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Embeddability of real analytic Cauchy-Riemann manifolds

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Embeddability of Real Analytic Cauchy-Riemann Manifolds (*).

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The local and global embeddability problems for Cauchy-Riemann (C-R) manifolds are as follows:

(I) Which C-R manifolds are locally isomorphic to a generic submanifold of C^n?

(II) Which C-R manifolds are globally isomorphic to a generic submanifold of some complex manifold?

It is well-known (see Rossi [10]) that real analytic C-R manifolds are locally embeddable as in (I). In § 2 we prove that real analytic C-R manifolds are also globally embeddable as in (II). This is a generalization of the theorem of Ehresmann-Shutrick (see Shutrick [12]) on the existence of complexifications—a theorem which was also proven independently by Haefliger [8] and by Bruhat and Whitney [4]. It also improves a result proved by Rossi [11]. In § 3 we give a notion of domination of real analytic C-R structures and prove a sort of functorial property concerning their complexifications. In the last section we prove a result about the convexity of the complexification which is a generalization of a theorem of Grauert [6].

1. - Preliminaries.

The two best references for the material presented in this section are [1] and Greenfield [7]. We will use the word « manifold » to mean an infinitely

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differentiable paracompact manifold and the word «submanifold» to mean a subspace of a manifold for which the inclusion map is an embedding.

For each m-dimensional manifold M, let $T(M)\otimes\boldsymbol{C}$ denote the complexified tangent bundle of M so that

$$T(M)\otimes\boldsymbol{C}_p = \textit{complex span of } \frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_m}(p),$$

if (x, U) is a chart of M with $p \in U$. A *Cauchy-Riemann* (C-R) *structure of type l on M* is an l-dimensional complex subbundle A of $T(M)\otimes\boldsymbol{C}$ such that

(*a*) $A \cap \bar{A} = \{0\}$ (zero section), and

(*b*) A is involutive, i.e. $[P, Q]$ is a section of A whenever P and Q are sections of A.

Note that the zero section of $T(M)\otimes\boldsymbol{C}$ defines a C-R structure of type 0 on M. This trivial C-R structure is called the *totally real structure of M*.

Observe from the definition of a C-R structure that $0 \leqslant l \leqslant m/2$ and also that if A is a C-R structure of type l on M, then so is $\bar{A}$.

Note also that condition (*a*) above is equivalent to the condition that if $p \in M$ and P, $Q \in A_p$ with $\operatorname{re} P = \operatorname{re} Q$ then $P = Q$. Thus we can define a complex structure on the $2l$-dimensional subbundle $\operatorname{re} A$ of $T(M)$, i.e. a bundle map $J\colon \operatorname{re} A \to \operatorname{re} A$ with $J^2 = -I$, by defining J on $\operatorname{re} A_p$ as follows:

$$JP = Q \quad \textit{if and only if } P + iQ \in A_p\,. \tag{1.1}$$

A C-R *manifold* is a pair (M, A) where A is a C-R structure on M. The C-R manifold (M, A) is said to be *of type* (m, l) if M is an m-dimensional manifold and A is a C-R structure of type l. A C-R manifold (M, A) is called *real analytic* if M is a real analytic manifold and, for each chart (x, U) of the real analytic atlas of M, there exist complex-valued vector fields $P_1, \ldots, P_l$ such that

(*a*) $A_p =$ *complex span of* $P_1(p), \ldots, P_l(p)$ *for each* $p \in U$, *and*

(*b*) $P_i = \sum_{j=1}^{m} c_{ij}(\partial/\partial x_j)$ *for* $i = 1, \ldots, l$ *with* $c_{ij}\colon U \to \boldsymbol{C}$ *real analytic.*

We will now examine in detail the C-R manifold $(X, HT(X))$, where X is an n-dimensional complex manifold and $HT(X)$ is its holomorphic tangent bundle. Note that if (z, U) is a (holomorphic) chart of X and $p \in U$, then

$$T(X)\otimes\boldsymbol{C}_p = \textit{complex span of } \frac{\partial}{\partial z_1}(p), \ldots, \frac{\partial}{\partial z_n}(p), \frac{\partial}{\partial \bar{z}_1}(p), \ldots, \frac{\partial}{\partial \bar{z}_n}(p),$$

so that each $P \in T(X) \otimes \boldsymbol{C}_p$ can be written uniquely in the form

$$P = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}(p) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial \bar{z}_i}(p) \qquad \textit{with } a_i, b_i \in \boldsymbol{C}\,.$$

Such a P is called *holomorphic* and an element of $HT(X)_p$ if all the b_i are zero. Similarly, such a P is called *antiholomorphic* and an element of $AT(X)_p$ if all the a_i are zero. Defining the holomorphic and antiholomorphic tangent bundles, $HT(X)$ and $AT(X)$, in the natural way, we see that $HT(X)$ is an n-dimensional complex subbundle of $T(X) \otimes \boldsymbol{C}$ with

$$T(X) \otimes \boldsymbol{C} = HT(X) \oplus AT(X) \qquad \textit{and} \qquad \overline{HT(X)} = AT(X)\,.$$

Now $HT(X) \cap AT(X) = \{0\}$ and it is easy to see that $HT(X)$ is involutive. Hence $\big(X, HT(X)\big)$ is a C-R manifold; in fact, a real analytic C-R manifold.

Henceforth, a complex manifold X will be considered to be a C-R manifold with C-R structure $HT(X)$. When there is no ambiguity we will simply write X for $\big(X, HT(X)\big)$. We remark, however, that it is customary to take $AT(X)$ as the C-R structure on X.

We will now give a detailed description of how a submanifold of a complex manifold can « inherit » a C-R structure. Since the two defining properties of a C-R structure are essentially local, we begin by considering a real submanifold M of $\boldsymbol{C}^n$ of (real) dimension m. Fixing $p \in M$ there exists a sufficiently small neighborhood U of p in $\boldsymbol{C}^n$ and $2n - m$ smooth functions

$$f_j\colon U \to \boldsymbol{R} \qquad \textit{for } j = 1, \ldots, 2n - m \tag{1.2}$$

such that

(a) $M \cap U = \{z \in U \mid f_j(z, \bar{z}) = 0 \textit{ for } j = 1, \ldots, 2n - m\}$,

(b) $df_1 \wedge \ldots \wedge df_{2n-m} \neq 0$ *on* U.

If U is sufficiently small we can also find *local parametric equations of* M *on* U, i.e. a set of n smooth functions

$$\varphi_i\colon D \to \boldsymbol{C} \qquad \textit{for } i = 1, \ldots, n \tag{1.3}$$

defined on an open set $D \subset \boldsymbol{R}^m$ such that

(a) $M \cap U = \varphi(D)$, where $\varphi\colon D \to \boldsymbol{C}^n$ is given by the equations

$$\varphi \equiv \{z = \varphi_i(t),\ \textit{for } i = 1, \ldots, n\}\,,$$

(b) $\displaystyle \operatorname{rk} \frac{\partial(\varphi_1, \ldots, \varphi_n, \bar{\varphi}_1, \ldots, \bar{\varphi}_n)}{\partial(t_1, \ldots, t_m)} = m \qquad \textit{on } D\,.$

If M is a real analytic submanifold of $\boldsymbol{C}^n$ the functions f_j and φ_i described above can be chosen to be real analytic.

