ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 6, nº 2 (1979), p. 305-334
<http://www.numdam.org/item?id=ASNSP_1979_4_6_2_305_0>

Littlewood-Paley a Priori Estimates for Parabolic Equations with Sub-Dini Continuous Coefficients.

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0. – Introduction.

Consider the initial value problem

$$(0.1) \qquad Lu = 0 \quad \text{in } \boldsymbol{R}^n \times (0, T) = S_T\,, \quad u(\cdot, 0) = g$$

where the coefficients $A_\alpha(x, t)$ of the parabolic operator

$$Lu(x, t) \equiv \sum_{|\alpha| \leq m} A_\alpha(x, t) D_x^\alpha u(x, t) - D_t u(x, t)$$

are assumed to be uniformly continuous for $|\alpha| = m$ and bounded and measurable for $|\alpha| < m$.

From the theory of singular integral operators, see [4] and [11], it follows that if the *initial data* g are smooth enough, say $g \in C_0^\infty$, then a solution of (0.1) exists. On the other hand, it is of course well known that if the *coefficients* are smooth, then (0.1) is solvable if $g \in L^p(\boldsymbol{R}^n)$; the initial values are then taken on in the sense that $\lim_{t\to 0} \|u(\cdot, t) - g\|_{L^p} = 0$.

In the last decade there has been a growing interest in parabolic equations with continuous-only coefficients, mostly in connection with diffusion processes. The standard reference is [14]. However, at present it is not known whether uniform continuity of the highest order coefficients is sufficient to guarantee the solvability of (0.1) with L^p data g.

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Pervenuto alla Redazione il 10 Febbraio 1978.

In this paper we consider operators whose coefficients are slightly more regular in that we assume that the modulus of continuity $\omega(\tau)$ satisfies

$$\int_0 \frac{\omega^2(\tau)}{\tau}\, d\tau < \infty ,$$

a property which we have dubbed « Dini ($\frac{1}{2}$) continuity ». For these operators we give a priori estimates which allow the whole existence and uniqueness program to be carried through.

The crucial necessary estimate, sufficient to imply the solvability of (0.1) with g in L^p, see [3], is

$$\sup_{0<t<T} \|u(\cdot, t)\|_{L^p} \leqslant k \|g\|_{L^p} , \tag{0.2}$$

valid for all solutions of (0.1) with $g \in C_0^\infty(\boldsymbol{R}^n)$, say. In fact we prove much more, namely so called *Littlewood-Paley estimates*, involving also the derivatives of the solution, as well as a supremum estimate, see Theorem 1 below. These estimates lead, via approximation arguments, to the *existence*, Theorem 2, and *uniqueness*, Theorem 3, of solutions. Note that the supremum estimate implies that the solution takes on the initial data also in the almost everywhere sense.

We also prove an *inverse Littlewood-Paley* inequality, i.e. an inequality where the L^p norm of the initial values are estimated by a weighted integral of derivatives of the solution, see Theorem 4. An immediate consequence of the Littlewood-Paley inequalities is a Fatou type result, Theorem 5.

The program for finding the key Littlewood-Paley inequality is the following. First the proof is reduced to the case when the coefficients depend on x only, and the elliptic part of the operator contains only highest order terms. Then a representation formula for solutions and their spatial derivatives is derived:

$$\begin{aligned} D^\alpha u(x,t) = \int_{\boldsymbol{R}^n} D^\alpha \Gamma_x(x-y,t) g(y)\, dy + \lim_{\varepsilon\to 0} \int_0^{t-\varepsilon}\int_{\boldsymbol{R}^n} D^\alpha \Gamma_x(x-y,t-s) Lu(y,s)\, dy\, ds\, + \\ + \sum_{|\beta|=m} \lim_{\varepsilon\to 0} \int_0^{t-\varepsilon}\int_{\boldsymbol{R}^n} D^\alpha \Gamma_x(x-y,t-s)[\mathcal{A}_\beta(x) - \mathcal{A}_\beta(y)] D^\beta u(y,s)\, dy\, ds\, . \end{aligned}$$

Here $D^\alpha \Gamma_z(y,s)$ is a derivative of the fundamental solution of the constant coefficient operator obtained when freezing the coefficients of L at z. For $|\alpha| = m$ the integrals in this representation become singular.

In order to be able to use the theory of (vector-valued) singular integral operators the potential integrals are modified, in section 6, by the splitting off of terms which can be treated e.g. by approximation, integration by parts, and the Hardy-Littlewood maximal theorem.

The remaining hard core terms involving genuinely singular integrals are handled in section 7, using a result on Hilbert space valued singular integrals by Benedeck, Calderon and Panzone.

We end this introduction by a short review of earlier work in the area. To facilitate the discussion of continuity conditions which are weaker than that of Dini we propose the label *Dini* (α) *continuous* for a function whose modulus of continuity satisfies

$$\int_0 \frac{\omega^{1/\alpha}(\tau)}{\tau}\, d\tau < \infty .$$

For operators L whose coefficients are Hölder continuous in either the x-direction or the t-direction the estimate (0.2) is known through the work of Eidelman [2], and Kato [8]. (See also [5], chapter IX). However, the Littlewood-Paley inequalities seem to be new even in this case.

One of the present authors showed in [3] that Dini ($\frac{3}{4}$) continuous coefficients is sufficient to guarantee the estimate (0.2) and in [12] the second named author extended this result to Dini ($\frac{1}{2}$) coefficients in the case $2 \le p < \infty$.

In the elliptic case, with Hölder continuous coefficients, the Littlewood-Paley inequalities for $p = 2$ as well as Fatou type theorems for $p \ge 1$ were proved by the third of the present authors in [15]. See also [10] for a related result.

The only relevant counterexample known to the authors is that of Il'in [7], which shows that (0.2) cannot hold in general for $p = 1$ if only Dini (α) continuity with $\alpha < \frac{1}{4}$ is assumed.

1. – Notations and statement of results.

We will let D^α denote differentiation with respect to $x = (x_1, \dots, x_n) \in R^n$, α being the usual multi-index $(\alpha_1, \dots, \alpha_n)$, while t-differentiation is denoted by D_t. The convenient abuse of the symbol D^j for an arbitrary, or all, derivatives of order j will also be continued in this paper.

The principal part of the operator, L,

$$Lu(x, t) \equiv \sum_{|\alpha| \le m} A_\alpha(x, t) D^\alpha u(x, t) - D_t u(x, t) ,$$

where $m \geq 2$ is an even integer, is assumed to be strongly elliptic, i.e. for all $(x, t) \in S_T = R^n \times (0, T)$ and for all $\xi \in R^n$ we have

$$\operatorname{Re}\Big(\sum_{|\alpha|=m} A_\alpha(x, t)(i\xi)^\alpha\Big) \leqslant -\lambda|\xi|^m$$

for some $\lambda > 0$. Also, all coefficients are assumed to be bounded and measurable with

$$\mu = \sup\{|A_\alpha(x, t)| : (x, t) \in S_T,\ 0 \leqslant |\alpha| \leq m\},$$

while the coefficients of order m are uniformly continuous in S_T. We put

$$\omega_1(\tau) = \sup\{|A_\alpha(x, t) - A_\alpha(y, t)| : |x - y| < \tau,\ 0 < t < T,\ |\alpha| = m\}$$
$$\omega_1'(\tau) = \sup\{|A_\alpha(x, 0) - A_\alpha(y, 0)| : |x - y| < \tau,\ |\alpha| = m\}$$
$$\omega_2(\tau) = \sup\{|A_\alpha(x, t) - A_\alpha(x, s)| : (x, t) \in S_T,\ (x, s) \in S_T,\ |s - t| < \tau,\ |\alpha| = m\}$$
$$\omega_2'(\tau) = \sup\{|A_\alpha(x, t) - A_\alpha(x, 0)| : x \in R^n,\ 0 < t < \tau,\ |\alpha| = m\},$$

and define

$$\omega(\tau) = \omega_1(\tau^{1/m}) + \omega_2(\tau)$$
$$\omega_0(\tau) = \omega_1'(\tau^{1/m}) + \omega_2'(\tau).$$

We will say that the m-th order coefficients are Dini $(\frac{1}{2})$ continuous when

$$\int_0 \frac{\omega^2(\tau)}{\tau}\, d\tau < \infty.$$

The space $L^p(L^2(S_{\delta,T}))$, $S_{\delta,T} = R^n \times (\delta, T)$, consists of those locally integrable functions for which

$$\|f\|_{L^p(L^2(S_{\delta,T}))} = \Big[\int_{R^n}\Big(\int_\delta^T |f(x, t)|^2 dt\Big)^{p/2} dx\Big]^{1/p} < \infty$$

and by $W^{m,1}_{p,2}(S_T)$ we denote the closure of $C_0^\infty(R^{n+1})$ with respect to the norm

$$\|u\|_{W^{m,1}_{p,2}(S_T)} = \sum_{|\alpha| \leqslant m} \|D^\alpha u\|_{L^p(L^2(S_T))} + \|D_t u\|_{L^p(L^2(S_T))}$$

while $W^{m,1}_p = W^{m,1}_{p,p}$.

Note that all elements of $W^{m,1}_{p,2}(S_T)$ have a locally integrable trace on $\{t = \tau\}$ for all $\tau \in [0, T]$.

THEOREM 1. *Suppose that* $\int_0 (\omega_0^2(\tau)/\tau)\, d\tau < \infty$. *Then for* $1 < p < \infty$ *and for all* $u \in W_{p,2}^{m,1}(S_T)$

$$\| \sup_{0<t<T} |u(\cdot, t)| \|_{L^p(R^n)} + \sum_{1 \leqslant |\alpha| \leqslant m} \| t^{|\alpha|/m - \frac{1}{2}} D^\alpha u \|_{L^p(L^2(S_T))} \leqslant C \| u(\cdot, 0) \|_{L^p(R^n)} + C_T \left\| \frac{t^{\frac{1}{2}}}{\theta(t)} Lu \right\|_{L^p(L^2(S_T))}$$

for every positive nondecreasing function θ *on* $(0, T]$ *such that* $\int_0 (\theta^2(\tau)/\tau)\, d\tau < \infty$ *and such that* $\theta(t) \geqslant \omega_2'(t) + t^{1/m}$. *The constants* C *and* C_T *depend on* p, T, n, λ, μ, m, ω_0, *and* θ. *Moreover, for fixed* p, C_T *tends to zero with* T.

