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On Bombieri's Asymptotic Sieve.

JOHN FRIEDLANDER (*) - HENRYK IWANIEC (*)

1. – Introduction.

Let (a_n) denote a sequence of non-negative reals and let

$$A(x) = \sum_{n \leqslant x} a_n \,,$$

for $x < X$ where $X \geqslant 2$ and the numbers a_n may depend on X. A basic goal of the sieve is the estimation of the contribution to the sum $A(x)$ of those terms for which n has relatively few prime divisors. This information is usually deduced from information about the sums

$$A(x, d) = \sum_{\substack{n \leqslant x \\ n \equiv O (\mathrm{mod}\, d)}} a_n \,.$$

ASSUMPTIONS. We assume that the latter sums may be written in the form

$$A(x, d) = \frac{A(x)}{f(d)} + R(x, d)$$

subject to the following:

(A_1) The function $f(d)$ is multiplicative, $f(d) > 1$ for $d > 1$, and $f(d) \gg d^{\frac{3}{4}}$.

(A_2) There exists θ_0 with $0 < \theta_0 \leqslant 1$ such that, for every B, for every $\varepsilon > 0$, and for $X \geqslant x \geqslant 2$,

$$\sum_{d < x^{\theta - \varepsilon}} \sup_{1 \leqslant y \leqslant x} |R(y, d)| \ll A(x)(\log x)^{-B} \,,$$

the implied constant depending on θ_0, ε and B.

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(A_3) There exists $C > 0$ such that we have, for $X \geqslant x \geqslant 2$ and $d < x^{\theta_0}$,

$$|R(x, d)| < C\frac{F(d)}{d} A(x)(\log x)^C$$

for some $F(d)$ satisfying

$$\sum_{d<x^{\theta_0}} \frac{F^2(d)}{d} < C(\log x)^C .$$

(A_4) We have $a_n \geqslant 0$.

(A_5) There exist a number field K, a real number η with $0 < \eta < \frac{1}{2}$, an integer N such that $1 \leqslant N \leqslant X$, and a function $\bar{f}(n)$, such that:

(i) $\bar{f}$ satisfies (A_1) and is independent of N and X,

(ii) we always have $f(n) \geqslant \bar{f}(n)$ and, if $(n, N) = 1$, then $f(n) = \bar{f}(n)$,

(iii) if $G(s)$ is defined by

$$\sum_{d\geqslant 1} \frac{1}{\bar{f}(d)\, d^s} = \zeta_K(s+1)\, G(s) ,$$

then the Dirichlet series for $G(s)$ is absolutely convergent for $\sigma \geqslant -\eta$.

(A_6) There exists an integer $g \geqslant 2$ such that, for $x \leqslant X$,

$$\sum_{m\leqslant x^{1/g}} |R(x, m^g)| \ll A(x)(\log x)^{-2} ,$$

where the implied constant may depend on g.

NOTE. We stress that, although (a_n) and f may depend on X, the parameters θ_0, C, $\bar{f}$, g do not, nor do the implied constants in the above assumptions.

EXAMPLES. It is a simple matter to construct examples of sequences which have been treated by the linear sieve and which satisfy the above assumptions. We mention only three.

EXAMPLE 1. Let b be a non-zero integer and define

$$a_n = \Lambda(n + b) ,$$

where Λ is the von Mangoldt function. This sequence satisfies the assumptions with $\theta_0 = \frac{1}{2}$ and $g = 2$. The only assumption not trivially verified is (A_2) and this is an immediate consequence of the Bombieri-Vinogradov theorem.

EXAMPLE 2. Let $H(x)$ denote an irreducible polynomial in $\boldsymbol{Z}[x]$ of degree h, with the coefficient of x^h being positive. Let $a_n = 1$ if $n = H(m)$ for some positive integer m and $a_n = 0$ otherwise. Here the assumptions hold with $\theta_0 = 1/h$ and $g = h+1$.

EXAMPLE 3. Let

$$a_n = \begin{cases} 1 & \text{if } n = qm + a \leqslant X\,, \quad m \in \boldsymbol{Z}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $(a, q) = 1$ and $1 < q < X^\alpha$. Here one may take $\theta_0 = 1 - \alpha$ and $g =$ any integer $> (1-\alpha)^{-1}$.

SOME DEFINITIONS. In a study of integers with few prime factors, it is often convenient to define the generalized von Mangoldt functions

$$\Lambda_k = \mu * L^k\,, \qquad \text{for } k = 0, 1, 2, \ldots$$

where μ is the Möbius function, $*$ denotes Dirichlet convolution and L is the function $L(n) = \log n$.

For a vector $(k) = (k_1, \ldots, k_r)$ of non-negative integers, we define

$$\Lambda_{(k)} = \Lambda_{k_1} * \ldots * \Lambda_{k_r}\,.$$

Letting $|k| = k_1 + \ldots + k_r$, we have (see Lemma 1):

If $\omega(n) > |k|$, then $\Lambda_{(k)}(n) = 0$.

This fact offers partial justification for the first statement of this paragraph. Further justification is offered (see the discussion in [2]) by the fact that linear combinations of the $\Lambda_{(k)}$ can be used in approximating a rather wide class of functions whose support is the set of integers with «few» prime factors.

In the sequel we shall denote by $\beta(x)$ a function satisfying

$$\int_1^x \frac{A(t)}{t}\,dt \leqslant \beta(x)\,A(x)\log x\,.$$

Trivially, we may choose $\beta(x) \leqslant 1$ and, in practice, we usually have much more, e.g. $\beta(x) \ll 1/\log x$.

STATEMENT OF RESULTS. Bombieri has proved [2] the following result.

THEOREM 1 (Bombieri). *Assume that* (a_n) *satisfies* $(A_1\text{-}A_5)$ *with* $\theta_0 = 1 = N$ *and* $K = Q$. *Assume that we may take* $\beta(x) \to 0$ *as* $x \to \infty$. *Let* (k) *be a fixed vector with* $\max k_\nu \geqslant 2$. *We then have*

$$\sum_{n\leqslant x} a_n \Lambda_{(k)}(n) \sim \gamma_{(k)} HA(x)(\log x)^{|k|-1},$$

where

$$H = \prod_p \left(1 - \frac{1}{f(p)}\right)\left(1 - \frac{1}{p}\right)^{-1},$$

$$\gamma_{(k)} = \frac{(k)!}{(|k|-1)!},$$

and, in turn, $(k)! = k_1! \ldots k_n!$.

Our goal is to weaken the assumption $\theta_0 = 1$, and to simultaneously give an estimation uniform in (k). This leads to the following result.

THEOREM 2. *Let* (a_n) *satisfy assumptions* $(A_1\text{-}A_6)$. *Let* $0 < \theta < \theta_0$ *and* $\Delta = (6r)^{16/\theta}$. *Let* $|k| \geqslant 2$ *and define* $a = \max k_\nu$. *There exists a positive constant* c *depending at most on* θ, θ_0, g, $\bar{f}$, K *and* C, *but not on* (k), r, *or* N, *such that, if*

$$\binom{|k|}{a}(1-\theta)^a \leqslant 1, \qquad |k| > c\Delta \text{ and } \log X > |k|^{c|k|}$$

then

$$\sum_{n\leqslant X} a_n \Lambda_{(k)}(n) = \gamma_{(k)} HA(X)(\log X)^{|k|-1}(1 + \emptyset E),$$

where $|\varphi| \leqslant 1$ *and*

$$E < \left(\left(\binom{|k|}{a}(1-\theta)^a\right)^{1/\Delta} + \left(\frac{2}{1-\theta}\right)^a \beta(X)\right)(c|k|)^\Delta.$$

REMARKS I.

1) The constant 16 can be improved (and perhaps replaced by $1 + \varepsilon$). It is probably sufficient to take $\log X > \exp(c|k|)$ but this would have required a strengthening of A_6 and a surprising amount of extra effort.

2) Although $(A_1\text{-}A_5)$ are essentially present in Theorem 1 we have found it is advisable to introduce the additional A_6 for Theorem 2. In the case where one assumes A_2 with $\theta_0 = 1$, the assumption A_3 with $\theta_0 = 1$ seems quite mild. Even in the case where A_2 is not known with $\theta_0 = 1$, it may be possible to show that A_3 holds uniformly for $d < x$. Such is the case for

Example 1 and this example has already been investigated by Bombieri in [1]. It may, however, happen as when $A(x) \ll x^{1-\varepsilon}$ for some ε, such as Example 2, that the assumption of A_3 uniformly for $d < x$ becomes unrealistic. It seems that, on the other hand, the assumption of A_3 with $d < x^{\theta_0}$, while much more reasonable, is insufficient to the completion of the proof and that A_6, an assumption which seems reasonable for sequences satisfying $A(x) \gg x^{\delta}$, is a suitable method for filling this gap. It might be mentioned that, were we to suppose that the support of a_n consisted only of square-free numbers, A_6 could be dispensed with and, indeed, the technical details of the proof of Theorem 2 could be greatly simplified.

3) The other changes in the axioms, the introduction of the field K and the parameters N, X, were motivated by the wish to include sequences such as given in Examples 2 and 3 respectively.

4) It should be mentioned that Bombieri [2] uses Theorem 1 to prove results on $\sum_{n \leqslant x} a_n g(n)$ for a wide class of functions $g(n)$ with support on « near primes ». Due to the less precise estimates in the case $\theta_0 < 1$, we make no attempt to discuss this.

5) The proof of Theorem 2 will make it clear that the conclusion of Theorem 1 remains true even if the assumptions $K = Q$ and $N = 1$ are dropped provided, say, that $X \geqslant N$ and we consider $\sum_{n \leqslant X} a_n \Lambda_{(k)}(n)$.

Consequences of Theorem 2. The import of Theorem 2 is that if (k) ranges through a sequence of vectors and $X \to \infty$, then, under suitable circumstances, one obtains an asymptotic formula for $\sum_{n \leqslant X} a_n \Lambda_{(k)}(n)$. This is illustrated by the following corollary.

Corollary. *Let* (a_n) *satisfy* $(A_1\text{-}A_6)$ *and assume* $\beta(X) = o(1)$. *Let* $0 < \theta < \theta_0$. *There exists a constant* c, *depending at most on* θ, θ_0, C, K, g *and* $\bar{f}$ *such that if* (k) *ranges through a sequence of vectors and* $|k| \to \infty$ *subject to*

(i) $|k| < c \dfrac{\log\log X}{\log\log\log X}$,

(ii) $\dbinom{|k|}{a} < (1-\theta)^{-a}$,

(iii) $r < a^{\theta/33}$

(iv) $a < c \log (1/\beta(X))$,

then we have

$$\sum_{n \leqslant X} a_n \Lambda_{(k)}(n) \sim \gamma_{(k)} H A(X) (\log X)^{|k|-1}.$$

REMARKS II.

1) If (k) is a scalar, so that $r=1$, conditions (ii) and (iii) are trivially satisfied.

2) If $M>1$ is fixed and $|k|<Ma$ then condition (ii) is satisfied provided that

$$\frac{(M-1)^{M-1}}{M^M}>1-\theta .$$

If θ approaches 0, then the best admissible value of M approaches 1. If θ approaches 1, then the best admissible value of M approaches ∞.

3) For each fixed r, there is a constant $\bar{\theta}(r)=1-(r-1)^{r-1}/r^r$ such that, if $\theta>\bar{\theta}$, $|k|<c\log\log X/\log\log\log X$ and $|k|<c\log\big(1/\beta(X)\big)$, then the asymptotic formula holds as $|k|\to\infty$.

NOTATION. We shall use $c, c_1, c_2, \ldots$ to denote positive constants, not necessarily the same at each occurrence. These, as well as all implied constants may depend on $\bar{f}$, g, K, θ, θ_0 and C, but not on (k) or N.

OUTLINE OF CONTENTS. In Section 2 we consider some special sequences associated with the field K. In this case the sum $\sum_{n\leqslant X} a_n \Lambda_{(k)}(n)$ may be evaluated without reference to the sieve, as an elementary consequence of the prime ideal theorem. The proof is complicated by the search for a result uniform in (k). Aside the obvious special interest of the result for these sequences, this result will be used in the proofs of the theorems.

In Section 3 we prove a fundamental lemma of Halberstam-Richert type (see [3]). This result will be used repeatedly throughout the paper.

In Section 4 we give a proof of Bombieri's Theorem 1. Our proof involves some modifications of that in [2], which we believe simplify the presentation. The proof also serves as an outline of the essential ideas of the more technically complicated proof of Theorem 2.

The remaining sections are devoted to the proof of Theorem 2. Section 5 consists of several lemmata to be used in the proof. The sum $\sum_{n\leqslant X} a_n \Lambda_{(k)}(n)$ is dissected into three parts and these are estimated in Sections 6, 7 and 8. In Section 9 these estimates are combined and the proof is concluded.

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2. – Some basic sequences.

Before, commencing with the proof of the main results of this section, we list some results from [2], that will be needed.