Let $F(M)$ denote the sheaf of germs of smooth, complex-valued functions on $\boldsymbol{C}^n$ which vanish on M. If U and f_i are as defined in (1.2) then

$$F(M)(U) = \mathcal{S}(U)(f_1, \dots, f_{2n-m}),$$

where $\mathcal{S}$ is the sheaf of germs of smooth, complex-valued functions on $\boldsymbol{C}^n$. If U and φ_j are as defined in (1.3) then

$$F(M)(U) = \{g \in \mathcal{S}(U) | g \circ \varphi = 0 \ \text{ on } D\}.$$

The *holomorphic tangent space to M at $p \in M$* is now defined by

$$HT(M, \boldsymbol{C}^n)_p = \{P \in HT(\boldsymbol{C}^n)_p | P(f) = 0, \text{ for every } f \in F(M)_p\}.$$

Note that $HT(M, \boldsymbol{C}^n)_p$ is a complex vector space, and set

$$l(p) = \dim_{\boldsymbol{C}} HT(M, \boldsymbol{C}^n)_p \quad \text{ for each } p \in M.$$

PROPOSITION 1.4. *The function $l(p)$ is an upper semicontinuous function of p along M which satisfies*

$$m - n \leqslant l(p) \leqslant \frac{m}{2}.$$

PROOF. From above $F(M)_p = \mathcal{S}_p(f_1, \dots, f_{2n-m})$ and hence

$$P = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}(p) \in HT(M, \boldsymbol{C}^n)_p$$

if and only if $P(f_j) = 0$ for $j = 1, \dots, 2n - m$. Thus $l(p)$ is the dimension of the subspace of vectors $a = (a_1, \dots, a_n) \in \boldsymbol{C}^n$ such that

$$\sum_{i=1}^{n} a_i \frac{\partial f_j}{\partial z_i}(p) = 0 \qquad \text{for } j = 1, \dots, 2n - m.$$

Hence

$$l(p) = n - \operatorname{rk} \frac{\partial(f_1, \dots, f_{2n-m})}{\partial(z_1, \dots, z_n)}(p)$$

and we see that $l(p)$ is upper semicontinuous since the rank of a matrix of smooth functions is lower semicontinuous. Moreover, the rank of the above matrix is less than or equal to $2n-m$ and hence $l(p)\geqslant m-n$. Since the functions f_j are real-valued we see that, in self-explanatory notation,

$$\operatorname{rk}\frac{\partial(f)}{\partial(z)}=\operatorname{rk}\frac{\partial(f)}{\partial(\bar z)}\qquad on\ U\,.$$

From (1.2) (*b*) it follows that

$$\operatorname{rk}\frac{\partial(f)}{\partial(z,\bar z)}=2n-m\qquad on\ U\,,$$

so that $\operatorname{rk}\partial(f)/\partial(z)\geqslant\frac{1}{2}(2n-m)$ on U and consequently that $l(p)\leqslant m/2$.

A submanifold M of C^n is said to be *generic at* $p\in M$ if $l(p)=m-n$.

PROPOSITION 1.5. *M is generic at $p\in M$ (and hence in a neighborhood of $p\in M$ by proposition 1.4) if and only if one of the following conditions is satisfied.*

$$(a)\quad \operatorname{rk}\frac{\partial(f_1,\dots,f_{2n-m})}{\partial(z_1,\dots,z_n)}(p)=2n-m\,.$$

$$(b)\quad \operatorname{rk}\frac{\partial(\bar\varphi_1,\dots,\bar\varphi_n)}{\partial(t_1,\dots,t_m)}(t_0)=n\,,\ where\ t_0\in D\ with\ \varphi(t_0)=p\,.$$

PROOF. The first statement follows from the proof of proposition 1.4. For the second statement note that $f_j\circ\varphi=0$ on D for $j=1,\dots,2n-m$ so that

$$\sum_{i=1}^{n}\frac{\partial\varphi_i}{\partial t_k}(t_0)\frac{\partial f_j}{\partial z_i}(p)+\sum_{i=1}^{n}\frac{\partial\bar\varphi_i}{\partial t_k}(t_0)\frac{\partial f_j}{\partial\bar z_i}(p)=0$$

for $j=1,\dots,2n-m$ and $k=1,\dots,m$. Thus

$$\varphi_*\frac{\partial}{\partial t_1}(p),\dots,\varphi_*\frac{\partial}{\partial t_m}(p)$$

defined by

$$\varphi_*\frac{\partial}{\partial t_k}(p)=\sum_{i=1}^{n}\frac{\partial\varphi_i}{\partial t_k}(t_0)\frac{\partial}{\partial z_i}(p)+\sum_{i=1}^{n}\frac{\partial\bar\varphi_i}{\partial t_k}(t_0)\frac{\partial}{\partial\bar z_i}(p)$$

are m linearly independent, complex-valued tangent vectors at p which span $T(M)\otimes C_p$. Moreover

$$P=\sum_{k=1}^{m}a_k\varphi_*\frac{\partial}{\partial t_k}(p)\qquad with\ (a_1,\dots,a_m)\in C^m$$

is an element of $HT(M, C^n)_p$ if and only if

$$\sum_{k=1}^{m} a_k \frac{\partial \bar{\varphi}_i}{\partial t_k}(t_0) = 0 \qquad \textit{for } i = 1, \ldots, n\,.$$

Consequently

$$l(p) = m - \operatorname{rk} \frac{\partial(\bar{\varphi}_1, \ldots, \bar{\varphi}_n)}{\partial(t_1, \ldots, t_m)}(t_0)$$

and (b) now follows.

REMARK 1.6. From the preceding proof we see that the complex-valued vector fields $\varphi_*(\partial/\partial t_1), \ldots, \varphi_*(\partial/\partial t_m)$ are defined on $\varphi(D) \subset M$ and span $T(M) \otimes C$ at each point of $\varphi(D)$. We also see that

$$l(\varphi(t)) = m - \operatorname{rk} \frac{\partial(\bar{\varphi}_1, \ldots, \bar{\varphi}_n)}{\partial(t_1, \ldots, t_m)}(t) \qquad \textit{for every } t \in D\,.$$

Hence if $l(q) = l$ (a constant) for all q in a neighborhood of p in M, we can find smooth (or real analytic if the φ_i's are real analytic) complex-valued functions c_{ik} for $i = 1, \ldots, l$ and $k = 1, \ldots, m$ which are defined near p in M so that the vector fields

$$P_i = \sum_{k=1}^{m} c_{ik} \varphi_* \frac{\partial}{\partial t_k} \qquad \textit{for } i = 1, \ldots, l$$

span $HT(M, C^n)$ at each point of M near p.

