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Some Regularity Results for a Family of Variational Inequalities (*).

ALESSANDRO TORELLI (**)

1. – Introduction.

In a recent paper Brézis and Stampacchia [3] have studied the regularity of the solution for some fourth order elliptic variational inequalities. To describe more precisely the result of [3], let Ω be a bounded and sufficiently « smooth » open set of $\boldsymbol{R}^n$ and

$$K_1 = \{v \in H_0^1(\Omega) \cap H^2(\Omega) : \alpha \leqslant \Delta v \leqslant \beta \ \ in \ \ \Omega\},$$
$$K_2 = \{v \in H_0^2(\Omega) : \alpha \leqslant \Delta v \leqslant \beta \ \ in \ \ \Omega\},$$

where α and β are « smooth » functions defined in Ω such that:

$$\alpha(x) < 0 < \beta(x), \qquad \forall x \in \bar{\Omega}.$$

We may state the result of [3] as follows:

if $u \in K_1$ (resp. K_2) satisfies the variational inequality:

$$\int_\Omega \Delta u \Delta(v-u)dx \geqslant \int_\Omega f(v-u)dx, \qquad \forall v \in K_1 \ \ (resp.\ \ K_2),$$

where $f \in L^p(\Omega)$ $(p > n)$, then $\Delta u \in W^{1,\infty}(\Omega)$ $\big($resp. $\Delta u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)\big)$ and $u \in W^{3,q}(\Omega)$ $\big($resp. $u \in W^{3,q}_{\mathrm{loc}}(\Omega)\big)$, with $q < +\infty$. In [3] Brézis and Stampacchia have shown that this regularity result is maximal, in the sense that (in general) $u \notin W^{4,q}_{\mathrm{loc}}(\Omega)$.

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It is well known that, for a wide family of second order variational inequalities, the optimum level of regularity is of the type $W^{2,p}$ $(p < +\infty)$. Similarly for the fourth order variational inequalities studied in [3], the optimum level of regularity is of the type $W^{3,p}$ $(p < +\infty)$. During a lecture at the Laboratorio di Analisi Numerica (C.N.R.) of Pavia, prof. Stampacchia put the following question: for a variational inequality of order $2r$ is the maximum level of regularity of the type $W^{r+1,p}$?

The present paper answers affermatively to this question for a special family of variational inequalities (see for instance Theorem 4.1). The generalization here considered consists mainly in replacing the operator Δ with an elliptic operator, the scalar product in $L^2(\Omega)$ with more general ones (see in particular the example 5.7), the convexes K_i with more general ones. The method used to obtain these results is based on the ideas of Brézis-Stampacchia [3] and on some new remarks. An important role will be played by an « abstract regularity result » for some variational inequalities (see Lemma 2.1 and the subsequent Remark 2.1) and a « representation lemma » for a special class of functionals defined in $L^\infty(\Omega)$ (see Section 3). In section 4 we describe some concrete examples in which the theory here considered may be useful.

Since this paper resulted from a suggestion of prof. Guido Stampacchia I would like to dedicate it to his memory.

2. – Preliminaries.

We state now the main result of the present section:

LEMMA 2.1. *Let X be a linear topological space, X' its dual, M a linear manifold of X, j a convex proper functional on X, Λ an operator defined in X with values in X'. Assume that:*

$$\exists x \in M \quad \text{in a neighborhood of which } j \text{ is bounded above}. \tag{2.1}$$

Let $\bar{x}$ be a solution of the following problem:

$$\bar{x} \in M\,; \quad \langle \Lambda\bar{x}, x - \bar{x}\rangle + j(x) \geqslant j(\bar{x})\,, \qquad \forall x \in M\,. \tag{2.2}$$

Then there exists $z \in M_0^\perp$ (that is z belongs to the subspace of X' orthogonal to M_0, where M_0 is the parallel subspace of M), such that:

$$\langle \Lambda\bar{x} + z, x - \bar{x}\rangle + j(x) \geqslant j(\bar{x})\,, \quad \forall x \in X\,. \tag{2.3}$$

REMARK 2.1. The interest of Lemma 2.1 is that, if for the variational inequality (2.3) is known a regularity result and if $M_0^\perp$ is a « regular » subspace of X', then it follows that $\bar{x}$ is « regular ».