THEOREM 2. *Let* $1 < p < \infty$ *and suppose that*

$$\int_0 \frac{\omega_0^2(\tau)}{\tau}\, d\tau < \infty .$$

Then to every $g \in L^p(R^n)$ *and every* f *such that*

$$\left\| \frac{t^{\frac{1}{2}}}{\theta(t)} f \right\|_{L^p(L^2(S_T))} < \infty \tag{1.1}$$

for some non-decreasing θ *such that* $\int_0 (\theta^2(\tau)/\tau)\, d\tau < \infty$, *there exists a function* u *with the following properties*:

(i) $\| \sup_{0<t<T} |u(\cdot, t)| \|_{L^p(R^n)} + \sum_{1 \leqslant |\alpha| \leqslant m} \| t^{|\alpha|/m - \frac{1}{2}} D^\alpha u \|_{L^p(L^2(S_T))} < \infty$;

(ii) $Lu = f$ *almost everywhere in* S_T;

(iii) $\lim_{t \to 0^+} u(x, t) = g(x)$ *for almost every* x *in* R^n.

Furthermore, if $f \in L^q$, $1 < q < \infty$, *locally in* S_T *then* u *belongs to* $W_q^{m,1}$ *locally.*

THEOREM 3. *Let* L *have Dini* $(\frac{1}{2})$ *continuous* m-th *order coefficients, i.e. assume that*

$$\int_0 \frac{\omega^2(\tau)}{\tau}\, d\tau < \infty ,$$

and let p, g, f *be as in Theorem* 2. *Then there is at most one function satisfying*:

(i) $u \in W_{p,2}^{m,1}(S_{\delta,T})$ *for every* $\delta > 0$;

(ii) $Lu = f$ *almost everywhere in* S_T;

(iii) $\lim_{t \to 0^+} u(\cdot, t) = g$ *weakly in* $L^p(R^n)$.

THEOREM 4. *Let L, f, and p be as in Theorem 3. If a solution u of $Lu = f$ satisfies*

$$u(\cdot, \tau) \in L^p(R^n)$$

for some $\tau \in (0, T]$, and if

$$\sum_{1 \leq |\alpha| \leq m} t^{|\alpha|/m - \frac{1}{2}} |D^\alpha u(x, t)| \in L^p\big(L^2(S_T)\big)$$

then, provided $\theta(t)/t^{\frac{1}{2}}$ is monotone,

(i) $u(\cdot, t) \in L^p(R^n)$ *for all* $t \in (0, T)$;

(ii) $u(\cdot, 0) \equiv \lim_{t \to 0^+} u(\cdot, t)$ *exists as a limit in L^p and as a pointwise limit almost everywhere*;

(iii) $$\|u(\cdot, 0)\|_{L^p(R^n)} \leq C\left[\|u(\cdot, \tau)\|_{L^p} + \sum_{1 \leq |\alpha| \leq m} \|t^{|\alpha|/m - \frac{1}{2}} D^\alpha u\|_{L^p(L^2(S_T))} + \left\|\frac{t^{\frac{1}{2}}}{\theta(t)} f\right\|_{L^p(L^2(S_T))}\right],$$

where C depends on ω and the parameters mentioned in Theorem 1.

THEOREM 5. *Let L, f, and p be as in Theorem 3. If u is a solution of $Lu = f$ satisfying*

$$u \in W^{m,1}_{p,2}(S_{\delta,T}) \qquad \text{for all } \delta > 0$$

and

$$\sup_{0 < t < T} \|u(\cdot, t)\|_{L^p(R^n)} < \infty$$

then $u(x, 0) \equiv \lim_{t \to 0^+} u(x, t)$ exists as a limit in $L^p(R^n)$ and pointwise almost everywhere in R^n.

2. – Proof of Theorem 1.

We will start by showing that the proof can be reduced to a simpler case. In fact, we first observe that it is sufficient to prove Theorem 1 for *some* $T = T_0 > 0$, depending on the parameters indicated.

Let $\psi \in C_0^\infty(0, \infty)$ be zero in $(0, T_0/2)$ and one in (T_0, T). By the results of [4] and [11], for $0 \leq j \leq m$

$$\|D^j(\psi u)\|_{L^p(L^2(S_T))} \leq C \|L(\psi u)\|_{L^p(L^2(S_T))}$$

and trivially

$$\|L(\psi u)\|_{L^p(L^2(S_T))} \leqslant \|\psi L u\|_{L^p(L^2(S_T))} + \|\psi' u\|_{L^p(L^2(S_T))} \leqslant C_{T_0}\left[\left\|\frac{t^{\frac{1}{2}}}{\theta(t)} L u\right\|_{L^p(L^2(S_T))} + \|\sup_{0<t<T_0} |u(\cdot, t)|\|_{L^p(R^n)}\right].$$

On the other hand,

$$\sup_{0<t<T} |u(x, t)| \leqslant \sup_{0<t<T_0} |u(x, t)| + \int_{T_0}^{T} |D_s u(x, s)| ds$$

and for $1 \leq j \leq m$

$$\|t^{j/m-\frac{1}{2}} D^j u\|_{L^p(L^2(S_T))} \leqslant \|t^{j/m-\frac{1}{2}} D^j u\|_{L^p(L^2 S_{T_0}))} + C_{T_0}\|D^j(\psi u)\|_{L^p(L^2(S_T))} .$$

Combining the above inequalities we find

$$\|\sup_{0<t<T} |u(x, t)|\|_{L^p(R^n)} + \sum_{j=1}^{m} \|t^{j/m-\frac{1}{2}} D^j u\|_{L^p(L^2(S_T))} \leqslant C_{T_0}\left[\|\sup_{0<t<T_0} |u(x, t)|\|_{L^p(R^n)} + \left\|\frac{t^{\frac{1}{2}}}{\theta(t)} L u\right\|_{L^p(L^2(S_T))}\right],$$

which obviously implies the assertion.

The next step is the observation that it is sufficient to consider operators of the form

$$L_0 \equiv \sum_{|\alpha|=m} A_\alpha(x) D^\alpha - D_t , \tag{2.1}$$

i.e. operators with derivations of the highest order only and whose coefficients are independent of t. In fact, if Theorem 1 is valid for this type of operator then for one of general type we write

$$L = L^0 + \sum_{|\alpha|=m} [A_\alpha(x, t) - A_\alpha(x, 0)] D^\alpha + \sum_{|\alpha|<m} A_\alpha(x, t) D^\alpha$$

where $L^0 = \sum_{|\alpha|= m} A_\alpha(x, 0) D_x^\alpha - D_t$. Then

$$\begin{aligned}\|\sup_{0<t<T} |u(x, t)|\|_{L^p(R^n)} &+ \sum_{j=1}^{m} \|t^{j/m-\frac{1}{2}} D^j u\|_{L^p(L^2(S_T))} \leqslant \\ &C\|u(\cdot, 0)\|_{L^p(R^n)} + C_T\left\|\frac{t^{\frac{1}{2}}}{\theta(t)} L^0 u\right\|_{L^p(L^2(S_T))} \leqslant \\ &C\|u(\cdot, 0)\|_{L^p(R^n)} + C_T\left\{\left\|\frac{t^{\frac{1}{2}}}{\theta(t)} L u\right\|_{L^p(L^2(S_T))} + \left\|\frac{t^{\frac{1}{2}}\omega_2'(t)}{\theta(t)} D^m u\right\|_{L^p(L^2(S_T))}\right. \\ &\left.+ \sum_{j=0}^{m-1}\left\|\frac{t^{\frac{1}{2}}}{\theta(t)} D^j u\right\|_{L^p(L^2(S_T))}\right\}.\end{aligned}$$

We need now only observe that for $1 \le j \le m-1$

$$\left\| \frac{t^{\frac{1}{2}}}{\theta(t)} D^j u \right\|_{L^p(L^2(S_T))} \leqslant T^{1-(j+1)/m} \| t^{j/m - \frac{1}{2}} D^j u \|_{L^p(L^2(S_T))}$$

while remembering that $\theta(t) \geqslant \omega_2'(t) + t^{1/m}$ and that C_T tends to zero with T, in order to realize that the undesireable terms in the right hand side may be absorbed by the left hand side for small enough T.

Thus, in the remaining parts of the proof we assume that the operator has the form (2.1). Also, it is obvious that it is sufficient to consider functions u belonging to $C_0^\infty(R^{n+1})$.

We define

$$\Gamma_z(x, t) = C_n t^{-n/m} \mathcal{F}_\xi\Big(\exp\Big(\sum_{|\alpha|=m} \mathcal{A}_\alpha(z)(i\xi)^\alpha\Big)\Big)(x/t^{1/m}), \qquad t > 0,$$

where $\mathcal{F}_\xi$ denotes the Fourier transform with respect to ξ and C_n is a suitable constant depending on n only, such that Γ_z becomes the traditional fundamental solution of L with coefficients frozen at z, i.e. for the operator

$$L_z = \sum_{|\alpha|=m} \mathcal{A}_\alpha(z) D^\alpha - D_t.$$

For $u \in C_0^\infty(R^{n+1})$ and $t > 0$ we have the following easily derivable representation formulas.

$$D^j u(x, t) = \int_{R^n} D^j \Gamma_x(x-y, t) u(y, 0)\, dy + \lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \int_{R^n} D^j \Gamma_x(x-y, t-s) L_x u(y, s)\, dy\, ds$$

where

$$D^j \Gamma_x(x-y, t-s) \equiv D_w^j \Gamma_x(w, t-s)\big|_{w=x-y}$$

and where the limit exists for all x and t. Now we write $L_x u = Lu + (L_x - L)u$ and so for $1 \le j \le m$

$$\begin{aligned} t^{j/m-\frac{1}{2}} D^j u(x, t) = {} & t^{j/m-\frac{1}{2}} \int_{R^n} D^j \Gamma_x(x-y, t) u(y, 0)\, dy \,+ \\ & \lim_{\varepsilon \to 0^+} t^{j/m-\frac{1}{2}} \int_0^{t-\varepsilon} \int_{R^n} D^j \Gamma_x(x-y, t-s) Lu(y, s)\, dy\, ds \,+ \\ & \sum_{|\alpha|=m} \lim_{\varepsilon \to 0} t^{j/m-\frac{1}{2}} \int_0^{t-\varepsilon} \int_{R^n} D^j \Gamma_x(x-y, t-s) \big(\mathcal{A}_\alpha(x) - \mathcal{A}_\alpha(y)\big) D^\alpha u(y, s)\, dy\, ds. \end{aligned}$$

In section 6 it is proved that the limit of the last two terms exists in $L^p(L^2(S_T))$.

