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An Inequality for the Rank of a Web and Webs of Maximum Rank (*).

SHIING-SHEN CHERN (**) - PHILLIP A. GRIFFITHS (***)

dedicated to Hans Lewy

1. – Statement of results.

A *web* is given in a neighborhood $U \subset R^N$ by a set of codimension k foliations in general position, a notion we shall make precise in a moment. *Web geometry* is the study of local diffeomorphism invariants of a web (1); for example, we may ask if it is equivalent to a standard web whose foliations consist of parallel linear spaces of dimension $N-k$. An invariant arises from the consideration of the *abelian q-equations* $(1 \leqslant q \leqslant k)$ (2) associated to the web. In this paper we will be concerned with the abelian equations when $q = k$. Specifically, we will find a bound on the *rank* or maximum number of linearly independent abelian k-equations, and will show that webs of maximal rank give a very special G-structure in the projectivized tangent spaces PT_x $(x \in U)$. In a future paper we hope to use this to show that such webs have a standard local form, generalizing our previous result in the codimension-one case (3).

For simplicity of notation we will carry out our study in detail only in the case $k = 2$. Therefore we now agree, until specified otherwise, that a

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(1) The basic reference is Blaschke-Bol [1].

(2) These are defined in general in [4]. The definitions relevant to our present discussion will be given below.

(3) Cf. [2], and also [3] for an outline of the main result from [2].

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web will be given in an open set $U \subset R^{2n}$ [4] by d foliations by codimension-two submanifolds. The leaves of the i-th foliation will be taken as level sets

$$u_i(x) = \text{const}, \qquad v_i(x) = \text{const};$$

these functions are defined up to a local diffeomorphism in (u_i, v_i)-space. The i-th *web normal* is

$$\Omega_i(x) = du_i(x) \wedge dv_i(x).$$

Under a diffeomorphism of (u_i, v_i), Ω_i is multiplied by a non-vanishing factor so that what is intrinsic is the point

$$\Omega_i(x) \in P(\Lambda^2 T_x^*),$$

the latter being the projective space associated to the vector space of 2-forms at $x \in U$.

We want to say what it means for the web to be in general position. For this some linear algebra preliminaries are required. It will be convenient not to distinguish between a point $u \in P^N = P(R^{N+1})$ and its homogeneous coordinate vector $u \in R^{N+1} - \{0\}$. A set of points $u_1, \ldots, u_d \in P^N$ is in general position in case any $k \leqslant N+1$ of them span a P^{k-1}; i.e. $u_{i_1} \wedge \ldots \wedge u_{i_k} \neq 0$ for $1 \leqslant i_1 < \ldots < i_k \leqslant d$, $k \leqslant N+1$.

When we come to the notion of general position of lines some care is necessary. Denote by $G(1, 2n-1)$ the Grassmannian of lines in $P^{2n-1} = P(R^{2n})$. We will identify $G(1, 2n-1)$ with its image under the *Plücker embedding*

$$G(1, 2n-1) \hookrightarrow P^{\binom{2n}{2}-1} = P(\Lambda^2 R^{2n})$$

given by sending the line spanned by points $u, v \in P^{2n-1}$ into $u \wedge v \in P(\Lambda^2 R^{2n})$. A first guess is that a set of lines

$$\Omega_1, \ldots, \Omega_d \in G(1, 2n-1)$$

should be said to be in general position if any $k \leqslant n$ of them span a P^{2k-1}; i.e. if all

$$\Omega_{i_1} \wedge \ldots \wedge \Omega_{i_k} \neq 0 \qquad 1 \leqslant i_1 < \ldots < i_k \leqslant d, \ k \leqslant n. \tag{1.1}$$

This condition is certainly necessary, but for some purposes may not be sufficient.

[4] The reason for taking the dimension $N = 2n$ will appear in § 4.

For example ([5]), consider a set of four lines $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ in P^3. The condition (1.1) is equivalent to those lines being pairwise skew. Now it is well-known that there are « in general » two lines meeting each of four skew lines in P^3. To see better what « in general » means recall that a non-singular quadric surface S in P^3 is doubly ruled by two families of lines, called the A-lines and B-lines. The A-lines (resp. B-lines) are pairwise skew, and any A-line meets any B-line exactly once. All of these facts follow easily by representing S as the image under the Segre embedding

$$P^1 \times P^1 \to P^3$$

given by

$$(s, t) \to [1, s, t, st] .$$

The A-lines and B-lines are given by $s = \text{const}$ and $t = \text{const}$ respectively. Now there is a unique non-singular quadric surface S containing $\Omega_1, \Omega_2, \Omega_3$ as A-lines.

For the remaining line Ω_4 there are the three possibilities:

Ω_4 meets S in distinct points u_1, u_2;

Ω_4 is tangent to S at u;

Ω_4 is an A-line lying in S.

In the first case each of the B-lines through u_1, u_2 meets all four Ω_i once, and the second possibility is the limiting case of the first when $u_1 = u_2$. But in the third case there are infinitely many lines meeting the four skew lines Ω_i.

