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Nonlinear Perturbations of Linear Elliptic and Parabolic Problems at Resonance: Existence of Multiple Solutions.

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1. – Introduction.

In this paper we are concerned with the existence of multiple solutions of the nonlinear equation

$$Lu + G(u) = f \tag{1}$$

in the real Hilbert space $H = L^2(\Omega)$, Ω a bounded domain in a finite-dimensional real Euclidean space. Here $L: H \supset D(L) \to H$ denotes a linear operator with dense domain $D(L)$ and compact resolvent; we assume that 0 is eigenvalue of L (and of the adjoint operator L^*), and that for the corresponding eigenspaces, $N(L) = N(L^*)$. Further G is the Nemytskii operator associated with the continuous function $g: \mathbf{R} \to \mathbf{R}$; we assume that the limits $g_\pm := \lim_{s \to \pm\infty} g(s)$ exist (in the proper sense), and that $g_- \leqq 0 \leqq g_+$. Then G maps H continuously into itself and has bounded range. Finally $f \in H$ is given.

By a well-known result which goes back to Landesman-Lazer [7], and for which various different proofs and extensions have been given (e.g. [4] and the comprehensive list of references therein), (1) is solvable at least for those $f \in H$ for which

$$\text{(LL)} \qquad (f, w) < \int_\Omega (g_+ w^+ - g_- w^-)\, dx \qquad \forall w \in N(L)\,,\ w \neq 0\,.$$

Here w^+ (w^-) denotes the positive (negative) part of the function w, respectively, i.e. $w = w^+ - w^-$. We remark that if $g_- = g_+$, no $f \in H$ will satisfy (LL).

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Under some additional assumptions on $N(L)$ and g we show that equation (1) is solvable for certain $f \in H$ which do *not* satisfy (LL), and admits multiple solutions. We impose the following further conditions:

(I) The eigenfunctions of L enjoy the unique continuation property: if $w \in N(L)$ vanishes on a set of positive measure in Ω, then $w = 0$.

(II) There exists $\delta > 0$ such that

$$g(s) \geqq g_+ \qquad \forall s \geqq \delta ,$$
$$g(s) \leqq g_- \qquad \forall s \leqq -\delta .$$

Note that (II) is opposed to the original assumption

$$g_- < g(s) < g_+ \qquad \forall s \in \mathbf{R} \tag{2}$$

made in the theorem of Landesman-Lazer. Set

$$\gamma_+ := \liminf_{s \to +\infty} \left(g(s) - g_+\right) s \qquad (\geqq 0) ,$$
$$\gamma_- := \liminf_{s \to -\infty} \left(g(s) - g_-\right) s \qquad (\geqq 0) .$$

The space H admits a decomposition $H = N(L) \oplus R(L)$. We set $H_1 := N(L)$, $H_2 := R(L)$ and denote by P_1 and P_2 the orthogonal projections on H_1 and H_2, respectively. For $f \in H$ we write $f_1 := P_1 f$ and $f_2 := P_2 f$.

DEFINITION. Let $\mathcal{S}$ be the nonempty, bounded, closed set in H_1 consisting of all functions f_1 for which

$$(f_1, w) \leqq \int_\Omega (g_+ w^+ - g_- w^-)\, dx \qquad \forall w \in N(L) = H_1 .$$

We remark that the set $\mathcal{S}$ is *independent* of $f_2 \in H_2$. Our main result is

THEOREM 1. *Let the mappings L and G be as described above, and suppose that either*

(α) *the functions in $N(L)$ have constant sign in Ω and both γ_+, γ_- are positive, or*

(β) *the functions in $N(L)$ change sign in Ω and at least one of γ_+, γ_- is positive.*

Then to each (fixed) $f_2 \in H_2$ *there exists a bounded open set $\mathcal{S}_{f_2} \subset H_1$ containing $\mathcal{S}$, such that*

(i) *equation* (1) *is solvable for all $f = f_1 + f_2$ with $f_1 \in \mathcal{S}_{f_2}$;*

(ii) *equation* (1) *has at least two different solutions for* $f = f_1 + f_2$ *if* $f_1 \in \mathcal{S}_{f_2} \setminus \mathcal{S}$.

As a consequence of Theorem 1 we further get

THEOREM 2. *Under the assumptions of Theorem* 1, *the mapping* $L + G$ *has closed range in* H.

Theorem 2 should be compared with the assertion that the range of $L + G$ is *open* under condition (2).

REMARK. If $N(L)$ is one-dimensional, it is readily seen that the results hold without hypothesis (I).

This research is related to two recent results concerning the particular situation where $g_- = 0 = g_+$. The first one is due to Fučik-Krbec [5, Theorem 3] (cf. also [6] for some simplifications and improvements), the second one to Ambrosetti-Mancini [2, Theorem 3.1]. In [5, 6] attention is restricted to existence, while in [2] a multiplicity result is obtained by a global Lyapunow-Schmidt method. In order that the equation in $R(L)$ is uniquely solvable with continuous dependence on the given data, Ambrosetti-Mancini need some boundedness condition on the derivative g'.

If $g_- < g_+$, a multiplicity result is given in [1, Prop. 6.4] for perturbations in the *first* eigenvalue and functions $f \in L^\infty(\Omega)$.

