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A Remark on Runge Approximation of Meromorphic Functions (*).

KLAUS HULEK (**)

0. – Introduction.

Let Ω_1 be an open subset of the complex manifold Ω_2 . In [2] Hirschowitz calls the pair (Ω_1, Ω_2) meromorphic-convex if every function holomorphic in Ω_1 may be uniformly approximated by functions meromorphic in Ω_2 . He calls the pair (Ω_1, Ω_2) μ -convex if even every function meromorphic in Ω_1 may be uniformly approximated by functions meromorphic in Ω_2 . In [2, Theorem 5.1] it is claimed that a meromorphic-convex pair of Stein manifolds is μ -convex if and only if the natural homomorphism $H_2(\Omega_1, \mathbf{R}) \rightarrow H_2(\Omega_2, \mathbf{R})$ is injective. In this paper I shall prove by means of a counter-example that this condition is not necessary.

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1. – An approximation theorem.

PROPOSITION 1. Let Ω_1 be an open and Stein subset of Ω_2 such that the pair (Ω_1, Ω_2) is meromorphic convex. Assume for each hypersurface $h \subset \Omega_1$ and for each $\alpha \in H_2(\Omega_1, \mathbf{Z}_n)$, $n \in \mathbf{N}_0$, that the intersection number $S(h, \alpha)$ vanishes. Then (Ω_1, Ω_2) is μ -convex.

PROOF. Let m be an in Ω_1 meromorphic function which is holomorphic in a neighbourhood of the compact set K . For a given $\varepsilon > 0$ we shall have to construct a meromorphic function \tilde{m} which is also holomorphic in a

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neighbourhood of K , such that $\|m - \tilde{m}\|_K < \varepsilon$. Since Ω_1 is Stein we can exhaust it by special analytic polyhedra. Therefore we can choose such polyhedra P_1, P_2 with

$$K \subset P_1 \subset \bar{P}_1 \subset P_2 \subset \Omega_1.$$

Let h be the set of poles of m . According to our hypothesis we have $S(h, \alpha) = 0$ for all $\alpha \in H_2(\Omega_1, \mathbf{Z}_n)$ where n is an arbitrary non-negative integer. It then follows from [7] that the Poincarè-problem has a solution for m on P_2 , i.e. there are functions $f, g \in \mathcal{O}(P_2)$ which are relatively prime, s. th. $m|_{P_2} = f/g$. In particular we have $M_\sigma := \inf_{x \in K} |g(x)| > 0$. According to [9] f and g can be uniformly approximated by functions holomorphic in Ω_1 . We can choose $f_1, g_1 \in \mathcal{O}(\Omega_1)$ with $\|f - f_1\|_K < \varepsilon$ and $\|g - g_1\|_K < \varepsilon$.

Since (Ω_1, Ω_2) is meromorphic-convex, there are functions m_f and m_g meromorphic in Ω_2 , which are holomorphic in a neighbourhood of K , s. th.

$$\|f_1 - m_f\|_K < \varepsilon \quad \text{and} \quad \|g_1 - m_g\|_K < \varepsilon.$$

Put $\tilde{m} := m_f/m_g$. For sufficiently small ε , \tilde{m} is holomorphic in a neighbourhood of K and for $\varepsilon < \text{Min}\{M_\sigma/4, \|f\|_K, \|g\|_K\}$ we have

$$\begin{aligned} \|m - \tilde{m}\|_K &= \left\| m - \frac{m_f}{m_g} \right\|_K \leq \left\| \frac{f}{g} - \frac{f_1}{g_1} \right\|_K + \left\| \frac{f_1}{g_1} - \frac{m_f}{m_g} \right\|_K \leq \\ &\leq \frac{1}{M_\sigma(M_\sigma - \varepsilon)} \|fg_1 - gf_1\|_K + \frac{1}{(M_\sigma - \varepsilon)(M_\sigma - 2\varepsilon)} \|f_1m_g - g_1m_f\|_K \leq \\ &\leq \frac{4}{M_\sigma^2} (\|fg_1 - fg\|_K + \|fg - f_1g\|_K + \|f_1m_g - f_1g_1\|_K + \|f_1g_1 - g_1m_f\|_K) \leq \\ &\leq \varepsilon \frac{12}{M_\sigma^2} (\|f\|_K + \|g\|_K). \quad *** \end{aligned}$$