DEFINITION.

$$\mathcal{L}(k) = \mu * \Lambda_{(k)}\,.$$

DEFINITION. Let $(k) = (k_1, \ldots, k_r)$ and $(h) = (h_1, \ldots, h_r)$. We say $(h) \leqslant (k)$ if $0 \leqslant h_\nu \leqslant k_\nu$ for each ν and denote by $\binom{(k)}{(h)}$, the « binomial » coefficient

$$\binom{(k)}{(h)} = \prod_{\nu=1}^{r} \binom{k_\nu}{h_\nu}\,.$$

LEMMA 1. *We have*

(i) $\Lambda_{k+1} = \Lambda_k L + \Lambda_k * \Lambda$,

(ii) $\Lambda_{(k)}(n) \leqslant \Lambda_{|k|}(n) \leqslant (\log n)^{|k|}$,

(iii) $|\mathcal{L}_{(k)}(n)| \leqslant \sum_{d|n} \Lambda_{(k)}(d) \leqslant (\log n)^{|k|}$.

(iv) *If* $(m, n) = 1$ *then*

$$\Lambda_{(k)}(mn) = \sum_{(h)\leqslant(k)} \binom{(k)}{(h)} \Lambda_{(h)}(m) \Lambda_{(k-h)}(n)\,.$$

(v) *If* $\omega(n) > |k|$ *then* $\Lambda_{(k)}(n) = 0$.

PROOF. All of these are to be found in [2]. The latter inequality of (ii), which is not specifically mentioned, follows from (iii).

We define the integers $b(n)$, $b_1(n)$ and $b_2(n)$ by their generating functions:

$$\zeta_K(s) = \sum \frac{b(n)}{n^s}\,,$$

$$\prod_{\deg \mathfrak{p}=1} \left(1 + (N\mathfrak{p})^{-s}\right) = \sum \frac{b_1(n)}{n^s}\,,$$

$$\prod_{\deg \mathfrak{p}=1} \left(1 - (N\mathfrak{p})^{-s}\right)^{-1} = \sum \frac{b_2(n)}{n^s}\,,$$

where the two products run over those prime ideals of K which are of the first degree.

The following properties will be frequently used.

(1) $b_1(p) = b_2(p) = b(p) \leq [K:Q]$,

(2) $\mu^2(n)b(n) \leq b_1(n) \leq b_2(n) \leq b(n) \ll n^\varepsilon$,

and all of these are multiplicative,

(3) For $i = 1, 2$, we have

$$b_i(p^{a+b}) \leq b_i(p^a)b_i(p^b)$$

and hence $b_i(mn) \leq b_i(m)b_i(n)$ for all $m, n \geq 1$.

LEMMA 2. *For $x \geq 1$ and $k \geq 1$ we have*

(A) $\sum_{d\leq x}\mu^2(d)b(d)\Lambda_k(d)d^{-1} > (\log x)^k + O\big(k\log 2k(\log cx)^{k-1}\big)$

(B) $\sum_{d\leq x} b_2(d)\Lambda_k(d)d^{-1} < (\log x)^k + O\big(k(\log cx)^{k-1}\big)$.

PROOF. For $k = 1$, these are well known elementary results on the distribution of prime ideals. We assume that both results hold for k. From Lemma 1,

$$\begin{aligned}\sum_{n\leq x} b_2(n)\Lambda_{k+1}(n)n^{-1} &\leq \sum_{n\leq x} b_2(n)\frac{\Lambda_k(n)}{n}\log n + \\ &+ \sum_{n\leq x} b_2(n)\Lambda_k(n)n^{-1}\sum_{m\leq x/n} b_2(m)\Lambda(m)m^{-1} \leq \\ &\leq \sum_{n\leq x} b_2(n)\frac{\Lambda_k(n)}{n}\log n + \sum_{n\leq x} b_2(n)\Lambda_k(n)n^{-1}\left(\log\frac{x}{n} + O(1)\right) = \\ &= \sum_{n\leq x} b_2(n)n^{-1}\Lambda_k(n)\big(\log x + O(1)\big) \leq (\log x)^{k+1} + O\big(k(\log cx)^k\big).\end{aligned}$$

$$\begin{aligned}\sum_{n\leq x}\mu^2(n)b(n)\Lambda_{k+1}(n)n^{-1} &= \\ &= \sum_{n\leq x}\mu^2(n)b(n)\frac{\Lambda_k(n)}{n}\log n + \sum_{mp\leq x}\mu^2(mp)b(mp)\frac{\Lambda_k(m)\log p}{mp} = \\ &= \sum_{n\leq x}\mu^2(n)b(n)\frac{\Lambda_k(n)}{n}\log n + \sum_{m\leq x}\mu^2(m)b(m)\frac{\Lambda_k(m)}{m}\sum_{p\leq x/m} b(p)\frac{\log p}{p} + \\ &\qquad + O\left(\sum_{m\leq x}\mu^2(m)b(m)\frac{\Lambda_k(m)}{m}\sum_{p|m} b(p)\frac{\log p}{p}\right).\end{aligned}$$

Since $b(p) \leq [K:Q]$ and $\omega(m) \leq k$, we have $\sum_{p|m} b(p)\log p/p \ll \log 2k$, and (A) follows from the induction hypothesis as did (B), but with slight additional complications.

LEMMA 3. *There exists a constant $c > 0$, depending only on the field K, such that*

$$(A)\quad \sum_{m\leqslant x}\mu^2(m)b(m)\Lambda_{(k)}(m) > \gamma_{(k)}x(\log x)^{|k|-1} - c^{|k|}x(\log xc)^{|k|-2},$$

$$(B)\quad \sum_{m\leqslant x}b_2(m)\Lambda_{(k)}(m) < \gamma_{(k)}x(\log x)^{|k|-1} + c^{|k|}x(\log xc)^{|k|-2}.$$

PROOF. We assume $k < \log\log x$, since otherwise the result is trivial. We first consider the case where k is scalar. For b, a positive integer we have

$$x(\log x)^b \geqslant \int_1^x (\log t)^b\,dt \geqslant x(\log x)^b - bx(\log x)^{b-1}.$$

Assume that $c_k > k\log 2k$, and we have

$$\sum_{n\leqslant x} b_2(n)\Lambda_k(n) = kx(\log x)^{k-1} + \varphi c_k x(\log xc)^{k-2},$$

and

$$\sum_{n\leqslant x}\mu^2(n)b(n)\Lambda_k(n) = kx(\log x)^{k-1} + \varphi c_k x(\log cx)^{k-2},$$

where $|\varphi| \leqslant 1$. For $k = 1$, the prime ideal theorem gives such a result. We have, by partial summation,

$$\begin{aligned}\sum_{n\leqslant x} b_2(n)\Lambda_k(n)\log n &= kx(\log x)^k + \varphi c_k x(\log xc)^{k-1}\\ &\quad -\int_1^x \big(k(\log t)^{k-1} + \varphi c_k(\log tc)^{k-2}\big)\,dt\\ &= kx(\log x)^k + O\big(c_k x(\log cx)^{k-1}\big)\end{aligned}$$

and the same estimate holds for $\sum_{n\leqslant x}\mu^2(n)b(n)\Lambda_k(n)$.

We have

$$\begin{aligned}\sum_{mn\leqslant x}\mu^2(mn)b(mn)\Lambda_k(m)\Lambda(n) &= \sum_{mn\leqslant x}\mu^2(m)\mu^2(n)b(m)b(n)\Lambda_k(m)\Lambda(n)\\ &\quad + O\Big(\sum_{m\leqslant x}\mu^2(m)b(m)\Lambda_k(m)\sum_{\substack{p\mid m\\ p\leqslant x/m}} b(p)\log p\Big).\end{aligned}$$

Since $b(p) \leqslant [K:Q]$ and $\sum_{p\mid m,\, p\leqslant x/m}\log p \ll k\log(x/m)$, the error term is

$$\ll k\sum_{m\leqslant x}\mu^2(m)b(m)\Lambda_k(m)\log\frac{x}{m} \ll k^2x(\log x)^{k-1} + kc_k x(\log xc)^{k-2}$$

by partial summation and the induction hypothesis. Also,

$$\sum_{mn\leqslant x} b_2(mn)\Lambda_k(m)\Lambda(n) \leqslant \sum_{mn\leqslant x} b_2(m)b_2(n)\Lambda_k(m)\Lambda(n)\,.$$

We write this last sum as

$$\sum_{m\leqslant\sqrt{x}} + \sum_{n\leqslant\sqrt{x}} - \sum_{m,n\leqslant\sqrt{x}} = S_1 + S_2 - S_3\,.$$

(The same is done for $\sum_{mn\leqslant x} \mu^2(m)\mu^2(n)b(m)b(n)\Lambda_k(m)\Lambda(n)$. We omit the details for this sum which are entirely similar).

By the prime ideal theorem,

$$S_1 = \sum_{m\leqslant\sqrt{x}} b_2(m)\frac{\Lambda_k(m)}{m}x\left(1 + O\left(\frac{1}{\log xc}\right)\right)$$

which, by Lemma 2 is

$$= 2^{-k}x(\log x)^k + O\big(k(\log 2k)x(\log cx)^{k-1}\big)\,.$$

By the induction hypothesis,

$$S_2 = \sum_{n\leqslant\sqrt{x}} b_2(n)\frac{\Lambda(n)}{n}\left(\log\frac{x}{n}\right)^{k-1} kx\left(1 + O\left(\frac{c_k}{k\log cx}\right)\right)$$
$$= (1-2^{-k})x(\log x)^k + O\big(c_k x(\log cx)^{k-1}\big)\,.$$

By the induction hypothesis

$$S_3 = \Big(\sum_{m\leqslant\sqrt{x}} b_2(m)\Lambda_k(m)\Big)\Big(\sum_{n\leqslant\sqrt{x}} \Lambda(n)\Big) \ll kx(\log x)^{k-1}\left(1 + O\left(\frac{c_k}{k\log cx}\right)\right).$$

Collecting together we get the result for $k+1$ with $c_{k+1} \ll c_k$, which completes the proof for scalar k.

Now, let $(k) = (k_1, \ldots, k_r)$, $(k') = (k_1, \ldots, k_{r-1})$ and $a = k_r$. Assume that

$$\sum_{n\leqslant x} \mu^2(n)\, b(n)\Lambda_{(k')}(n) = \frac{(k')!}{(|k'|-1)!}x(\log x)^{|k'|-1} + O\big(c^{|k'|}x(\log cx)^{|k'|-2}\big)\,,$$

and that $\sum_{n\leqslant x} b_2(n)\Lambda_{(k')}(n)$ satisfies the same estimate.

We have

$$\sum_{n\leqslant x}\mu^2(n)b(n)\Lambda_{(k)}(n) = \sum_{mn\leqslant x}\mu^2(m)b(m)\Lambda_{(k')}(m)\mu^2(n)b(n)\Lambda_a(n) + O\Big(\sum_{m\leqslant x}\mu^2(m)b(m)\Lambda_{(k')}(m)\sum_{\substack{(n,m)>1\\ n\leqslant x/m}}\mu^2(n)b(n)\Lambda_a(n)\Big).$$

If $a=1$, the error term is estimated as in the scalar case. If $a\geqslant 2$, we write $n=ps$, where $p|m$, and Lemma 1 gives

$$\sum_{\substack{(n,m)>1\\ m\leqslant x/m}} \leqslant \sum_{j=1}^{a}\binom{a}{j}\sum_{s\leqslant x/m}\mu^2(s)b(s)\Lambda_{a-j}(s)\sum_{\substack{p|m\\ p\leqslant x/ms}} b(p)\Lambda_j(p)\leqslant$$

$$\leqslant [K:Q]\sum_{j=1}^{a}\binom{a}{j}\sum_{s\leqslant x/m}\mu^2(s)b(s)\Lambda_{a-j}(s)\Big(\log\frac{x}{ms}\Big)^j\leqslant$$

$$\leqslant [K:Q]\sum_{j=1}^{a}\binom{a}{j}\Big(\log\frac{x}{m}\Big)^{j-1}\sum_{s\leqslant x/m}\mu^2(s)b(s)\Lambda_{a-j}(s)\log\frac{x}{ms}.$$

The sum over s was estimated in the scalar case and so the above is $\leqslant c^a(x/m)(\log cx)^{a-2}$. The error term is thus

$$\ll c^a x(\log cx)^{a-2}\sum_{m\leqslant x}\mu^2(m)b(m)\Lambda_{(k')}(m)m^{-1}.$$

This sum is estimated by partial summation and the induction hypothesis, making the error

$$\ll c^a x(\log xc)^{|k|-2}(1+c^{|k'|}(\log cx)^{-1}).$$

We turn now to the main term. (We omit the details for $b_2(m)$. The main term is similar, using $b_2(mn)\leqslant b_2(m)b_2(n)$, and there is no error term corresponding to the above).

By partial summation,

$$\sum_{n\leqslant x}\mu^2(n)b(n)\frac{\Lambda_a(n)}{n}\Big(\log\frac{x}{n}\Big)^j = j\int_1^x\Big((\log t)^a + O\big(a^2(\log ct)^{a-1}\big)\Big)\Big(\log\frac{x}{t}\Big)^{j-1}\frac{dt}{t} =$$

$$= j(\log x)^{a+j}\int_0^1 u^a(1-u)^{j-1}du + O\big(a^2 j(\log cx)^{a+j-1}\big) =$$

$$= \frac{a!\,j!}{(a+j)!}(\log x)^{a+j} + O\big(a^2 j(\log cx)^{a+j-1}\big)$$

(see p. 56 of [5]).

