if

$$\frac{\partial g}{\partial(\partial_\nu^m u^+)} + f_{nn}(0) \frac{\partial g}{\partial(\partial_\nu^m u^-)}\, \partial_\nu^m u^-(0) \neq 0 ;$$

by (2 5) this is condition (7.1).

8. – A first order elliptic system.

The elliptic systems which we have treated involve only a few functions and are rather special in nature. We are still far from being able to treat general elliptic systems.

As a first step in attacking such systems we consider in this section a first order (possibly overdetermined) elliptic system with rather simple boundary conditions.

We consider a system of N real functions

$$u = (u^1, \dots, u^N)\,,$$

in $C^1(\Omega^+ \cup \Gamma)$ satisfying a first order system

$$Lu \equiv Au + Bu = f\,. \tag{8.1}$$

Here u and f are N and M column vector valued functions respectively, and $B(x)$ is an $M \times N$ matrix, and A is the first order elliptic differential operator

$$A = \sum_1^n A^j \partial_j$$

whose coefficients $A^j(x)$ are $M \times N$ matrices $\{A_k^{jr}\}$, $r = 1, \dots, M$, $k = 1, \dots, N$. The system is assumed to be elliptic as in section 3 with weights: all $s_r = 0$ and all $t_j = 1$.

We shall now require that at each point of Γ, u is to satisfy m linear relations or boundary conditions. At each point x near the origin let $P(x)$ be an $N \times N$ projection matrix whose range $R(x)$ is $m+1$ dimensional. We impose the boundary condition

$$Pu = 0 \quad \text{on } \Gamma\,.$$

Question. Assume $P(0)\,u_n(0) \neq 0$ and suppose that P, f and the coefficients in (8.1) are analytic (C^∞) in a full neighborhood of the origin. Under what conditions is Γ necessarily analytic (C^∞)?

We shall present a result in this direction. First of all there is no loss of generality in supposing that the boundary conditions $Pu = 0$ on Γ take the form

$$u^1 = \dots = u^m = u^N = 0 \quad \text{on } \Gamma \tag{8.2}$$

and the condition $P(0)\,u_n(0) \neq 0$ the form

$$u_n^N(0) \neq 0\,. \tag{8.3}$$

We may use a zero order partial hodograph transformation

$$x \mapsto y = (x_1, \dots, x_{n-1}, u^N)$$

mapping $\Omega^+(\Gamma)$ near the origin into a neighborhood $U(S)$ of the origin in $y_n > 0$ $(y_n = 0)$; set

$$x_n = \psi(y) , \quad u^j(x) = w^j(y) , \quad j < N .$$

A familiar calculation shows that in U these satisfy the M equations, here α is summed from 1 to $n-1$ and j from 1 to $N-1$,

$$A_j^{\alpha r}\left(\partial_\alpha - \frac{\psi_\alpha}{\psi_n}\,\partial_n\right) w^j + A_j^{nr}\,\frac{w_n^j}{\psi_n} - A_N^{\alpha r}\,\frac{\psi_\alpha}{\psi_n} + A_N^{nr}\,\frac{1}{\psi_n} = f^r \tag{8.4}$$

$r = 1, \ldots, M$ and on $y_n = 0$ the boundary conditions

$$w^j = 0 , \quad j \leqslant m . \tag{8.5}$$

CLAIM. The system (8.4) is elliptic.

We have only to check this at the origin, where $\psi_\alpha = 0$ for $\alpha < n$. There, the linearized equations for $\bar{w}^j, \bar{\psi}$ are (here j is summed from 1 to $N-1$),

$$A_j^{\alpha r}(0)\,\partial_\alpha\big(\bar{w}^j - w_n^j(0)\,\psi_n^{-1}(0)\,\bar{\psi}\big) + A_j^{nr}(0)\,\psi_n^{-1}(0)\,\partial_n\big(\bar{w}^j - w_n^j(0)\,\psi_n^{-1}(0)\,\bar{\psi}\big)$$
$$- A_N^{\alpha r}(0)\,\psi_n^{-1}(0)\,\bar{\psi}_\alpha - A_N^{nr}(0)\,\psi_n^{-2}(0)\,\bar{\psi}_n = 0 .$$

For the N-column vector $v(x)$:

$$v^j = \bar{w}^j - w_n^j(0)\,\psi_n^{-1}(0)\,\bar{\psi} , \quad j < N \qquad \text{and } v^n = -\,\psi_n^{-1}(0)\,\bar{\psi} , \tag{8.6}$$

after y_n is replaced by $\psi_n(0)y_n$, the system takes the form

$$\sum_{\beta=1}^{n} A^\beta(0)\,\partial_\beta v = 0 \tag{8.7}$$

which is clearly elliptic. The claim is proved.