COROLLARY. *Again let Ω_1 be open and Stein in Ω_2 , such that (Ω_1, Ω_2) is meromorphic-convex. If $H_2(\Omega_1, \mathbf{Z})$ is divisible and $H_1(\Omega_1, \mathbf{Z})$ is torsion free, then (Ω_1, Ω_2) is μ -convex.*

PROOF. Because of Proposition 1 it suffices to prove that for each hypersurface $h \subset \Omega_1$ and each $\alpha \in H_2(\Omega_1, \mathbf{Z}_n)$ we have $S(h, \alpha) = 0$. For $n = 0$ this is an immediate consequence of the divisibility of $H_2(\Omega_1, \mathbf{Z})$. For $n \neq 0$ the universal coefficient theorem and our hypothesis yield

$$H_2(\Omega_1, \mathbf{Z}_n) \cong H_2(\Omega_1, \mathbf{Z}) \otimes \mathbf{Z}_n \oplus \text{Tor}(H_1(\Omega_1, \mathbf{Z}), \mathbf{Z}_n) = 0. \quad ***$$

2. – Construction of a counterexample.

Let $D := \{(z_1, z_2) \in \mathbf{C}^2; |z_1| < 1, |z_2| < 1\}$ be the standard dicylinder in \mathbf{C}^2 .

PROPOSITION 2. *There exists a domain of holomorphy $G \subset D$ with the following properties:*

- (i) (G, D) is meromorphic-convex.
- (ii) $H_1(G, \mathbf{Z}) \cong \mathbf{Q}$.
- (iii) $H_2(G, \mathbf{Z}) = 0$.

PROOF. In carrying out this construction we follow ideas of Pontrjagin, Stein and Ramspott (see [4], [8] and [5]). We shall construct a sequence of biholomorphic mappings $f_n: \mathbf{C}^2 \rightarrow \mathbf{C}^2$ with $f_n(0) = 0$, of smooth analytic sets $B_n \subset D_n := f_n(D)$ and of neighbourhoods V_n of B_n such that with $A_n := f_n^{-1}(B_n)$ and $U_n := f_n^{-1}(V_n)$ the following conditions are fulfilled:

- (1) $B_n = \{(z_1, z_2) \in D_n; z_2^n - c_n z_1 = 0\}$ for some $c_n \in \mathbf{R}_+$. There is a smooth neighbourhood of $\{z_2^n - c_n z_1 = 0\} \cap \partial D_n$ in ∂D_n and the two manifolds intersect transversally.
- (2) 0 is deformation retract of B_n .
- (3) $D_n - V_n$ and $D_n - B_n$ have the same homotopy type.
- (4) $\bar{U}_n \subset U_{n-1}$ where \bar{U}_n is the closure of U_n in D .
- (5) $d(D - U_n, A_n) > 0$ where d is the Euclidean distance.
- (6) $\check{d}(A_n, A_{n+1}) < 1/2^n$ where \check{d} denotes the Hausdorff metric (see [1], [2]).
- (7) $\check{d}(\bar{U}_n, A_n) < 1/2^n$.

The conditions (6) and (7) imply that the sequences $(A_n)_{n \in \mathbf{N}}$ and $(U_n)_{n \in \mathbf{N}}$ converge to a common limit A . (2), (3) and (4) will enable us to compute the homology of $G := D - A$, the other conditions are necessary for the induction.