In our study we will give a definition of general position motivated by webs arising from non-degenerate algebraic surfaces in P^{n+1}. For this we assume first the condition (1.1). Given any $n-1$ of the Ω_i, say $\Omega_1, \ldots, \Omega_{n-1}$, spanning a P^{2n-3} we consider any P^{2n-5} contained in this P^{2n-3} and the linear projection

$$\pi : P^{2n-1} - P^{2n-5} \to P^3 .$$

Our second requirement is:

(1.2) the lines $\pi(\Omega_n), \ldots, \pi(\Omega_d)$ do not all pass through a common point.

([5]) This observation is due to Ran Donagi.

A set of lines satisfying (1.1) and (1.2) will be said to be in *general position.* It is *not* the case that lines in general position have as Plücker images points in general position in $P^{\binom{2n}{2}-1}$. The linear algebra subtlety here is crucial in our study.

A set of foliations is in general position in case the normals $\Omega_i(x)$ are lines in general position in PT^*_x.

An *abelian equation* is a relation

$$\sum_i f_i(u_i(x), v_i(x))\,\Omega_i(x) = 0\,, \tag{1.3}$$

and the *rank* r of the web is the maximum number of linearly independent abelian equations. Our first result is a bound on this rank. Namely, define the integer t uniquely by the conditions

$$t \equiv -d+1 \bmod n-1\,, \qquad 0 \leqslant t \leqslant n-2 \tag{1.4}$$

and set

$$p_g(d, n) = \frac{1}{6(n-1)^2}(d-2n+1+t)(d-n+t)(d-1-2t)\,. \tag{1.5}$$

THEOREM I. *The rank of a d-web in* $U \subset R^{2n}$ *satisfies*

$$r \leqslant p_g(d, n) \tag{1.6}$$

In particular, $r = 0$ *when* $d < 2n$.

This bound may be seen to be sharp. Webs for which equality holds in (1.6) will be said to be of *maximal rank.* Our remaining results will, in this case, give a particular type of G-structure in the projectivized cotangent spaces PT^*_x.

Before stating the next theorem we recall a little algebraic geometry. A *ruled surface* S in P^N may be constructed by taking two skew subspaces P^m, $P^{m'}$ $(m+m' = N-1)$ spanning P^N together with rational curves $C \subset P^m$, $C' \subset P^{m'}$ in projective correspondence and letting S be the surface of ∞^1 lines obtained by joining corresponding points. In case $N = 2n-1$, $m = m' = n-1$, and C together with C' are rational normal curves we obtain what will be termed a *special ruled surface.* In suitable coordinates it is the image of $P^1 \times P^1$ under the map

$$(t, s) \to [t, \ldots, t^n; st, \ldots, st^n]\,. \tag{1.7}$$

The lines $\Omega(t)$ on S are obtained by holding t fixed. They all have a common linear parameter s, and are therefore all in projective correspondence with corresponding points spanning a $P^{n-1}(s)$ (where $P^{n-1}(0) = P^{n-1}$, $P^{n-1}\cdot(\infty) = P'^{n-1}$).

THEOREM II. *Assume given a d-web of maximal rank in* $U \subset R^{2n}$ *where*

$$d \geqslant 2n + 1 , \qquad n \geqslant 3 \text{ (6)}$$

Then there is defined a field of special ruled surfaces

$$S(x) \subset PT_x^*$$

such that the web normals are lines belonging to this surface. In particular the web normals are all in projective correspondence, written

$$\Omega_i(x) \overline{\wedge} \Omega_j(x) \qquad 1 \leqslant i , j \leqslant d \tag{1.9}$$

with corresponding points spanning a P^{n-1} *in* PT_x^*.

2. – Proof of the bound on the rank.

Suppose that

$$\sum_{i=1}^{d} f_i^\lambda(u_i(x), v_i(x))\, du_i(x) \wedge dv_i(x) = 0 \qquad \lambda = 1, \ldots, r \tag{2.1}$$

are r linearly independent abelian equations.

Set $\Omega_i = du_i \wedge dv_i$ and

$$Z_i(x) = [\ldots, f_i^\lambda(u_i(x), v_i(x)), \ldots] \in P^{r-1} .$$

The abelian equations (2.1) become

$$\sum_i Z_i(x) \otimes \Omega_i(x) = 0 . \tag{2.2}$$

As x varies over $U \subset R^{2n}$, $Z_i(x)$ traces out a piece of two-dimensional surface S_i in P^{r-1}. We may take (u_i, v_i) as local coordinates on S_i. There

(6) The rank is zero when $d < 2n$ and is $\leqslant 1$ when $d = 2n$. These cases are excluded.

is a 1-to-d correspondence

$$x \to Z_1(x), \ldots, Z_d(x),$$

but because of (2.2) corresponding points $Z_i(x)$ are not in general position. At first glance it might appear that there are in fact $\binom{2n}{2}$ independent linear relations among the Z_i, but this is not so. Letting $\{Z_1, \ldots, Z_d\}$ denote the linear span in P^{r-1} of $Z_1(x), \ldots, Z_d(x)$ we shall prove that

$$\dim \{Z_1, \ldots, Z_d\} \leqslant d - 2n, \tag{2.3}$$

an estimate which will turn out to be sharp.