We now define for $0 \le j \le m$

$$D^j I_0(g)(x,t) = \int_{R^n} D^j \Gamma_x(x-y,t) g(y)\,dy$$

$$D^j V(f)(x,t) = \lim_{\varepsilon\to 0} \int_0^{t-\varepsilon}\int_{R^n} D^j \Gamma_x(x-y,t-s) f(y,s)\,dy\,ds\,,$$

and in terms of these operators we have

$$u = I_0(u(\cdot,0)) + V(Lu) + \sum_{|\alpha|=m} (\mathcal{A}_\alpha V - V\mathcal{A}_\alpha)(D^\alpha u) \tag{2.2o}$$

and for $1 \le j \le m$

$$\begin{aligned} t^{j/m-\frac12} D^j u = t^{j/m-\frac12} D^j I_0(u(\cdot,0)) + t^{j/m-\frac12} D^j V(Lu) + \\ + t^{j/m-\frac12} \sum_{|\alpha|=m} (\mathcal{A}_\alpha D^j V - D^j V \mathcal{A}_\alpha)(D^\alpha u)\,. \end{aligned} \tag{2.2j}$$

Theorems 6, 7, and 8 show that

$$\begin{aligned} \|\sup_{0<t<T} u(x,t)\|_{L^p(R^n)} + \sum_{j=1}^m \|t^{j/m-\frac12} D^j u\|_{L^p(L^2(S_T))} \le \\ C\|u(\cdot,0)\|_{L^p(R^n)} + C\left\|\frac{t^{\frac12}}{\theta(t)} Lu\right\|_{L^p(L^2(S_T))} + C_p C(T) \sum_{j=1}^m \|t^{j/m-\frac12} D^j u\|_{L^p(L^2(S_T))} \end{aligned}$$

where

$$C(T) = \omega(T) + \left(\int_0^T \frac{\omega^2(\tau)}{\tau}\,d\tau\right)^{\frac12}.$$

To complete the proof we note that we can choose T_0 such that $C_p C(T_0) \le \frac12$ and then absorb the final sum on the right side into the left hand side of the above inequality.

3. – Proof of Theorem 2.

For an arbitrary $g \in L^p(R^n)$ choose a sequence $g_\nu \to g$ in L^p such that $g_\nu \in C_0^\infty(R^n)$, and let $f^\nu \in C_0^\infty(R^n \times (0,\infty))$ be such that

$$\frac{t^{\frac12}}{\theta(t)} f^\nu \to \frac{t^{\frac12}}{\theta(t)} f$$

in $L^p(L^2(S_T))$ as ν tends to infinity. By the results of [4] we know there is a solution, u_ν, of the initial value problem

$$Lu_\nu = f^\nu \quad \text{in } S_T \qquad u_\nu(x, 0) = g_\nu(x)$$

lying in $W_q^{m,1}(S_T)$ for every $1 < q < \infty$.

Set $\mathring{W}_{p,2}^{m,1}(S_T) = \{u \in W_{p,2}^{m,1}(S_T) : u(x, 0) = 0\}$. Now L is an isomorphism from $\mathring{W}_{p,2}^{m,1}(S_T)$ onto $L^p(L^2(S_T))$ (see [4]). Hence if $\varphi(t) \in C_0^\infty(0, \infty)$ we have $\|\varphi(t)(u_\nu - u_\mu)\|_{W_{p,2}^{m,1}(S_T)} \leqslant C(\|\varphi(t)(f_\nu - f_\mu)\|_{L^p(L^2(S_T))} + \|\varphi'(t)(u_\nu - u_\mu)\|_{L^p(L^2(S_T))})$. By Theorem 1 the last term is bounded by

$$C \sup |\varphi'| \left[\|g_\nu - g_\mu\|_{L^p(R^n)} + \left\| \frac{t^{\frac{1}{2}}}{\theta(t)} (f_\nu - f_\mu) \right\|_{L^p(L^2(S_T))} \right].$$

We conclude that the sequence $\{u_\nu\}$ is convergent in $W_{p,2}^{m,1}(S_{\delta,T})$, for every $\delta > 0$, to a function u which clearly satisfies $Lu = f$. By selecting a subsequence we may also assume that

$$D^j u_\nu \to D^j u \quad \text{and} \quad D_t u_\nu \to D_t u$$

pointwise almost everywhere in S_T for $0 \leq j \leq m$.

To prove (i) we note by Fatou and Theorem 1 that

$$\begin{aligned}
&\| \sup_{0<t<T} |u(x, t)| \|_{L^p(R^n)} + \sum_{j=1}^{m} \|t^{j/m - \frac{1}{2}} D^j u\|_{L^p((L^2 S_T))} \leqslant \\
&\quad \limsup_{\nu \to \infty} \| \sup_{0<t<T} |u_\nu(x, t)| \|_{L^p(R^n)} + \sum_{j=1}^{m} \limsup_{\nu \to \infty} \|t^{j/m - \frac{1}{2}} D^j u_\nu\|_{L^p(L^2(S_T))} \\
&\quad \leqslant C \limsup_{\nu \to \infty} \|g_\nu\|_{L^p(R^n)} + C_T \limsup_{\nu \to \infty} \left\| \frac{t^{\frac{1}{2}}}{\theta(t)} f^\nu \right\|_{L^p(L^2(S_T))} \\
&\qquad\qquad \leqslant C \|g\|_{L^p(R^n)} + C_T \left\| \frac{t^{\frac{1}{2}}}{\theta(t)} f \right\|_{L^p(L^2(S_T))}.
\end{aligned}$$

To show (iii) we first set

$$\Delta_u(x) = \limsup_{t \to 0+} u(x, t) - \liminf_{t \to 0+} u(x, t).$$

Now for a fixed positive number δ we have

$$|\{x \in R^n : \Delta_u(x) > \delta\}| = |\{x \in R^n : \Delta_{u-u_\nu}(x) > \delta\}|$$

since $\Delta_{u_\nu}(x) = 0$ for all x. Hence

$$|\{x\colon \Delta_u(x) > \delta\}| \leqslant \frac{2^p}{\delta^p} \left\| \sup_{0<t<T} |u(x,t) - u_\nu(x,t)| \right\|^p_{L^p(R^n)}$$

which, by Theorem 1, tends to zero as $\nu \to \infty$: We conclude that $\lim_{t\to 0^+} u(x,t)$ exists pointwise almost everywhere. But also

$$\|u(x,t) - g(x)\|_{L^p(R^n)} \leqslant$$
$$\leqslant \|u(x,t) - u_\nu(x,t)\|_{L^p(R^n)} + \|g_\nu - g\|_{L^p(R^n)} + \|u_\nu(x,t) - g_\nu(x)\|_{L^p(R^n)} .$$

The first two terms on the right side are small for all ν large enough uniformly in $t \in (0, T)$, while for ν fixed the final term tends to zero as t tends to $0+$. In conclusion we have shown that

$$\lim_{t\to 0+} \|u(x,t) - g(x)\|_{L^p(R^n)} = 0$$

and, therefore, also

$$\lim_{t\to 0+} u(x,t) = g(x) \quad \text{pointwise almost everywhere} .$$

Finally, if $f \in L^q_{\text{loc}}$ we may choose the approximating functions f^ν above so that they converge to f in L^q_{loc}. By Theorem 1 we know that $u_\nu \in W^{m,1}_{q_0}(S_T)$ locally, uniformly in ν, with $q_0 = \min(2, p)$. This implies that $D^j u_\nu$ converges in $L^{q_1}_{\text{loc}}(S_T)$ for

$$\frac{1}{q_1} > \frac{1}{q_0} - \frac{1}{n+m} \qquad \text{and} \qquad 0 \le j \le m-1 .$$

Also, since L is an isomorphism from

$$\mathring{W}^{m,1}_{q_1}(S_T) \equiv \{u \in W^{m,1}_{q_1}(S_T)\colon u(x,0) = 0\} \qquad \text{onto } L^{q_1}(S_T) ,$$

we have for $\varphi \in C_0^\infty\ R^n \times (0, T))$

$$\|\varphi(u_\nu - u_\mu)\|_{W^{m,1}_{q_1}(S_T)} \leqslant K \|L(\varphi u_\nu - \varphi u_\mu)\|_{L^{q_1}(S_T)}$$
$$\leqslant K \sum_{j=0}^{m-1} \|D^j(u_\nu - u_\mu)\|_{L^{q_1}(\text{supp}\,\varphi)} + K\|\varphi(f^\nu - f^\mu)\|_{L^{q_1}(S_T)} .$$

Hence the sequence u_ν converges in $W^{m,1}_{q_1}(S_T)$ locally if $q_1 \le q$.

Now the above process can be repeated, with q_1 instead of q_0, so that in the j-th step we obtain convergence in $W^{m,1}_{q_j}(S_T)$ locally with any $q_j < \infty$ such that $1/q_j > 1/q_0 - j/(n+m)$, $q_j \leq q$.

This proves the assertion.

4. - Proof of Theorem 3.

We start by proving the implicit proposition that (i) and (ii) imply that $u(\cdot, \tau) \in L^p(R^n)$ for every $\tau \in (0, T]$. In fact, let $\varphi \in C_0^\infty(R)$ be one in (τ, T) and zero in $(0, \tau/2)$. Then for almost every x

$$|u(x,\tau)| = \Big|\int_{\tau/2}^{\tau} D_t u(x,t)\varphi(t)\,dt + \int_{\tau/2}^{T} u(x,t)\varphi'(t)\,dt\Big| \\ \leqslant K_\tau\{\|D_t u(x,\cdot)\|_{L^2(\tau/2,\tau)} + \|u(x,\cdot)\|_{L^2(\tau/2,\tau)}\}$$

and the observation is completed by taking L^p norms over R^n.