Next we have to see whether the boundary conditions (8.5) are covering for (8.4) at the origin. If they are, then, by applying the regularity theorem in the appendix, we may conclude that Γ is analytic (C^∞). For convenience we denote the stretched y_n variable, $\psi_n(0)y_n$ by t. For any vector $\xi' \in R^{n-1}\setminus 0$ we have to consider the system of ordinary differential equations

$$\left(i\sum_{\alpha=1}^{n-1}\xi_\alpha A^\alpha(0) + A^n(0)\,\partial_t\right)v = 0 , \quad t \geqslant 0 \tag{8.8}$$

subject to the conditions at $t = 0$:

$$v^r(0) = cu_n^r(0)\,, \qquad r \leqslant m$$
$$v^N(0) = cu_n^N(0)$$

for some constant c (if c is eliminated this comes to m conditions). These boundary conditions follow from (8.5) and (8.6), with $c = -\bar{\psi}(0)$.

These conditions mean: at $t = 0$

(8.9) $$Pv = cPu_n(0) \quad \text{for some constant } c\,,$$

and we consider solutions of (8.8) and (8.9) which are decaying as $t \to \infty$. If this condition is covering then we conclude that $c = 0$, *i.e.* $\bar{\psi}(0) = 0$ and hence $\bar{w}^j \equiv \bar{\psi} \equiv 0$. We have therefore proved

THEOREM 8.1. *Let* $u \in C^1(\Omega^+ \cup \Gamma)$ *be a solution of* (8.1) *satisfying the boundary conditions* $Pu = 0$ *on* Γ *with* $P(0)u_n(0) \neq 0$. *Assume that* P, f, *and the coefficients in* (8.1), *are analytic* (C^∞) *in a full neighborhood of the origin. Then* Γ *is analytic* (C^∞) *provided*:

(8.10) *for the system* $Lw = g$ *the following boundary condition is covering at the origin*:

$$Pw \text{ is a multiple of } Pu_n(0)\,.$$

REMARKS. (i) Condition (8.10) is rather unusual. We had expected to obtain the regularity result under a different condition:

(8.11) *For some projection operator* $Q(x)$ *having m-dimensional range contained in the range of* $P(x)$, *the boundary condition* $Qu = 0$ *on* Γ *is covering for* (8.1).

We have an example with $M = N = 2m = 4$, in which this condition is satisfied but (8.10) is not. Whether (8.11) is sufficient for regularity is not known.

(ii) It is clear that the theorem holds also for a general overdetermined nonlinear elliptic system in place of (8.1) provided it has weights: $s_r = 0$ and all $t_j = 1$.

(iii) Since every elliptic system may be rewritten as an overdetermined first order system it might be thought that many of our earlier results follow simply from Theorem 8.1. This is not the case since in Theorem 8.1 the overdetermined system has very special weights. It's worth remarking that the argument extends to a first order system with nontrivial weights satisfying rather special conditions. We state the result for a linear system though it applies as well to a nonlinear one. Consider a (possibly overdetermined) first order system (8.1) with weight s_r attached to the r-th equation, and weight t_j attached to u^j. We assume no equation is identically zero. As in Theorem 8.1 suppose $u \in C^1(\Omega^+ \cup \Gamma)$ is a solution of (8.1) satisfying the boundary conditions on Γ:

$$u^N = 0\,, \qquad \operatorname{grad} u^N \neq 0\,,$$

$$\sum_{j<N} b_j^k(x)\, u^j = \varphi^k\,, \qquad k = 1, \ldots, \mu\,.$$

with weight r_k attached to the k-th such condition.