To see this we note that since the lines $\Omega_1(x), \ldots, \Omega_d(x) \in G_x(1, 2n-1)$ are in general position we may choose points

$$\omega_1 = du_1 + \lambda_1 dv_1 \in \Omega_1, \ldots, \omega_{2n-2} = du_{2n-2} + \lambda_{2n-2} dv_{2n-2} \in \Omega_{2n-2}$$

such that

$$\omega_1 \wedge \ldots \wedge \omega_{2n-2} \wedge \Omega_{2n-1} \neq 0 .$$

If we multiply (2.2) by $\omega_1 \wedge \ldots \wedge \omega_{2n-2}$ the first $2n-2$ terms drop out and Z_{2n-1} appears with a non-zero coefficient, i.e.

$$Z_{2n-1} \in \{Z_{2n}, \ldots, Z_d\} . \tag{2.4}$$

By symmetry it follows that at most $d - 2n + 1$ of $Z_i(x)$ are linearly independent, which proves (2.3).

Now the argument proceeds as in the codimension-one case. By (2.4)

$$Z_{2n-1} = \varrho_{2n} Z_{2n} + \ldots + \varrho_d Z_d . \tag{2.5}$$

Choose $(u_{2n-1}, v_{2n-1}, u_{2n}, v_{2n}, \ldots, u_{3n-2}, v_{3n-2})$ as coordinates on U and differentiate (2.5) to obtain

$$\frac{\partial Z_{2n-1}}{\partial u_{2n-1}}, \frac{\partial Z_{2n-1}}{\partial v_{2n-1}} \equiv \frac{\partial Z_{3n-1}}{\partial u_{3n-1}}, \frac{\partial Z_{3n-1}}{\partial v_{3n-1}}, \ldots, \frac{\partial Z_d}{\partial u_d}, \frac{\partial Z_d}{\partial v_d} \bmod Z_1, \ldots, Z_d . \tag{2.6}$$

Let

$$P^{k(0)} = \{Z_1, \ldots, Z_d\}, \qquad P^{k(1)} = \left\{Z_1, \frac{\partial Z_1}{\partial u_1}, \frac{\partial Z_1}{\partial v_1}, \ldots, Z_d, \frac{\partial Z_d}{\partial u_d}, \frac{\partial Z_d}{\partial v_d}\right\} \qquad , \ldots ,$$

$P^{k(\mu)} =$ span of the μ-th osculating spaces to the surfaces S_i at corresponding points. By (2.4) and (2.6)

$$\dim P^{k(0)} \leqslant (d - 2n + 1) - 1\,,$$
$$\dim P^{k(1)} \leqslant (d - 2n + 1) + 2(d - 3n + 2) - 1\,,$$

and in general

$$\dim P^{k(\mu)} \leqslant (d - 2n + 1) + 2(d - 3n + 2) + \ldots + (\mu + 1)(d - (\mu + 2)n + \mu + 1) - 1\,. \tag{2.7}$$

Here we agree that zero is put in whenever one of the first $\mu + 1$ terms on the right becomes $\leqslant 0$, which obviously happens for large μ.

The $P^{k(\mu)}(x)$ give an increasing sequence of linear subspaces of P^{r-1}, which eventually terminates at say $P^{k(\infty)}(x)$. Since $P^{k(\infty)}(x)$ does not change by differentiation, it must be a constant linear subspace, and hence is all of P^{r-1} since the equations (2.1) were assumed linearly independent. By (2.7)

$$r \leqslant \sum_{\mu \geqslant 0} \max\left[(\mu + 1)\{d - n(\mu + 2) + (\mu + 1)\}, 0\right]. \tag{2.8}$$

It remains to identify this sum with the expression (1.5). Write

$$\tau = \frac{d - 2n + 1}{n - 1} + \frac{t}{n - 1}$$

where t is determined by (1.4); τ is an integer. Put

$$a = d - 2n + 1\,, \qquad s = n - 1\,.$$

Then the R.H.S. of (2.8) is

$$a + 2(a - s) + 3(a - 2s) + \ldots + \tau(a - (\tau - 1)s) =$$
$$= \tfrac{1}{2}\tau(\tau + 1)a - \tfrac{1}{3}(\tau - 1)\tau(\tau + 1)s = \tfrac{1}{6}\tau(\tau + 1)\{3a - 2(\tau - 1)s\}\,.$$

Now

$$(n - 1)\tau = d - 2n + 1 + t\,,$$
$$(n - 1)(\tau + 1) = d - n + t\,,\ (n - 1)(\tau - 1) = d - 3n + 2 + t\,,$$
$$3a - 2(\tau - 1)s = 3d - 6n + 3 - 2(d - 3n + 2 + t) = d - 1 - 2t\,,$$

so that the R.H.S. of (2.8) is

$$\frac{1}{6(n-1)^2}(d-2n+1+t)(d-n+t)(d-1-2t) = p_g(d, n)$$

according to (1.5). This completes the proof of Theorem I.