Now let u satisfy $Lu = 0$ with a weak L^p limit equal to zero at $t = 0$. We set

$$L^r = \sum_{|\alpha|=m} \mathcal{A}^r_\alpha(x) D^\alpha_x - D_t \qquad \text{where } \mathcal{A}^r_\alpha(x) = \mathcal{A}_\alpha(x, r)\,.$$

Since we have assumed Dini $(\frac{1}{2})$ continuity of the coefficients in the whole slab S_T we may apply the representation formula (2.2j) in $S_{r,T}$ and with respect to L^r:

$$(4.1)\qquad u(x,t) = I_{0,r}\big(u(\cdot, r)\big)(x,t) + V_r[(L^r - L)u](x,t) \\ + \sum_{|\alpha|=m} (\mathcal{A}^r_\alpha V_r - V_r \mathcal{A}^r_\alpha)(D^\alpha u)(x,t)$$

where the kernel of $I_{0,r}$ and V_r is $\Gamma_x(\cdot, t-r)$ and $\Gamma_x(\cdot, \cdot - r)$ respectively. As in section 2 we find from Theorems 6, 7, and 8, and from Theorem 1,

$$\|V_r[(L^r - L)u](\cdot, t) + \sum_{|\alpha|=m} (\mathcal{A}^r_\alpha V_r - V_r \mathcal{A}^r_\alpha)(D^\alpha u)(\cdot, t)\|_{L^p(R^n)} \leqslant \\ \leqslant C_{t-r}\|u(\cdot, r)\|_{L^p(R^n)} + \sum_{j=1}^{m} \|(t-r)^{j/m - \frac{1}{2}} D^j u\|_{L^p(L^2(S_{r,T}))} \leqslant C_{t-r}\|u(\cdot, r)\|_{L^p(R)},$$

where $C_{t-r} \to 0$ as $(t-r) \to 0$.

If $\varphi \in C_0^\infty(R^n)$ we multiply (4.1) by φ and integrate to find

$$\Big|\int_{R^n} u(x,t)\varphi(x)\,dx\Big| \leqslant \Big|\int_{R^n} u(y,r)\int_{R^n} \Gamma_x(x-y,t-r)\varphi(x)\,dx\,dy\Big| + C_{t-r}\|\varphi\|_{L^q(R^n)}\|u(\cdot,r)\|_{L^p(R^n)},$$

$1/q + 1/p = 1$. Since the weak L^p limit of $u(x,r)$ as $r \to 0+$ is zero we find that the integral on the right converges to zero when r tends to zero. Hence we have $\|u(\cdot,t)\|_{L^p(R^n)} \leqslant C_t$ where $\lim\limits_{t\to 0+} C_t = 0$, i.e. $\lim\limits_{t\to 0+} u(x,t) = 0$ where the limit is taken in $L^p(R^n)$. Hence, by Theorem 1, we conclude that u vanishes identically.

5. – Proofs of Theorems 4 and 5.

In view of the monotonicity property of θ it is clearly sufficient to show the inequality (iii) for τ small, dependent on n, m, and μ only.

The first step is to see that (i) follows from the inequality

$$\begin{aligned}\sup_{\varrho<t<\tau} |u(x,t)| &\leqslant |u(x,\tau)| + \int_\varrho^\tau |D_t u(x,t)|\,dt \\ &\leqslant |u(x,\tau)| + \int_\varrho^\tau \Big|\sum_{|\alpha|\leqslant m} A_\alpha D^\alpha u(x,t)\Big|\,dt + \int_\varrho^\tau |f(x,t)|\,dt \\ &\leqslant |u(x,\tau)| + \mu\tau \sup_{\varrho<t<\tau} |u(x,t)| + K_\varrho \sum_{j=1}^m \|t^{j/m-\frac12} D^j u(x,t)\|_{L^2(\varrho,\tau)} \\ &\qquad + K\Big\|\frac{t^{\frac12}}{\theta(t)} f(x,t)\Big\|_{L^2(\varrho,\tau)}\end{aligned}$$

which for $\mu\tau \leq \frac12$ gives

$$\begin{aligned}(5.1)\qquad \sup_{\varrho<t<\tau} \|u(\cdot,t)\|_{L^p(R^n)} &\leqslant \Big\|\sup_{\varrho<t<\tau} |u(\cdot,t)|\Big\|_{L^p(R^n)} \\ &\leqslant K\|u(\cdot,\tau)\|_{L^p(R^n)} + K_\varrho \sum_{j=1}^m \|t^{j/m-\frac12} D^j u\|_{L^p(L^2(S_{\varrho,\tau}))} + K\Big\|\frac{t^{\frac12}}{\theta(t)} f\Big\|_{L^p(L^2(S_{\varrho,\tau}))}.\end{aligned}$$

To prove (ii) and (iii) it is sufficient to prove that

$$(5.2)\qquad \|u(\cdot,r)\|_{L^p(R^n)} \leqslant C\Big[\|u(\cdot,\tau)\|_{L^p(R^n)} + \sum_{j=1}^m \|t^{j/m-\frac12} D^j u\|_{L^p(L^2(S_{r,\tau}))} + \Big\|\frac{t^{\frac12}}{\theta(t)} f\Big\|_{L^p(L^2(S_{r,\tau}))}\Big],$$

for all r less than some $\varrho > 0$ which depends on the parameters in the theorem. In fact, we may then choose a sequence $u(\cdot, t_k)$, $t_k \to 0$, which converges weakly in $L^p(R^n)$ to some function $u(\cdot, 0)$ satisfying (iii). Borrowing from the proof and notation of Theorem 3 we see that

$$\int_{R^n} u(x, t)\varphi(x)\,dx = \int_{R^n} [u(x, t) - I_{0,t_k}(u(\cdot, t_k))(x, t)]\,\varphi(x)\,dx + \int_{R^n} I_{0,t_k}(u(\cdot, t_k))(x, t)\,\varphi(x)\,dx$$

and $\|u(\cdot, t) - I_{0,t_k}(u(\cdot, t_k))(\cdot, t)\|_{L^p} \leqslant C_{t-t_k}$ where $C_t \to 0$ as $t \to 0+$. As $t_k \to 0$ the final integral on the right side converges to

$$\int_{R^n} I_0(u(\cdot, 0))(x, t)\varphi(x)\,dx\,.$$

We conclude that u is the solution, in the sense of Theorem 3, of the initial value problem with data $u(x, 0)$, and then (ii) follows from Theorem 2.

As in the proof of Theorem 1 it is no restriction to assume that the operator has the form

$$\sum_{|\alpha|=m} \mathcal{A}_\alpha(x)D^\alpha - D_t\,.$$

In fact, if we set $L^0 = \sum_{|\alpha|=m} \mathcal{A}_\alpha(x, 0)D^\alpha - D_t$ and assume that (5.2) holds for L^0, we find that (5.2) holds in the general case with $(L^0 - L)u + f$ substituted for f. All the terms of $(L^0 - L)u$ except $\mathcal{A}_0 u$ are trivially swallowed by the remaining part of the right hand side of (5.2). To estimate

$$\|t^{\frac{1}{2}-1/m}\mathcal{A}_0 u\|_{L^p(L^2(S_{r,\tau}))}$$

we write, with K depending only on n, m, θ and μ

$$|u(x, t)| \leqslant |u(x, \tau)| + K\sum_{j=0}^{m}\int_t^\tau |D^j u(x, s)|\,ds + \int_t^\tau |f(x, s)|\,ds \leqslant$$

$$\leqslant |u(x, \tau)| + K\sum_{j=1}^{m-1}\|s^{j/m-\frac{1}{2}}D^j u(x, s)\|_{L^2(t,\tau)}\tau^{j/m} + K\left\|\frac{s^{\frac{1}{2}}}{\theta(s)}f(x, s)\right\|_{L^2(t,\tau)} +$$

$$+ K\tau^{1/m}\|s^{\frac{1}{2}-1/m}u(x, s)\|_{L^2(t,\tau)} + K\left(\log\frac{\tau}{t}\right)^{\frac{1}{2}}\|s^{\frac{1}{2}}D^m u(x, s)\|_{L^2(t,\tau)}\,.$$

This last estimate yields the inequality

$$\|t^{\frac{1}{2}-1/m}u(x,t)\|_{L^2(r,\tau)}\leqslant K\Big[\tau^{1-1/m}|u(x,\tau)|+\sum_{j=1}^{m-1}\tau^{1+(j-1)/m}\|s^{j/m-\frac{1}{2}}D^j u(x,s)\|_{L^2(r,\tau)}+$$
$$+\tau^{1-1/m}\Big\|\frac{s^{\frac{1}{2}}}{\theta(s)}f(x,s)\Big\|_{L^2(r,\tau)}+\tau\|s^{\frac{1}{2}-1/m}u(x,s)\|_{L^2(r,\tau)}+\tau^{\frac{1}{2}}\|s^{\frac{1}{2}}D^m u(x,s)\|_{L^2(r,\tau)}\Big].$$

Choosing τ so that $K\tau=\frac{1}{2}$ we see that the desired estimate follows.

Let $\varphi\in C_0^\infty(R^n)$ be arbitrary, and denote by φ_r the solution in $S_{r,\tau}$ of the initial value problem

$$L\varphi_r=0\,,\qquad \varphi_r(\cdot,r)=\varphi\,.$$

By Theorem 2 we know that

$$\tag{5.3} \Big\|\sup_{r<t<\tau}|\varphi_r(\cdot,t)|\Big\|_{L^q(R^n)}+\sum_{j=1}^{m}\|(t-r)^{j/m-\frac{1}{2}}D^j\varphi_r\|_{L^q(L^2(S_{r,\tau}))}\leqslant K\|\varphi\|_{L^q(R^n)}\,.$$

Also we have

$$\tag{5.4} \begin{aligned}\int_{R^n}u(x,r)\varphi(x)\,dx&=\int_{R^n}u(x,\tau)\varphi_r(x,\tau)\,dx-\iint_{S_{r,\tau}}D_t(u\varphi_r)\,dx\,dt=\\ &=\int_{R^n}u(x,\tau)\varphi_r(x,\tau)\,dx-\sum_{|\alpha|=m}\iint_{S_{r,\tau}}(\mathcal{A}_\alpha D^\alpha u)\varphi_r\,dx\,dt-\sum_{|\alpha|=m}\iint_{S_{r,\tau}}u\mathcal{A}_\alpha D^\alpha\varphi_r\,dx\,dt\\ &\qquad+\iint_{S_{r,\tau}}f\varphi_r\,dx\,dt\,.\end{aligned}$$

Now put $\mathcal{A}_\alpha(x)=\mathcal{A}_\alpha^\sigma(x)+[\mathcal{A}_\alpha(x)-\mathcal{A}_\alpha^\sigma(x)]$ with $\sigma=(t-r)^{1/m}$, where $\mathcal{A}_\alpha^\sigma$ is a suitable regularization of $\mathcal{A}_\alpha$, see section 6. The third (and worst) integral on the right hand side of (5.4) is then a sum of terms of the type

$$\tag{5.5} \iint_{S_{r,\tau}}\mathcal{A}^\sigma uD^m\varphi_r\,dx\,dt+\iint_{S_{r,\tau}}[\mathcal{A}-\mathcal{A}^\sigma]uD^m\varphi_r\,dx\,dt\,.$$