We now consider M to be an m-dimensional submanifold of an n-dimensional complex manifold X. Let $F(M)$ denote the sheaf of germs of smooth real-valued functions on X which vanish on M and define

$$HT(M, X)_p = \{P \in HT(X)_p | P(f) = 0, \textit{ for every } f \in F(M)_p\}$$

for each $p \in M$. Set $l(p) = \dim_C HT(M, X)_p$. Note that the complex structure J determined by $HT(X)$ as in (1.1) is defined on $\operatorname{re} HT(X)_p = T(X)_p$ by

$$J\left(\sum_{i=1}^{u} a_i \frac{\partial}{\partial x_i}(p) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial y_i}(p)\right) = -\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}(p) + \sum_{i=1}^{n} a_i \frac{\partial}{\partial y_i}(p)$$

where (z, U) is a chart of X with $p \in U$, $z_j = x_j + iy_j$ and $a_i, b_i \in \boldsymbol{R}$. Now, for each $p \in M$, we see that $\operatorname{re} HT(M, X)_p = T(M)_p \cap JT(M)_p$ and that

$$\operatorname{re} HT(M, X)_p \subset T(M)_p \subset T(M)_p + JT(M)_p \subset T(X)_p\,. \tag{1.7}$$

Hence $\operatorname{re} HT(M, X)_p$ is the maximal complex subspace of $T(X)_p$ (with complex structure J) contained in $T(M)_p$ and $T(M)_p + JT(M)_p$ is the minimal complex subspace of $T(X)_p$ containing $T(M)_p$. Moreover,

$$\dim_{\boldsymbol{C}} \big(T(M)_p + JT(M)_p\big) = m - l(p) \,. \tag{1.8}$$

PROPOSITION 1.9. *If $l(p)$ is constant and equal to l on M, then $HT(M, X)$ is a* C-R *structure of type l on M. Moreover, if M is real analytic, then $\big(M, HT(M, X)\big)$ is a real analytic* C-R *manifold.*

PROOF. From the definition of $HT(M, X)$ we see that $HT(M, X) \cap \cap \overline{HT(M, X)} = \{0\}$ and also that a complex-valued vector field P is a section of $HT(M, X)$ if and only if it is holomorphic and satisfies $PF(M) \subset F(M)$. It thus follows that if P and Q are two such vector fields, then so is $[P, Q]$. From remark (1.6) we see that we can choose a real analytic, local basis for $HT(M, X)$ if M is real analytic.

If $HT(M, X)$ is a C-R structure of type l on M we say that M *inherits a* C-R *structure of type l from X*. When there is no ambiguity we will write $HT(M)$ for the inherited C-R structure $HT(M, X)$. Note that $\boldsymbol{R}^n \subset \boldsymbol{C}^n$ inherits the totally real structure from $\boldsymbol{C}^n$ and that every real hypersurface in $\boldsymbol{C}^n$ inherits a C-R structure of type $n - 1$ from $\boldsymbol{C}^n$.

We remark again that, classically (as in the study of tangential Cauchy-Riemann equations), one considers $\mathrm{AT(M)} = \overline{HT(M)}$ as the inherited C-R structure on M.

An m-dimensional submanifold M of an n-dimensional complex manifold X is *generic* if it inherits a C-R structure of type $m - n$ from X. A generic submanifold of X will always be considered to be a C-R manifold with its inherited C-R structure from X.

Two C-R manifolds, (M, A) and (N, B), are said to be *isomorphic* if there exists a diffeomorphism $\psi\colon M \to N$ such that $\psi_* A = B$, where $\psi_*\colon T(M) \otimes \boldsymbol{C} \to T(N) \otimes \boldsymbol{C}$ is the natural map induced by ψ. Note that two isomorphic C-R manifolds are necessarily of the same type.

PROPOSITION 1.10. *If M is an m-dimensional submanifold of X which inherits a* C-R *structure of type l from X, then $\big(M, HT(M)\big)$ is locally isomorphic to a generic submanifold of $\boldsymbol{C}^{m-l}$. If M is real analytic the local isomorphism is also real analytic.*

PROOF. Since the result is local, consider M to be a submanifold of $\boldsymbol{C}^n$ and fix $p \in M$. Recall from (1.7) and (1.8) that $T(M)_p + JT(M)_p$ is the minimal complex subspace of $T(\boldsymbol{C}^n)_p$ containing $T(M)_p$ and that it has dimension $m - l$. Let π_1 and π_2 be complex planes of dimensions $m - l$

and $n-m+l$ respectively, for which

(a) $T(\pi_1)_p \oplus T(\pi_2)_p = T(C^n)_p$ (transversal at p);

(b) $T(\pi_1)_p = T(M)_p + JT(M)_p$.

Define $\lambda\colon C^n \to C^{m-l}$ by projecting C^n onto π_1 parallel to π_2 and then identifying π_1 with C^{m-l} by some biholomorphic map. Then λ is holomorphic and, by (a) and (b), there is a sufficiently small neighborhood W of p in M on which $\tau = \lambda|_W$ is a diffeomorphism (real analytic if M is real analytic) onto its image. Suppose now that $z=(z_1, \ldots, z_n)$ are holomorphic coordinates on C^n and $w=(w_1, \ldots, w_{m-l})$ are holomorphic coordinates on C^{m-l}, and that

$$w_j = g_j(z_1, \ldots, z_n) \qquad \textit{for } j = 1, \ldots, m-l$$

are the equations of the holomorphic projection $\lambda\colon C^n \to C^{m-l}$. Also assume that, after possibly choosing a smaller W, the smooth functions (real analytic if M is real analytic) $\varphi_i\colon D \to C$ for $i=1, \ldots, n$ give local parametric equations of M on an open set U in C^n, as in (1.3), for which $U \cap M = W$. Thus

$$w_j = g_j\big(\varphi_1(t), \ldots, \varphi_n(t)\big) \qquad \textit{for } j=1, \ldots, m-l \textit{ and } t \in D$$

gives parametric equations of τW on $\tau U \subset C^{m-l}$ and we see by the construction of τ that

$$\operatorname{rk} \frac{\partial(g)}{\partial(t)} = m - l \qquad on\, D\,.$$

Since this rank is maximal we conclude from proposition 1.5 that τW is a generic submanifold of C^{m-l}. Since λ is holomorphic we have that

$$\lambda_*\left(\frac{\partial}{\partial z_i}(q)\right) = \sum_{j=1}^{m-l} \frac{\partial g_j}{\partial z_i}(q) \frac{\partial}{\partial w_j}(\lambda(q)) \qquad \textit{for every } q \in C^n\,,$$

and hence $\lambda_* HT(C^n) \subset HT(C^{m-l})$. Since $\tau_* = \lambda_{*|T(W)\otimes C}$ we now conclude that

$$\tau_* HT(W, C^n) \subset HT(\tau W, C^{m-l})\,.$$

In fact we have equality in the preceding line since $\tau\colon W \to \tau W$ is a diffeomorphism and τW is a generic submanifold of C^{m-l} so both are C-R structures of type l on τW. Thus $\big(W, HT(W)\big)$ and $\big(\tau W, HT(\tau W)\big)$ are isomorphic and the proof is complete.