When $n=2$, we have $t=0$ and

$$(2.9)\qquad p_g(d, n) = \tfrac{1}{6}(d-1)(d-2)(d-3) = $$
$$= \text{geometric genus of a smooth surface of degree } d \text{ in } P^3 .$$

Exactly the same considerations can be carried out for a d-web of codimension k in a neighborhood $U \subset R^{kn}$, $k \leqslant n$. Let r_k be its rank, i.e., the maximum number of linearly independent abelian k-equations. Then we have

$$(2.10)\qquad r_k \leqslant \pi(d, n, k)$$

where

$$(2.11)\qquad \pi(d, n, k) = \sum_{\mu \geqslant 0} \max\left(\binom{k+\mu-1}{\mu}\{d-(k+\mu)n+k-1+\mu\}, 0\right).$$

The first term in $\pi(d, n, k)$ is

$$d - kn + k - 1 .$$

Hence we have

$$(2.12)\qquad \pi(d, n, k) = 0 \text{ when and only when } d \leqslant kn - k + 1 .$$

The first two terms in $\pi(d, n, k)$ are

$$d - kn + k - 1 + k\{d-(k+1)n + k\} =$$
$$= (k+1)d - k(k+2)n + k^2 + k - 1 .$$

Hence we have

$$(2.13a)\qquad \pi(d, n, k) = n + k ,$$

when and only when

$$(2.13b)\qquad d = (k+1)n - (k-1) .$$

3. – Proof of Theorem II.

By the assumption of maximal rank

$$\{Z_1, \ldots, Z_d\} = \{Z_{2n}, \ldots, Z_d\} = P^{d-2n}(x)\,, \tag{3.1}$$

i.e. there are exactly $(2n-1)$ independent relations among the $Z_i(x)$. On the other hand, (2.2) gives what appears to be $\binom{2n}{2}$ relations, and consequently some of these must be dependent. We will see that the geometrical consequence of this is the presence in $P(T_x^*)$, $x \in U \subset R^{2n}$, of a field of special ruled surfaces. An intermediate step is the normal form (3.16), which we will derive first.

To carry this out we choose $u_1, v_1, \ldots, u_n, v_n$ as coordinate system and write, at a fixed point $x_0 \in U$,

$$\begin{cases} du_s = du_1 + \sum\limits_{\lambda=2}^{n} A_{s\lambda}\, du_\lambda + \sum\limits_{\lambda=2}^{n} B_{s\lambda}\, dv_\lambda\,, \\ dv_s = dv_1 + \sum\limits_{\lambda=2}^{n} C_{s\lambda}\, du_\lambda + \sum\limits_{\lambda=2}^{n} D_{s\lambda}\, dv_\lambda\,, \qquad s = n+1, \ldots, d\,. \end{cases} \tag{3.2}$$

This is possible since by general position all

$$\frac{\partial(u_s, v_s)}{\partial(u_1, v_1)} \neq 0\,.$$

We set

$$A_\lambda = (A_{n+1,\lambda}, \ldots, A_{d,\lambda}) \in R^{d-n}$$

and similarly for B_λ, C_λ, D_λ.

(3.3) LEMMA. *The vectors A_λ, B_λ are multiples of a vector E_λ, and the E_λ are linearly independent* (*here* $\lambda = 2, \ldots, n$).

PROOF. The abelian equations (2.2) are

$$\sum_{\alpha=1}^{n} Z_\alpha\, du_\alpha \wedge dv_\alpha + \sum_{s=n+1}^{d} Z_s\, du_s \wedge dv_s = 0\,. \tag{3.4}$$

The coefficient of $du_\alpha \wedge dv_\alpha$ gives Z_α, $\alpha = 1, \ldots, n$, as a linear combination of Z_s, $s = n+1, \ldots, d$, so that by (3.1) there are at most $n-1$ independent relations among Z_s. In particular, the coefficient of $du_1 \wedge dv_1$ gives

$$Z_1 + \sum_{s=n+1}^{d} Z_s = 0 , \tag{3.5}$$

and the coefficients of $dv_1 \wedge du_\lambda$ and $dv_1 \wedge dv_\lambda$ give

$$(3.6)_\lambda \qquad \begin{cases} \sum_{s=n+1}^{d} A_{s\lambda} Z_s = 0 , \\ \sum_{s=n+1}^{d} B_{s\lambda} Z_s = 0 , \qquad \lambda = 2, \ldots, n . \end{cases}$$