The first integral in (5.5) is treated to an $(m-1)$-fold integration by parts and yields a sum of terms of the form

$$\iint_{S_{r,\tau}}(D^{m-1-j}\mathcal{A}^\sigma)D^j uD\varphi_r\,dx\,dt\,,\qquad j=0,1,\ldots,m-1\,.$$

For $0 < j < m-1$ we divide the area of integration in two, $S_{r,2r}$ and $S_{2r,\tau}$. In the former we get, using (6.1),

$$\int_{R^n}\int_r^{2r} \frac{\omega\big((t-r)^{1/m}\big)}{(t-r)^{m-1-j/m}}\,|D^j u|\,|D\varphi_r|\,dt\,dx \leqslant \omega(r^{1/m})\big\|(t-r)^{j/m-\frac12}D^j u\big\|_{L^p(L^2(S_{r,\tau}))}\,\big\|(t-r)^{1/m-\frac12}D\varphi_r\big\|_{L^q(L^2(S_{r,\tau}))},$$

$1/p + 1/q = 1$. By (5.3) and Theorem 1 applied to u in $S_{r,\tau}$ this is less than or equal to

$$K\omega(r^{1/m})\left[\|u(\cdot,r)\|_{L^p(R^n)} + \left\|\frac{t^{\frac12}}{\theta(t)}f\right\|_{L^p(L^2(S_{r,\tau}))}\right]\|\varphi\|_{L^q(R^n)}.$$

In the latter case we get, again using (5.3), the estimate

$$K\big\|t^{j/m-\frac12}D^j u\big\|_{L^p(L^2(S_{r,\tau}))}\|\varphi\|_{L^q(R^n)}.$$

For $j = 0$ we divide the area of integration into $S_{r,\varrho}$ and $S_{\varrho,\tau}$ with ϱ to be chosen, and estimate the L^p norms of $\sup\limits_{r<t<\varrho}|u(\cdot,t)|$ and $\sup\limits_{\varrho<t<\tau}|u(\cdot,t)|$ by Theorem 1 and (5.1) respectively, and after applying Hölder's inequality and (5.3) we see that these integrals are estimated by

$$\int_{R^n}\sup_{r<t<\varrho}|u(x,t)|\left|\int_r^{\varrho}\frac{\omega\big((t-r)^{1/m}\big)}{(t-r)^{\frac12}}(t-r)^{1/m-\frac12}|D\varphi_r|\,dt\,dx\right. \leqslant K\left(\int_0^{\varrho}\frac{\omega^2(S^{1/m})}{s}\,ds\right)^{\frac12}\left\{\|u(\cdot,r)\|_{L^p(R^n)} + \left\|\frac{t^{\frac12}}{\theta(t)}f\right\|_{L^p(L^2(S_{r,\tau}))}\right\}\|\varphi\|_{L^q(R^n)}$$

and

$$K_\varrho\left\{\|u(\cdot,\tau)\|_{L^p(R^n)} + \sum_{j=1}^{m}\big\|t^{1/m-\frac12}D^j u\big\|_{L^p(L^2(S_{r,\tau})} + \left\|\frac{t^{\frac12}}{\theta(t)}f\right\|_{L^p(L^2(S_{r,\tau}))}\right\}\|\varphi\|_{L^q(R^n)}$$

respectively.

In the case $j = m-1$, finally, a simple application of the Hölder inequality and (5.3) yields the estimate

$$\big\|t^{\frac12-1/m}D^{m-1}u\big\|_{L^p(L^2(S_{r,\tau}))}\|\varphi\|_{L^p(R^n)}.$$

Returning now to (5.5) we divide the second integral into two by considering $S_{r,\varrho}$ and $S_{\varrho,\tau}$ separately and proceeding as in the case $j = 0$ above we find that

$$\left|\iint\limits_{S_{r,\tau}}(\mathcal{A}-\mathcal{A}^{\sigma})uD^m\varphi_r\,dt\,dx\right|\leqslant\iint\limits_{S_{r,\tau}}\omega((t-r)^{1/m})\,|u|\,|D^m\varphi_r|\,dt\,dx$$

$$\leqslant K\left(\int\limits_0^{\varrho}\frac{\omega^2(s^{1/m})}{s}\,ds\right)^{\frac{1}{2}}\left[\|u(\cdot,r)\|_{L^p(R^n)}+\left\|\frac{t^{\frac{1}{2}}}{\theta(t)}f\right\|_{L^p(L^2(S_{r,\tau}))}\right]\|\varphi\|_{L^q(R^n)}$$

$$+K_{\varrho}\left[\|u(\cdot,\tau)\|_{L^p(R^n)}+\sum_{j=1}^{m}\|t^{j/m-\frac{1}{2}}D^ju\|_{L^p(L^2(S_{r,\tau}))}+\left\|\frac{t^{\frac{1}{2}}}{\theta(t)}f\right\|_{L^p(L^2(S_{r,\tau}))}\right]\|\varphi\|_{L^q(R^n)}.$$

Collecting all our estimates we have for (5.5) the majorization

$$(5.6)\qquad K\left[\omega(\varrho^{1/m})+\left(\int\limits_0^{\varrho}\frac{\omega^2(s^{1/m})}{s}\,ds\right)^{\frac{1}{2}}\right]\|u(\cdot,r)\|_{L^p(R^n)}\|\varphi\|_{L^q(R^n)}+$$

$$K_{\varrho}\left[\|u(\cdot,\tau)\|_{L^p(R^n)}+\sum_{j=1}^{m}\|t^{j/m-\frac{1}{2}}D^ju\|_{L^p(L^2(S_{r,\tau}))}+\left\|\frac{t^{\frac{1}{2}}}{\theta(t)}f\right\|_{L^p(L^2(S_{r,\tau}))}\right]\|\varphi\|_{L^q(R^n)}.$$

Going back to (5.4) we realize that there are still three terms to estimate. The second term on the right hand side is treated in a manner similar to that of the third one; in fact, our task is simpler here in that, after having set $\mathcal{A}_{\alpha}=\mathcal{A}_{\alpha}^{\sigma}+[\mathcal{A}_{\alpha}-\mathcal{A}_{\alpha}^{\sigma}]$, we need integrate by parts only once, and it is not necessary to divide the area of integration. We end up with a majorization of these terms by (5.6).

The first and last terms of (5.4) yield, via Hölder's inequality and (5.3), an estimate which again can be subsumed under (5.6).

All in all, the left hand side of (5.4) is less than or equal to (5.6) and the following inequality holds for $r<\varrho$

$$(5.7)\qquad \|u(\cdot,r)\|_{L^p(R^n)}\leqslant K\left\{\omega(\varrho^{1/m})+\left(\int\limits_0^{\varrho}\frac{\omega^2(s^{1/m})}{s}\,ds\right)^{\frac{1}{2}}\right\}\|u(\cdot,r)\|_{L^p(R^n)}$$

$$+K_{\varrho}\left\{\|u(\cdot,\tau)\|_{L^p(R^n)}+\sum_{j=1}^{m}\|t^{j/m-\frac{1}{2}}D^ju\|_{L^p(L^2(S_{r,\tau}))}+\left\|\frac{t^{\frac{1}{2}}}{\theta(t)}f\right\|_{L^p(L^2(S_{r,\tau}))}\right\}$$

where K depends only on the parameters of the theorem and K_{ϱ} in addition on ϱ. It is now easy to see that (5.2) follows.

The proof of Theorem 4 is now complete.

Proof of Theorem 5. By Theorem 1 and the assumption on $\|u(\cdot,t)\|_{L^q(R^n)}$ we find that

$$\limsup_{\varrho\to 0}\sum_{j=1}^{m}\|(t-\varrho)^{j/m-\frac{1}{2}}D^ju\|_{L^p(L^2(S_{r,\tau}))}<\infty$$

and then it follows from Fatou's lemma that

$$\sum_{j=1}^{m} \|t^{j/m-\frac{1}{2}} D^j u\|_{L^p(L^2(S_T))} < \infty .$$

Hence the assumptions of Theorem 4 are fulfilled, and from this the statement of the theorem follows immediately.

6. – Estimates for the potentials.

In this section $\mathcal{A}$ will always denote a Dini $(\frac{1}{2})$ continuous function on R^n i.e. a function whose modulus of continuity satisfies $\int\limits_0 \big(\omega^2(\tau)/\tau\big)\, d\tau < \infty$. By $\mathcal{A}_\sigma$ we denote a regularization of $\mathcal{A}$,

$$\mathcal{A}_\sigma(x) = \sigma^{-n} \int\limits_{R^n} \eta\big((x-y)\sigma^{-1}\big)\, \mathcal{A}(y)\, dy$$

where η is an arbitrary but fixed non-negative C^∞ function with support in the open unit ball and integral one. It will be convenient to set

$$\mathcal{A}(x) - \mathcal{A}(y) = [\mathcal{A}(x) - \mathcal{A}_\sigma(x) + \mathcal{A}_\sigma(x) - \mathcal{A}_\sigma(y)] + [\mathcal{A}_\sigma(y) - \mathcal{A}(y)] \equiv$$
$$\equiv \psi_\sigma(x, y) + \varphi_\sigma(y) .$$

The following inequalities are immediate:

$$(6.1) \qquad \begin{aligned} &|\mathcal{A}(x) - \mathcal{A}(y)| \leqslant \omega(\sigma)\left(1 + \frac{|x-y|}{\sigma}\right)\\ &|D_y^k \mathcal{A}_\sigma(y)| \leqslant C_k \frac{\omega(\sigma)}{\sigma^k}, \qquad k > 0,\\ &|\psi_\sigma(x, y)| \leqslant \omega(\sigma)\left[1 + \frac{|x-y|}{\sigma}\right]\\ &|\varphi_\sigma(y)| \leqslant \omega(\sigma)\\ &|D_y \psi_\sigma(x, y)| \leqslant \frac{\omega(\sigma)}{\sigma}. \end{aligned}$$

We will also have occasion to use the Hardy-Littlewood maximal function,

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |f(y)|\, dy$$

which satisfies

$$\|Mf\|_{L^p(R^n)} \leqslant C\|f\|_{L^p(R^n)}, \qquad 1 < p \leqslant \infty .$$

THEOREM 6. *For* $1 < p < \infty$ *the potential*

$$I_0(g)(x, t) = \int_{R^n} \Gamma_x(x - y, t) g(y) dy$$

satisfies

$$\| \sup_{t<0} |I_0(g)(x, t)| \|_{L^p(R^n)} + \sum_{j=1}^{m} \|t^{j/m - \frac{1}{2}} D^j I_0(g)\|_{L^p(L^2(S_\infty))} \leqslant C\|g\|_{L^p(R^n)} .$$

Here the constant C_p *can be chosen independently of the modulus of continuity of the coefficients* $\mathcal{A}_\alpha$.