Hence

$$\sum_{n\leqslant x}\mu^2(n)b(n)\Lambda_a(n)\sum_{m\leqslant x/n}\mu^2(m)b(m)\Lambda_{(k')}(m)\geqslant$$

$$\geqslant\frac{(k)!}{(|k|-1)!}x(\log x)^{|k|-1}+O\left(\frac{(k')!}{(|k'|-1)!}a^2|k'|x(\log cx)^{|k|-2|}\right)+$$

$$+O\left(c^{|k'|}\frac{a!(|k'|-2)!}{(a+|k'|-2)!}x(\log cx)^{|k|-2}\right)+O(c^{|k'|}|k'|a^2x(\log cx)^{|k|-3}).$$

Collecting the estimates together gives the final result.

LEMMA 4. *The sequence $b_1(n)$ satisfies the assumptions (A_1-A_6) for every X, with $\theta_0 = N = 1$.*

PROOF. We denote by $\mathfrak{a}$ an ideal of K (all our ideals will be integral), $\mathfrak{g}$ an ideal which is square-free and free of prime ideals of degree >1, $\sum^*$ a sum over ideals of this type, $\mathfrak{b}$ an ideal free of primes of degree >1, $\sum^{**}$ a sum over ideals of this type, and $\mathfrak{p}$ a prime ideal.

We note that there exists $\delta>0$ such that

$$\sum_{N\mathfrak{a}\leqslant x}^{*}1=V_1x+O(x^{1-\delta}),$$

and

$$\sum_{N\mathfrak{a}\leqslant x}^{**}1=V_2x+O(x^{1-\delta}),$$

where the implied constants depend only on K and δ, where

$$V_1=\prod_{\deg\mathfrak{p}=1}(1-(N\mathfrak{p})^{-2})\prod_{\deg\mathfrak{p}>1}(1-(N\mathfrak{p})^{-1})\operatorname{res}\zeta_K,$$

$$V_2=\prod_{\deg\mathfrak{p}>1}(1-(N\mathfrak{p})^{-1})\operatorname{res}\zeta_K,$$

and $\operatorname{res}\zeta_K$ is the residue at $s=1$ of the simple pole of $\zeta_K(s)$. (We omit the proof of these estimates which follow from a routine application of Cauchy's theorem, using the estimate

$$|\zeta_K(s)|<|t|^{c(1-\sigma)+\varepsilon},\quad\text{for }|t|>1\geqslant\sigma).$$

Now, fixing $\mathfrak{g}$ as above

$$\sum_{\substack{N\mathfrak{a}\leqslant Y\\(\mathfrak{a},\mathfrak{g})=1}}^{**}1=\sum_{\nu|\mathfrak{g}}\mu(\nu)\sum_{\substack{N\mathfrak{a}\leqslant Y\\\nu|\mathfrak{a}}}^{**}1=\sum_{\nu|\mathfrak{g}}\mu(\nu)\left\{V_2\frac{Y}{N\nu}+O\left(\frac{Y}{N\nu}\right)^{1-\delta}\right\}=$$

$$=V_2Y\prod_{\mathfrak{p}|\mathfrak{g}}\left(1-\frac{1}{N\mathfrak{p}}\right)+O\left(Y^{1-\delta}\prod_{\mathfrak{p}|\mathfrak{g}}(1+(N\mathfrak{p})^{\delta-1})\right).$$

Thus

$$\sum_{\substack{N\mathfrak{a}\leqslant Y\\(\mathfrak{a},\mathfrak{g})=1}}^{*} 1 = \sum_{\substack{N\mathfrak{a}\leqslant Y\\(\mathfrak{a},\mathfrak{g})=1}}^{**}\sum_{\mathfrak{b}^2|\mathfrak{a}}\mu(\mathfrak{b}) = \sum_{(\mathfrak{b},\mathfrak{g})=1}^{**}\mu(\mathfrak{b})\sum_{\substack{N\mathfrak{a}\leqslant Y\\(\mathfrak{a},\mathfrak{g})=1\\\mathfrak{b}^2|\mathfrak{a}}}^{**} 1 =$$

$$= \sum_{(\mathfrak{b},\mathfrak{g})=1}^{**}\mu(\mathfrak{b})\left\{V_2\frac{Y}{(N\mathfrak{b})^2}\prod_{\mathfrak{p}|\mathfrak{g}}\left(1-\frac{1}{N\mathfrak{p}}\right)+O\left(\frac{Y^{1-\delta}}{(N\mathfrak{b})^{2(1-\delta)}}\prod_{\mathfrak{p}|\mathfrak{g}}(1+(N\mathfrak{p})^{\delta-1})\right)\right\} =$$

$$= V_2 Y\prod_{\mathfrak{p}|\mathfrak{g}}\left(1+\frac{1}{N\mathfrak{p}}\right)^{-1}\prod_{\deg\mathfrak{p}=1}\left(1-\frac{1}{(N\mathfrak{p})^2}\right)+O\left(Y^{1-\delta}\prod_{\mathfrak{p}|\mathfrak{g}}(1+(N\mathfrak{p})^{\delta-1})\right) =$$

$$= V_1 Y\prod_{\mathfrak{p}|\mathfrak{g}}\left(1+\frac{1}{N\mathfrak{p}}\right)^{-1}+O\left(Y^{1-\delta}\prod_{\mathfrak{p}|\mathfrak{g}}(1+(N\mathfrak{p})^{\delta-1})\right).$$

We are now in a position to evaluate the function $f(d)$ for the special sequence $b_1(n)$ and to estimate $R(x, d)$, thus verifying the assumptions. We may take d to have no prime factors which are not norms in the field, since for other d we take $f(d) = \infty$ and have $R(x, d) = 0$.

Let $d = \prod_{l=1}^{r} p_l^{\beta_l}$, and let $\mathfrak{g}_l$ be the product of all g_l prime ideals of norm p_l. Note that $g_l = b(p_l) \leqslant [K:Q]$. Let $\mathfrak{g} = \prod_{l=1}^{r}\mathfrak{g}_l$. For a vector $(\alpha) = (\alpha_1, \ldots, \alpha_r)$, we have

$$U_{(\alpha)} = \sum_{\substack{N\mathfrak{a}\leqslant x\\p_1^{\alpha_1}\|N\mathfrak{a}\\\cdots\cdots\\p_r^{\alpha_r}\|N\mathfrak{a}}}^{*} 1 = \sum_{(\mathfrak{p})_{(\alpha)}}\sum_{\substack{N\mathfrak{a}\leqslant x\\(\mathfrak{a},\mathfrak{g})=1\\\prod_{l=1}^{r}\mathfrak{p}_1^{(l)}\cdots\mathfrak{p}_{\alpha_l}^{(l)}|\mathfrak{a}}}^{*} 1 = \sum_{(\mathfrak{p})_{(\alpha)}}\sum_{\substack{N\mathfrak{a}\leqslant x/d_{(\alpha)}\\(\mathfrak{a},\mathfrak{g})=1}}^{*} 1,$$

where $(\mathfrak{p})_{(\alpha)}$ denotes a subset of primes $\mathfrak{p}_1^{(l)}, \ldots, \mathfrak{p}_{\alpha_l}^{(l)}$, for each $l = 1, \ldots, r$ and $d_{(\alpha)} = p_1^{\alpha_1}\ldots p_r^{\alpha_r}$. We have

$$U_{(\alpha)} = V_1\frac{x}{d_{(\alpha)}}\prod_{\mathfrak{p}|\mathfrak{g}}\left(1+\frac{1}{N\mathfrak{p}}\right)^{-1}\sum_{(\mathfrak{p})_{(\alpha)}} 1 + O\left(\left(\frac{x}{d_{(\alpha)}}\right)^{1-\delta}\sum_{(\mathfrak{p})_{(\alpha)}}\prod_{\mathfrak{p}|\mathfrak{g}}(1+(N\mathfrak{p})^{\delta-1})\right) =$$

$$= V_1 x\prod_{l=1}^{r}\binom{g_l}{\alpha_l}\left(1+\frac{1}{p_l}\right)^{-g_l}p_l^{-\alpha_l} + O\left(x^{1-\delta}\prod_{l=1}^{r}\binom{g_l}{\alpha_l}(1+p_l^{\delta-1})^{g_l}p_l^{(\delta-1)\alpha_l}\right)$$

and

$$\sum_{\substack{N\mathfrak{a}\leqslant x\\N\mathfrak{a}\equiv O(\mathrm{mod}\, d)}}^{*} 1 = \sum_{(\alpha)\geqslant(\beta)} U_{(\alpha)} = V_1 x\prod_{l=1}^{r}\sum_{\alpha_l\geqslant\beta_l}\binom{g_l}{\alpha_l}\left(1+\frac{1}{p_l}\right)^{-g_l}p_l^{-\alpha_l} +$$

$$+ O\left(x^{1-\delta}\prod_{l=1}^{r}\sum_{\alpha_l\geqslant\beta_l}(1+p_l^{\delta-1})^{-g_l}p_l^{(\delta-1)\alpha_l}\binom{g_l}{\alpha_l}\right).$$

This formula makes it clear that f is multiplicative, that

$$\frac{1}{f(p^\alpha)} \leqslant \frac{1}{f(p^{\alpha-1})} \leqslant \ldots \leqslant \frac{1}{f(p)},$$

that

$$\frac{1}{f(p)} = 1 - \left(1 + \frac{1}{p}\right)^{-b(p)} < 1$$

and that

$$\frac{1}{f(p^\beta)} = \sum_{\alpha \geqslant \beta} \left(1 + \frac{1}{p}\right)^{-b(p)} p^{-\alpha} \binom{b(p)}{\alpha} \leqslant \sum_{\alpha \geqslant \beta} \binom{b(p)}{\alpha} p^{-\alpha} \leqslant$$
$$\leqslant 2^{[K:Q]} p^{-\beta} \left(1 + \frac{1}{p-1}\right) \ll p^{-\frac{3}{4}\beta}.$$

Noting that the error satisfies

$$|R(x, d)| \ll x^{1-\delta} \prod_{l=1}^{r} 4^{g_l} p_l^{(\delta-1)\beta_l} \ll 4^{[K:Q]\omega(d)} (x/d)^{1-\delta}$$

it is now easy to check the remaining assumptions. We give only the details for (A_2). Letting $q = [K:Q]$,

$$\sum_{d < x^{1-\varepsilon}} 4^{q\omega(d)} (x/d)^{1-\delta} \leqslant x^{1-\delta} x^{(1-\varepsilon)\delta} \sum_{d<x} \frac{4^{q\omega(d)}}{d} \leqslant$$
$$\leqslant x^{1-\varepsilon\delta} \prod_{p<x} \left(1 + 4^q \left(\frac{1}{p} + \frac{1}{p^2} + \ldots\right)\right) = x^{1-\varepsilon\delta} \prod_{p<x} \left(1 + \frac{4^q}{p-1}\right) \ll x^{1-\varepsilon\delta} (\log cx)^{4^q}.$$

3. – A fundamental lemma.

The following lemma is essentially due to Halberstam and Richert (see p. 82 of [3]). We, however, shall have need of a somewhat different formulation of their result.

LEMMA 5. *Let $z \geqslant 2$. Let P^* be a set of primes and $P^*(z) = \prod_{\substack{p<z \\ p \in P^*}} p$. Let $s \geqslant 2$ and $y = z^s$. There exist two sequences $\{\lambda_d^\pm\}_{d | P^*(z)}$ of real numbers such that*

(i) $\lambda_1^\pm = 1$, $|\lambda_d^\pm| \leqslant 1$ *and* $\lambda_d^\pm = 0$ *for* $d \geqslant y$,

(ii) *for all* $D | P^*(z)$, $D > 1$

$$\sum_{d|D} \lambda_d^+ \geqslant 0, \quad \sum_{d|D} \lambda_d^- \leqslant 0.$$

(iii) *for all multiplicative functions* $f(d)$ *satisfying the condition*

$$(1)\qquad \prod_{\substack{w\leqslant p<z\\ p\in P^*}}\left(1-\frac{1}{f(p)}\right)\geqslant\frac{1}{K_0}\left(\frac{\log w}{\log z}\right)^{\varkappa},\qquad \textit{for some } K_0\geqslant 1\,,$$

some $\varkappa>0$, *and all* z, w *with* $z>w>1$

we have

$$(2)\qquad \sum_{d|P^*(z)}\frac{\lambda_d^{\pm}}{f(d)}=\prod_{p|P^*(z)}\left(1-\frac{1}{f(d)}\right)\{1+O(e^{-s})\}\,,$$

where the implied constant depends only on K_0 *and* $\varkappa$.

REMARK. As in [3], the error term e^{-s} can be improved.