By the above remark at most $(n-1)$ of the $2(n-1)$ equations $(3.6)_\lambda$ can be independent. In other words, if $R_\lambda \subset R^{d-n}$ is the span of A_λ, B_λ then

$$\dim \Big(\sum_{\lambda=2}^{n} R_\lambda \Big) \leqslant n-1 .$$

The lemma amounts to

$$\sum_{\lambda=2}^{n} R_\lambda = \bigoplus_{\lambda=2}^{n} R_\lambda \cong R^{n-1} ,$$

which is implied by

$$R_\lambda \not\subset \sum_{\lambda \neq \gamma} R_\gamma$$

for fixed λ. If, on the contrary, the equations $(3.6)_\lambda$ are linear combinations of $(3.6)_\gamma$ for $\gamma \neq \lambda$, then taking $\lambda = n$ we will have

$$A_n = \sum_{\gamma=2}^{n-1} a_\gamma A_\gamma + b_\gamma B_\gamma ,$$

$$B_n = \sum_{\gamma=2}^{n-1} c_\gamma A_\gamma + d_\gamma B_\gamma .$$

By (3.2)

$$du_s = du_1 + \sum_{\gamma=2}^{1-n} A_{s\gamma}(du_\gamma + a_\gamma \, du_n + c_\gamma \, dv_n) + \sum_{\gamma=2}^{n} B_{s\gamma}(dv_\gamma + b_\gamma \, du_n + d_\gamma \, dv_n) .$$

In the R^4 defined by

$$(3.7) \qquad \begin{cases} du_\gamma + a_\gamma \, du_n + c_\gamma \, dv_n = 0 , \\ dv_\gamma + b_\gamma \, du_n + d_\gamma dv_n = 0 , \qquad \gamma = 2, \ldots, n-1 , \end{cases}$$

we have

$$(3.8) \qquad du_s = du_1\,, \qquad s = n+1, \dots, d\,,$$

contradicting general position. More precisely, the $2(n-2)$ one-forms (3.7) span a P^{2n-5} in $P^{2n-1} = P(T_x^*)$. This P^{2n-5}, does not meet any of the web normals $\Omega_1, \Omega_{n+1}, \dots, \Omega_d$. Under the linear projection

$$P^{2n-1} - P^{2n-5} \xrightarrow{\pi} P^3$$

(3.8) says exactly the lines $\pi(\Omega_1), \pi(\Omega_{n+1}), \dots, \pi(\Omega_d)$ all pass through a common point, and this contradicts general position. Thus Lemma (3.3) is proved.

By the lemma we have $A_\lambda = \alpha_\lambda E_\lambda$, $B_\lambda = \beta_\lambda E_\lambda$, α_λ, β_λ not both zero. Replacing du_λ by $\alpha_\lambda\, du_\lambda + \beta_\lambda\, dv_\lambda$ we obtain

$$A_\lambda = E_\lambda\,, \qquad B_\lambda = 0$$

in (3.2). After a similar argument applied to C_λ, D_λ we may assume

$$C_\lambda = 0\,, \qquad D_\lambda = F_\lambda,$$

so that (3.2) is now

$$(3.9) \qquad \begin{cases} du_s = du_1 + \sum_{\lambda=2}^{n} E_{s\lambda}\, du_\lambda\,, \\ dv_s = dv_1 + \sum_{\lambda=2}^{n} F_{s\lambda}\, dv_\lambda\,. \end{cases}$$

(3.10) LEMMA. *E_λ is a non-zero multiple of F_λ.*

PROOF. By the proof of Lemma (3.3) the $2(n-1)$ vectors E_γ, F_γ span an R^{n-1} in R^{d-n}. Thus, if R_λ is the span of E_λ, F_λ

$$\dim \sum_{\lambda=2}^{n} R_\lambda = n-1\,.$$

If some E_λ and F_λ are linearly independent, i.e.

$$\dim R_\lambda = 2\,,$$

then for some other γ we must have

$$R_\gamma \subset \sum_{\lambda \neq \gamma} R_\lambda .$$

Taking $\gamma = n$ we obtain a relation

$$(3.11) \qquad \begin{cases} E_n = \sum_{\gamma=2}^{n-1} a_\gamma E_\gamma + b_\gamma F_\gamma , \\ F_n = \sum_{\gamma=2}^{n-1} c_\gamma E_\gamma + d_\gamma F_\gamma , \end{cases}$$

which we will show leads to a contradiction.

Using (3.9) the coefficients of $du_\gamma \wedge dv_n$ and $du_n \wedge dv_\gamma$ in (3.4) give

$$(3.12) \qquad \begin{cases} \sum_{s=n+1}^{d} (E_{s\gamma} \ F_{sn}) \ Z_s = 0 , \\ \sum_{s=n+1}^{d} (E_{sn} F_{s\gamma}) Z_s = 0 , \qquad 2 \leqslant \gamma \leqslant n-1 . \end{cases}$$

The coefficient of $du_\lambda \wedge dv_\gamma$ gives

$$(3.13) \qquad \delta_\gamma^\lambda Z_\gamma + \sum_{s=n+1}^{d} (E_{s\gamma} F_{s\lambda}) Z_s = 0 , \qquad 2 \leqslant \gamma , \ \lambda \leqslant n-1 .$$

Finally the coefficient of $du_n \wedge dv_n$ gives, after we plug in (3.11),

$$(3.14) \qquad Z_n + \sum_{\lambda,\gamma=2}^{n-1} \ \sum_{s=n+1}^{d} (a_\lambda E_{s\lambda} + b_\lambda F_{s\lambda})(c_\gamma E_{s\gamma} + d_\gamma F_{s\gamma}) Z_s = 0 .$$