PROOF. The proof of the supremum estimate is of course standard (see e.g. [13, p. 62]) since

$$|I_0(g)(x, t)| \leqslant Kt^{-n/m} \int_R h\left(\frac{x - y}{t^{1/m}}\right) |g(y)| dy$$

where

$$h(x) = \frac{1}{[1 + |x|^2]^{(n+2)/2}} .$$

The estimates for $D^j I_0$ are proved in Theorem 9.

THEOREM 7. *For* $1 < p < \infty$

$$\Big\| \sup_{0<t<T} |Vf(x, t)| \Big\|_{L^p(R^n)} + \sum_{j=1}^{m} \|t^{j/m - \frac{1}{2}} D^j Vf\|_{L^p(L^2(S_T))}$$
$$\leqslant C\left(\theta^2(T) + \int_0^T \frac{\theta^2(s)}{s} ds\right)^{\frac{1}{2}} \left\| \frac{t^{\frac{1}{2}}}{\theta(t)} f \right\|_{L^p(L^2(S_T))},$$

where $C = C(p, n, m, \lambda, \mu)$.

PROOF. If we set $\tilde{f}(y, s) = s^{\frac{1}{2}} \theta^{-1}(s) f(y, s)$ we realize for the sum it is sufficient to show that for $1 \leqslant j \leqslant m$ the limit

$$K_j(\tilde{f})(x, t) = \lim_{\varepsilon \to 0} t^{j/m - \frac{1}{2}} \int_0^{t-\varepsilon} \int_{R^n} D^j \Gamma_x(x - y, t - s) \frac{\theta(s)}{s^{\frac{1}{2}}} \tilde{f}(y, s) dy ds$$

exists in $L^p(L^2(S_T))$ and that the operator K_j maps $L^p(L^2(S_T))$ continuously into itself. This is done in Theorem 10.

It remains to prove the supremum estimate. Note that

$$Vf(x,t) = -\int_0^t\int_{R^n} \Gamma_x(x-y,t-s)\left(\frac{d}{ds}\int_s^t f(y,r)\,dr\right)dy\,ds$$
$$=\int_{R^n} \Gamma_x(x-y,t)\left(\int_0^t f(y,r)\,dr\right)dy + \int_0^t\int_{R^n}\frac{d}{dt}\Gamma_x(x-y,t-s)\left(\int_s^t f(y,r)\,dr\right)dy\,ds\,.$$

Hence, if once again we set $h(x) = \left(1/[1+|x|^2]^{(n+2/2)}\right)$,

$$|Vf(x,t)| \leqslant K\int_{R^n} t^{-n/m} h\left(\frac{x-y}{t^{1/m}}\right)\left|\int_0^t f(y,r)\,dr\right|dy$$
$$+ K\theta(T)\int_0^t (t-s)^{-1}\left(\log\frac{t}{s}\right)^{\frac12}\int_{R^n}(t-s)^{-n/m}h\left(\frac{x-y}{(t-s)^{n/m}}\right)\left(\int_0^t \frac{r}{\theta^2(r)}\,|f(y,r)|^2dr\right)^{\frac12}dy\,ds$$
$$\leqslant K\left(\int_0^T \frac{\theta^2(r)}{r}\,dr\right)^{\frac12} M\left[\left(\int_0^T \frac{r}{\theta^2(r)}\,|f(\cdot,r)|^2dr\right)^{\frac12}\right](x)$$
$$+ K\theta(T)\,M\left[\left(\int_0^T \frac{r}{\theta^2(r)}\,|f(\cdot,r)|^2dr\right)^{\frac12}\right](x)\,.$$

The desired estimate now follows from the Hardy-Littlewood maximal theorem.

THEOREM 8. *For* $1<p<\infty$

$$\Big\|\sup_{0<t<T}|(V(\mathcal{A}D^m u) - \mathcal{A}V(D^m u))(x,t)|\Big\|_{L^p(R^n)} +$$
$$+\sum_{j=1}^m \left\|t^{j/m-\frac12}[D^jV(\mathcal{A}D^m u) - \mathcal{A}\,D^jV(D^m u)]\right\| \leqslant$$
$$\leqslant c\left[\omega(T^{1/m}) + \left(\int_0^{T^{1/m}} \frac{\omega^2(s)}{s}\,ds\right)^{\frac12}\right]\sum_{j=1}^m \left\|t^{j/m-\frac12}D^j u\right\|_{L^p(L^2(S_T))}$$

where $C = C(n,m,p,\lambda,\mu)$.

PROOF. We start by proving that

$$(6.2)\qquad \Big\|\sup_{0<t<T}|V(\mathcal{A}D^m u) - \mathcal{A}V(D^m u)|(x,t)\Big\|_{L^p(R^n)} \leqslant$$
$$\leq C\omega(T^{1/m})\sum_{j=1}^m \left\|t^{j/m-\frac12}D^j u\right\|_{L^p(L^2(S_T))}\,.$$

By an integration by parts

$$-\int_0^t\int_{R^n}\Gamma_x(x-y,t-s)[\mathcal{A}(x)-\mathcal{A}(y)]\left(\frac{d}{ds}\int_s^t D^m u(y,r)\,dr\right)dy\,ds=$$
$$=\int_0^t\int_{R^n}\Gamma_x(x-y,t)[\mathcal{A}(x)-\mathcal{A}(y)]D^m u(y,r)\,dy\,dr+$$
$$+\int_0^t\int_{R^n}D_t\Gamma_x(x-y,t-s)[\mathcal{A}(x)-\mathcal{A}(y)]\left(\int_s^t D^m u(y,r)\,dr\right)dy\,ds\,.$$

In the first term we use the decomposition

$$\mathcal{A}(x)-\mathcal{A}(y)=\psi_{t^{1/m}}+\varphi_{t^{1/m}}$$

and after an integration by parts in the y-variable

$$\int_0^t\int_{R^n}\Gamma_x(x-y,t)\,\psi_{t^{1/m}}D^m u\,dr\,dy=\int_0^t\int_{R^n}D\Gamma_x(x-y,t)\,\psi_{t^{1/m}}D^{m-1}u\,dr\,dy$$
$$-\int_0^t\int_{R^n}\Gamma_x(x-y,t)\,D\psi_{t^{1/m}}D^{m-1}u\,dr\,dy\,.$$

By (6.1) we find, using the technique of the proof of Theorem 6, that the final two integrals are majorized by

$$K\omega(t^{1/m})\int_{R^n}t^{-n/m}h\left(\frac{x-y}{t^{1/m}}\right)\left(\int_0^T r^{1-2/m}|D^{m-1}u(y,r)|^2dr\right)^{\frac12}dy\leqslant$$
$$\leqslant K\omega(t^{1/m})\,M\left[\left(\int_0^T r^{1-2/m}|D^{m-1}u(\cdot,r)|^2dr\right)^{\frac12}\right](x)\,.$$

For the second term we use (6.1) directly with $\sigma=(t-s)^{1/m}$ and obtain the estimate

$$K\omega(t^{1/m})\int_0^t\frac{(\log t/s)^{\frac12}}{(t-s)}\int_{R^n}\frac{1}{(t-s)^{n/m}}\left(1+\frac{|x-y|}{(t-s)^{1/m}}\right)h\left(\frac{x-y}{(t-s)^{1/m}}\right)$$
$$\left(\int_0^T r|D^m u(y,r)|^2dr\right)^{\frac12}dy\leqslant K\omega(t^{1/m})\,M\left[\left(\int_0^T r|D^m u(\cdot,r)|^2dr\right)^{\frac12}\right](x)\,.$$

Together with the Hardy-Littlewood maximal theorem this proves (6.2). Next we prove for $1 \leq j \leq m$

$$(6.3) \qquad \left\| t^{j/m-\frac{1}{2}}[D^m V(\mathcal{A}D^j u) - \mathcal{A}D^m V(D^j u)] \right\|_{L^p(L^2(S_T))} \leq \leq C\left[\omega(T^{1/m}) + \left(\int_0^{T^{1/m}} \frac{\omega^2(s)}{s} ds\right)^{\frac{1}{2}}\right] \left\| t^{j/m-\frac{1}{2}} D^j u \right\|_{L^p(L^2(S_T))} .$$

By writing $t^{j/m-\frac{1}{2}} = s^{j/m-\frac{1}{2}} + s^{-1}(st^{j/m-\frac{1}{2}} - s^{j/m+\frac{1}{2}})$ we divide the integral to be estimated into two parts the first of which is seen to satisfy

$$\|(\mathcal{A}D^m V - D^m V\mathcal{A})(s^{j/m-\frac{1}{2}} D^j u)\|_{L^p(L^2(S_T))} \leq K\omega(T^{1/m}) \|s^{j/m-\frac{1}{2}} D^j u\|_{L^p(L^2(S_T))}$$

after we use the decomposition $\mathcal{A}(x) - \mathcal{A}(y) = \psi_\sigma + \varphi_\sigma$ with $\sigma = T^{1/m}$ and the known fact that the operator $f \to D^m Vf$ is continuous from $L^p(L^2(S_T))$ into itself (see [9]).

The second term is handled by an integration by parts in the s-direction, giving

$$\int_0^t \int_{R^n} D^m \Gamma_x(x-y, t-s)\big(\mathcal{A}(x) - \mathcal{A}(y)\big)(st^{j/m-\frac{1}{2}} - s^{j/m+\frac{1}{2}})\left(-\frac{d}{ds}\int_s^t \frac{1}{r} D^j u(y, r)\, dr\right) ds dy$$
$$= \int_0^t \int_{R^n} D_j[D^m \Gamma_x(x-y, t-s)(st^{j/m-\frac{1}{2}} - s^{j/m+\frac{1}{2}})\big(\mathcal{A}(x) - \mathcal{A}(y)\big) \cdot \cdot \left(\int_s^t r^{-1} D^j u(y, r)\, dr\right) ds\, dy$$

and using (6.1) with $\sigma = (t-s)^{1/m}$ we find the pointwise estimate

$$K\omega(t^{1/m}) \int_0^t \left[\frac{(t^{j/m-\frac{1}{2}} - s^{j/m-\frac{1}{2}})(s^{-2j/m} - t^{-2j/m})s}{(t-s)^2} + \frac{(t^{j/m-\frac{1}{2}} + s^{j/m-\frac{1}{2}})(s^{-2j/m} - t^{2-j/m})}{(t-s)}\right] \cdot$$
$$\cdot \left[\int_{R^n} \frac{1}{(t-s)^{n/m}}\left(1 + \frac{|x-y|}{(t-s)^{1/m}}\right) h\left(\frac{x-y}{(t-s)^{1/m}}\right)\left(\int_0^T r^{2j-1} |D^j u(y, r)|^2 dr\right)^{\frac{1}{2}} dy\right] ds$$
$$< K \frac{\omega(t^{1/m})}{t^{\frac{1}{2}}} M\left[\left(\int_0^T s^{2j/m-1} |D^j u(\cdot, s)|^2 ds\right)^{\frac{1}{2}}\right](x)$$

and the proof of (6.3) is completed by squaring, integrating over t, and using the Hardy-Littlewood maximal theorem.