To start the induction we choose

$$\begin{aligned} f_1 &:= id, \\ A_1 &:= B_1 := \{(z_1, z_2) \in D; z_2 - \frac{1}{2}z_1 = 0\}. \\ U_1 &:= V_1 := \{(z_1, z_2) \in D; |z_2 - \frac{1}{2}z_1| < \frac{1}{4}\}. \end{aligned}$$

Now we assume that f_n, B_n and V_n are given.

Put $g_n: \mathbf{C}^2 \rightarrow \mathbf{C}^2; (z_1, z_2) \mapsto (z_2, z_2^n - c_n z_1)$.

We take $f_{n+1} := g_n \circ f_n$ and get $D_{n+1} = f_{n+1}(D) = g_n(D_n)$.

Furthermore

$$f_{n+1}(A_n) = g_n(B_n) = \{(z_1, z_2) \in D_{n+1}; z_2 = 0\}.$$

The plane $\{z_2 = 0\}$ intersects ∂D_{n+1} transversally. We define

$$B_{n+1} := \{(z_1, z_2) \in D_{n+1}; z_2^{n+1} - c_{n+1}z_1 = 0\}.$$

For sufficiently small $c_{n+1} \in \mathbf{R}_+$ the conditions (1) and (6) are clearly fulfilled. A retraction of B_n to 0 gives a retraction of $g_n(B_n)$ to 0 and out of this we can construct a retraction for B_{n+1} , hence (2) is valid. We also can choose c_{n+1} sufficiently small such that $B_{n+1} \subset g_n(V_n)$ and $d(B_{n+1}, D_{n+1} - g_n(V_n)) > 0$. We now have to find a suitable neighbourhood V_{n+1} of B_{n+1} . To do this we look at

$$g_{n+1}: \mathbf{C}^2 \rightarrow \mathbf{C}^2; \quad (z_1, z_2) \mapsto (z_2, z_2^{n+1} - c_{n+1}z_1).$$

Again we have

$g_{n+1}(B_{n+1}) = \{(z_1, z_2) \in D_{n+2}; z_2 = 0\}$ where the plane $\{z_2 = 0\}$ intersects ∂D_{n+2} transversally. Put

$$W_{n+1} := \{(z_1, z_2) \in D_{n+2}; |z_2| < \varrho_{n+1}\} \quad \text{for some } \varrho_{n+1} \in \mathbf{R}_+.$$

For sufficiently small ϱ_{n+1} we have according to the above $\overline{W_{n+1}} \subset g_{n+1}(g_n(V_n))$ and $d(D_{n+2} - g_{n+1}(B_{n+1}), D_{n+2} - W_{n+1}) > 0$.

Moreover we can acquire

$$\tilde{d}(\overline{f_{n+1}^{-1}(W_{n+1})}, A_{n+1}) < \frac{1}{2^{n+1}}.$$

The sets

$$D_{n+2} - g_{n+1}(B_{n+1}) = \{(z_1, z_2) \in D_{n+2}; z_2 = 0\}$$

and

$$D_{n+2} - W_{n+1} = \{(z_1, z_2) \in D_{n+2}; |z_2| < \varrho_{n+1}\}$$

have the same homotopy type. If we put $V_{n+1} := g_{n+1}^{-1}(W_{n+1})$ then the conditions (3), (4), (6) and (7) are fulfilled, i.e. V_{n+1} is a suitable neighbourhood of B_{n+1} .