The hypothesis (2.1) is very restrictive: this obliges us to choose $X = L^\infty(\Omega)$ in the most part of the concrete examples here considered (see the section 4). This means that $M_0^\perp$ is a subspace of $(L^\infty(\Omega))'$ and then $M_0^\perp$ does not look a « regular » space. We avoid this difficulty proving that, in several concrete cases, the elements of $M_0^\perp$ have locally a representation as « regular » distributions (see the subsequent section 3).

REMARK 2.2. – We shall now outline a proof of Lemma 2.1 based on a result of Rockafellar (see [4], Theorem 20). In any case different proofs can be considered using other techniques, for instance employing the geometric form of the Hahn-Banach theorem.

First of all we prove a variant of Lemma 2.1:

LEMMA 2.2. *Let X be a linear topological space, X' its dual, M a linear manifold of X, j a proper convex functional on X. If the hypothesis* (2.1) *is fulfilled and if $a \in \mathbf{R}$ and $x' \in X$ verify the following relation*

$$\langle x', x\rangle - j(x) \leqslant a\,, \qquad \forall x \in M\,, \tag{2.4}$$

then there exists $z \in M_0^\perp$, such that:

$$\langle x' - z, x\rangle - j(x) \leqslant a - \langle z, m\rangle\,, \qquad \forall x \in X\,, \tag{2.5}$$

where m is an arbitrary element of M.

PROOF. Let I_M be the indicator function of M, that is

$$I_M(x) = 0 \quad \text{if } x \in M\,, \qquad I_M(x) = +\infty \quad \text{if } x \notin M\,.$$

Let also I_M^* the conjugate function of I_M (for the definition of the conjugate of a proper convex function see [4]). Then we have ($y' \in X'$):

$$I_M^*(y') = \langle y', m\rangle \quad \text{if } y' \in M_0^\perp\,, \qquad I_M^*(y') = +\infty \quad \text{if } y' \notin M_0^\perp\,. \tag{2.6}$$

The relation (2.4) can be equivalently written as

$$(I_M + j)^*(x') \leqslant a\,. \tag{2.7}$$

Using the over mentioned result of Rockafellar it follows that (recalling also the hypothesis (2.1)):

$$(I_M + j)^*(x') = \min_{Y'} [I_M^*(y') + j^*(x' - y')] . \tag{2.8}$$

If $z \in X'$ minimizes the right member of (2.8), we have (by (2.6) and (2.7)):

$$z \in M_0^\perp \quad \textit{and then } I_M^*(z) = \langle z, m\rangle , \tag{2.9}$$

$$\langle z, m\rangle + \sup \big(\langle x' - z, x\rangle - j(x)\big) \leqslant a , \tag{2.10}$$

that is the relation (2.5).

PROOF OF LEMMA 2.1. The second relation (2.2) can be written as

$$- \langle \Lambda\bar{x}, x\rangle - j(x) \leqslant - j(\bar{x}) - \langle \Lambda\bar{x}, \bar{x}\rangle , \qquad \forall x \in M .$$

If we put:

$$x' = - \Lambda\bar{x} , \qquad a = - j(\bar{x}) - \langle \Lambda\bar{x}, \bar{x}\rangle , \tag{2.11}$$

it follows (by Lemma 2.2 and by the fact that $\bar{x} \in M$)

$$\langle x' - z, x\rangle - j(x) \leqslant a - \langle z, \bar{x}\rangle , \qquad \forall x \in X ,$$

where $z \in M_0^\perp$. Recalling the positions (2.11) we obtain soon the relation (2.3).