Substituting (3.11) into (3.12) and using (3.13) gives

$$(3.15) \qquad \begin{cases} \sum_{\lambda=2}^{n-1} \ \sum_{s=n+1}^{d} (c_\lambda E_{s\gamma} E_{s\lambda}) Z_s = - d_\gamma Z_\gamma , \\ \sum_{\lambda=2}^{n-1} \ \sum_{s=n+1}^{d} (b_\lambda F_{s\lambda} F_{s\gamma}) Z_s = - a_\gamma Z_\gamma , \qquad 2 \leqslant \gamma \leqslant n-1 . \end{cases}$$

Expanding out (3.14) and plugging in (3.15) and (3.13) we obtain

$$Z_n - \sum_{\gamma=2}^{n-1} (3a_\gamma d_\gamma + b_\gamma c_\gamma) Z_\gamma = 0 ,$$

which contradicts the maximal rank assumption (3.1). This proves Lemma (3.10).

We now arrive at our *normal form* for the du_i, dv_i. Namely, we may multiply by a scale factor and assume $E_\lambda = F_\lambda$. If we relabel and define $A_{i\alpha}$ by

$$A_{s\lambda} = E_{s\lambda} , \qquad n+1 \leqslant s \leqslant d , \ 2 \leqslant \lambda \leqslant n ,$$

$$A_{s1} = 1 , \qquad n+1 \leqslant s \leqslant d ,$$

$$A_{\alpha\beta} = \delta_{\alpha\beta} , \qquad 1 \leqslant \alpha , \ \beta \leqslant n ,$$

then (3.9) becomes

(3.16)
$$\begin{cases} du_i = \sum_{\alpha=1}^{n} A_{i\alpha} du_\alpha , \\ dv_i = \sum_{\alpha=1}^{n} A_{i\alpha} dv_\alpha , \qquad 1 \leqslant i \leqslant d . \end{cases}$$

(3.17) LEMMA. *The vectors* $A_i = [\ldots, A_{i\alpha}, \ldots] \in P^{n-1}$ *lie on* $\infty^{(n-1)(n-2)/2}$ *linearly independent quadrics.*

PROOF. The basic abelian equation (3.4) gives upon substituting in (3.16)

(3.18)
$$\sum_{i=1}^{d} A_{i\alpha} A_{i\beta} Z_i = 0 , \qquad 1 \leqslant \alpha , \ \beta \leqslant n .$$

These are $n(n+1)/2$ relations among the Z_i, and by (3.1) only $2n-1$ of the equations (3.18) can be independent. In other words we have

$$\frac{n(n+1)}{2} - (2n-1) = \frac{(n-1)(n-2)}{2}$$

linearly independent relations

$$\sum_{\alpha,\beta=1}^{n} k_{\alpha\beta} A_{i\alpha} A_{i\beta} = 0 , \qquad k_{\alpha\beta} = k_{\beta\alpha} ,$$

among the coefficients in (3.18), and this gives the lemma.

Now we can complete the proof of Theorem II. Namely, by (3.17) the A_i lie on a rational normal curve C in P^{n-1}. After a linear change of coordinates we may assume that C is given parametrically by

$$t \to [t, t^2, \dots, t^n]\,. \tag{3.19}$$

According to (3.16) we have now written $P^{2n-1} = P(T_x^*)$ as the span of the P^{n-1} determined by the du_α and P'^{n-1} determined by the dv_α, and in each of P^{n-1}, P'^{n-1} we have the rational normal curve (3.19) such that setting $t = t_i$ gives $du_i \in P^{n-1}$ and $dv_i \in P'^{n-1}$ respectively. The i-th web normal Ω_i is the line $du_i + s\,dv_i$, which is just the line $t = t_i$ on the standard ruled surface given parametrically by

$$(s, t) \to [t, t^2, \dots, t^n; st, st^2, \dots, st^n]\,.$$

4. – Webs defined by algebraic varieties.

A projective algebraic variety of dimension k, $V_k \subset P^m$ is non-degenerate in case it does not lie in a P^{m-1}. The degree d is the number of intersections with a generic P^{m-k}, written

$$V \cdot P^{m-k} = p_1 + \dots + p_d\,. \tag{4.1}$$

(Here and in what follows, we frequently omit the index of V_k.) For non-degenerate V, which we will always assume, the $p_i \in P^{m-k}$ are in general position (c.f. Lemma 1.8 in [2]).

We continue to denote by $G(m-k, m)$ the Grassmannian of P^{m-k}'s in P^m, and for fixed $p \in P^m$ we let $\Sigma(p)$ designate the Schubert variety of all P^{m-k}'s which pass through the point p. Note that $\Sigma(p) \cong G(m-k-1, m-1)$ and has codimension k in $G(m-k, m)$. The algebraic variety V defines a web in open sets $U \subset G(m-k, m)$ by specifying the i-th web leaf through P^{m-k} to be $\Sigma(p_i)$ where the p_i are given by (4.1). The basic geometric object here is the incidence correspondence

$$I_V \subset V \times G(m-k, m) \tag{4.2}$$

defined by V, where $I_V = \{(p, A) \colon p \in V,\ A \in G(m-k, m),\ p \in A \cap V\}$. By taking V and the p_i to be defined over the real numbers, we have associated to a projective variety $V_k \subset P^m$ a d ($=$ degree V) web of codimension k ($=$ dim V) submanifolds in $U \subset R^{k(m-k+1)}$.