THEOREM 8.2. *Assume that the coefficients in* (8.1), f, *and the functions* b_j^k, φ^k *are analytic* (C^∞) *in a full neighborhood of the origin. Then* Γ *is analytic* (C^∞) *provided*

(i) $t_N = \max t_i$,

(ii) $s_r + t_j \neq 0 \ \forall r, j$,

(iii) *for the system* $Lw = g$ *the following boundary condition is covering at the origin:*

$$\sum_{\substack{j<N \\ r_k + t_j = 0}} b_j^k u^j = c \sum_{\substack{j<N \\ r_k + t_j = 0}} b_j^k u_n^j(0)\,, \qquad k = 1, \ldots, \mu\,.$$

for some multiplier c.

Appendix. Regularity for overdetermined elliptic systems.

Consider a nonlinear overdetermined elliptic system (3.4)

$$\varphi_r\left(y, \left(\frac{1}{i}\,\partial\right)^{\alpha} u\right) = 0\,, \qquad r = 1, \ldots, M \tag{A.1}$$

for a system of N functions $u(y) = \left(u^1(y), \ldots, u^N(y)\right)$ in $y_n \geq 0$ near the origin and satisfying a system of covering boundary conditions, as on page 24,

$$\chi_q\left(y, \left(\frac{1}{i}\,\partial\right)^{\alpha} u\right) = 0\,, \qquad q = 1, \ldots, \mu\,, \tag{A.2}$$

with all weights as described in those sections. With

$$l = \max(0, r_q + 1)$$

we suppose that $u^j \in C^{l+t_j}$ in $y_n \geqslant 0$. All our considerations are local, near the origin in $y_n \geqslant 0$, and we will not bother to restate this.

The regularity result on which we have relied throughout this paper is the following. It was described in broad terms in the Remark on page 289 in [7].

THEOREM A. *Under the preceding conditions, if the functions φ_r, χ_q, are analytic (C^∞) in their arguments then the solution u is analytic (C^∞) in $y_n \geqslant 0$.*

It is proved by combining a number of known results and standard procedures and which we now describe briefly. The core of the proof consists of the estimates in L_p and Hölder norms for corresponding linear over-determined systems of Solonnikov [11-12]. To state these briefly (his results are still more general), consider solutions of linear equations (3.2)

$$L_{rj}\left(y, \frac{1}{i}\,\partial\right) u^j(y) = f_r(y)\,, \qquad r = 1, \dots, M$$

satisfying linear boundary conditions as in Definition 4.1,

$$B_{qj}\left(y, \frac{1}{i}\,\partial\right) u^j = g_q \qquad \text{on} \qquad y_n = 0\,, \qquad q = 1, \dots, \mu,$$

with $u^j \in C^{l+t_j}$ in $y_n \geqslant 0$.

(i) If the coefficients of L_{rj} belong to C^{l-s_r} in $y_n \geqslant 0$ and those of B_{qj} to C^{l-r_q} then there is an estimate for the L_p norm, $1 < p < \infty$, of $\partial^\alpha u^j$ for $|\alpha| \leqslant l + t_j$, $j = 1, \dots, N$ in $y_n \geqslant 0$ in terms of suitable norms of $L_{rj} u$ and of the g_q:

(ii) If for some α, $0 < \alpha < 1$ the coefficients of L_{rj} belong to $C^{l-s_r+\alpha}$ in $y_n \geqslant 0$ and those of B_{qj} to $C^{l-r_q+\alpha}$, and if $u^j \in C^{l+t_j+\alpha}$ in $y_n \geqslant 0$ then there is an estimate for the $C^{l+t_j+\alpha}$ norm of u_j in $y_n \geqslant 0$ in terms of suitable Hölder norms for the f_r and g_q.

These estimates are generalizations of the corresponding well known estimates for determined systems, i.e. $M = N$ and $2\mu = \sum (s_j + t_j)$.

The C^∞ assertion in Theorem A follows easily from these estimates by a standard procedure as in [1], [10]. Taking difference quotients of the equations (A.1) and (A.2) at two points on the boundary, distance h apart, one obtains linear looking equations for the difference quotients of u. The estimates of (i) then yield estimates on the L_p norms, of the corresponding derivatives of the difference quotient independent of h. Letting $h \to 0$ one finds that for $j = 1, \ldots, N$, u^j has derivatives of order $l + t_j + 1$ in L_p $(y_n \geqslant 0)$. For p large this implies that $u^j \in C^{l+t_j+\alpha}$ $(y_n \geqslant 0)$ $\alpha < 1$. Applying the same difference quotient technique and the estimates of (ii) one obtains bounds for the $C^{l+t_j+\alpha}$ norms of the difference quotients of u^j independent of h. It then follows that $u^j \in C^{l+t_j+1+\alpha}$. One then differentiates the equations and uses the difference quotient procedure and the estimates of (ii) to show that $u^j \in C^{l+t_j+2+\alpha}$ and so on.