PROOF. Let $\beta\geqslant 2$. We choose $\mu(d)\lambda_d^{\pm}$ as the characteristic functions of the sets $D^{\pm}$;

$$D^+=\left\{d|P^*(z);\, d=p_1\dots p_r,\, p_r<\dots<p_1<z,\, p_{2l+1}^{\beta}p_{2l}\dots p_1<y \text{ for all } 1\leqslant l\leqslant\frac{r-1}{2}\right\},$$

$$D^-=\left\{d|P^*(z);\, d=p_1\dots p_r,\, p_r<\dots<p_1<z,\, p_{2l}^{\beta}p_{2l-1}\dots p_1<y \text{ for all } 1\leqslant l\leqslant\frac{r}{2}\right\}.$$

It is easy to see that

$$d\in D^{\pm}\Rightarrow d<y\,.$$

For $D|P^*(z)$, $D>1$ we have

$$\sum_{d|D}\lambda_d^{\pm}=\sum_{d|(D/p(D))}(\lambda_d^{\pm}+\lambda_{dp(D)}^{\pm})$$

where $p(D)$ is the smallest prime divisor of D. Hence, we get (ii).

We have

$$(3)\qquad \sum_{d|P^*(z)}\frac{\lambda_d^+}{f(d)}=V(z)+\sum_{r\geqslant 1}\sum_{\substack{(y/p_1\dots p_{2r})^{1/\beta}\leqslant p_{2r+1}<\dots<p_1<z\\ p_{2l+1}<(y/p_1\dots p_{2l})^{1/\beta}\text{ for } 1\leqslant l<r}}\frac{V(p_{2r+1})}{f(p_1\dots p_{r2+1})}$$

and

$$(4)\qquad \sum_{d|P^*(z)}\frac{\lambda_d^-}{f(d)}=V(z)-\sum_{r\geqslant 1}\sum_{\substack{(y/p_1\dots p_{2r-1})^{1/\beta}\leqslant p_{2r}<\dots<p_1<z\\ p_{2l}<(y/p_1\dots p_{2l-1})^{1/\beta}\text{ for } 1\leqslant l<r}}\frac{V(p_{2r})}{f(p_1\dots p_{2r})}$$

where

$$V(z)=\prod_{p\mid P^*(z)}\left(1-\frac{1}{f(p)}\right).$$

If $d=p_1\dots p_{2r+1}$, $p_{2r+1}<\dots<p_1$ appears in the sum (3), then $p_1\dots p_{2l}<y^{1-(1-2/\beta)^l}$ for $l=1,\dots,r$ (by induction on l) and hence

$$p_{2r+1}>y^{(1/\beta)(1-2/\beta)^r}\quad\text{and}\quad p_1\geqslant y^{1/(\beta+2r)}.$$

Similarly, if $d=p_1\dots p_{2r}$, $p_{2r}<\dots<p_1$ appears in the sum (4), then

$$p_1\cdots p_{2l-1}<y^{1-\frac12(1-2/\beta)^l}\quad\text{for } l=1,\dots,r$$
$$p_{2r}>y^{1/2\beta(1-2/\beta)^r}\quad\text{and}\quad p_1\geqslant y^{1/(\beta+2r-1)}.$$

Hence

(5) $$\sum V(p_{2r+1})/f(p_1\cdots p_{2r+1})\leqslant\frac{V(y^{(1/\beta)(1-2/\beta)^r})}{(2r+1)!}\left(\sum_{y^{(1/\beta)(1-2/\beta)^r}\leqslant p<y^{1/s}}\frac{1}{f(p)}\right)^{2r+1}$$

and

(6) $$\sum V(p_{2r})/f(p_1\cdots p_{2r})\leqslant\frac{V(y^{(1/2\beta)(1-2/\beta)^r})}{(2r)!}\left(\sum_{y^{(1/2\beta)(1-2/\beta)^r}\leqslant p<y^{1/s}}\frac{1}{f(p)}\right)^{2r}.$$

From (1) we have

$$\sum_{w\leqslant p<z}\frac{1}{f(p)}\leqslant\varkappa\log\left(K_0\frac{\log z}{\log w}\right)$$

so, the sums (5) and (6) are less than

$$V(z)\,K_0\frac{\beta}{s}\left(1-\frac{2}{\beta}\right)^{-r}\frac{1}{(2r+1)!}\left(\varkappa r\log\frac{\beta}{\beta-2}+\varkappa\log\frac{\beta K_0}{s}\right)^{2r+1}$$

and

$$V(z)\,K_0\frac{2\beta}{s}\left(1-\frac{2}{\beta}\right)^{-r}\frac{1}{(2r)!}\,\varkappa r\log\frac{\beta}{\beta-2}+\varkappa\log\frac{2\beta K_0}{s}\Big)^{2r}$$

respectively. Note that, for $s>\beta+2r$, the sum (5) and, for $s>\beta+2r-1$, the sum (6) are empty. Thus,

$$\sum_{\beta+2r\leqslant s}\sum\frac{V}{f}\leqslant$$
$$\leqslant\frac{\beta K_0}{s}V(z)\sum_{\beta+2r\geqslant s}\left(1-\frac{2}{\beta}\right)^{-r}\frac{1}{(2r+1)!}\left(\varkappa r\log\frac{\beta}{\beta-2}+\varkappa\log\beta K_0\right)^{2r+1}\leqslant$$

$$\leqslant \beta K_0 V(z) e^{\beta - s} \sum_{r \geqslant 1} \left(1 - \frac{2}{\beta}\right)^{-r} \frac{1}{(2r+1)!} \left(\varkappa e r \log \frac{\beta}{\beta - 2} + \varkappa e \log \beta K_0\right)^{2r+1} \ll$$

$$\ll V(z) e^{-s},$$

since for sufficiently large absolute β the series $\sum_{r \geqslant 1}$ converges.

We can do the same with (4). This completes the proof.

4. – A Theorem of Bombieri.

In this section we shall give a modified proof of Theorem 1 of Bombieri [2]. Hence, we assume that axioms $(A_1\text{-}A_5)$ hold for $N = \theta_0 = 1$. Axiom A_6 will not be needed. We also assume $\beta(x) \to 0$ as $x \to \infty$ and $K = Q$.

We begin by dividing the sum $\sum_{n \leqslant x} a_n \Lambda_{(k)}(n)$ into three parts as follows: Let $(k) = \big((k'), a\big)$, where $a = \max k_\nu$ (The order of the components is clearly immaterial).

$$\Sigma_0 = \sum_{\substack{n \leqslant x \\ (n, P(z)) > 1}} a_n \Lambda_{(k)}(n),$$

$$\Sigma_1 = \sum_{\substack{n \leqslant x \\ (n, P(z)) = 1}} a_n \sum_{\substack{d | n \\ d < y}} \mathfrak{L}_{(k')}(d) \left(\log \frac{n}{d}\right)^a,$$

$$\Sigma_2 = \sum_{\substack{n \leqslant x \\ (n, P(z)) = 1}} a_n \sum_{\substack{d | n \\ d \geqslant y}} \mathfrak{L}_{(k')}(d) \left(\log \frac{n}{d}\right)^a.$$

Here, y and z will be chosen later and $P(z) = \prod_{p < z} p$.

Remark. In the case when (k) is scalar, $\mathfrak{L}_{(k')}$ is just the Möbius function.

The division of the sum into Σ_1 and Σ_2 corresponds to Bombieri's division, while the introduction of Σ_0 allows us to avoid Bombieri's use of a supplementary set of coefficients λ_d, at the same time enabling us to estimate Σ_2 much more simply. Each device makes it possible to estimate Σ_2 at the right order of magnitude. In comparison with Σ_1 of [2], ours is complicated slightly by the condition $\big(n, P(z)\big) = 1$. In practice the choice of z will be sufficiently small so that, with the aid of the fundamental lemma, we can see that the asymptotic formula is not destroyed.

Lemma 6. *Let g_δ be defined by*

$$\frac{1}{f(m)} = \frac{1}{m} \sum_{\delta d = m} b_2(d) g_\delta .$$

Then, for $z \geqslant 1$,

$$\sum_{\delta \geqslant z} |g_\delta| \delta^{-1} \ll z^{-\eta} .$$

PROOF. One checks readily that, by (A_5),

$$G_2(s) = \sum_{\delta \geqslant 1} g_\delta \delta^{-1-s}$$

is absolutely convergent for $\sigma \leqslant -\eta$, so

$$\sum_{\delta \geqslant z} |g_\delta| \delta^{-1} \leqslant z^{-\eta} \sum |g_\delta| \delta^{\eta-1} \ll z^{-\eta} .$$

LEMMA 7. *For* $z \geqslant 2 + (\log N)^2$, *we have*

$$\prod_{p<z} \left(1 - \frac{1}{f(p)}\right) = H' \prod_{p<z} \left(1 - \frac{1}{p}\right)^{b(p)} \{1 + O(z^{-\eta})\} \ll (\log z)^{-1}$$

where

$$H' = \prod_p \left(1 - \frac{1}{f(p)}\right)\left(1 - \frac{1}{p}\right)^{-b(p)} .$$

REMARK. The same method shows that, if f is replaced by $\bar{f}$, this result holds for $z \geqslant 2$. From this, restriction (iii) of Lemma 5 follows easily for P, the set of all primes, and, a fortiori, for P^* any subset thereof, with $\varkappa = 1$.

PROOF. We have

$$S = \frac{1}{H'} \prod_{p<z} \left(1 - \frac{1}{f(p)}\right)\left(1 - \frac{1}{p}\right)^{-b(p)} = \prod_{p \geqslant z} \left(1 - \frac{1}{p}\right)^{b(p)} \left(1 + \frac{1}{f(p)} + \frac{1}{f(p^2)} + \ldots\right) =$$
$$= \prod_{p \geqslant z} \left(1 + \frac{1}{f(p)} - \frac{b(p)}{p} + O(p^{\varepsilon - \frac{3}{2}})\right).$$

We assume $z > c(\bar{f}, K)$, so $\log S$ is meaningful (for smaller z the result is trivial) and hence

$$|\log S| \ll \sum_{p \geqslant z} \left| \frac{1}{f(p)} - \frac{b(p)}{p} \right| + z^{\varepsilon - \frac{1}{2}} \ll \sum_{p \geqslant z} |g| p^{-1} + \sum_{\substack{p|N \\ p \geqslant z}} \frac{b(p)}{p} + z^{\varepsilon - \frac{1}{2}} \ll z^{-\eta}$$

if ε is sufficiently small, since the second sum is $\ll \log N/(z \log z)$.

This gives the first half of the result, the latter half being well-known.

LEMMA 8. *For* $2 \leqslant z < x$, *we have*

$$\sum_{\substack{d \leqslant x \\ (d, P(z))=1}} \frac{1}{f(d)} \ll \frac{\log x}{\log z}.$$

PROOF. We have $f(d) \geqslant \bar{f}(d)$ and

$$\sum_{\substack{d \leqslant x \\ (d, P(z))=1}} \frac{1}{\bar{f}(d)} \ll \prod_{z \leqslant p \leqslant x} \left(1 + \frac{1}{\bar{f}(p)} + \frac{1}{\bar{f}(p^2)} + \ldots\right) \ll \prod_{z \leqslant p \leqslant x} \left(1 + \frac{1}{p}\right)^{b(p)} \ll \frac{\log x}{\log z}.$$

LEMMA 9. *Denote*

$$S_d = \frac{1}{f(d)} \prod_{p | P(z)} \left(1 - \frac{1}{f(p)}\right) - H' \frac{b_2(d)}{d} \prod_{p | P(z)} \left(1 - \frac{1}{p}\right)^{b(p)}.$$

For $2 + (\log N)^2 \leqslant z \leqslant x$, *we have*

$$\sum_{\substack{d \leqslant x \\ (d, P(z))=1}} |S_d| \ll z^{-\eta} \frac{\log x}{\log z}.$$

PROOF.

$$S_d = \frac{1}{f(d)} \left(\prod_{p | P(z)} \left(1 - \frac{1}{f(p)}\right) - H' \prod_{p | P(z)} \left(1 - \frac{1}{p}\right)^{b(p)}\right) + \\ + H' \left(\frac{1}{f(d)} - \frac{b_2(d)}{d}\right) \prod_{p | P(z)} \left(1 - \frac{1}{p}\right)^{b(p)}.$$

Moreover,

$$\sum_{\substack{d \leqslant x \\ (d, P(z))=1}} \left|\frac{1}{f(d)} - \frac{b_2(d)}{d}\right| \ll \sum_{\substack{d \leqslant x \\ (d, P(z))=1}} \frac{1}{d} \sum_{\substack{\delta\delta' = d \\ \delta > 1}} |g_\delta| b_2(\delta') + \sum_{\substack{d \leqslant x \\ (d, P(z))=1 \\ (d, N) > 1}} \frac{b_2(d)}{d}.$$

The former sum is $\ll \sum_{\delta \geqslant z} |g_\delta| \delta^{-1} \sum_{\delta' \leqslant x} b_2(\delta')(\delta')^{-1} \ll z^{-\eta} \log x$ by lemma 6.

The latter sum is

$$\leqslant -1 + \prod_{\substack{p | N \\ p \geqslant z}} \left(1 + \frac{b(p)}{p} + \frac{b(p^2)}{p^2} + \ldots\right) \leqslant$$

$$\leqslant -1 + O\left(\prod_{\substack{p | N \\ p \geqslant z}} \left(1 + \frac{1}{p}\right)^{[K:Q]}\right) \leqslant$$

$$\leqslant -1 + \exp\left(\frac{c \log N}{z \log z}\right) \ll z^{-\eta}.$$

The result now follows from Lemmata 7 and 8.