REMARK 2.3. Already when $\dim(X) < +\infty$, the hypothesis (2.1) is necessary for the validity of Lemma 2.1. Indeed if we put

$$X = \boldsymbol{R}^2 , \qquad M = \{(x_1, x_2) \in \boldsymbol{R}^2 : x_2 = 0\} ,$$

$$\Lambda = \textit{identity operator in } \boldsymbol{R}^2 ,$$

$$j(x) = I_K(x) + x_1 ,$$

where I_K is the indicator function of the following convex set of $\boldsymbol{R}^2$:

$$K = \{(x_1, x_2) \in \boldsymbol{R}^2 : x_1^2 + (x_2 - 1)^2 \leqslant 1\} ,$$

then the point $\bar{x} = (0, 0)$ is a solution of problem (2.2), but does not exist $z \in M^\perp = \{(x_1, x_2) \in \boldsymbol{R}^2 : x_1 = 0\}$ such that z verifies the relation (2.3).

3. – A representation remark.

Let Ω be an open set of $\boldsymbol{R}^n$ and Λ a strongly elliptic operator with indefinitely differentiable coefficients in Ω. Put now:

$$Z(\Lambda) = \{v \in L^1(\Omega) : \Lambda v = 0 \textit{ in the sense of } \mathcal{D}'(\Omega)\}, \tag{3.1}$$

$$L^\infty_c(\Omega) = \{g \in L^\infty(\Omega) : \operatorname{supp}(g) \textit{ is compact in } \Omega\}, \tag{3.2}$$

where supp (g) is the support of g. Let now $\mathfrak{Z}(\Lambda)$ be the subspace of $(L^\infty(\Omega))'$ so defined: $T \in (L^\infty(\Omega))'$ belongs to $\mathfrak{Z}(\Lambda)$ if $\exists z_T \in Z(\Lambda)$ such that

$$_{(L^\infty)'}\langle T, g\rangle_{L^\infty} = {}_{L^\infty}\langle g, z_T\rangle_{L^1}, \quad \forall g \in L^\infty_c(\Omega), \tag{3.3}$$

that is $\mathfrak{Z}(\Lambda)$ is the subspace of $T \in (L^\infty(\Omega))'$ which admits (locally) a representation in $Z(\Lambda)$.

LEMMA 3.1. – *$\mathfrak{Z}(\Lambda)$ is closed in the w^*-topology of $(L^\infty(\Omega))'$.*

PROOF. Thanks to a lemma of Banach (see for instance [5], Lemma 2, §. 4 Appendix to chapter V) it is sufficient to prove that: *if $B \subset \mathfrak{Z}(\Lambda)$, B strongly bounded in $(L^\infty(\Omega))'$, then the w^*-adherence $\bar{B}^*$ of B is contained, in $\mathfrak{Z}(\Lambda)$.*

Let now $T \in \bar{B}^*$: we must prove that $T \in \mathfrak{Z}(\Lambda)$. Since B is strongly bounded, there exists a generalized sequence $T_\lambda \in (L^\infty(\Omega))'$, such that:

$$T_\lambda \to T \text{ in } (L^\infty(\Omega))' \quad \textit{weakly*} \tag{3.4}$$

$$T_\lambda \in \mathfrak{Z}(\Lambda), \quad \forall \lambda, \tag{3.5}$$

$$\{T_\lambda\} \textit{ belongs to a bounded set of } (L^\infty(\Omega))'. \tag{3.6}$$

By the definition of $\mathfrak{Z}(\Lambda)$, there exists a generalized sequence $z_\lambda \in L^1(\Omega)$, such that:

$$\langle T_\lambda, g\rangle = \langle g, z_\lambda\rangle, \quad \forall g \in L^\infty_c(\Omega), \tag{3.7}$$

$$z_\lambda \in Z(\Lambda), \tag{3.8}$$

$$\{z_\lambda\} \textit{ belongs to a bounded set of } L^1(\Omega). \tag{3.9}$$

Let now C be a compact set of Ω. Since $\Lambda z_\lambda = 0$, it follows (thanks to [1]) that

$$\{z_\lambda\} \textit{ belongs to a strongly bounded set of } \mathcal{C}^1(C), \tag{3.10}$$

hence

$$\{z_\lambda\} \text{ belongs to a strongly compact set of } \mathrm{C}^0(C)\,. \tag{3.11}$$