An inspection of the proof of (6.3) shows that we may substitute any function $\tilde{\mathcal{A}}(y, s)$ satisfying $|\tilde{\mathcal{A}}(y, s)| \leqslant C\omega(s^{1/m})$ for $(\mathcal{A}(x) - \mathcal{A}(y))$, and we obtain

$$(6.4)\qquad \|t^{j/m-\frac{1}{2}} D^m V(\tilde{\mathcal{A}} D^j u)\|_{L^p(L^2(S_T))} \leqslant C\left(\omega(T^{1/m}) + \left(\int_0^{T^{1/m}} \frac{\omega^2(s)}{s}\, ds\right)^{\frac{1}{2}}\right) \|t^{j/m-\frac{1}{2}} D^j u\|_{L^p(L^2(S_T))}\,.$$

Finally we show that for $1 \leqslant j \leqslant m-1$

$$(6.5)\qquad \|t^{j/m-\frac{1}{2}}[D^j V(\mathcal{A} D^m u) - \mathcal{A} D^j V(D^m u)]\|_{L^p(L^2(S_T))} \leqslant c\left[\omega(T^{1/m}) + \left(\int_0^{T^{1/m}} \frac{\omega^2(s)}{s}\, ds\right)^{\frac{1}{2}}\right] \sum_{j=1}^{m} \|t^{j/m-\frac{1}{2}} D^j u\|_{L^p(L^2(S_T))}\,.$$

Using that

$$\mathcal{A} D^j V(D^m u) = \mathcal{A} D^m V(D^j u)$$

and

$$\mathcal{A}_{s^{1/m}} D^m u = D^{m-j}(\mathcal{A}_{s^{1/m}} D^j u) - \sum_{k=1}^{m-j} \binom{m-j}{k} (D^k \mathcal{A}_{s^{1/m}})(D^{m-k} u)$$

one finds

$$\begin{aligned}\mathcal{A} D^j V(D^m u) - D^j V(\mathcal{A} D^m u) &= (\mathcal{A} D^m V - D^m V \mathcal{A})(D^j u) + \\ &+ D^m V\big((\mathcal{A} - \mathcal{A}_{s^{1/m}}) D^j u\big) + D^j V\big((\mathcal{A}_{s^{1/m}} - \mathcal{A}) D^m u\big) + \\ &+ \sum_{k=1}^{m-j} \binom{m-j}{k} D^j V(D^k \mathcal{A}_{s^{1/m}} D^{m-k} u)\,.\end{aligned}$$

Now (6.3) and (6.4) gave the desired estimate for the first two terms here, while (6.1) and Theorem 7 do the same for the remaining ones.

Theorem 8 is proved.

7. – Singular integral estimates.

We recall the potentials

$$D^j I_0(g)(x, t) = \int_{R^n} D^j \Gamma_x(x - y, t)\, g(y)\, dy\,, \qquad 1 \leq j \leq m\,,$$

and

$$K_j f(x, t) = \lim_{\varepsilon \to 0} t^{j/m-\frac{1}{2}} \int_0^{t-\varepsilon} \int_{R^n} D^j \Gamma_x(x - y, t - s) \frac{\theta(s)}{s^{\frac{1}{2}}} f(y, s)\, dy\, ds\,, \qquad 1 \leq j \leq m\,.$$

Our main tool in showing that these operators are continuous from $L^p(R^n) \to L^p(L^2(S_\infty))$ and $L^p(L^2(S_T)) \to L^p(L^2(S_T))$, respectively, will be the following theorem by Benedek, Calderón, and Panzone [1] on vector valued singular integral operators.

Let H_1 and H_2 be Hilbert spaces and $k\colon R^n \to \mathcal{L}(H_1, H_2)$ (= the space of bounded linear operators $H_1 \to H_2$) be such that

$$T(f)(x) \equiv \int_{R^n} k(x-y)f(y)\,dy$$

is defined for all simple functions f with compact support in R^n and values in H_1. Assume

(i) $\big\| \|Tf(x)\|_{H_2} \big\|_{L^2(R^n)} \leqslant C_1 \big\| \|f(x)\|_{H_1} \big\|_{L^2(R^n)}$, *$C_1$ independent of f,*

(ii) $\int\limits_{|x|>4|y|} \|k(x-y) - k(x)\|_{\mathcal{L}(H_1,H_2)}\,dx \leqslant C_2$, *$C_2$ independent of y.*

Then T can be extended to a continuous mapping from $L^p(R^n, H_1) \to L^p(R^n, H_2)$ for $1 < p < \infty$ and

$$\big\| \|Tf\|_{H_2} \big\|_{L^p(R^n)} \leqslant C_p(C_1 + C_2) \big\| \|f\|_{H_1} \big\|_{L^p(R^n)}\,.$$

It is obvious that (ii) is implied by the condition

(ii)′ $\|\nabla_x k(x)\|_{\mathcal{L}(H_1,H_2)} \leqslant C_2 |x|^{-1-n}$.

THEOREM 9. *For $1 < p < \infty$ and for all $g \in L^p(R^n)$*

$$\sum_{j=1}^{m} \|t^{j/m-\frac{1}{2}} D^j I_0(g)\|_{L^p(L^2(S_\infty))} \leqslant C \|g\|_{L^p(R^n)}$$

where $C = C(n, m, p, \lambda, \mu)$.

PROOF. Since

$$\Gamma_y(x, 1) = \mathcal{F}_\xi\Big(\exp \sum_{|\alpha|=m} \mathcal{A}_\alpha(y)(i\xi)^\alpha\Big)(x)\,,$$

$$D^j\Gamma_y(x, 1) = \int_{R^n} (-i\xi)^j \exp\Big(\sum_{|\alpha|=m} \mathcal{A}_\alpha(y)(i\xi)^\alpha - i\xi\cdot x\Big)\,d\xi\,.$$

Because of the parabolicity there is a constant $c > 0$ such that

$$D^j\Gamma_y(x, 1) = \int_{R^n} h(y, \xi)(-i\xi)^j \exp\big(-c|\xi|^2\big)\exp(-i\xi\cdot x)\,d\xi$$

where $h(y, \xi)$ is rapidly decreasing in ξ, uniformly in y, i.e. for all multi-indices α and β,

$$\sup_{y,\xi} |\xi^\beta D_\xi^\alpha h(y, \xi)| < \infty .$$

Hence

$$D^j \Gamma_y(x, 1) = \int_{R^n} H(y, z)\, \Omega_j(x-z)\, dz$$

where

$$H(y, z) = \mathcal{F}_\xi\big(h(y, \xi)\big)(z) \qquad \text{and} \qquad \Omega_j(z) = \mathcal{F}_\xi\big((-i\xi)^j \exp\left(-c|\xi|^2\right)\big)(z) .$$

Finally we have the representation

$$t^{j/m-\frac{1}{2}} D^j I_0(g)(x, t) = \int_{R^n} H(x, z) \left(t^{-\frac{1}{2}} \int_{R^n} \frac{\Omega_j\big((x-y)/t^{1/m} - z\big)}{t^{n/m}}\, g(y)\, dy \right) dz$$

$$= \int_{R^n} H(x, z)\, I_z(g)(x, t)\, dz$$

where

$$I_z(g)(x, t) = t^{-\frac{1}{2}} \int_{R^n} \frac{\Omega_j\big((x-y)/t^{1/m} - z\big)}{t^{n/m}}\, g(y)\, dy .$$

We will now apply the theory of vector valued singular integral operators mentioned above to show that for fixed z $I_z\colon L^p(R^n) \to L^p\big(L^2(S_\infty)\big)$ continuously for $1 < p < \infty$ with norm bounded by a constant times $\big(1 + |z|^N\big)$.

Let $H = L^2(0, \infty)$ and for $x \in R^n$ fixed define $k(x)\colon C \to H$ (C = complex numbers) by

$$k(x)\alpha = \alpha t^{-\frac{1}{2}-n/m}\, \Omega_j\left(\frac{x}{t^{1/m}} - z\right),$$

and for $f \in L^p(R^n)$ we define the operator

$$Kf(x) = \int_{R^n} k(x-y)\, f(y)\, dy .$$

Of course Kf is nothing more than the function $I_z(g)$. Using Parseval's theorem it is easy to see that

$$\|Kf\|^2_{L^2(L^2(S_\infty))} = C \int_{R^n} |\mathcal{F}(\varphi)(x)|^2 \int_0^\infty \left| \frac{\mathcal{F}(\Omega_j)(x/t^{1/m})}{t} \right|^2 dt \leqslant C \|g\|^2_{L^2(R^n)}$$

with C independent of z. Also for $x \neq 0$ $\nabla k(x)$ is a continuous mapping from $C \to H$ defined by

$$\nabla k(x)(\alpha) = \alpha t^{-\frac{1}{2}-(n+1)/m} \nabla \Omega_j\left(\frac{x}{t^{1/m}} - z\right).$$

As such a mapping its norm

$$= \left(\int_0^\infty \frac{|\nabla \Omega_j(x/t^{1/m} - z)|^2}{t^{2(n+1)/m}} \frac{dt}{t}\right)^{\frac{1}{2}} = \frac{1}{|x|^{n+1}} \left(\int_0^\infty \frac{|\nabla \Omega_j(x'/t^{1/m} - z)|^2}{t^{2(n+1)/m}} \frac{dt}{t}\right)^{\frac{1}{2}}$$

where $x' = x/|x|$. Observe now that

$$\int_0^\infty \frac{|\nabla \Omega_j(x'/t^{1/m} - z)|^2}{t^{2(n+1)/m}} \frac{dt}{t} = \int_{t^{1/m}<1/2|z|} + \int_{t^{1/m}>1/2|z|} \left(\frac{|\nabla \Omega_j(x'/t^{1/m} - z)|^2}{t^{2(n+1)/m}}\right) \frac{dt}{t} \leqslant C + C|z|^{2(n+1)}$$

with again C independent of z.