The local embeddability problem can be stated as follows.

(I) *Given a* C-R *manifold* (M, A) *can one find, for each* $p \in M$, *a neighborhood* U *of* p *in* M *and an embedding* $\tau\colon U \to \boldsymbol{C}^n$ *such that* τU *is a generic submanifold of* $\boldsymbol{C}^n$ *and* $\tau_*(A_{|U}) = HT(\tau U)$?

Note that if (M, A) is of type (m, l) then $n = m - l$.

As we remarked earlier, (I) is solvable for real analytic C-R manifolds. Indeed, one can prove (see [2])

THEOREM 1.11. *If* (M, A) *is a real analytic* C-R *manifold, then* (I) *is solvable with real analytic embeddings* τ.

Note that Nirenberg [9] has given an example of a C-R structure of type one on a neighborhood of the origin in $\boldsymbol{R}^3$ for which (I) is not solvable.

We now turn to the global embeddability problem which is the following:

(II) *Given a* C-R *manifold* (M, A) *for which* (I) *is solvable, can one find a complex manifold* X *and an embedding* $\tau\colon M \to X$ *such that* τM *is a generic submanifold of* X *and* $\tau_* A = HT(\tau M)$?

Note again that if (M, A) is of type (m, l), then X is a complex manifold of dimension $m - l$.

We now define a *complexification of the real analytic* C-R *manifold* (M, A) to be a pair (X, τ) where X is a complex manifold and $\tau\colon M \to X$ is a real analytic, closed embedding such that τM is a generic submanifold of X and $\tau_* A = HT(\tau M)$. In the next section we will prove

THEOREM 1.12: (*a*) *Each real analytic* C-R *manifold has a complexification.* (*b*) *If* (X, τ) *and* (Y, σ) *are two complexifications of the real analytic* C-R *manifold* (M, A), *then there exist neighborhoods* U *of* τM *in* X *and* V *of* σM *in* Y *and a biholomorphic map* $h\colon U \to V$ *such that the diagram*

$$\begin{tikzcd} & U \arrow[dd, "h"] \\ M \arrow[ur, "\tau"] \arrow[dr, "\sigma"'] & \\ & V \end{tikzcd}$$

commutes. Moreover, h *is uniquely defined if* U *is sufficiently small and connected with* τM.

Thus (II) is solvable for real analytic C-R manifolds and the germ of the complexification of (M, A) along M is unique. The totally real case of theorem 1.12, i.e. the case when A is the totally real structure of M, is the theorem of Ehresmann-Shutrick (see Shutrick [12]) which was referred to in the introduction.

2. – Global embeddability of real analytic C-R manifolds.

Before proving theorem 1.12 we would like to collect some results concerning real analytic C-R manifolds.

PROPOSITION 2.1. *Suppose that $M = \varphi(D)$ is an m-dimensional, generic, real analytic submanifold of $\boldsymbol{C}^{m-l}$ with real analytic parametric equations*

$$\varphi \equiv \{z_j = \varphi_j(t) \text{ for } j = 1, \ldots, m-l \text{ and } t \in D \subset \boldsymbol{R}^m .$$

There exist neighborhoods $\tilde{D}$ of D in $\boldsymbol{C}^m$ and W of M in $\boldsymbol{C}^{m-l}$ such that the map $\varphi\colon D \to M$ extends to a holomorphic, open, surjective map $\tilde{\varphi}\colon \tilde{D} \to W$ with the property that

$$\operatorname{rk} \frac{\partial(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{m-l})}{\partial(w_1, \ldots, w_m)} = m - l \qquad \text{on } \tilde{D},$$

where $w_k = t_k + is_k$ for $k = 1, \ldots, m$ give holomorphic coordinates on $\tilde{D}$.

PROOF. The functions $\tilde{\varphi}_j(w_1, \ldots, w_m) = \varphi_j(t_1 + is_1, \ldots, t_m + is_m)$ are holomorphic in a neighborhood $\tilde{D}$ of D in $\boldsymbol{C}^m$ and

$$\operatorname{rk} \frac{\partial(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{m-l})}{\partial(w_1, \ldots, w_m)} = m - l \qquad \text{on } D,$$

since M is generic and $\tilde{\varphi}_j$ extends φ_j. Hence the above matrix has maximal rank on a neighborhood of D in $\boldsymbol{C}^m$ which we take to be $\tilde{D}$. The map $\tilde{\varphi}\colon \tilde{D} \to \boldsymbol{C}^{m-l}$ defined by the functions $\tilde{\varphi}_j$ is thus holomorphic and of maximal rank. It is therefore open and we can take $W = \tilde{\varphi}\tilde{D}$ to complete the proof.

PROPOSITION 2.2 (Tomassini [13]). *If M is as in proposition 2.1 and $f\colon M \to \boldsymbol{C}$ is a real analytic function with $f_*\colon HT(M) \to HT(\boldsymbol{C})$, then there exists a holomorphic function F defined in a neighborhood W of M in $\boldsymbol{C}^{m-l}$ which extends f. Moreover, if F_1 and F_2 are holomorphic extensions of f defined on neighborhoods W_1 and W_2 of M in $\boldsymbol{C}^{m-l}$, then $F_1 = F_2$ on every neighborhood of M in $\boldsymbol{C}^{m-l}$ which is connected with M and contained in $W_1 \cap W_2$.*

PROOF. By proposition 2.1 we can select a neighborhood $\tilde{D}$ of D in C^m so that $\zeta_1 = \tilde{\varphi}_1, \dots, \zeta_{m-l} = \tilde{\varphi}_{m-l}$ are part of a system of holomorphic coordinates $(\zeta_1, \dots, \zeta_m)$ on $\tilde{D}$. Since $f \circ \varphi$ is real analytic we can find a holomorphic function G on $\tilde{D}$ (sufficiently small) such that $G = f \circ \varphi$ on D. Since for each $P \in T(M) \otimes C_p$ we have

$$f_* P = Pf \frac{\partial}{\partial z}(f(p)) + P\bar{f} \frac{\partial}{\partial \bar{z}}(f(p)),$$

we see that the condition $f_*: HT(M) \to HT(C)$ is equivalent to the condition

$$P\bar{f} = 0 \quad \textit{whenever } P \in HT(M)_p \textit{ and } p \in M,$$

or equivalently

$$Qf = 0 \quad \textit{whenever } Q \in AT(M)_p \textit{ and } p \in M.$$

From the proof of proposition 1.5 the latter condition is in turn equivalent to the condition

$$\left(\sum_{k=1}^{m} b_k \varphi_* \frac{\partial}{\partial t_k}(p)\right) f = 0$$

whenever $p \in M$ *and* $b \in C^m$ *satisfying*

$$\sum_{k=1}^{m} b_k \frac{\partial \varphi_j}{\partial t_k}(t_0) = 0 \qquad \textit{for } j = 1, \dots, m - l.$$