Let now H the adherence set in $\mathrm{C}^0(C)$ of the generalized sequence z_λ, that is

$$H = \{x \in \mathrm{C}^0(C) : x \in \bar{H}_\lambda,\ \forall\lambda\}\,,$$

where $H_\lambda = \{z_\mu : \mu > \lambda\}$. The relation (3.11) implies that

$$H \neq \emptyset\,. \tag{3.12}$$

Let now $y_i \in H$ $(i = 1, 2)$. By (3.7) we have

$$\int_C y_j g \, \mathrm{d}m = {}_{(L^\infty)'}\langle T, g\rangle_{L^\infty}, \qquad g \in L^\infty(\Omega) \text{ such that } \operatorname{supp}(g) \subset C\,. \tag{3.13}$$

where m is the Lebesgue measure. This implies that $y_1 = y_2$, that is the generalized sequence z_λ has exactly one adherence point. This means that $\exists z_C \in \mathrm{C}^0(C)$ such that

$$z_\lambda \to z_C \quad \text{in } \mathrm{C}^0(C) \qquad \textit{strongly}\,. \tag{3.14}$$

We can now define a function $z \in \mathrm{C}^0(\Omega)$ such that

$$z_\lambda \to z \quad \text{a.e. } \textit{in } \Omega \textit{ and uniformly on any compact of } \Omega\,. \tag{3.15}$$

Recalling also (3.9) we have

$$z \in L^1(\Omega)\,, \qquad \Lambda z = 0 \qquad (\textit{that is } z \in Z(\Lambda))\,. \tag{3.16}$$

By (3.4), (3.7) and (3.15) we have

$${}_{(L^\infty)'}\langle T, g\rangle_{L^\infty} = {}_{L^\infty}\langle g, z\rangle_{L^1}\,, \qquad \forall g \in L^\infty_c(\Omega)\,.$$

that is $T \in \mathfrak{Z}(\Lambda)$. The Lemma is then completely proved.

An immediate consequence of Lemma 3.1 is

COROLLARY 3.1. *The w*-adherence of* $Z(\Lambda)$ *in* $(L^\infty(\Omega))'$ *is contained in* $\mathfrak{Z}(\Lambda)$.

4. – Applications.

a) Let

(4.1) $$\Omega \text{ be a bounded open set of } \boldsymbol{R}^n, \quad n \geqslant 2 .$$

Let also E be a Banach space such that (all the embedding being continuous):

(4.2) $$W_0^{2m,p}(\Omega) \subset E \subset W^{2m,p}(\Omega), \quad p \geqslant 2 ,$$

where m is a positive integer. Put also

(4.3) $$A_1 \text{ is a strongly elliptic operator of order } 2m .$$

We assume that, for sake of simplicity, A_1 has $\mathcal{C}^\infty(\bar{\Omega})$ coefficients and that

(4.4) $$A_1(E) \quad \text{is closed in } L^p(\Omega) .$$

We have that

(4.5) $$A_1 \in \mathcal{L}(E, L^p(\Omega)) ,$$

that is A_1 is a linear and continuous operator between the spaces E and $L^p(\Omega)$.
Let also

(4.6) $$A_2 \text{ be an operator defined in } L^p(\Omega) \text{ with values in } L^{p'}(\Omega) ,$$

where $p' = p/(p-1)$. Let us consider two functions α and β in $\mathcal{C}^\infty(\bar{\Omega})$ such that