We now wish to verify that the web defined by a non-degerate algebraic variety is non-degenerate according to our definition, which we shall do for a surface $S \subset P^{n+1}$. For this consider the linear projection

$$\pi\colon P^{n+1} - P^{n-3} \to P^3 \tag{4.3}$$

with center P^{n-3} defined by

$$\pi(p) = (p \wedge P^{n-3}) \cdot P^3$$

where $p \wedge P^{n-3}$ is the P^{n-2} spanned by $p \in P^{n+1} - P^{n-3}$ and the center. Under such a projection, $\pi(S) = S'$ is a non-degenerate surface in P^3 of degree

$$d' = d - \# \text{ of points in } S \cap P^{n-3}\,.$$

The projection induces an inclusion

$$\pi^{-1}\colon G(1,3) \to G(n-1, n+1) \tag{4.4}$$

whose image is the Schubert cycle of all P^{n-1}'s containing the center P^{n-3}. Our first observation is that the *web in* $G(1,3)$ *defined by* S' *is the intersection of* $\pi^{-1}G(1,3)$ *with the web in* $G(n-1, n+1)$ *defined by* S, even in case there are finitely many points in $S \cap P^{n-3}$.

Now consider the web in $G(1,3)$ defined by a non-degenerate surface $S' \subset P^3$. For a generic line P^1 in P^3 the intersection

$$P^1 \cdot S' = p_1 + \dots + p_{d'}$$

where the p_i are distinct. The Schubert cycle $\Sigma(p)$ consists of all lines passing through $p \in P^3$, and under the Plücker embedding

$$G(1,3) \to P^5$$

$\Sigma(p)$ is a plane. If $\Sigma(p)$ and $\Sigma(q)$ fail to intersect transversely, then they must have in common a line in P^5. Any line on $G(1,3)$ is the $\Sigma(p, P^2)$ $(p \in P^2 \subset P^3)$ of lines in P^3 passing through p and contained in P^2. Consequently $\Sigma(p)$ and $\Sigma(q)$ meet transversely unless $p = q$. From this we deduce that the normals to the web defined by $S' \subset P^3$ are skew lines in the projectivized cotangent spaces to $G(1,3)$, these being P^3's.

Finally, for the web defined by a non-degenerate surface S in P^{n+1}, the projection

$$P^{2n-1} - P^{2n-5} \to P^3 \qquad (P^{2n-1} = P(T^*_x))$$

in the definition of web non-degeneracy corresponds to the transposed differential of the inclusion (4.4) induced by the linear projection (4.3) whose center P^{n-3} contains $p_1, \dots, p_{n-1}$. But since $S' = \pi(S)$ is still non-degenerate we deduce that the web defined by S in the neighborhood of a generic $P^{n-1} \in G(n-1, n+1)$ is non-degenerate according to the definition used in Theorems I and II.

Given $V_k \subset P^m$ we consider a meromorphic k-form ω on V and define the *trace* $\boldsymbol{\omega}$, a meromorphic k-form on the Grassmannian $G(m-k, m)$, by

$$\boldsymbol{\omega}(A) = \sum_{i=1}^{d} \omega(p_i(A)) , \qquad A \in G(m-k, m) ,$$

where the intersection

$$A \cdot V = p_1(A) + \dots + p_d(A)$$

for a variable $(m-k)$-plane A. In terms of the diagram (4.2)

$$\boldsymbol{\omega} = (\pi_2)_* \pi_1^* \omega$$

where π_1, π_2 are respectively the projections $I_V \to V$, $I_V \to G(m-k, m)$. The form ω is a *differential of the first kind* (d.f.k.) if $\boldsymbol{\omega}$ is holomorphic (cf. § II of [4]). The space of d.f.k. will be denoted by $H^{k,0}(V)$ and its dimension by $h^{k,0}(V)$. In case V is non-singular $H^{k,0}(V)$ are just the holomorphic k-forms and $h^{k,0}(V)$ is the usual Hodge number.

Since there are no holomorphic forms on $G(m-k, m)$, for ω a d.f.k. we have $\boldsymbol{\omega} = 0$, which is *Abel's theorem*

$$\sum_{i=1}^{d} \boldsymbol{\omega}(p_i(A)) = 0 . \tag{4.5}$$

Clearly (4.5) gives an abelian k-equation on the web defined by V. Conversely, it is not difficult to see that every abelian k-equation is of this form, and consequently the rank of the web is equal to $h^{k,0}(V)$. From Theorem I we deduce the bound

$$h^{k,0}(V) \leqslant \pi(d, m-k+1, k) \tag{4.6}$$

on the number of linearly independent d.f.k. of a non-degenerate $V_k \subset P^m$.