Thus $d\varphi_1 \wedge \dots \wedge d\varphi_{m-l} \wedge d(f \circ \varphi) = 0$ on D, or $d\zeta_1 \wedge \dots \wedge d\zeta_{m-l} \wedge dG = 0$ on D. Since

$$dG = \sum_{k=1}^{m} \frac{\partial G}{\partial \zeta_k} d\zeta_k$$

we see that the holomorphic functions $\partial G / \partial \zeta_k$ for $k = m - l + 1, \dots, m$ vanish on D and hence on $\tilde{D}$ (assuming $\tilde{D}$ is connected with D). Thus G is independent of $\zeta_{m-l+1}, \dots, \zeta_m$ and so G factors through $\tilde{\varphi}$, i.e. there exists a holomorphic function F defined on a neighborhood W of M in C^{m-l} such that $G = F \circ \tilde{\varphi}$. Now $F|_M = f$ and the unicity of F follows from the above considerations.

Since the restriction of a holomorphic map is real analytic and maps holomorphic tangent vectors to holomorphic tangent vectors we have

COROLLARY 2.3. *If F_1 and F_2 are holomorphic functions defined on open sets W_1 and W_2 in C^{m-l} and $F_1 = F_2$ on a generic submanifold M of C^{m-l},* *then* $F_1 = F_2$ *on every neighborhood of* M *in* C^{m-l} *which is connected with* M *and contained in* $W_1 \cap W_2$.

We now establish theorem 1.12 (b), i.e. the unicity of the germ of the complexification of (M, A) along M. From corollary 2.3 it suffices to prove this result locally. Fixing $p \in M$ we apply proposition 2.2 to the components of $\sigma \circ \tau^{-1}: \tau M \to \sigma M$ near $\tau(p)$ to obtain a holomorphic function h which extends $\sigma \circ \tau^{-1}$ near $\tau(p)$. Similarly, we also obtain a holomorphic function g which extends $\tau \circ \sigma^{-1}$ near $\sigma(p)$. Now $g \circ h$ is a holomorphic function in a sufficiently small neighborhood U of $\tau(p)$ in X and $g \circ h$ is the identity on $U \cap \tau M$. Assuming that U is connected with τM we conclude from corollary 2.3 that $g \circ h$ is the identity on U. Similarly, $h \circ g$ is the identity on the neighborhood $V = h(U)$ of $\sigma(p)$ in Y which is connected with $\sigma(M)$ and thus h is the desired biholomorphic map. The unicity of h follows from another application of corollary 2.3.

The proof of theorem 1.12 (a) is established by a construction similar to that given by Bruhat and Whitney [4] in the case of the totally real structure. (See also [3].)

Let (M, A) be a real analytic C-R manifold of type (m, l) and let $n = m - l$.

Part 1. We can find three locally finite open covers of M with the same index set I, say $\{V_i'\}$, $\{U_i'\}$ and $\{T_i'\}$ such that

$$V_i' \subset\subset U_i' \subset\subset T_i' \quad \textit{for every } i \in I .$$

For each $i \in I$ we can find local complexifications of T_i' by theorem 1.11, i.e. there exist real analytic isomorphisms $\varphi_i: T_i' \to T_i$, where T_i is an m-dimensional, generic, real analytic submanifold of C^n and

$$\varphi_{i*}(A_{|T_i'}) = HT(T_i) .$$

We now set

$$U_i = \varphi_i U_i', \qquad V_i = \varphi_i V_i'$$

$$U_{ij} = \varphi_i(U_i' \cap U_j') , \qquad V_{ij} = \varphi_i(V_i' \cap V_j')$$

$$T_{ij} = \varphi_i(T_i' \cap T_j') .$$

The isomorphism

$$\varphi_j \circ \varphi_i^{-1}: T_{ij} \to T_{ji}$$

extends by proposition 2.2 to a biholomorphic map $\psi_{ji}: \tilde{T}_{ij} \to \tilde{T}_{ji}$ of a neighborhood $\tilde{T}_{ij}$ of T_{ij} in C^n to a neighborhood $\tilde{T}_{ji}$ of T_{ji} in C^n. We can assume that $\tilde{T}_{ij}$ is empty if T_{ij} is empty and that $\psi_{ij} = \psi_{ji}^{-1}$. For every ordered pair (i, j) we can select open sets $\tilde{U}_{ij}$ in $\tilde{T}_{ij}$ such that $\tilde{U}_{ij} \subset\subset \tilde{T}_{ij}$, $\psi_{ji}\tilde{U}_{ij} = \tilde{U}_{ji}$ and

$$\tilde{U}_{ij} \cap T_i = U_{ij} \quad \textit{and} \quad \bar{\tilde{U}}_{ij} \cap T_i = \bar{U}_{ij}\,, \tag{2.4}$$

where the bar denotes the closure of the set. Since $\bar{V}_i \cap \psi_{ij}\ (\bar{V}_j \cap \bar{U}_{ji})$ is a compact subset of U_{ij} we can choose open sets $\tilde{W}_{ij} \subset \tilde{T}_{ij}$ such that $\tilde{W}_{ij} \subset\subset \tilde{U}_{ij}$, $\tilde{W}_{ij} = \psi_{ij}(\tilde{W}_{ji})$ and

$$\bar{V}_i \cap \psi_{ij}(\bar{V}_j \cap \bar{U}_{ji}) \subset \tilde{W}_{ij}\,.$$

The subsets $\bar{V}_i - \tilde{W}_{ij}$ and $\psi_{ij}(\bar{V}_j \cap \bar{U}_{ji}) - \tilde{W}_{ij}$ of T_i are compact and disjoint, and therefore contained in disjoint open sets $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ of C^n respectively, so we have

$$\bar{V}_i \subset \tilde{A}_{ij} \cup \tilde{W}_{ij}\,, \qquad \psi_{ij}(\bar{V}_j \cap \bar{U}_{ji}) \subset \tilde{B}_{ij} \cup \tilde{W}_{ij}\,. \tag{2.5}$$

We now choose $\tilde{A}_i$ open in C^n such that

$$\tilde{A}_i \cap T_i = V_i\,, \qquad \bar{\tilde{A}}_i \cap T_i = \bar{V}_i \tag{2.6}$$

and

$$\tilde{A}_i \subset \tilde{A}_{ij} \cup \tilde{W}_{ij} \quad \textit{for all } j \in I \quad \textit{such that } T_{ij} \textit{ is nonempty.} \tag{2.7}$$

The last condition can be satisfied since there are only a finite number of $j \in I$ such that T_{ij} is nonempty.