LEMMA 10. *For fixed* (k)

$$\sum\nolimits_0 \ll A(x)(\log x)^{|k|-2}\log z + o\big(A(x)(\log x)^{|k|-1}\big) .$$

PROOF. This follows from Lemmata 1 and 2 of [2] where the same bound is given for an obviously larger sum.

This proof can also be simplified by means of the fundamental lemma. We shall not do this as, in the sequel, we shall be giving a bound uniform in (k).

LEMMA 11. *If* $zy < x$ *and* $zx^{\frac{1}{2}} < y < x^{1-\varepsilon}$, *then*

$$\Sigma_2 \ll A(x)\left(\log\frac{x}{y}\right)^{a+1}(\log x)^{|k'|}(\log z)^{-2} ,$$

where the implied constant may depend on f, C, *and* ε.

PROOF. Using Lemma 1 we get

$$\Sigma_2 \leqslant \left(\log\frac{x}{y}\right)^{a}(\log x)^{|k'|}\sum_{\substack{d\leqslant x/y\\ (d,P(z))=1}}\ \sum_{\substack{n\leqslant x\\ (n,P(z))=1\\ n\equiv 0 (\mathrm{mod}\ d)}} a_n .$$

By Lemma 5 we get

$$\sum_{\substack{n\leqslant x\\ (n,P(z))=1\\ n\equiv 0 (\mathrm{mod}\ d)}} a_n \leqslant \sum_{\substack{n\leqslant x\\ n\equiv 0(\mathrm{mod}\ d)}} a_n \sum_{\substack{\nu|n\\ \nu|P(z)\\ \nu\leqslant z^2}}\lambda_\nu^+ = \sum_{\substack{\nu|P(z)\\ \nu\leqslant z^2}}\lambda_\nu^+ A(x,\nu d) \ll$$

$$\ll \frac{A(x)}{f(d)}\prod_{p|P(z)}\left(1-\frac{1}{f(p)}\right) + \sum_{\substack{\nu|P(z)\\ \nu\leqslant z^2}}|R(x,\nu d)| .$$

The result follows from (A_2), (A_3) and Lemmata 7 and 8.

LEMMA 12. *Let* $\varepsilon > 0$. *If* $s \geqslant 2$, $z \geqslant 2$ *and* $z_\nu y \leqslant x^{1-\varepsilon}$, *then*

$$\Sigma_1 = HA(x)F + O\left\{\left(e^{-s}+\beta(x)+\frac{c(\varepsilon)}{\log x}+z^{-\eta}\log x\right)A(x)(\log x)^{|k|-1}\left(\frac{\log x}{\log z}\right)^2\right\}$$

where $c(\varepsilon)$ *depends only on* ε *and*

$$F = \prod_{p|P(z)}\left(1-\frac{1}{p}\right)\sum_{\substack{d<y\\ (d,P(z))=1}}\frac{\mathfrak{L}_{(k')}(d)}{d}\left(\log\frac{x}{d}\right)^a .$$

PROOF. We have

$$\Sigma_1 = \sum_{b=0}^{a}(-1)^{a-b}\binom{a}{b}\sum_{\substack{d<y\\ (d,P(z))=1}}\mathfrak{L}_{(k')}(d)(\log d)^{a-b}\sum_{\substack{n\leqslant x\\ (n,P(z))=1\\ n\equiv 0(\mathrm{mod}\ d)}} a_n(\log n)^b .$$

The sum over n has upper and lower bounds given by

$$\sum_{\substack{n\leqslant x\\ n\equiv 0(\mathrm{mod}\, d)}} a_n(\log n)^b \sum_{\substack{\nu|P(z)\\ \nu|n\\ \nu\leqslant z^s}} \lambda_\nu^\pm = \sum_{\substack{\nu|P(z)\\ \nu\leqslant z^s}} \lambda_\nu^\pm \sum_{\substack{n\leqslant x\\ n\equiv 0(\mathrm{mod}\,\nu d)}} a_n(\log n)^b .$$

Partial summation (twice) gives

$$\sum_{\substack{n\leqslant x\\ n\equiv 0(\mathrm{mod}\, d\nu)}} a_n(\log n)^b = A(x,\nu d)(\log x)^b - \int_1^x A(t,\nu d)\, d(\log t)^b =$$

$$= \left(\frac{A(x)}{f(d\nu)} + R(x,\nu d)\right)(\log x)^b - \int_1^x \left(\frac{A(t)}{f(\nu d)} + R(t,\nu d)\right) d(\log t)^b =$$

$$= \frac{1}{f(\nu)f(d)} \sum_{n\leqslant x} a_n(\log n)^b + O\big((\log x)^b \sup_{1\leqslant t\leqslant x} |R(t,\nu d)|\big) .$$

By Lemmata 5 and 7

$$\sum_{\substack{n\leqslant x\\ (n,P(z))=1\\ n\equiv 0(\mathrm{mod}\, d)}} a_n(\log n)^b = \frac{1}{f(d)} \prod_{p|P(z)} \left(1 - \frac{1}{f(p)}\right) \sum_{n\leqslant x} a_n(\log n)^b + O\left(e^{-s}\frac{A(x)}{f(d)}\frac{(\log x)^b}{\log z}\right) +$$

$$+ O\Big((\log x)^b \sum_{\substack{\nu|P(z)\\ \nu<z^s}} \sup_{t\leqslant x} |R(t,\nu d)|\Big) .$$

By partial summation

$$\sum_{n\leqslant x} a_n(\log n)^b = A(x)(\log x)^b\big(1 + O(b\beta(x))\big)$$

and hence

$$\Sigma_1 = A(x) \prod_{p|P(z)} \left(1 - \frac{1}{f(p)}\right) \sum_{\substack{d<y\\ (d,P(z))=1}} \frac{\mathfrak{L}_{(k')}(d)}{f(d)} \left(\log \frac{x}{d}\right)^a +$$

$$+ O\left(\big(e^{-s} + a\beta(x)\big) A(x)(\log x)^{|k|-1} \left(\frac{\log x}{\log z}\right)^2 + c(\varepsilon) A(x)(\log x)^{|k|-2}\right) .$$

Using Lemma 9 we can remove the dependence on f from the main term, getting

$$\Sigma_1 = HA(x)F + O\left\{\left(e^{-s} + \beta(x) + \frac{c(\varepsilon)}{\log x} + z^{-\eta}\log x\right) A(x)(\log x)^{|k|-1} \left(\frac{\log x}{\log z}\right)^2\right\} .$$

This completes the proof of Lemma 12.

CONCLUSION OF PROOF OF THEOREM 1. Choose $y = x^{1-2\varepsilon}$, $z = x^{\varepsilon^{4/3}}$ and $s = \varepsilon^{-1/3}$. (This is clearly not an optimal choice, but is sufficient). We have

$$\Sigma_0 \ll \varepsilon^{4/3} A(x)(\log x)^{|k|-1},$$

$$\Sigma_1 = HA(x)F + O\big(\varepsilon^{-8/3} e^{-\varepsilon^{-1/3}} A(x)(\log x)^{|k|-1}\big)$$

and

$$\Sigma_2 \ll \varepsilon^{1/3} A(x)(\log x)^{|k|-1}$$

(since $a \geqslant 2$) provided $x > x_0 \ (\varepsilon, (k))$. Thus

$$\sum_{n \leqslant x} a_n \Lambda_{(k)}(n) = HA(x)F + O\big(\varepsilon^{1/3} A(x)(\log x)^{|k|-1}\big)$$

Since the sequence $a_n \equiv 1$ satisfies the axioms, the above result holds also for it and with the same F. Combining this with Lemma 3,

$$F = \gamma_{(k)}(\log x)^{|k|-1}(1 + O(\varepsilon^{1/3}))$$

and hence the result follows.

5. – Auxiliary lemmata.

This section contains various lemmata whih will be needed for the proof of Theorem 2.

LEMMA 13. *There exists a constant c such that, if $x > 1$ and $k < \frac{1}{4}\eta \log x$ then*

$$\sum_{n \leqslant x} \frac{\Lambda_k(n)}{f(n)} \leqslant (\log x)^k + ck(\log cx)^{k-1} .$$

PROOF. We have $f(n) \geqslant \bar{f}(n)$ and

$$S = \sum_{n \leqslant x} \frac{\Lambda_k(n)}{\bar{f}(n)} = \sum_{m \leqslant x} \frac{\Lambda_k(m)}{m} \sum_{\delta d = m} g_\delta b_2(d) = \sum_{\delta} g_\delta \sum_{\substack{m \leqslant x \\ m \equiv 0 (\mathrm{mod}\ \delta)}} b_2\left(\frac{m}{\delta}\right) \frac{\Lambda_k(m)}{m} .$$

Let $m = \delta h u$, where $(u, \delta) = 1$ and $h | \delta^\infty$, i.e. h divides some power of δ. This gives

$$S = \sum_{j=0}^{k} \binom{k}{j} \sum_{\delta} \frac{g_\delta}{\delta} \sum_{h | \delta^\infty} b_2(h) \frac{\Lambda_j(h\delta)}{h} \sum_{\substack{u < x/\delta h \\ (u, \delta) = 1}} b_2(u) \frac{\Lambda_{k-j}(u)}{u} .$$

For the term $j = 0$, the only non-zero contribution comes when $\delta = h = 1$ and by Lemma 2,

$$S \leqslant (\log x + c)^k + \sum_{j=1}^{k} \binom{k}{j} (\log x + c)^{k-j} \sum_{\delta \leqslant x} \frac{|g_\delta|}{\delta} \sum_{h | \delta^\infty} b_2(h) \frac{\Lambda_j(\delta h)}{h} .$$

Since $e^u > uj/j!$, choosing $u = \frac{1}{4}\eta \log \delta\, h$ we get

$$b_2(h)\Lambda_j(\delta h) \ll h^\varepsilon(\log \delta h)^j < h^\varepsilon \left(\frac{4}{\eta}\right)^j j!(\delta h)^{\eta/4} < \left(\frac{4}{\eta}\right)^j j!(\delta h)^{\eta/3}$$

for $\varepsilon < \eta/12$. Hence

$$S < (\log x + c)^k \left\{1 + c(\varepsilon) \sum_{j=1}^{k} \left(\frac{4k/\eta}{\log x + c}\right)^j \sum_\delta |g_\delta| \delta^{-1+\eta/3} \sum_{h|\delta^\infty} h^{-1+\eta/3}\right\}.$$

Since

$$\sum_{h|\delta^\infty} h^{-1+\eta/3} \leqslant \sum_{h_1|\delta} h_1^{-1+\eta/3} \sum_{h_2|\delta^\infty} h_2^{-2+2\eta/3} < \delta^{2\eta/3} \sum_{h_1} h_1^{-1-\eta/3} \sum_{h_2} h_2^{-2+2\eta/3} \ll \delta^{2\eta/3},$$

by (A_5) and the assumption on k, S is less than

$$(\log x)^k + ck(\log cx)^{k-1}.$$

LEMMA 14. *There exists a constant c such that, for $x > 1$,*

$$\sum_{n \leqslant x} \mu^2(n)\Lambda_k(n)/f(n) > (\log x)^k - ck(\log 2k)(\log\log 3N)(\log cx)^{k-1}.$$

PROOF. For any $c_1(\bar{f})$ and $k > c_1 \log x/(\log\log 3x)(\log\log 3N)$, the result is trivial. For smaller k it is clearly sufficient to prove that, for some $c_2(\bar{f})$ we have

$$\sum_{n \leqslant x} \mu^2(n)\,\Lambda_k(n)/f(n) \geqslant \big(\log x - c_2(\log 2k)(\log\log 3N)\big)^k.$$

We have

$$\sum_{p \leqslant x} \frac{\log p}{f(p)} \geqslant \sum_{p \leqslant x} \frac{b(p)\log p}{p} + O\left(\sum_p \frac{g_p \log p}{p}\right) - \sum_{p|N} \frac{b(p)\log p}{p} \geqslant$$
$$\geqslant \log x + O(\log\log 3N)$$

which gives the result for $k = 1$.

Assuming the result for k, we get

$$\sum_{n \leqslant x} \mu^2(n)\,\Lambda_{k+1}(n)/f(n) = \sum_{n \leqslant x} \mu^2(n)\Lambda_k(n)\log n/f(n) +$$
$$+ \sum_{n \leqslant x} \mu^2(n)\Lambda_k(n)/f(n) \sum_{\substack{m \leqslant x/n \\ (m,n)=1}} \mu^2(m)\Lambda(m)/f(m) \geqslant$$
$$\geqslant \sum_{n \leqslant x} \mu^2(n)\frac{\Lambda_k(n)}{f(n)}\left\{\log n + \sum_{p \leqslant x/n} \frac{b(p)\log p}{p} - \sum_{p|nN} \frac{b(p)\log p}{p} - \right.$$
$$\left. - \sum_p |g_p| \frac{\log p}{p}\right\}.$$

Since n is divisible by at most k distinct primes, the expression in parentheses is at least $\log x - c_2(\log 2k)(\log\log 3N)$ which we may assume is positive. This completes the proof.

LEMMA 15. *For $x > 1$, we have*

$$\sum_{n\leqslant x}\mu^2(n)\Lambda_{(k)}(n)/f(n)\leqslant\frac{\gamma_{(k)}}{|k|}(\log x)^{|k|}+c_{(k)}(\log x)^{|k|-1},$$

where $c_{(k)} < (c|k|)^{3r}$.