(4.7) $$\alpha(x) < 0 < \beta(x) , \quad \forall x \in \bar{\Omega} .$$

Put now

(4.8) $$K = \{v \in L^p(\Omega) : \alpha \leqslant v \leqslant \beta \text{ a.e. in } \Omega\} ,$$

(4.9) $$\tilde{K} = \{v \in E : A_1 v \in K\} .$$

Let now u be a solution of the following variational inequality:

(4.10) $$u \in \tilde{K} ,$$

(4.11) $$(A_1 A_2 u, A_1(v-u)) \geqslant {}_{E'}\langle f, v-u \rangle_E , \quad \forall v \in \tilde{K} ,$$

where $f \in E'$ and $(\cdot,\cdot)$ denotes the duality between $L^{p'}(\Omega)$ and $L^p(\Omega)$. We assume that:

$$f \in \Lambda_1'(L^{p'}(\Omega)), \tag{4.12}$$

where $\Lambda_1' \in \mathfrak{L}(L^{p'}(\Omega), E')$ is the transposed operator of Λ_1.

In the present section, if A (resp. B) is a subspace of $L^p(\Omega)$ (resp. $L^{p'}(\Omega)$), we denote by $A^\perp$ (resp. ${}^\perp B$) the subspace of $L^{p'}(\Omega)$ (resp. $L^p(\Omega)$(orthogonal to A (resp. B). Similarly if A (resp. B) is a subspace of $L^\infty(\Omega)$ (resp. $(L^\infty(\Omega))'$), we denote by $A^\top$ (resp. ${}^\top B$) the subspace of $(L^\infty(\Omega))'$ (resp. $L^\infty(\Omega)$) orthogonal to A (resp. B).

Lemma 4.1. *Under the hypotheses* (4.1)-(4.4), (4.6)-(4.9), *and* (4.12), *if u is a solution of the problem* (4.10) *and* (4.11), *then there exists T in the w^*-adherence in $(L^\infty(\Omega))'$ of $N(\Lambda_1')$ (where $N(\Lambda_1')$ is the kernel of Λ_1') such that*:

$$U \in K, \tag{4.13}$$

$$(\Lambda_2 U, V - U) \geqslant {}_{(L^\infty)'}\langle F + T, V - U\rangle_{L^\infty}, \quad \forall V \in K, \tag{4.14}$$

where $U = \Lambda_1 u$ and $\Lambda_1' F = f$.

Proof. We can verify easily that

$$U \in K \cap \Lambda_1(E), \tag{4.15}$$

$$(\Lambda_2 U, V - U) \geqslant (F, V - U), \quad \forall V \in K \cap \Lambda_1(E). \tag{4.16}$$

The relation (4.16) can be equivalently written as

$$(\Lambda_2 U, V - U) + I_K(V) \geqslant I_K(U) + (F, V - U), \quad \forall V \in \Lambda_1(E), \tag{4.17}$$

where I_K is the indicator function of K. If $p < +\infty$, we can not apply directly Lemma 2.1 since the interior $\mathring{K}$ of K in $L^p(\Omega)$ is empty. We remark that

$$\Lambda_1(E) \cap L^\infty(\Omega) = {}^\top(N(\Lambda_1')). \tag{4.18}$$

Since $U \in K \subset L^\infty(\Omega)$, it follows that:

$$U \in {}^\top(N(\Lambda_1')),$$

$$\langle \Lambda_2 U, V - U\rangle + I_K(V) \geqslant I_K(U) + \langle F, V - U\rangle, \quad \forall V \in {}^\top(N(\Lambda_1')),$$

where the dualities are between $(L^\infty(\Omega))'$ and $L^\infty(\Omega)$. If we put $j(V) = I_K(V) - \langle F, V\rangle$, we can apply Lemma 2.1, since the identically zero function belongs to

$$\mathring{K} \cap {}^\top(N(\Lambda_1'))\,.$$

Hence there exists $T \in \big({}^\top(N(\Lambda_1'))\big)^\top$ $\big(=$ w^*-adherence of $N(\Lambda_1')$ in $(L^\infty(\Omega))'\big)$ verifying the relation (4.14). The Lemma 4.1 is completely proved.