Since $\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij}$ is compact and contained in $\tilde{T}_{ij}$ we have

$$\overline{\psi_{ji}(\tilde{A}_i \cap \tilde{U}_{ij})} \subset \psi_{ji}(\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij})\,.$$

By (2.4) and (2.6) we have

$$\begin{aligned} \psi_{ji}(\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij}) \cap T_j &= \psi_{ji}(\bar{\tilde{A}}_i \cap \bar{\tilde{U}}_{ij} \cap T_i) \\ &= \psi_{ji}(\bar{V}_i \cap \bar{U}_{ij}) \end{aligned}$$

and hence

$$\overline{\psi_{ji}(\tilde{A}_i \cap \tilde{U}_{ij})} \cap T_j \subset \psi_{ji}(\bar{V}_i \cap \bar{U}_{ij})\,. \tag{2.8}$$

PART 2. For any point $x \in U_i$ there exists an open set $\tilde{U}_i(x)$ in C^n containing x and satisfying the following five conditions:

(1) $\tilde{U}_i(x) \subset \tilde{U}_{ij}$ for every index j such that $x \in U_{ij}$.

(2) $\tilde{U}_i(x) \subset \tilde{B}_{ij} \cup \tilde{W}_{ij}$ for every index j such that $x \in \psi_{ij}(\overline{V}_j \cap \overline{U}_{ji})$. (Compare (2.5).)

(3) $\tilde{U}_i(x) \cap \psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ is empty for every index j such that $\varphi_i^{-1}(x) \notin \overline{V}'_j$.

(4) $\tilde{U}_i(x) \subset \psi_{ij}(\tilde{U}_{ji} \cap \tilde{U}_{jk}) \cap \psi_{ik}(\tilde{U}_{ki} \cap \tilde{U}_{kj})$ for every pair of indices (j, k) such that $x \in U_{ij} \cap U_{ik}$ (i.e. $\varphi^{-1}(x) \in U'_i \cap U'_j \cap U'_k$).

(5) $\psi_{ji} = \psi_{jk} \circ \psi_{ki}$ on $\tilde{U}_i(x)$ for all (j, k) as in (4).

The conditions (1), (2) and (4) are satisfied because the number of indices involved is finite. Condition (5) is satisfied on $U_{ij} \cap U_{ik}$ and hence in a neighborhood. Condition (3) is trivially satisfied if $\tilde{U}_{ij}$ is empty. Otherwise there are at most finitely many j's such that $\tilde{U}_{ij}$ is nonempty and $\varphi_i^{-1}(x) \notin \overline{V}'_j$ implies $x \notin \psi_{ij}(\overline{V}_j \cap \overline{U}_{ji})$ so $x \notin \psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ by (2.8).

PART 3. Set

$$\tilde{U}_i = \bigcup_{x \in U_i} \tilde{U}_i(x)$$

and let $\tilde{V}_i$ be a neighborhood of V_i in $\mathbf{C}^n$ which is contained in $\tilde{A}_i$ and relatively compact in $\tilde{U}_i$. Note from (2.6) that $\tilde{V}_i \cap T_i = V_i$ and $\overline{\tilde{V}}_i \cap T_i = \overline{V}_i$. Setting

$$\tilde{V}_{ij} = \tilde{V}_i \cap \psi_{ij}(\tilde{V}_j \cap \tilde{U}_{ji})\,, \qquad \tilde{V}_{ijk} = \tilde{V}_{ij} \cap \tilde{V}_{ik} \tag{2.9}$$

we see that $\tilde{V}_{ij} \subset \tilde{U}_{ij}$ and $\psi_{ij}\colon \tilde{V}_{ji} \xrightarrow{\sim} \tilde{V}_{ij}$.

Let $y \in \tilde{V}_{ijk}$, so $y \in \tilde{U}_i(x)$ for some $x \in U_i$. Since $\tilde{V}_{ijk}$ intersects $\psi_{ij}(\tilde{V}_j \cap \tilde{U}_{ji})$ and $\psi_{ik}(\tilde{V}_k \cap \tilde{U}_{ki})$ it also meets $\psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ and $\psi_{ik}(\tilde{A}_k \cap \tilde{U}_{ki})$ and hence $x \in \tilde{U}_{ij} \cap \tilde{U}_{ik}$ by (3). Therefore $\psi_{ki}(y) \in \tilde{V}_k \cap \tilde{U}_{kj}$ and $\psi_{jk} \circ \psi_{ki}(y) = \psi_{ji}(y)$ by (2), so

$$\begin{aligned} z = \psi_{ji}(y) &\in \tilde{V}_j \\ &\in \psi_{ji}(\tilde{V}_i \cap \tilde{U}_{ij}) \\ &\in \psi_{jk}(\tilde{V}_k \cap \tilde{U}_{kj}) \\ &\in \tilde{V}_{jik}\,. \end{aligned}$$

We therefore conclude that $\psi_{ji}(\tilde{V}_{ijk}) \subset \tilde{V}_{jik}$ and by symmetry $\psi_{ij}(\tilde{V}_{jik}) \subset \tilde{V}_{ijk}$, so

$$\psi_{ji}\colon \tilde{V}_{ijk} \xrightarrow{\sim} \tilde{V}_{jik}\,.$$

We also have $\psi_{ij} = \psi_{ji}^{-1}$ and $\psi_{ji} = \psi_{jk} \circ \psi_{ki}$ on $\tilde{V}_{ijk}$ so $\{\tilde{V}_i, \psi_{ij}\}$ is an amalgamation system and we can construct the amalgamated sum

$$\{\tilde{\Omega}, \psi_i\} = \lim \{\tilde{V}_i, \psi_{ij}\} .$$

This sum is obtained from the disjoint union $\mathring{\bigcup} \tilde{V}_i$ by dividing out by the equivalence relation

$$x \in \tilde{V}_i \sim y \in \tilde{V}_j \quad \textit{if } x \in \tilde{V}_{ij},\ y \in \tilde{V}_{ji} \textit{ and } y = \psi_{ji}(x) .$$

The space $\tilde{\Omega}$ is a complex manifold if it is Hausdorff. The natural maps ψ_i: $\tilde{V}_i \to \tilde{\Omega}$ are given by the canonical open embeddings

$$\tilde{V}_i \to \mathring{\bigcup} \tilde{V}_i \to \tilde{\Omega} ,$$

and the isomorphisms φ_i merge into an isomorphism φ of M onto a generically embedded, real analytic, closed submanifold of $\tilde{\Omega}$.

PART 4. The proof will be complete if we show that $\tilde{\Omega}$ has a Hausdorff topology.