PROOF. We have the trivial estimate

$$T=\sum_{n\leqslant x}\mu^2(n)\Lambda_{(k)}(n)/f(n)<(\log x)^{|k|}\sum_{n\leqslant x}\frac{1}{f(n)}\ll(\log x)^{|k|+1},$$

so the result holds for $|k|\geqslant\frac14\eta\log x$. For smaller $|k|$ the result is proved by induction on r. For $r=1$, it follows from Lemma 13. We have

$$T\leqslant\sum_{mn\leqslant x}\mu^2(m)\mu^2(n)\frac{\Lambda_a(m)}{f(m)}\frac{\Lambda_{(k')}(n)}{f(n)}\leqslant\sum_{m<x}\frac{\Lambda_a(m)}{f(m)}\sum_{n<x/m}\mu^2(n)\frac{\Lambda_{(k')}(n)}{f(n)}\leqslant$$
$$\leqslant\sum_{m<x}\frac{\Lambda_a(m)}{f(m)}\left(\frac{\gamma_{(k')}}{|k'|}\left(\log\frac{x}{m}\right)^{|k'|}+c_{(k')}\left(\log\frac{x}{m}\right)^{|k'|-1}\right).$$

Also

$$\sum_{m\leqslant x}\frac{\Lambda_a(m)}{f(m)}\left(\log\frac{x}{m}\right)^l=\int_1^x\left(\log\frac{x}{u}\right)^l dS(u)=-\int_1^x S(u)\,d\left(\log\frac{x}{u}\right)^l<$$
$$<-\int_1^x\{(\log u)^a+O\big(a^2(\log u)^{a-1}\big)\}\,d\left(\log\frac{x}{u}\right)^l=$$
$$=a\int_1^x\left(\log\frac{x}{u}\right)^l(\log u)^{a-1}\frac{du}{u}+O\big(a^2(\log x)^{a+l-1}\big)=$$
$$=\frac{a!\,l!}{(a+l)!}(\log x)^{a+l}+O\big(a^2(\log x)^{a+l-1}\big).$$

Hence

$$T\leqslant\frac{\gamma_{(k)}}{|k|}(\log x)^{|k|}+O\left(\left(a^2\frac{\gamma_{(k')}}{|k'|}+c_{(k')}\frac{a!(|k'|-1)!}{(a+|k'|-1)!}\right)(\log x)^{|k|-1}\right)+$$
$$+O\big(c_{(k')}a^2(\log x)^{|k|-2}\big),$$

which gives the result.

COROLLARY. *For* $\log x > (|k|!/(k)!)(c|k|)^{3r}$ *we have*

$$\sum_{n\leqslant x} \mu^2(n)\frac{\Lambda_{(k)}(n)}{f(n)} \ll \frac{\gamma_{(k)}}{|k|}(\log x)^{|k|}.$$

PROOF. Immediate.

REMARK. The same method works for $\sum_{n\leqslant x} b_2(n)\Lambda_{(k)}(n)/n$ giving the same estimate.

LEMMA 16. *Let* $q\geqslant 2$ *and* $l\geqslant 1$. *We have*

(i) $\sum_{m\leqslant x}(\tau_q(m))^l m^{-1} < e^l(\log x + q^l)^{q^l}$,

(ii) $\tau_q(m)\leqslant q^{\Omega(m)}$,

(iii) $\tau_{a+b} = \tau_a * \tau_b$.

PROOF. In the proof of (i) we use a result of C. Mardjanichvili [4],

$$\sum_{m\leqslant x}(\tau_q(m))^l < A_q^{(l)}x(\log x + q^l)^{q^l-1},$$

where

$$A_q^{(l)} = \frac{q^l}{(q!)^{q^l-1/q-1}} \leqslant \frac{q^l}{q!} < e^l.$$

Hence, by partial summation we get (i). The parts (ii) and (iii) are immediate.

LEMMA 17. *We have*

$$\left(\frac{n}{e}\right)^n < n! < 2\left(\frac{n+1}{e}\right)^{n+1}.$$

LEMMA 18. *Let* $(k) = (k_1, \ldots, k_r)$, $L_{(k)} = L^{k_1}\ldots * L^{k_r}$ *and*

$$c_{(k)} = \frac{k_1^{k_1}\ldots k_r^{k_r}}{|k|^{|k|}}.$$

Then

$$c_{(k)} < 2\gamma_{(k)} \quad \textit{and} \quad L_{(k)}(n)\leqslant c_{(k)}\tau_r(n)(\log n)^{|k|}.$$

PROOF. By Lemma 17 we have

$$c_{(k)} = \frac{(k_1/e)^{k_1}\ldots(k_r/e)^{k_r}}{(|k|/e)^{|k|}} < 2\gamma_{(k)}.$$

Also

$$L_{(k)}(n) \leqslant \tau_r(n) \sup_{\substack{u_1 \dots u_r = n \\ u_i \geqslant 1,\ u_i \in R}} (\log u_1)^{k_1} \dots (\log u_r)^{k_r} .$$

We use Lagrange multipliers. Consider

$$F(u_1, \dots, u_r, \lambda) = (\log u_1)^{k_1} \dots (\log u_r)^{k_r} - \lambda(u_1 \dots u_r - n) .$$

Setting $\partial F / \partial u_i = 0$, we get

$$\frac{k_i}{\log u_i} \prod_{j=1}^{r} (\log u_j)^{k_j} = \lambda u_1 \dots u_r .$$

Hence $\log u_i / k_i =$ constant (independent of i). We have also

$$\Sigma \log u_i = \log n .$$

The result follows from these.

COROLLARY. *We have*

(i) $|\mathfrak{L}_{(k)}(n)| \leqslant z\gamma_{(k)} \tau_{r+2}(n)(\log n)^{|k|}$,

(ii) $\Lambda_{(k)}(n) \leqslant 2\gamma_{(k)} \tau_{r+1}(n)(\log n)^{|k|}$.

REMARK. We define $\gamma_{(0)} = 1$.

PROOF. Let $\mu_r = \mu * \dots * \mu$ (r times). Clearly $|\mu_r(n)| \leqslant \tau_r(n)$;

$$\mathfrak{L}_{(k)} = \mu_{r+1} * L_{(k)} ,$$
$$\Lambda_{(k)} = \mu_r \quad * L_{(k)} .$$

Now the result follows from Lemma 18.

LEMMA 19. *Let P^* and $P^*(z)$ be as in the fundamental lemma. For $(d, P^*(z)) = 1$, we have*

$$\sum_{\substack{n \leqslant x \\ (n, P^*(z)) = 1 \\ n \equiv 0 (\mathrm{mod}\ d)}} a_n \ll \frac{A(x)}{f(d)} \prod_{p | P^*(z)} \left(1 - \frac{1}{f(p)}\right) + \sum_{\substack{\nu | P^*(z) \\ \nu \leqslant z^2}} |R(x, \nu d)| .$$

PROOF. This follows from the fundamental lemma in the same way as in the proof of Lemma 11.

DEFINITION. For simplicity we write

$$R^*(x, d) = \sup_{1 \leqslant y \leqslant x} |R(y, d)| .$$

LEMMA 20. *Let* $q \geqslant 2$, $\alpha \geqslant 2$, $z > 16(\alpha q)^3$ *and*

$$\delta_n = \tau_q(n) \sum_{\substack{d|n \\ (d, P(z))=1}} \alpha^{\Omega(d)} .$$

We have

$$\sum_{n \leqslant x^\theta} \delta_n R^*(x, n) \ll \left(c \frac{\log x}{\log z}\right)^{6(\alpha q)^3} A(x)(\log x + q^3)^{q^3 + C - (B/3)}$$

where B is arbitrary and the implied constant depends on B.

PROOF. By Hölder's inequality we have

$$\sum_{n \leqslant x^\theta} \delta_n R^*(x, n) \leqslant \left(\sum_{n \leqslant x^\theta} \frac{\delta_n^3}{n}\right)^{\frac{1}{3}} \left(\sum_{n \leqslant x^\theta} n^{\frac{1}{2}} (R^*(x, n))^{\frac{3}{2}}\right)^{\frac{2}{3}} .$$

Each n can be uniquely represented in the form $n = n' n''$, where $(n', P(z)) = 1$ and all the prime divisors of n'' are $\leqslant z$. We have

$$\delta_n \leqslant \tau_q(n'') \tau_q(n') \tau(n') \alpha^{\Omega(n')} \leqslant \tau_q(n'')(2\alpha q)^{\Omega(n')} .$$

Hence

$$\sum_{n \leqslant x^\theta} \frac{\delta_n^3}{n} \leqslant \left(\sum_{n \leqslant x^\theta} \frac{(\tau_q(n))^3}{n}\right) \left(\sum_{\substack{n \leqslant x^\theta \\ (n, P(z))=1}} (2\alpha q)^{3\Omega(n)} n^{-1}\right) \leqslant$$

$$\leqslant e^3 (\log x + q^3)^{q^3} \prod_{z \leqslant p \leqslant x} \left(1 - \frac{(2\alpha q)^3}{p}\right)^{-1} .$$

For $0 < x < \frac{1}{2}$ we have $(1 - x)^{-1} < e^{2x}$. We have also

$$\sum_{z \leqslant p \leqslant x} \frac{1}{p} < \log\left(\frac{\log x}{\log z}\right) + c .$$

Hence

$$\sum_{n \leqslant x^\theta} \frac{\delta_n^3}{n} \leqslant \left(c \frac{\log x}{\log z}\right)^{16(\alpha q)^3} (\log x + q^3)^{q^3} .$$

We have

$$R^*(x, n) < C \frac{F(n)}{n} A(x)(\log x)^C .$$

Thus

$$\sum_{n<x^\theta} n^{\frac12}(R^*(x,n))^{\frac34} < CA(x)(\log x)^C \sum_{n\leqslant x^\theta} \frac{F(n)}{\sqrt{n}}\,(R^*(x,n))^{\frac14} <$$

$$\leqslant CA(x)(\log x)^C\Bigl(\sum_{n\leqslant x^\theta}\frac{F^2(n)}{n}\Bigr)^{\frac12}\Bigl(\sum_{n\leqslant x^\theta} R^*(x,n)\Bigr)^{\frac12} \ll$$

$$\ll (A(x))^{\frac32}(\log x)^{(C-B/3)\frac32}.$$

This completes the proof.

LEMMA 21. *There exists $c>0$ such that, for $\alpha\geqslant 1$ and $z>(c\alpha)^{8/\theta}$ we have*

$$\sum_{\substack{n\leqslant x\\ (n,P^*(z))=1}} \alpha^{\Omega(n)} a_n \ll \alpha^{4/\theta} A(x) \prod_{p|P^*(z)}\Bigl(1-\frac{1}{f(p)}\Bigr)\Bigl(c\frac{\log x}{\log z}\Bigr)^{\alpha^{4/\theta}} + \alpha^{4/\theta} \sum_{\substack{d\leqslant x^{\theta/4}\\ (d,P^*(z))=1}} \alpha^{(4/\theta)\Omega(d)} \sum_{\substack{\nu|P^*(z)\\ \nu\leqslant z^2}} |R(x,\nu d)|,$$

where the implied constant is independent of α.

PROOF. Let $n=p_1\dots p_l$ where $p_1\leqslant\dots\leqslant p_l$. Put $d=p_1\dots p_t$ where $t=[\theta l/4]$. If $t=0$ then $d=1\leqslant x^{\theta/4}$. If $t>0$ then $p_t\geqslant d^{1/t}$, so $x\geqslant n\geqslant d(d^{1/t})^{l-t}=d^{l/t}$ giving $d\leqslant x^{t/l}\leqslant x^{\theta/4}$.

Also

$$\alpha^{\Omega(n)}=\alpha^l\leqslant\alpha^{(4/\theta)(t+1)}=\alpha^{(4/\theta)(1+\Omega(d))}.$$

Hence

$$S=\sum_{\substack{n\leqslant x\\ (n,P^*(z))=1}}\alpha^{\Omega(n)}a_n\leqslant\alpha^{4/\theta}\sum_{\substack{d\leqslant x^{\theta/4}\\ (d,P^*(z))=1}}\alpha^{(4/\theta)\Omega(d)}\sum_{\substack{n\leqslant x\\ (n,P^*(z))=1\\ n\equiv 0 (\mathrm{mod}\ d)}} a_n.$$

Substituting in Lemma 19 we have $S\ll S_1+S_2$, where

$$S_1=\alpha^{4/\theta}\Bigl(\sum_{\substack{d\leqslant x^{\theta/4}\\ (d,P^*(z))=1}}\frac{\alpha^{(4/\theta)\Omega(d)}}{f(d)}\Bigr)A(x)\prod_{p|P^*(z)}\Bigl(1-\frac{1}{f(p)}\Bigr)$$

and

$$S_2=\alpha^{4/\theta}\sum_{\substack{d\leqslant x^{\theta/4}\\ (d,P^*(z))=1}}\alpha^{(4/\theta)\Omega(d)}\sum_{\substack{\nu|P^*(z)\\ \nu\leqslant z^2}}|R(x,\nu d)|.$$

The sum occuring in S_1 can be majorized by

$$\prod_{z\leqslant p\leqslant x}\Bigl(1+\frac{\alpha^{4/\theta}}{f(p)}+\frac{\alpha^{8/\theta}}{f(p^2)}+\dots\Bigr)<\Bigl(c\frac{\log x}{\log z}\Bigr)^{\alpha^{4/\theta}}$$

by the lower bound for z. This completes the proof.