b) Let us now introduce the following condition of regularity

$$f \in L^q(\Omega) \cap N(\Lambda_1)^\perp\,, \qquad q > \max\left(\frac{n}{2m-1}, 1\right), \tag{4.19}$$

where $N(\Lambda_1)$ is the kernel of Λ_1. Hence the relations (4.4) and (4.19) imply the relation (4.12). Put also:

$$\Lambda_2 = \textit{inclusion of } L^p(\Omega) \textit{ in } L^{p'}(\Omega)\,. \tag{4.20}$$

THEOREM 4.1. *Under the hypotheses* (4.1)-(4.4), (4.7)-(4.9), (4.19) *and* (4.20), *if u is a solution of Problem* (4.10) *and* (4.11), *then $\Lambda_1 u \in W^{1,\infty}_{\text{loc}}(\Omega)$ and $u \in W^{2m+1,r}_{\text{loc}}(\Omega)$ $(\forall r < +\infty)$. We have also that this result of regularity is maximal in the sense that (in general) $u \notin W^{2(m+1),r}_{\text{loc}}(\Omega)$.*

PROOF. – Using Corollary 3.1, it is easy to prove that the functional T (of Lemma 4.1) belongs to $\mathfrak{Z}(\Lambda_1^*)$, where Λ_1^* is the *formal* adjoint of Λ_1. Then there exists a function z such that

$$z \in L^1(\Omega)\,, \quad \Lambda_1^* z = 0 \qquad in\ \mathcal{D}'(\Omega)\,, \tag{4.21}$$

$$\langle T, g\rangle = \langle g, z\rangle\,, \quad \forall g \in L^\infty_c(\Omega)\,. \tag{4.22}$$

Recalling (4.19), we can see that the function F (in (4.14)) satisfies the relation: $\Lambda_1^* F = f$. Then

$$F \in W^{2m,q}_{\text{loc}}(\Omega)\,. \tag{4.23}$$

Let now C be a compact set of Ω. Put also

$$K(U, C) = \{v \in K \colon U = v \ on\ \Omega - C\}\,. \tag{4.24}$$

Then we have

$$U \in K(U, C)\,, \tag{4.25}$$

$$(U, V - U) \geqslant (F + z, V - U)\,, \quad \forall V \in K(U, C)\,. \tag{4.26}$$

This means that for every compact C of Ω, we have

$$U = \begin{cases} \beta & \text{if} \quad F + z > \beta \\ F + z & \text{if } \alpha \leqslant F + z \leqslant \beta \\ \alpha & \text{if} \quad F + z < \alpha \,. \end{cases}$$

Recalling that F and z are locally « regular » in Ω (see the relations (4.21) and (4.23)), we can conclude the proof of the theorem.

REMARK 4.1. The local regularity result of Theorem 4.1 may become a global regularity result in some special case. Indeed if:

$$N(\Lambda_1') \subset W^{2,q}(\Omega)\,, \quad q > n\,, \tag{4.27}$$

$$\dim N(\Lambda_1') < +\infty \tag{4.28}$$

and if

$$\exists F \in W^{2,q}(\Omega)\,, \quad q > n\,, \quad \textit{such that } \Lambda_1' F = f\,,$$

then the proof of Theorem 5.1 says that $\Lambda_1 u \in W^{1,\infty}(\Omega)$. Moreover if the space E is « smooth » enough, we have that $u \in W^{2m+1,r}(\Omega)$ $(r < +\infty)$ (see Examples 4.2 and 4.4). Also here, this result of regularity is maximal.

REMARK 4.2. Lemma 4.1 and Theorem 4.1 consider only the case in which the convex functional j of Lemma 1.1 is of the form:

$$j = I_K + f\,,$$

where I_K is the indicator function of a convex set and f is a linear and continuous term. We could consider a more general situation employing more general functionals provided that the condition (1.1) be again fulfilled.

c) Now we shall describe some examples related to the theory over explained. Unless otherwise stated, α and β are « regular » functions verifing (4.7).