From the above we conclude that for $1 < p < \infty$

$$\|t^{j/m-\frac{1}{2}} D^j I_z(g)\|_{L^p(L^2(S_\infty))} = \|K(g)\|_{L^p(L^2(S_\infty))} \leqslant C(1 + |z|)^{n+1} \|g\|_{L^p(R^n)}.$$

From our representation of $t^{j/m-\frac{1}{2}} D^j I_0(g)$ and with the use of Minkowski's inequality we have

$$\|t^{j/m-\frac{1}{2}} D^j I_0(g)\|_{L^p(L^2(S_\infty))} \leqslant C \int_R \sup_x \left|H(x, z)(1 + |z|)^{n+1}\right| dz \|g\|_{L^p(R^n)},$$

and this concludes the proof of Theorem 9.

THEOREM 10. *For $1 < p < \infty$, $1 \leq j \leq m$, and each $T < \infty$, K_j maps $L^p(L^2(S_T))$ into itself continuously and*

$$\|K_j\| \leqslant C\left[\theta^2(T) + \int_0^T \frac{\theta^2(s)}{s} ds\right]^{\frac{1}{2}}$$

where $C = C(n, m, p, \lambda, \mu)$.

PROOF. A standard argument in singular integral theory reduces the proof to showing that the norms of the operator, $K_{j,\varepsilon}$, defined by

$$K_{j,\varepsilon} f(x, t) = t^{j/m-\frac{1}{2}} \int_0^{t-\varepsilon} \int_{R^n} D^j \Gamma_x(x - y, t - s) \frac{\theta(s)}{s^{\frac{1}{2}}} f(y, s)\, dy\, ds$$

is bounded independently of ε by $C[\theta^2(T) + \int_0^T (\theta^2(s)/s)\, ds]^{\frac{1}{2}}$.

Using the same functions as defined in the proof of Theorem 9, we can represent the operator $K_{j,\varepsilon}$ as follows:

$$K_{j,\varepsilon}(f)(x,t) = \int_{R^n} H(x,z)\left(t^{j/m-\frac12}\int_0^{t-\varepsilon}\int_{R^n}\frac{\Omega_j\big((x-y)/(t-s)^{1/m}-z\big)}{(t-s)^{n/m+j/m}}\,\frac{\theta(s)}{s^{\frac12}}\,f(y,s)\,dy\,ds\right)dz$$
$$= \int_{R^n} H(x,z)\,K_z(f)(x,t)\,dz$$

where K_z is the (vector-valued) convolution operator

$$f \to t^{j/m-\frac12}\int_0^{t-\varepsilon}\int_{R^n}\frac{\Omega_j\big((x-y)/(t-s)^{1/m}-z\big)}{(t-s)^{(n+j)/m}}\,\frac{\theta(s)}{s^{\frac12}}\,f(y,s)\,dy\,ds\,.$$

Again we use the theory of vector-valued singular integral operators to prove that $K_z\colon L^p\big(L^2(S_T)\big) \to L^p\big(L^2(S_T)\big)$ continuously with norm bounded by

$$C\left(\theta(T) + \left(\int_0^T \frac{\theta^2(s)}{s}\,ds\right)^{\frac12}\right)(1+|z|)^N\,.$$

We first set $H = L^2(0,T)$ and for $x \in R^n$ we define $k(x)\colon H \to H$ by

$$k(x)(f)(t) = t^{j/m-\frac12}\int_0^{t-\varepsilon}\frac{\Omega_j\big(x/(t-s)^{1/m}-z\big)}{(t-s)^{(n+j)/m}}\,\frac{\theta(s)}{s^{\frac12}}\,f(s)\,ds\,.$$

For $f \in L^2(S_T)$ we set $\hat f(\xi,t) = \mathcal{F}_x\big(f(x,t)\big)(\xi)$. Then from Parseval's relation we have

$$\|K_z f\|^2_{L^2(S_T)} = \int_{R^n} d\xi \int_0^T t^{2j/m-1}\left|\int_0^{t-\varepsilon}\frac{\mathcal{F}(\Omega_j)\big(\xi(t-s)^{1/m}\big)\exp\big(iz\cdot\xi(t-s)^{1/m}\big)}{(t-s)^{j/m}}\,\cdot\right.$$
$$\left.\cdot\frac{\theta(s)}{s^{\frac12}}\,\hat f(\xi,s)\,ds\right|^2 dt \leqslant C\theta(T)^2\int_{R^n}\int_0^T\left|t^{j/m-1}\int_{t/2}^{t}\frac{|\mathcal{F}(\Omega_j)\big(\xi(t-s)^{1/m}\big)}{(t-s)^{j/m}}\,|\hat f(\xi,s)|\,ds\right|^2 dt\,d\xi$$
$$+\,C\int_{R^n}\int_0^T\left|t^{-\frac12}\int_0^{t/2}|\mathcal{F}(\Omega_j)\big(\xi(t-s)^{1/m}\big)|\;\frac{\theta(s)}{s^{\frac12}}\,|\hat f(\xi,s)|\,ds\right|^2 dt\,d\xi\,.$$

Recalling that $\mathcal{F}(\Omega_j)(\xi) = (i\xi)^j\exp\big(-c|\xi|^2\big)$, the second integral above is bounded by

$$C\left(\int_0^T \frac{\theta^2(s)}{s}\,ds\right)\int_0^T ds\int_{R^n}|\hat f(\xi,s)|^2|\xi|^{2j}\int_0^T \frac{\exp(-c|\xi|^2t^{2/m})}{t^{1-2j/m}}\,dt\,d\xi < C\int_0^T \frac{\theta^2(s)}{s}ds\|f\|_{L^2(L^2(S_T))}\,.$$

In the case $1\leqslant j<m$ we note that

$$t^{j/m-1}\int_{t/2}^{t}\frac{|(\mathcal{F}\Omega_j)(\xi(t-s)^{1/m})}{(t-s)^{j/m}}|\hat f(\xi,s)|ds\leqslant Ct^{j/m-1}\int_{t/2}^{t}\frac{1}{(t-s)^{j/m}}|\hat f(\xi,s)|ds\,,$$

and using a well-known lemma of Hardy ([6] p. 227) we see that

$$\int_{R^n}\int_0^T\left(t^{j/m-1}\int_{t/2}^{t}(t-s)^{-j/m}|\hat f(\xi,s)|d\xi\right)^2dt\,d\xi\leqslant C\int_{R^n}\int_0^T|\hat f(\xi,s)|^2ds\,d\xi = C\|f\|_{L^2(L^2(S_T))}\,.$$

In the case $j=m$ we note that

$$\int_{t/2}^{t}\frac{|\mathcal{F}\Omega_m(\xi(t-s)^{1/m})|}{(t-s)}|\hat f(\xi,s)|ds\leqslant C|\xi|^m\int_0^t\exp\left(-c|\xi|^2(t-s)^{2/m}\right)(|\hat f(\xi,s)|ds\,.$$

Since we now have convolution in the time variable with an L^1-kernel, it is easily seen that

$$\int_{R^n}|\xi|^{2m}\int_0^T\left(\int_0^t\exp\left(-c|\xi|^2(t-s)^{2/m}\right)|\hat f(\xi,s)|ds\right)^2dt\,d\xi\leqslant C\int_{R^n}\int_0^T|\hat f(\xi,t)|^2d\xi\,dt\,.$$

We have now completed the proof that

$$\|K_\varepsilon f\|_{L^2(L^2(S_T))}\leqslant C\left(\theta(T)+\left(\int_0^T\frac{\theta^2(s)}{s}\,ds\right)^{\frac12}\right)\|f\|_{L^2(L^2(S_T))}$$

with C independent of z and ε. For $x\neq 0$ the mapping $\nabla k(x): H\to H$ defined by

$$\nabla k(x)(f)(t) = t^{j/m-\frac12}\int_0^{t-\varepsilon}\frac{\nabla\Omega_j(x/(t-s)^{1/m}-z)}{(t-s)^{(n+j+1)/m}}\,\frac{\theta(s)}{s^{\frac12}}\,f(s)\,ds$$

has norm bounded by a constant, independent of ε, times

$$\left(\theta(T)+\left(\int_0^T\frac{\theta^2(s)}{s}\right)^{\frac12}\right)(1+|z|)^{n+1}|x|^{-n-1}.$$

To see this fact observe first that

$$t^{j/m-\frac12}\int_{t/2}^{t}\frac{|\nabla\Omega_j(x/(t-s)^{1/m}-z)}{(t-s)^{(n+j+1)/m}}\frac{\theta(s)}{s^{\frac12}}|f(s)|ds\leqslant C\theta(T)\int_0^t\frac{|\nabla\Omega_j(x/(t-s)^{1/m}-z)|}{(t-s)^{(n+1)/m+1}}|f(s)|ds$$

and hence the L^2-norm over $(0, T)$ of this function is bounded by

$$C\theta(T)\int_0^\infty\frac{|\nabla\Omega_j(x/t^{1/m}-z)|}{t^{(n+1)/m}}\frac{dt}{t}\|f\|_{L^2(0,T)}\leqslant C\theta(T)(1+|z|)^{n+1}|x|^{-(n+1)}\|f\|_{L^2(0,T)}.$$

Secondly by Minkowski's inequality we have

$$\left\|t^{j/m-\frac12}\int_0^{t/2}\frac{|\nabla\Omega_j(x/(t-s)^{1/m}-z)}{|(t-s)^{(n+j+1)/m}}\frac{\theta(s)}{s^{\frac12}}|f(s)|ds\right\|_{L^2(0,T)}$$
$$\leqslant C\int_0^T\frac{\theta(s)}{s^{\frac12}}|f(s)|\left(\int_s^\infty\frac{|\nabla\Omega_j(x/t^{1/m}-z)|^2}{t^{2(n+1)/m}}\frac{dt}{t}\right)^{\frac12}ds.$$

This last expression is bounded by

$$C(1+|z|)^{n+1}|x|^{-n-1}\int_0^T\frac{\theta(s)}{s^{\frac12}}|f(s)|ds\leqslant C(1+|z|)^{n+1}\left(\int_0^T\frac{\theta^2(s)}{s}ds\right)^{\frac12}|x|^{-n-1}\|f\|_{L^2(0,T)}.$$

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