Let A be the limit of the sequence $(A_n)_{n \in \mathbf{N}}$. A is non-empty. We claim that $G := D - A$ has the desired properties. We shall first prove that G is connected. Take two points $(z_1^{(1)}, z_2^{(1)})$, $(z_1^{(2)}, z_2^{(2)}) \in G$. For some big n_0 we

have $(z_1^{(1)}, z_2^{(1)}) \notin U_{n_0} \not\subseteq (z_1^{(2)}, z_2^{(2)})$. Because of $A \subset \overline{U_{n_0+1}} \subset U_{n_0}$ it is sufficient to prove that $D - U_{n_0}$ is pathwise connected. But this is a consequence of the fact that $D - A_{n_0}$ is connected and that both sets have the same homotopy-type. G is a domain of holomorphy. To see this, consider

$$G_m := D - A_m \quad \text{and} \quad \hat{G}_n := \bigcap_{m \geq n} \overset{\circ}{G}_m = D - \overline{\bigcup_{m \geq n} A_m}.$$

As above, one sees that \hat{G}_n is connected. Being the open kernel of an intersection of domains of holomorphy \hat{G}_n is a domain of holomorphy itself. Moreover $\hat{G}_n \subset \hat{G}_{n+1}$ and $G = \bigcup_{n \in \mathbb{N}} \hat{G}_n$. Hence G is a domain of holomorphy.

(See [3, p. 38]). A is a limit of hypersurfaces, hence it is a limace in the terminology of Hirschowitz. It follows from [2; Theorem 3.5] that (G, D) is meromorphic conex. The next step will be to prove $H_2(G, \mathbb{Z}) = 0$. Let β be a 2-cycle in G . For sufficiently big n_0 , β is contained in $D - U_{n_0}$. Thus it suffices to prove $H_2(D - U_{n_0}, \mathbb{Z}) \cong H_2(D - A_{n_0}, \mathbb{Z}) = 0$. The exact homology sequence of the pair $(D, D - A_{n_0})$ yields

$$\dots \rightarrow H_3(D, D - A_{n_0}, \mathbb{Z}) \rightarrow H_2(D - A_{n_0}, \mathbb{Z}) \rightarrow H_2(D, \mathbb{Z}) \rightarrow \dots$$

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On the other hand Alexander-Pontrjagin duality implies $H_3(D, D - A_{n_0}, \mathbb{Z}) \cong \cong H_*^1(A_{n_0}, \mathbb{Z})$, where the star denotes cohomology with compact support. Since A_{n_0} has no singularities Poincarè duality gives $H_*^1(A_{n_0}, \mathbb{Z}) \cong \cong H_1(A_{n_0}, \mathbb{Z}) = 0$, since A_{n_0} is contractible. Hence $H_3(D, D - A_{n_0}, \mathbb{Z}) = 0$, and this clearly implies $H_2(D - A_{n_0}, \mathbb{Z}) = 0$.

It remains to prove $H_1(G, \mathbb{Z}) \cong \mathbb{Q}$. According to [5, Satz 2] we have $H_1(D - U_n, \mathbb{Z}) \cong H_1(D - A_n, \mathbb{Z}) \cong \mathbb{Z}$. We want to construct a generating cycle for these homology groups. Therefore consider

$$f_{n+1}(A_n) = g_n(B_n) = \{(z_1, z_2) \in D_{n+1}; z_2 = 0\}$$

and

$$W_n = f_{n+1}(U_n) = \{(z_1, z_2) \in D_{n+1}; |z_2| < \varrho_n\}.$$

As a generating cycle for $H_1(D_{n+1} - f_{n+1}(A_n), \mathbb{Z}) \cong H_1(D_{n+1} - W_n, \mathbb{Z})$ we can choose

$$\alpha_n := \{(0, \varrho_n \exp [2\pi it]); 0 \leq t < 1\}.$$

Put $t_n := f_{n+1}^{-1}(\alpha_n)$, denote by \bar{t}_n the homology class in $H_1(D - U_n, \mathbb{Z})$ and by $\bar{\bar{t}}_n$ the homology class in $H_1(G, \mathbb{Z})$. The classes $\bar{\bar{t}}_n$ generate $H_1(G, \mathbb{Z})$.