In case $k = 1$ and $m = n$ we obtain *Castelnuovo's bound* (cf. [2])

$$\pi(C) \leqslant \pi(d, n) = \pi(d, n, 1) \tag{4.7}$$

on the genus of a curve of degree d in P^n. The curves for which equality holds in (4.1) were extensively discussed in our previous paper [2], where in fact we proved that their properties could be deduced by web-theoretic methods.

When $k = 2$ we set $m = n + 1$ so that our variety is a surface $S \subset P^{n+1}$ corresponding to a codimension-2 web in $U \subset R^{2n}$. We denote by $p_g(S)$ the number $h^{2,0}(S)$ of d.f.k.; for smooth S this is the *geometric genus*. Theorem I gives the bound

$$(4.8) \qquad p_g(S) \leqslant p_g(d, n) = \pi\,(d, n, 2)\,.$$

This inequality has been proved algebro-geometrically by Joe Harris in his Harvard thesis, which contains general methods of estimating the superabundance (= « number of relations among conditions imposed by ») of linear systems with base conditions imposed.

A special case of (4.7) is (7)

$$p_g(S) = 0 \qquad \text{for degree } S < 2n\,.$$

The general statement for a non-degenerate $V_k \subset P^m$ is, by (2.12) and with $n = m - k + 1$,

$$(4.8) \qquad h^{k,0}(V) = 0 \qquad \text{for } d < k(m-k) + 2\,.$$

These bounds are sharp. For example, for each $n \geqslant 2$ there are $K3$ surfaces $S \subset P^{n+1}$ of degree $2n$, characterized by having as hyperplane sections canonical curves of genus $n + 1$. In general

$$h^{k,0}(V) \leqslant 1 \qquad \text{for } d = k(m-k) + 2\,,$$

and if V is smooth and if $h^{k,0}(V) = 1$, then V is simply-connected (for $k \geqslant 2$) with trivial canonical bundle.

To give another application we first observe that, by (2.13*a*) and (2.13*b*), there is, for each k a unique function $m \to d(m)$ satisfying

$$\pi\big(d(m), m - k + 1, k\big) = m + 1\,.$$

(7) After we mentioned this result to R. Hartshorne, he showed us an algebraic-geometric proof, together with the result that S must be a $K3$-surface, if deg $S = 2n$, $p_g(S) = 1$.

(Notice that $n = m - k + 1$). For example we have

$$(4.9) \qquad \begin{cases} d(m) = 2m & \text{when } k = 1 , \\ d(m) = 3m - 4 & \text{when } k = 2 , \end{cases}$$

and in general

$$(4.10) \qquad d(m) = (k+1)m - (k-1)(k+2) .$$

Next we remark that (4.6) can be inverted to

$$(4.11) \qquad d \geqslant d(h^{k,0}, m)$$

bounding from below the degree of a non-degenerate $V_k \subset P^m$ with fixed $h^{k,0}$.

In particular we consider *canonical algebraic varieties*, defined by the property that their canonical linear system $|K|$ gives a birational and biregular mapping of the abstract variety onto its image in P^m ($m + 1 = h^{k,0}$). For such varieties the degree of the canonical image is

$$(-1)^k c_1^k$$

where c_1 is the 1-st Chern class, and by combining (4.10) and (4.11) we deduce the bound

$$(4.12) \qquad h^{k,0}(V) \leqslant \frac{1}{k+1} [(-1)^k c_1^k + k^2 + 2k - 1]$$

on the Hodge number of a canonical variety. For $k = 1, 2$ we may use (4.9) to obtain

$$\pi(C) \leqslant \frac{-c_1}{2} + 1 , \quad p_a(S) \leqslant \frac{c_1^2}{3} + \frac{7}{3} .$$

The first is an equality due to $c_1 = 2 - 2\pi$, but the second is in general an inequality. It may be compared with Max Noether's estimate

$$(4.13) \qquad p_g(S) \leqslant \frac{c_1^2}{2} + 2$$

valid for any surface. We remark that here the factor $\frac{1}{2}$ ultimately comes from the 2 in

CLIFFORD'S THEOREM:

$$\dim |L| \leqslant \frac{\deg L}{2}$$

for a special linear series $|L|$ *on a curve*, and consequently the generalization of (4.13) to higher dimension is

$$h^{k,0} \leqslant \frac{(-1)^k c_1^k}{2} + \text{const}\,,$$

which is sharp for suitable double coverings of P^k.

The estimates (4.6)-(4.8), (4.11)-(4.13) were consequences of Theorem I. It is of course, interesting to ask whether or not these bounds are sharp, and if so to determine the structure of the *extremal varieties* defined as those for which equality holds. Now Theorem II gives at least the infinitesimal structure of extremal surfaces $S \subset P^{n+1}$ where the degree $d > 2n$. By continuing the reasoning in the proof of that result we may show that an extremal surface lies in a very special way as a divisor on a threefold $V \subset P^{n+1}$ of minimal degree $n-1$, and this leads to an effective determination of all extremal surfaces. These matters will be taken up in a future paper.

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