We will first show that $\tilde{V}_{ij} \subset \tilde{U}_{ij}$ and more precisely that $\tilde{V}_{ij} \subset \tilde{W}_{ij}$ (when T_{ij} is nonempty). If $y \in \tilde{V}_{ij}$ then there exists $x \in U_i$ such that $y \in \tilde{U}_i(x)$ since $\tilde{V}_{ij} \subset \tilde{V}_i \subset \tilde{U}_i$.

If $x \notin \psi_{ij}(\overline{V}_j \cap \overline{U}_{ji})$ then $\varphi_i^{-1}(x) \notin V_j'$ and hence $y \notin \psi_{ij}(\tilde{A}_j \cap \tilde{U}_{ji})$ by (3). Since this contradicts the fact that $y \in \tilde{V}_{ij}$ in (2.9), we have that $x \in \psi_{ij}(\overline{V}_j \cap \overline{U}_{ji})$ and hence by (2) that $y \in \tilde{B}_{ij} \cup \tilde{W}_{ij}$. Since $\tilde{V}_{ij} \subset \tilde{V}_i \subset \tilde{A}_i$ we see from (2.7) that $y \in \tilde{A}_{ij} \cup \tilde{W}_{ij}$ and hence that $y \in \tilde{W}_{ij}$ as $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ are disjoint. In conclusion we have

$$\tilde{V}_{ij} \subset \tilde{W}_{ij} \subset \tilde{U}_{ij} .$$

Suppose now that x' and y' are two points of $\tilde{\Omega}$ with $x' \neq y'$. Let $x \in \tilde{V}_i$ and $y \in \tilde{V}_j$ such that $\psi_i(x) = x'$ and $\psi_j(y) = y'$. It suffices to show that there exist neighborhoods A of x in $\tilde{V}_i$ and B of y in $\tilde{V}_j$ such that no point of A is equivalent to a point of B.

If this was not the case we could find sequences $\{x_k\}$ and $\{y_k\}$ in C^n converging to x and y respectively with $x_k \in \tilde{V}_{ij}$, $y_k \in \tilde{V}_{ji}$ and $x_k = \psi_{ij}(y_k)$ for every k. Since $\tilde{V}_{ij} \subset \tilde{U}_{ij}$ we see that $x \in \tilde{U}_{ij}$ and by symmetry $y \in \tilde{U}_{ji}$, so

$x = \psi_{ij}(y)$ by continuity. Thus

$$y \in \tilde{V}_j \cap \tilde{U}_{ji} \subset \tilde{V}_{ji}\,,$$
$$x \in \tilde{V}_i \cap \psi_{ij}(\tilde{V}_j \cap \tilde{U}_{ji}) = \tilde{V}_{ij}$$

and $y = \psi_{ji}(x)$, so x is equivalent to y which is contrary to our assumption.
This completes the proof of theorem 1.12.

3. – Domination of C-R structures.

Let M be a real analytic, m-dimensional manifold and let A and B be two real analytic C-R structures of types k and l on M respectively. We say that B *dominates* A if $A_p \subset B_p$ for every $p \in M$ (and thus $l \geqslant k$). Note that every real analytic C-R structure on M dominates the totally real structure of M.

THEOREM 3.1. *If B dominates A as above and (X, τ) and (Y, σ) are complexifications of (M, A) and (M, B) respectively, then there exist open neighborhoods U of τM in X and V of σM in Y and a holomorphic surjective map $h\colon U \to V$ of maximal rank such that the diagram*

$$\begin{tikzcd} & U \subset X \arrow[dd, "h"] \\ M \arrow[ur, "\tau"] \arrow[dr, "\sigma"'] & \\ & V \subset Y \end{tikzcd}$$

commutes. Moreover, h is uniquely defined if U is sufficiently small and connected with τM.

PROOF. Note that the unicity of h and the sufficiency to prove the result locally both follow from corollary 2.3. We also have

$$\dim_{\mathbf{C}} X = m - k \geqslant m - l = \dim_{\mathbf{C}} Y\,.$$

PART 1. As a consequence of proposition 2.1 the result holds if A is the totally real structure of M.

PART 2. Since (X, τ) and (Y, σ) are complexifications of (M, A) and (M, B) respectively, and B dominates A, we have

$$\sigma_* \circ \tau_*^{-1}\colon HT(\tau M, X) \to HT(\sigma M, Y)\,. \tag{3.2}$$

Assume, from the remark at the beginning of this proof, that M is an open set $D \subset \boldsymbol{R}^m$ which is sufficiently small so that τD and σD are contained in local holomorphic coordinate systems $z = (z_1, \dots, z_{m-k})$ of X and $w = (w_1, \dots, w_{m-l})$ of Y respectively. The equations

$$\tau \equiv \{z_1 = \tau_i(t) \ \textit{for} \ i = 1, \dots, m-k ,$$

and

$$\sigma \equiv \{w_j = \sigma_j(t) \ \textit{for} \ j = 1, \dots, m-l$$

with $t \in D$ are real analytic, local parametric equations (see (1.3)) of τM and σM respectively, and thus, from (3.2) and the proof of proposition 1.5, we have

$$\sum_{r=1}^{m} a_r \frac{\partial \bar{\sigma}_j}{\partial t_r}(t) = 0 \qquad \textit{for} \ j = 1, \dots, m-l$$

whenever $t \in D$ *and* $a \in \boldsymbol{C}^m$ *satisfying*

$$\sum_{r=1}^{m} a_r \frac{\partial \bar{\tau}_i}{\partial t_r}(t) = 0 \qquad \textit{for} \ i = 1, \dots, m-k .$$

That is (after conjugating),

$$d\sigma_j \wedge d\tau_1 \wedge \dots \wedge \ \tau_{m-k} = 0 \qquad \textit{on} \ D \ \textit{for} \ j = 1, \dots, m-l .$$

Let $\tilde{\tau}$ and $\tilde{\sigma}$ be holomorphic extensions of τ and σ respectively, to a sufficiently small neighborhood $\tilde{D}$ of D in $\boldsymbol{C}^m$ such that $\tilde{D} \cap \boldsymbol{R}^m = D$ and

$$d\tilde{\sigma}_j \wedge d\tilde{\tau}_1 \wedge \dots \wedge d\tilde{\tau}_{m-k} = 0 \qquad \textit{on} \ \tilde{D} \ \textit{for} \ j = 1, \dots, m-l .$$

It now follows that the map $\tilde{\sigma}$ factors through the map $\tilde{\tau}$, i.e. there exists a holomorphic function $h\colon \tilde{\tau}\tilde{D} \to \tilde{\sigma}\tilde{D}$ such that $\tilde{\sigma} = h \circ \tilde{\tau}$ on $\tilde{D}$. Since $\tilde{D}$ is a complexification of M with its totally real structure, it follows from part 1 that $\tilde{\tau}$ and $\tilde{\sigma}$ are surjective and of maximal rank if $\tilde{D}$ is sufficiently small. Hence h is also a surjective map of maximal rank, and the proof is complete.