Finally for analytic data, one has to show that the solution u is analytic in $y_n \geqslant 0$. That this is true in $y_n > 0$ follows from the known analyticity result for solutions of determined nonlinear elliptic equations ([10], Theorem 6.8.2). For u is a solution of the *determined* elliptic system:

$$\sum_{r=1}^{M} L_{rj}^{*}\left(\frac{1}{i}\,\partial\right)\varphi_r\left(y, \left(\frac{1}{i}\,\partial\right)^{\alpha} u\right) = 0\,, \qquad j = 1, \ldots, N,$$

where L_{rj}^{*} is the formal adjoint of the operator with constant coefficients

$$L_{rj}\bar{u} = \frac{d}{dt}\left|\,\varphi_r\left(y, \left(\frac{1}{i}\,\partial\right)^{\alpha}(u + t\bar{u})\right)\right|_{t=0,\,y=0}.$$

This simple device does not work at the boundary. In [7], however, there is a proof of local boundary analyticity for second order nonlinear elliptic equations (parabolic equations are treated there but the results apply of course to elliptic equations by considering time independent functions). Analyticity is proved by establishing L_2 estimates for all the derivatives of u. As pointed out there in the Remark on page 289, the proof is based entirely on a single L^2 coercive estimate up to the boundary for linear equations. It applies equally well to equations of higher order, and to overdetermined systems. The appropriate estimates are available, they are those of (i) above with $p = 2$. We thus consider Theorem A as proved.

REFERENCES

[1] S. Agmon - A. Douglis - L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, II*, Comm. Pure Appl. Math., **12** (1959), pp. 623-727 and **17** (1964), pp. 35-92.

[2] L. Caffarelli - A. Friedman, *The obstacle problem for the biharmonic operator*, to appear.

[3] J. Frehse, *On the regularity of the solution of the biharmonic variational inequality*, Manuscripta Math., **9** (1973), pp. 91-103.

[4] D. Kinderlehrer, *Variational inequalities and free boundary problems*, Bull. Amer. Math. Soc., **84** (1978), pp. 7-26.

[5] D. Kinderlehrer - L. Nirenberg, *Regularity in free boundary value problems*, Ann. Scuola Norm. Sup. Pisa, Ser. IV, **4** (1977), pp. 373-391.

[6] D. Kinderlehrer - L. Nirenberg - J. Spruck, *Regularité dans les problèmes elliptiques à frontière libre*, C. R. Acad. Sci. Paris, **286** (1978), pp. 1187-1190.

[7] D. Kinderlehrer - L. Nirenberg, *Analyticity at the boundary of solutions of nonlinear second-order parabolic equations*, Comm. Pure Appl. Math., **31** (1978), pp. 283-338.

[8] D. Kinderlehrer - L. Nirenberg - J. Spruck, *Regularity in elliptic free boundary problems, I*, Journal d'Analyse Math., **34** (1978), pp. 86-119.

[9] H. Lewy, *On the reflection laws of second order differential equations in two independent variables*, Bull. Amer. Math. Soc., **65** (1959), pp. 37-58.

[10] C. B. Morrey Jr., *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, New York - Berlin, 1966.

[11] V. A. Solonnikov, *Overdetermined elliptic boundary value problems*, Zap. Nauch. Sem. Leningrad. Otdel Mat. Inst. Steklov. Akad. Nauk SSSR, **21** (1971), pp. 112-158 (J. Soviet Math., **1** (1973), pp. 477-512).

[12] V. A. Solonnikov, *Investigation of overdetermined elliptic boundary value problems in the fractional spaces of K. K. Golovkin*, Trudy Mat. Inst. Steklov., **127** (1975), pp. 93-114 (Boundary value problems in Math. Physics, O. A. Ladyzhenskaya ed., A.M.S. (1977), pp. 109-134).

[13] G. Stampacchia, *Formes bilinéaires coercitives sur les ensembles convexes*. C. R. Acad. Sci. Paris, **258** (1964), pp. 4413-4416.