LEMMA 22. *We have*

$$\sum_{(h)\leqslant(k)} \binom{(k)}{(h)} \gamma_{(h)} \leqslant |k| 2^{|k|} .$$

PROOF. We have $\gamma_{(h)} \leqslant |k|$ and

$$\sum_{(h)\leqslant(k)} \binom{(k)}{(h)} = 2^{|k|} .$$

This completes the proof.

LEMMA 23. *For $x > 1$ we have, uniformly for $\theta > 0$,*

$$\sum_{x^{\theta/4g} < m \leqslant x^{1/g}} A(x, m^g) \ll A(x)(\theta \log x)^{-2} .$$

PROOF. We have

$$\sum_{x^{\theta/4g} < m \leqslant x^{1/g}} A(x, m^g) \leqslant A(x) \sum_{m > x^{\theta/4g}} \frac{1}{f(m^g)} + \sum_{m \leqslant x^{1/g}} |R(x, m^g)| .$$

By (A_6) the second sum has the required bound and since $g \geqslant 2$ and $f(m) \gg m^{\frac{3}{4}}$ the first sum is

$$\ll \int_{x^{\theta/4g}}^{\infty} t^{-\frac{3}{4}g}\, dt < 2x^{(\theta/4g)(1-\frac{3}{4}g)} < 2x^{-\theta/16} \ll (\theta \log x)^{-2} .$$

This completes the proof.

6. – Sieving out small primes.

We divide the sum $\sum_{n \leqslant X} a_n \Lambda_{(k)}(n)$ into three parts precisely as in Section 4. In this section we estimate Σ_0 uniformly in (k). From now on we take $x = X$.

LEMMA 24. *There exists $c > 0$ such that, if*

$$c \log \log X + (c|k|)^{(4r)^{12/\theta}} |k|^{|k|} < \log z < \frac{\log X}{\mathrm{cr}|k|} ,$$

then

$$\sum_{\substack{n \leqslant X \\ (n, P(z)) > 1}} a_n \Lambda_{(k)}(n) < (c|k|)^{(4r)^{4/\theta}} \gamma_{(k)} A(X) (\log X)^{|k|-2} \log z .$$

PROOF. Let $\zeta = X^{1/cr|k|}$, so $z < \zeta$. The constant c is to be chosen (as will be apparent) so that some of the previous lemmata may be applied.

For n a positive integer, write $n = n_1 n_2$ where all the prime factors of n_1 are $\leqslant \zeta$, while those of n_2 are $> \zeta$.

Let

$$\sum_{\substack{n\leqslant X\\(n,P(z))>1}} a_n \Lambda_{(k)}(n) = \sum_{n_1\geqslant X^{\theta/2}} + \sum_{n_1<X^{\theta/2}} = S_1 + S_2 .$$

If $n_1 \geqslant X^{\theta/2}$, let m_1^g be the largest g-th power dividing n_1. Thus $n_1 m_1^{-g}$ has at most $|k|$ distinct prime factors, each $\leqslant \zeta$, and each occuring with multiplicity $\leqslant g-1$. Hence

$$n_1 m_1^{-g} \leqslant \zeta^{(g-1)|k|} \quad \text{and} \quad m_1^g \geqslant n_1 \zeta^{-(g-1)|k|} > X^{\theta/4}$$

(assuming $\theta c > 4g$). Thus

$$S_1 \leqslant (\log X)^{|k|} \sum_{m^g > X^{\theta/4}} A(X, m^g) \ll A(X)(\log X)^{|k|-2} ,$$

by Lemma 23.

Let $\mathcal{N}_l$ denote the set of positive integers divisible by precisely l distinct primes $\leqslant z$. We have

$$S_2 = \sum_{l=1}^{|k|} \sum_{\substack{n\leqslant X\\ n\in\mathcal{N}_l\\ n_1<X^{\theta/2}}} a_n \Lambda_{(k)}(n) = \sum_{l=1}^{|k|} S_2(l) ,$$

and

$$S_2(l) = \sum_{\substack{(h)\leqslant(k)\\ |h|\geqslant l}} \binom{(k)}{(h)} \sum_{\substack{n_1 n_2\leqslant X\\ n_1\in\mathcal{N}_l\\ n_1<X^{\theta/2}}} a_{n_1 n_2} \Lambda_{(h)}(n_1) \Lambda_{(k-h)}(n_2) .$$

By Lemma 18 (Corollary) we have

$$\sum_{\substack{n_2\leqslant X/n_1\\ (n_2,P(\zeta))=1}} a_{n_1 n_2} \Lambda_{(k-h)}(n_2) \leqslant 2\gamma_{(k-h)} (\log X)^{|k-h|} \sum_{\substack{n_2\leqslant X/n_1\\ (n_2,P(\zeta))=1}} a_{n_1 n_2} \tau_{r+1}(n_2)$$

which, by the argument used in the proof of Lemma 21, is

$$\leqslant 2\gamma_{(k-h)} (\log X)^{|k-h|} (r+1)^{4/\theta} \sum_{\substack{d\leqslant X^{\theta/4}\\ (d,P(\zeta))=1}} (r+1)^{(4/\theta)\Omega(d)} \sum_{\substack{n_2\leqslant X/n_1\\ n_2\equiv 0 (\mathrm{mod}\ d)\\ (n_1 n_2, P^*(\zeta))=1}} a_{n_1 n_2}$$

where P^* is the set of primes not dividing n_1. By Lemma 19 this is

$$\ll \gamma_{(k-h)}(\log X)^{|k-h|}(r+1)^{4/\theta} \sum_{\substack{d\leqslant X^{\theta/4} \\ (d,P((\zeta))=1}} (r+1)^{(4/\theta)\Omega(d)}\left\{\frac{A(x)}{f(d)f(n_1)}\prod_{p|P^*(\zeta)}\left(1-\frac{1}{f(p)}\right)+\right.$$
$$\left.+\sum_{\substack{\nu|P^*(\zeta) \\ \nu\leqslant\zeta^2}}|R(X, n_1\nu d)|\right\}.$$

Since n_1 is divisible by at most $|k|$ distinct prime factors, we have

$$\prod_{p|P^*(\zeta)}\left(1-\frac{1}{f(p)}\right)\leqslant\prod_{p|P(\zeta)}\left(1-\frac{1}{f(p)}\right)\prod_{p|n_1}\left(1-\frac{1}{f(p)}\right)^{-1}\ll\frac{\log 2|k|}{\log\zeta}.$$

Now, summing over n_1 and then over (h) we get

$$S_2(l)\ll A(X)\frac{\log 2|k|}{\log\zeta}(r+1)^{4/\theta}\left(c\frac{\log X}{\log\zeta}\right)^{(r+1)^{4/\theta}}\sum_{\substack{(h)\leqslant(k) \\ |h|\geqslant l}}\binom{(k)}{(h)}\gamma_{(k-h)}(\log X)^{|k-h|}\cdot$$
$$\cdot\sum_{\substack{n_1<x \\ n_1\in\mathcal{N}_l}}\frac{\Lambda_h(n_1)}{f(n_1)}+(r+1)^{4/\theta}(\log X)^{|k|}\sum_{(h)\leqslant(k)}\binom{(k)}{(h)}\gamma_{(k-h)}\cdot$$
$$\cdot\sum_{m\leqslant X^\theta}\tau_3(m)\sum_{\substack{d|m \\ (d,P(\zeta))=1}}(r+1)^{(4/\theta)\Omega(d)}|R(X,m)|=T_1(l)+T_2(l)\,,\qquad\text{say}.$$

By Lemma 20

$$T_2(l)\ll(r+1)^{4/\theta}\sum_{(h)\leqslant(k)}\binom{(k)}{(h)}\gamma_{(k-h)}\left(c\frac{\log X}{\log\zeta}\right)^{(4r)^{12/\theta}}A(X)(\log X)^{|k|-2}$$

so, by Lemma 22

$$\sum_{l=1}^{|k|}T_2(l)\ll(r+1)^{4/\theta}|k|^2 2^{|k|}\left(c\frac{\log X}{\log\zeta}\right)^{(4r)^{12/\theta}}A(X)(\log X)^{|k|-2}.$$

Before we estimate $T_1(l)$ we change the last sum over n_1 as follows

$$\sum_{\substack{n_1<X \\ n_1\in\mathcal{N}_l}}\frac{\Lambda_{(h)}(n_1)}{f(n_1)}\leqslant\sum_{\substack{n_1<X \\ n_1\in\mathcal{N}_l}}\mu^2(n_1)\frac{\Lambda_{(h)}(n_1)}{f(n_1)}+\sum_{n_1<X}(1-\mu^2(n_1))\frac{\Lambda_{|h|}(n_1)}{f(n_1)}=$$
$$\sum_{\substack{n_1<X \\ n_1\in\mathcal{N}_l}}\mu^2(n_1)\frac{\Lambda_{(h)}(n_1)}{f(n_1)}+O\big(|h|(\log 2|h|)(\log X)^{|h|-1}\log\log X\big)$$

by Lemmata 13 and 14.

Now, $T_1(l)$ splits up in two parts corresponding to the sum over square-free n_1 and to the error term $|h|(\log 2|h|)(\log X)^{|h|-1}(\log\log X)$

$$T_1(l) = U_1(l) + U_2(l).$$

By Lemma 22 we have

$$\sum_{l=1}^{|k|} U_2(l) \ll |k|^{4+(4/\theta)} 2^{|k|}\left(c\frac{\log X}{\log\zeta}\right)^{1+(r+1)^4/\theta} A(X)(\log X)^{|k|-2}\log\log X.$$

For the estimation of $U_1(l)$ we write $n_1 = n_3 n_4$, where the prime divisors of n_3 are $\leqslant z$ while those of n_4 are $> z$. Using Lemmata 1 and 15 and interchanging the order of summation we obtain

$$U_1(l) \ll |k|^{2+4/\theta}\left(c\frac{\log X}{\log\zeta}\right)^{1+(r+1)^4/\theta}\frac{A(X)}{\log X}\sum_{\substack{(h)\leqslant(k)\\|h|\geqslant l}}\binom{(k)}{(h)}\gamma_{(k-h)}(\log X)^{|k-h|}\cdot$$

$$\cdot\sum_{\substack{(j)\leqslant(h)\\|j|\geqslant l}}\binom{(h)}{(j)}\sum_{n_3\leqslant z^l}\mu^2(n_3)\frac{\Lambda_{(j)}(n_3)}{f(n_3)}\sum_{n_4\leqslant\zeta^{|h-j|}}\mu^2(n_4)\frac{\Lambda_{(h-j)}(n_4)}{f(n_4)} \ll$$

$$\ll |k|^{2+4/\theta}\left(c\frac{\log X}{\log\zeta}\right)^{1+(r+1)^4/\theta}\frac{A(X)}{\log X}\sum_{\substack{(j)\leqslant(k)\\|j|\geqslant l}}\frac{(j)!}{|j|!}(l\log z)^{|j|}\cdot$$

$$\cdot\sum_{(j)\leqslant(h)\leqslant(k)}\binom{(k)}{(h)}\binom{(h)}{(j)}\gamma_{(k-h)}\frac{(h-j)!}{|h-j|!}(|h-j|\log\zeta)^{|h-j|}(\log X)_{|k-h|}.$$

In the summation over (h) we examine the ratio of the terms in going from (h) to (h') where (h') adds one to one component, say the r-th one. This ratio is

$$\frac{W(h')}{W(h)} = (|k|-|h|-1)\left(1+\frac{1}{|h|-|j|}\right)^{|h|-|j|}\frac{\log\zeta}{\log X} < e|k|\frac{\log\zeta}{\log X}.$$

If the maximum of this ratio is $\leqslant\varrho<1$ then the sum is $\leqslant M$. (Term for $(h)=(j)$), where

$$M \leqslant (1+\varrho+\ldots+\varrho^{|k|-1})^r < (1-\varrho)^{-r}.$$

The choice of ζ can be made to ensure that $\varrho < 1/2r$ and hence $M < e$. Therefore we have

$$U_1(l) \ll |k|^{2+4/\theta}\left(c\frac{\log X}{\log\zeta}\right)^{1+(r+1)^4/\theta}\frac{A(X)}{\log X}\sum_{\substack{(j)\leqslant(k)\\|j|\geqslant l}}\frac{(j)!}{|j|!}\binom{(k)}{(j)}\gamma_{(k-j)}(l\log z)^{|j|}(\log X)^{|k-j|}.$$

Going from (j) to (j') and computing as with (h), the ratio is $< |k| \log z/\log X$ and since $z \leqslant \zeta$, the terms for which $|j| = l$ dominate. The sum over these terms is easily seen to be

$$\leqslant \frac{(k)!}{l!(|k|-l-1)!} (lr \log z)^l (\log X)^{|k|-l} .$$

Considering the ratio of consecutive terms in $\sum_{l=1}^{k} U_1(l)$, it is easily seen that the term $l = 1$ dominates, giving the result,

$$\sum_{l=1}^{|k|} U_1(l) \ll |k|^{2+4/\theta} \left(c \frac{\log X}{\log \zeta} \right)^{1+(r+1)^{4/\theta}} r \frac{(k)!}{(|k|-2)!} A(X)(\log X)^{|k|-2} \log z \ll$$
$$\ll (c|k|)^{(4r)^{4/\theta}} \gamma_{(k)} A(X)(\log X)^{|k|-2} \log z .$$

Using the lower bound for z, it is easily shown that S_1, $\sum_l T_2(l)$ and $\sum_l U_2(l)$ also have this upper bound (with some constant c) and the result follows from this.