4. – Convexity of the complexification.

By a theorem of Grauert [6] we know that there is a complexification (X, τ) of a real analytic manifold M with its totally real structure in which X is Stein. Since the totally real structure is the C-R structure of type zero and Stein is the same as 0-complete we are lead to

CONJECTURE 4.1. *If (M, A) is a real analytic* C-R *manifold of type (m, l), then there exists a complexification (X, τ) of (M, A) for which X is l-complete* (*).

The following theorem states that the above conjecture holds if M is compact.

THEOREM 4.2. *If (M, A) is a compact, real analytic* C-R *manifold of type (m, l) and (X, τ) is a complexification of (M, A), then there exists a neighborhood U of τM in X which is an l-complete manifold.*

REMARK. Since (U, τ) is a complexification of (M, A) whenever U is an open neighborhood of τM in X, we see that τM has a fundamental system of neighborhoods in X which are l-complete.

PROOF OF (4.2). Let $\{U_i, z^i\}$ be a locally finite covering of τM by holomorphic charts of X such that for each i there exist (see (1.2)) real analytic functions $f_j^i\colon U_i \to \boldsymbol{R}$ for which

(*a*) $\tau M \cap U_i = \{z \in U_i \mid f_j^i(z, \bar{z}) = 0 \ \ for\ \ j = 1, \ldots, m - 2l\}$, *and*

(*b*) $\operatorname{rk} \dfrac{\partial(f_1^i, \ldots, f_{m-2l}^i)}{\partial(z_1^i, \ldots, z_{m-l}^i)} = m - 2l \qquad on\ \ U_i\,.$

For each index i define $v^i\colon U_i \to \boldsymbol{R}$ by

$$v^i = \sum_{j=1}^{m-2l} |f_j^i|^2$$

and note from (*a*) that

$$\partial\bar{\partial} v^i(z) = 2 \sum_{j=1}^{m-2l} |\partial f_j^i(z)|^2 \geqslant 0 \qquad \text{for each } z \in \tau M \cap U_i\,.$$

It now follows from (*b*) that $\mathcal{L}(v^i)_z$ has at least $m - 2l$ positive eigenvalues for each $z \in \tau M \cap U_i$.

(*) An n-dimensional complex manifold X is *l-complete* if there exists a smooth function $\varphi\colon X \to \boldsymbol{R}$ such that

(1) $\{z \in X \mid \varphi(z) < c\}$ is relatively compact for every $c \in \boldsymbol{R}$, and

(2) at each point $z_0 \in X$ the Levi form of φ,

$$\mathcal{L}(\varphi)_{z_0}(u) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z_0)\, u_i \bar{u}_j\,,$$

has at least $n - l$ positive eigenvalues.

Let $\{\varrho_i\colon U_i \to \boldsymbol{R}\}$ be a partition of unity subordinate to the cover $\{U_i\}$ with $\varrho_i \geqslant 0$ and set

$$\theta = \sum_i \varrho_i v_i .$$

There exists a sufficiently small open neighborhood W of τM in X such that θ is smoothly defined in W with $\theta \geqslant 0$, $\theta(z) = 0$ if and only if $z \in \tau M$, and with $\mathfrak{L}(\theta)_z$ (the Levi form of θ at z) having at least $m - 2l$ positive eigenvalues for each $z \in W$.

Since τM is a compact subset of W there exists an open neighborhood U of τM in W with U relatively compact in W. Hence $\partial U = \overline{U} - U$ is a compact subset of W which does not meet τM and we set

$$\delta = \min_{z \in \partial U} \theta(z) > 0 .$$

If $V = \{z \in U | \theta(z) < \delta\}$, then $\overline{V} \subset \overline{U}$ and $g(z) = (1/\delta)\theta(z)$ has the following properties on V:

(1) $0 \leqslant g(z) < 1$ and $g(z) = 0$ if and only if $z \in \tau M$.

(2) $\mathfrak{L}(g)_z$ has at least $m - 2l$ positive eigenvalues for each $z \in V$.

We now define $\varphi\colon V \to R$ by

$$\varphi(z) = \frac{1}{1 - g(z)} .$$

Thus $\{z \in V | \varphi(z) < c\}$ is relatively compact in V for every $c \in \boldsymbol{R}$ and also $\mathfrak{L}(\varphi)_z$ has at least $m - 2l$ positive eigenvalues for each $z \in V$. Since V is an $(m - l)$-dimensional complex manifold we see that it is l-complete and (4.2) is established.

REFERENCES

[1] A. Andreotti, *Introduzione all'analisi complessa*. Contributi del Centro Linceo Interdisciplinare di Scienze Matematiche e loro Applicazioni, no. 24. Accademia Nazionale dei Lincei, Roma, 1976.

[2] A. Andreotti - C. D. Hill, *Complex characteristic coordinates and tangential Cauchy-Riemann equations*, Ann. Scuola Norm. Sup. Pisa, (3) **26** (1972), pp. 299-324.

[3] A. Andreotti - P. Holm, *Quasi-analytic and parametric spaces*, in *Singularities of Real/Complex Maps*, Oslo, 1976, Noordhoof, Leyden, The Netherlands, 1977.

[4] F. Bruhat - H. Whitney, *Quelques propriétés fondamentales des ensembles analytiques-réels*, Comment. Math. Helv., **33** (1959), pp. 132-160.

[5] G. A. Fredricks, *Some remarks on Cauchy-Riemann structures*, to appear in *Differential Topology*. Centro Internazionale Matematico Estivo (C.I.M.E.), III Ciclo, Varenna, Italy, 1976.

[6] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math., **68** (1958), pp. 460-472.

[7] S. J. Greenfield, *Cauchy-Riemann equations in several variables*, Ann. Scuola Norm. Sup. Pisa, (3) **22** (1968), pp. 275-314.

[8] A. Haefliger, *Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes*, Comment. Math. Helv., **32** (1958), pp. 248-329.

[9] L. Nirenberg, *Lectures on Linear Partial Differential Equations*. Regional Conf. Series in Math., no. 17, American Math. Society, Providence, 1973.

[10] H. Rossi, *Differentiable manifolds in complex Euclidean space*, in Proc. Internat. Congr. Math., Moscow, 1966 (Moscow, 1968).

[11] H. Rossi, *Homogeneous strongly pseudoconvex hypersurfaces*, Rice Univ. Studies, **59** (1) (1973), pp. 131-145.

[12] H. B. Shutrick, *Complex extensions*, Quart. J. Math. Oxford Ser., (2) **9** (1958), pp. 181-201.

[13] G. Tomassini, *Tracce delle funzioni olomorfe sulle sottovarietà analitiche reali d'una varietà complessa*, Ann. Scuola Norm. Sup. Pisa, (3) **20** (1966), pp. 31-44.