EXAMPLE 4.1. Let $f \in L^q(\Omega)$, $q > n$. Put also:

$$\tilde{K} = \{v \in H_0^2(\Omega) \colon \alpha \leqslant \Delta v \leqslant \beta\}\,.$$

With this position the compatibility condition contained in (4.19) disappears, because the kernel of Δ in $H_0^2(\Omega)$ is $\{0\}$. Let now u be a solution of the following problem:

$$(4.29) \qquad u \in \tilde{K}\,; \qquad (\Delta u, \Delta(v-u)) \geqslant (f, v-u)\,, \qquad \forall v \in \tilde{K}\,.$$

Then $\Delta u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and $u \in W^{3,p}_{\mathrm{loc}}(\Omega)$, $p < +\infty$. This is the result stated in Theorem 2 of Brézis-Stampacchia [3].

EXAMPLE 4.2. Let (Ω sufficiently «smooth»):

$$\tilde{K} = \{v \in H^2(\Omega) \cap H^1_0(\Omega) : \alpha \leqslant \Delta v \leqslant \beta\}\,.$$

In this case if $f \in L^q(\Omega)$ $(q > n)$, if u is a solution of the problem (4.29), then (applying Theorem 4.1 and Remark 4.1) we have that: $\Delta u \in W^{1,\infty}(\Omega)$ and $u \in W^{3,p}(\Omega)$, $p < +\infty$. This is the result stated in Theorem 1 of Brézis-Stampacchia [3].

EXAMPLE 4.3. Suppose Ω «smooth» and:

$$\tilde{K} = \{v \in H^2(\Omega) : \alpha \leqslant \Delta v \leqslant \beta\}\,.$$

In this case if $f \in L^q(\Omega)$ $(q > n)$ is orthogonal to the space

$$(4.30) \qquad N(\Delta) = \{v \in H^2(\Omega) : \Delta v = 0\}\,,$$

then a solution u of problem (4.29) satisfies the following relation of regularity: $\Delta u \in W^{1,\infty}(\Omega)$, $u \in W^{3,p}_{\mathrm{loc}}(\Omega)$ $(p < +\infty)$.

EXAMPLE 4.4. Suppose Ω «smooth» and

$$\tilde{K} = \left\{v \in H^2(\Omega) : \frac{\partial v}{\partial n} = 0 \qquad \textit{on } \partial\Omega \textit{ and } \alpha \leqslant \Delta v \leqslant \beta\right\}.$$

If $f \in L^q(\Omega)$ $(q > n)$ is orthogonal to every constant then $\Delta u \in W^{1,\infty}(\Omega)$ and $u \in W^{3,p}(\Omega)$ $(p < +\infty)$, where u is a solution of problem (4.29). In this case the condition (4.7) may be weakened assuming that α (resp. β) has average strictly positive (resp. negative). Indeed in this case we have that $\mathring{K} \cap M \neq \emptyset$ where

$$K = \{v \in L^\infty(\Omega) : \alpha \leqslant v \leqslant \beta\}\,, \qquad M = \{v \in L^\infty(\Omega) : v \textit{ has average zero in } \Omega\}\,.$$

Let us now consider an example in which the operator A_1 has order 4.

EXAMPLE 4.5. Put:

$$\tilde{K} = \{v \in H^4(\Omega) \cap H_0^3(\Omega) : \alpha \leqslant \Delta v \leqslant \beta\}.$$

If $f \in L^q(\Omega)$ $(q > \max(n/3, 1))$, then a solution u of the following problem

$$u \in \tilde{K}; \qquad (\Delta^2 u, \Delta^2(v-u)) \geqslant (f, v-u), \qquad \forall v \in \tilde{K},$$

satisfies the following condition of regularity: $\Delta^2 u \in W^{1,\infty}_{\text{loc}}(\Omega)$, $u \in W^{5,p}_{\text{loc}}(\Omega)$ $(p < +\infty)$.