7. – Estimation of Σ_2.

Recall that

$$\Sigma_2 = \sum_{\substack{n \leqslant X \\ (nP,(z))=1}} a_n \sum_{\substack{d|n \\ d \geqslant y}} \mathfrak{L}_{(k')}(d) \left(\log \frac{n}{d} \right)^a .$$

LEMMA 25. *If* $1 < y < X$ *and* $X^{\theta/4} > z > (6r)^{12/\theta}$, *then*

$$|\Sigma_2| < \gamma_{(k')} \left(\frac{\log X/y}{\log X} \right)^a \left(c \frac{\log X}{\log z} \right)^{(6r)^{12/\theta}} A(X)(\log X)^{|k|-1} .$$

PROOF. We have

$$|\Sigma_2| \leqslant \left(\log \frac{X}{y} \right)^a \sum_{\substack{n \leqslant X \\ (n,P(z))=1}} a_n \sum_{d|n} |\mathfrak{L}_{(k')}(d)| \leqslant$$
$$\leqslant 2 \left(\log \frac{X}{y} \right)^a \gamma_{(k')} \sum_{\substack{n \leqslant X \\ (n,P(z))=1}} (r+2)^{\Omega(n)} a_n (\log X)^{|k'|} .$$

and by Lemma 21 this is

$$\ll \gamma_{(k')} \left(\frac{\log X/y}{\log X} \right)^a A(X)(\log X)^{|k|-1}(r+2)^{4/\theta} \left(c \frac{\log X}{\log z} \right)^{1+(r+2)^{4/\theta}} +$$
$$+ \gamma_{(k')} \left(\frac{\log X/y}{\log X} \right)^a (\log X)^{|k|}(r+2)^{4/\theta} \sum_{\substack{d \leqslant X^{\theta/4} \\ (d,P(z))=1}} (r+2)^{(4/\theta)\Omega(d)} \sum_{\substack{\nu | P(z) \\ \nu \leqslant z^2}} |R(X, \nu d)| .$$

The last double sum is

$$\leqslant \sum_{m\leqslant X^\theta} \tau(m) \sum_{\substack{d|m\\(d,P(z))=1}} (r+2)^{(4/\theta)\Omega(d)}|R(X,m)|$$

and by Lemma 20 this is

$$\ll \left(c\,\frac{\log X}{\log z}\right)^{48(r+2)^{12/\theta}} A(X)(\log X)^{-2}$$

giving the result, since $48(r+2)^{12/\theta} < (6r)^{12/\theta}$.

8. – Estimation of Σ_1.

Recall that

$$\Sigma_1 = \sum_{\substack{d<y\\(d,P(z))=1}} \mathfrak{L}_{(k')}(d) \sum_{\substack{n\leqslant X\\(n,P(z))=1\\n\equiv 0 (\mathrm{mod}\ d)}} a_n \left(\log\frac{n}{d}\right)^a$$

and define

$$F = \prod_{p|P(z)} \left(1-\frac{1}{p}\right)^{b(p)} \sum_{\substack{d<y\\(d,P(z))=1}} b_2(d)\ \frac{\mathfrak{L}_{(k')}(d)}{d}\left(\log\frac{X}{d}\right)^a .$$

LEMMA 26. *Let* $0<\varepsilon<\theta_0$, $\theta=\theta_0-2\varepsilon$, $y=X^\theta$, $z^s=X^\varepsilon$ *(z will be chosen small enough to ensure* $s\geqslant 2$*) and assume* $\beta(X)\leqslant 1$. *If*

$$\log z > |k|^{c|k|} \quad \textit{and} \quad z^\eta > \frac{c|k|}{\gamma(k)}(\log X)^3$$

then we have

$$\Sigma_1 = H_{\prime}A\,(X)F + O\left(2^a\gamma_{(k')}\left(c\,\frac{\log X}{\log z}\right)^{(8(r+1))^s} A(X)(\log X)^{|k|-2}\right) +$$

$$+\,O\left((e^{-s}+a\beta(X))\,2^a\frac{\gamma_{(k')}}{|k'|}\left(\frac{\log X}{\log z}\right)^2 A(X)(\log X)^{|k|-1}\right).$$

PROOF. Following precisely the same argument as in Lemma 12 we have

$$\Sigma_1 = \sum_{\substack{d<y\\(d,P(z))=1}} \mathfrak{L}_{(k')}(d)\left(\log\frac{X}{d}\right)^a \frac{A(X)}{f(d)} \prod_{p|P(z)}\left(1-\frac{1}{f(p)}\right) +$$

$$+\,O\left((e^{-s}+a\beta(X))(2\log X)^a\frac{A(X)}{\log z}\sum_{\substack{d<y\\(d,P(z))=1}}\frac{|\mathfrak{L}_{(k')}(d)|}{f(d)}\right) +$$

$$+\,O\left(2^a\gamma_{(k')}(\log X)^{|k|}\sum_{m<X^{\theta_0-\varepsilon}}\tau(m)\sum_{\substack{d|m\\(d,P(z))=1}}(r+1)^{\Omega(d)}R^*(X,m)\right) =$$

$$= E_1 + E_2 + E_3\,, \qquad \text{say}\,.$$

Applying Lemma 20 we get

$$E_3 < 2^a \gamma_{(k')} \left(c \frac{\log X}{\log z}\right)^{(8(r+1))^3} A(X)(\log X)^{|k|-2} .$$

By Lemma 9 we have

$$E_1 = H' A(X) \prod_{p|P(z)} \left(1 - \frac{1}{p}\right)^{b(p)} \sum_{\substack{d<y \\ (d,P(z))=1}} b_2(d) \frac{\mathfrak{L}_{(k')}(d)}{d} \left(\log \frac{X}{d}\right)^a +$$
$$+ O\big(A(X)(\log X)^{|k|+1} z^{-\eta}\big) = H' A(X) F + O\big(\gamma_{(k')} A(X)(\log X)^{|k|-2}\big) .$$

To estimate E_2 we write

$$\sum_{\substack{d<y \\ (d,P(z))=1}} \frac{|\mathfrak{L}_{(k')}(d)|}{f(d)} \leqslant \sum_{\substack{d<y \\ (d,P(z))=1}} b_2(d) \frac{|\mathfrak{L}_{(k')k}(d)|}{d} + \sum_{\substack{d<y \\ (d,P(z))=1}} \left| \frac{1}{f(d)} - \frac{b_2(d)}{d} \right| |\mathfrak{L}_{(k')}(d)| .$$

Using the simple estimate $|\mathfrak{L}_{(k')}(d)| \leqslant (\log d)^{|k'|}$ of Lemma 1 and the argument of Lemma 9 the latter sum is

$$\ll (\log X)^{|k'|+1} z^{-\eta} \ll \frac{\gamma_{(k')}}{|k'|} \frac{\log X}{\log z} (\log X)^{|k'|} .$$

Moreover, since

$$|\mathfrak{L}_{(k')}(d)| \leqslant \sum_{\delta|d} \Lambda_{(k')}(\delta)$$

by Lemma 1, the former sum is bounded by

$$\sum_{\substack{d\leqslant y \\ (d,P(z))=1}} \frac{b_2(d)}{d} \sum_{\delta|d} \Lambda_{(k')}(\delta) \ll \frac{\log X}{\log z} \sum_{\delta\leqslant X} b_2(\delta) \frac{\Lambda_{(k')}(\delta)}{\delta} \ll \frac{\gamma^{(k')}}{|k'|} \frac{\log X}{\log z} (\log X)^{|k'|}$$

by the Corollary and Remark after Lemma 15. Thus

$$E_2 \ll (e^{-s} + a\beta(X))\, 2^a \frac{\gamma_{(k')}}{|k'|} \left(\frac{\log X}{\log z}\right)^2 A(X)(\log X)^{|k|-1} .$$

Collecting these estimates, we get the result.

9. – Conclusion of proof.

Collecting together the estimates of the last three sections we obtain

$$\sum_{n\leqslant X} a_n \Lambda_{(k)}(n) - H' A(X) F \ll 2^a \gamma_{(k')}\left(c\frac{\log X}{\log z}\right)^{(8(r+1))^3} A(X)(\log X)^{|k|-2} +$$

$$+ 2^a \gamma_{(k')}\big(e^{-s} + a\beta(X)\big)\left(\frac{\log X}{\log z}\right)^2 A(X)(\log X)^{|k|-1} +$$

$$+ (1-\theta)^a \gamma_{(k')}\left(c\frac{\log X}{\log z}\right)^{(6r)^{12/\theta}} A(X)(\log X)^{|k|-1} +$$

$$+ \gamma_{(k)}\big(c|k|\big)^{(4r)^{4/\theta}}(\log z) A(X)(\log X)^{|k|-2} .$$

Since $2^a(1-\theta)^{-a} < \log X$ the first term is less than the third one. Substituting $z = X^{\varepsilon/s}$ we arrive at

$$\sum_{n\leqslant X} a_n \Lambda_{(k)}(n) - H' A(X) F \ll$$

$$\ll \{2^a \gamma_{(k')}\big(e^{-s} + a\beta(X)\big) s^2 + \gamma_{(k')}(1-\theta)^a (cs)^{(6r)^{12/\theta}} + \gamma_{(k)} s^{-1}\big(c|k|\big)^{(4r)^{4\theta}}\} \cdot$$

$$\cdot A(X)(\log X)^{|k|-1} .$$

Substituting

$$s = \big(c_1|k|\big)^{(4r)^{4/\theta}} u ,$$

where $u \geqslant 1$ and $c_1 > c$ is some sufficiently large constant we get $s > 2a \log\big(2(1-\theta)^{-1}\big)$ and so the term containing e^{-s} is less than the one containing $(1-\theta)^a$. Thus we have

$$\{...\} \ll a 2^a s^2 \gamma_{(k')} \beta(x) + \gamma_{(k')}(1-\theta)^a \big(cc_1|k|\big)^{(6r)^{16/\theta}} u^{(6r)^{16/\theta}} + \gamma_k u^{-1} .$$

For

$$u = \left(\frac{\gamma_{(k)}}{\gamma_{(k')}}(1-\theta)^{-a}\right)^{(6r)^{-16/\theta}}$$

the last two terms are less than

$$\gamma_{(k)}\left(\frac{\gamma_{(k')}}{\gamma_{(k)}}(1-\theta)^a\right)^{(6r)^{-16/\theta}} \big(cc_1|k|\big)^{(6r)^{16/\theta}}.$$

Since

$$u^2 < \frac{\gamma_{(k)}}{\gamma_{(k')}}(1-\theta)^{-a} \quad \text{and} \quad \frac{\gamma_{(k')}}{\gamma_{(k)}} \leqslant \binom{|k|}{a}$$

$\left(\text{for } r \geqslant 2 \text{ we have even } \gamma_{(k')}/\gamma_{(k)} = \binom{|k|-1}{a}\right)$ we obtain

$$\sum_{n \leqslant X} a_n \Lambda_{(k)}(n) = H' A(X) F + E^* \gamma_{(k)} A(X) (\log X)^{|k|-1},$$

where

$$E^* \ll \left(\frac{2}{1-\theta}\right)^a (c(k))^{(6r)^{4/\theta}} \beta(X) + \left(\binom{|k|}{a}(1-\theta)^a\right)^{(6r)^{-16/\theta}} |k|^{(6r)^{16/\theta}}.$$

Now, it remains to estimate F. Since by Lemma 4 the above result is true for the sequence $b_1(n)$, with

$$A(X) = V_1 X + O(X^{1-\delta}) \quad \text{and} \quad H_1' = \prod_p \left(1 - \frac{1}{f_1(p)}\right)\left(1 - \frac{1}{p}\right)^{1-b(p)},$$

where f_1 is the f of Lemma 4, we get

$$\sum_{n \leqslant X} b_1(n) \Lambda_{(k)}(n) = H_1' V_1 X F + E^{**} \gamma_{(k)} A(X) (\log X)^{|k|-1}$$

with a different E^{**} satisfying the same bound as E^*. Also, from Lemma 3

$$\sum_{n \leqslant X} b_1(n) \Lambda_{(k)}(n) = \gamma_{(k)} X (\log X)^{|lk|-1} + O\big(c^{|k|} X (\log X)^{|k|-2}\big).$$

Comparing these two results we obtain

$$F = \frac{1}{V_1 H_1'} \gamma_{(k)} (\log X)^{|k|-1} - \gamma_{(k)} E^{**} (\log X)^{|k|-1} + O\big(c^{|k|} (\log X)^{|k|-2}\big),$$

which completes the proof, since a simple computation shows that

$$H' = V_1 H_1' H.$$

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