To see this take a 1-cycle α with homology class $\bar{\alpha}$. Then for some n_0 , α is contained in $D - U_{n_0}$ and there it is homologous to some $m \cdot \bar{t}_{n_0}$. In particular $\bar{\alpha} = m \cdot \bar{t}_{n_0}$. Now we have to find the relations between the \bar{t}_n . Therefore consider

$$g_{n+2}(\alpha_n) = \{(\varrho_n \exp [2\pi i t], \varrho_n^{(n+1)} \exp [2\pi i(n+1)t]); 0 \leq t \leq 1\}.$$

In $D_{n+2} - W_{n+1}$ the cycle $g_{n+2}(\alpha_n)$ is homologous to $(n+1) \cdot \alpha_{n+1}$. This implies $\bar{t}_n = (n+1) \cdot \bar{t}_{n+1}$. Moreover $m \cdot \bar{t}_n \neq 0$ for all $m \neq 0$. Because, if we assumed $m \cdot \bar{t}_n = 0$ this would imply that $m \cdot t_n$ was homologous to 0 in some set $D - U_{n_0}$, $n_0 \geq n$. But this would mean $m \cdot (n+1) \dots n_0 \cdot \bar{t}_{n_0} = 0$, a contradiction to $H_1(D - U_{n_0}, \mathbf{Z}) \cong \mathbf{Z}$. This also means that apart from the relations $\bar{t}_n = (n+1) \cdot \bar{t}_{n+1}$ there are no other relations between the \bar{t}_n . The map $\bar{t}_n \mapsto 1/n!$ gives an isomorphism $H_1(G, \mathbf{Z}) \cong \mathbf{Q}$. ***

We can now deliver our counterexample. Take $\dot{E} := \{z \in \mathbf{C}; 0 < |z| < 1\}$ to be the punctured unit-disc in \mathbf{C} . The pair (\dot{E}, \mathbf{C}) is meromorphic-convex. $(G \times \dot{E}, D \times \mathbf{C})$ is meromorphic-convex since it is the product of meromorphic-convex pairs. The Künneth formula yields

$$H_2(G \times \dot{E}, \mathbf{Z}) \cong \mathbf{Q}.$$

$$H_1(G \times \dot{E}, \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Q}.$$

By virtue of our corollary $(G \times \dot{E}, D \times \mathbf{C})$ is μ -convex. On the other hand it follows from the universal coefficient theorem that

$$H_2(G \times \dot{E}, \mathbf{R}) \cong \mathbf{R}.$$

Since

$$H_2(D \times \mathbf{C}, \mathbf{R}) = 0$$

the canonical homomorphism $H_2(G \times \dot{E}, \mathbf{R}) \rightarrow H_2(D \times \mathbf{C}, \mathbf{R})$ cannot be injective.

3. - Remarks

As A. Hirschowitz has pointed out in a discussion, it is the first sentence that contains the mistake in the proof of [2; Theorem 5.1]. There it is assumed that the mapping $H_2(\Omega_1, \mathbf{R}) \rightarrow \text{Hom}(H^2(\Omega_1, \mathbf{Z}), \mathbf{R})$ is injective. This is not true in general. If however the homology of Ω_1 is of finite type there is an exact sequence

$$0 \rightarrow \text{Ext}(H^3(\Omega_1, \mathbf{Z}), \mathbf{R}) \rightarrow H_2(\Omega_1, \mathbf{R}) \rightarrow \text{Hom}(H^2(\Omega_1, \mathbf{Z}), \mathbf{R}) \rightarrow 0.$$

(Cf. [6, p. 248]). Since \mathbf{R} is divisible $\text{Ext}(H^2(\Omega_1, \mathbf{Z}), \mathbf{R}) = 0$. Under this condition as well as under other conditions which imply the injectivity of $H_2(\Omega_1, \mathbf{R}) \rightarrow \text{Hom}(H^2(\Omega_1, \mathbf{Z}), \mathbf{R})$ the arguments given in [2] remain true.

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