EXAMPLE 4.6. By a slight modification of Theorem 4.1, we can adapt the theory to convex set of the type:

$$\tilde{K} = \{v \in \mathcal{M} : \alpha \leqslant \Lambda_1 v \leqslant \beta\}$$

where $\mathcal{M}$ is not a linear subspace of $W^{2m,p}(\Omega)$, but only a linear manifold. Let us now consider an example of this situation. Let $\bar{v} \in C^\infty(\bar{\Omega})$. We assume also that the functions α and β are in $C^\infty(\bar{\Omega})$ and verify the following inequalities:

$$\alpha(x) < \Delta\bar{v}(x) < \beta(x), \qquad \forall x \in \bar{\Omega}.$$

Put

$$\mathcal{M} = H_0^2(\Omega) + \bar{v} = \{v + \bar{v} : v \in H_0^2(\Omega)\},$$

$$\tilde{K} = \{v \in \mathcal{M} : \alpha \leqslant \Delta v \leqslant \beta\}, \qquad K = \{v \in L^\infty(\Omega) : \alpha \leqslant v \leqslant \beta\}.$$

Let now u be a solution of the following problem:

$$u \in \tilde{K}; \qquad (\Delta u, \Delta(v-u)) \geqslant (f, v-u), \qquad \forall v \in \tilde{K},$$

where $f \in L^q(\Omega)$ $(q > n)$, f orthogonal to the space defined in (4.30). If we put $U = \Delta u$ and

$$M = \Delta(H_0^2(\Omega)) + \Delta\bar{v},$$

it follows:

$$U \in K \cap M; \qquad (U, V-U) \geqslant (F, V-U), \qquad \forall v \in K \cap M,$$

where F is a solution of the problem

$$(f, v) = (F, \Delta v), \qquad \forall v \in H^2(\Omega).$$

Adapting the proof of Theorem 4.1, we obtain again that:

$$\Delta u \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \quad \textit{and} \quad u \in W^{3,p}_{\mathrm{loc}}(\Omega) \quad (p < +\infty)\,.$$

d) We can consider also some example in which the scalar product between $L^p(\Omega)$ and $L^{p'}(\Omega)$ is replaced by a different one. We now outline a very easy example of this situation.

EXAMPLE 4.7. We choose (Ω sufficiently « smooth »):

$$\begin{aligned} E &= H^3(\Omega) \cap H^1_0(\Omega)\,, \\ K &= \{v \in H^1_0(\Omega) \colon |\mathrm{grad}\, v| \leqslant 1\}\,, \\ \tilde{K} &= \{v \in E \colon \Delta v \in K\}\,. \end{aligned}$$

Let u be a solution of the following variational inequality $\big(f \in L^p(\Omega)$, $p > \max(n/2, 1)\big)$:

$$u \in \tilde{K}\,,$$

$$\big(\mathrm{grad}\, \Delta u, \mathrm{grad}\, \Delta(v-u)\big) \geqslant (f, v-u)\,, \qquad \forall v \in \tilde{K}\,.$$

If we put $U = \Delta u$, it follows

$$(4.31) \qquad U \in K\,; \quad \big(\mathrm{grad}\, U, \mathrm{grad}\,(V-U)\big) \geqslant (F, V-U)\,, \qquad \forall V \in K\,,$$

where $F \in W^{2,p}(\Omega)$ $\big(\subset C^0(\Omega)\big)$ verifies the relation:

$$\int_\Omega f v \,\mathrm{d}x = (F, \Delta v)\,, \qquad \forall v \in E\,.$$

Thanks to a result of Brézis and Stampacchia [2], the relation (4.31) implies that $U = \Delta u \in W^{2,q}(\Omega)$ and then $u \in W^{4,q}(\Omega)$ ($\forall q < +\infty$).

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