Our approach to multiplicity results is similar to that in [2] in as much as degree theory is used. By employing the Leray-Schauder degree in suitable rectangles in H we are however able to avoid any local restriction on g.

The paper is organized as follows: Section 2 contains the proof of Theorem 1, Section 3 that of Theorem 2, while in Section 4 two examples are given of mappings L which satisfy the hypotheses of this paper: (*a*) an elliptic differential operator, (*b*) a parabolic differential operator with a periodicity condition in time.

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2. – Proof of Theorem 1.

(i) Let $f = f_1 + f_2$ with $f_1 \in \mathcal{S}$ and (fixed) $f_2 \in H_2$. Equation (1) is equivalent to the equation

$$(L + P_1)u + \big(G(u) - P_1 u - f\big) = 0$$

which, since $L+P_1$ is invertible on H, is in turn equivalent to

$$u+(L+P_1)^{-1}(G(u)-P_1u-f)=0 . \tag{3}$$

Note that $(L+P_1)^{-1}: H\to H$ is a compact linear operator, and that G has bounded range in H. For $t\in[0,1]$ and $u\in H$ we define the homotopy mapping

$$\mathcal{K}(t, u)=u+t(L+P_1)^{-1}(G(u)-P_1u-f) .$$

Considering only the component in H_2 we see immediately that

$$\mathcal{K}(t, u)=0 \qquad \text{for } t\in[0,1], u\in H \Rightarrow \|P_2u\|<b, \tag{4}$$

with some constant $b>0$. For $n\in\boldsymbol{N}$ let

$$\mathfrak{B}_n=\{u\in H: \|P_1u\|<n, \|P_2u\|<b\} .$$

We claim that there exists $n_0\in\boldsymbol{N}$ such that

$$\mathcal{K}(t, u)\neq 0 \qquad \forall t\in[0,1], \ \forall u\in\partial\mathfrak{B}_{n_0} . \tag{5}$$

Let us assume for the moment that (5) holds. By the homotopy invariance of the Leray-Schauder degree,

$$\begin{aligned}\deg(\mathcal{K}(1,\cdot), \mathfrak{B}_{n_0}, 0)&=\deg(\mathcal{K}(0,\cdot), \mathfrak{B}_{n_0}, 0)\\ &=\deg(I, \mathfrak{B}_{n_0}, 0)=1 .\end{aligned} \tag{6}$$

Since the degree is moreover invariant in components of $H\setminus\mathcal{K}(1,\partial\mathfrak{B}_{n_0})$, there exists an open neighborhood $\mathcal{U}(f_1)$ of f_1 in H_1 such that the degree $=1$ also for $\tilde{f}\in H$ of the form $\tilde{f}=\tilde{f}_1+f_2$ with $\tilde{f}_1\in\mathcal{U}(f_1)$. For those $\tilde{f}$ there exists a solution of (1) in $\mathfrak{B}_{n_0}$.

We set $\mathcal{S}_{f_2}:=\bigcup_{f_1\in\mathcal{S}}\mathcal{U}(f_1)$. Then assertion (i) of Theorem 1 is proved.

It remains to establish (5). We argue by contradiction. Suppose for each $n\in\boldsymbol{N}$ we find $t_n\in[0,1]$ and $u_n\in\partial\mathfrak{B}_n$ such that

$$\mathcal{K}(t_n, u_n)=0 . \tag{7}$$

By (4) it follows that

$$\|P_1u_n\|=n .$$

Applying the linear operator $L + P_1$ on both sides of (7) we get

$$(L + P_1)u_n + t_n(G(u_n) - P_1 u_n - f) = 0 . \tag{8}$$

Hence $t_n \neq 0$, $\forall n \in \boldsymbol{N}$. We take the inner product of (8) with $P_1 u_n$ and obtain

$$(1 - t_n)\|P_1 u_n\|^2 + t_n(G(u_n) - f, P_1 u_n) = 0 .$$

We conclude that

$$(G(u_n) - f, P_1 u_n) \leqq 0 \qquad \forall n ,$$

or, writing $P_1 u_n = n w_n$ with $w_n \in H_1$, $\|w_n\| = 1$,

$$\int_\Omega (g(u_n) - f_1) n w_n \, dx \leqq 0 . \tag{9}$$

Since $f_1 \in \mathcal{S}$, we know on the other hand that

$$\int_\Omega f_1 n w_n \, dx \leqq \int_\Omega (g_+(n w_n)^+ - g_-(n w_n)^-) \, dx . \tag{10}$$

Adding (9) and (10) we get

$$\int_\Omega (g(u_n) - g_+)(n w_n)^+ \, dx - \int_\Omega (g(u_n) - g_-)(n w_n)^- \, dx \leqq 0 , \qquad \forall n \in \boldsymbol{N}. \tag{11}$$

We investigate the first integral in (11); the second one is handled similarly. In the following limiting arguments we pass to subsequences repeatedly; in order not to complicate the notation we however do not change the indices thereby.

Considering the components of (8) in H_2 and recalling that $(L/_{D(L) \cap H_2})^{-1}$: $H_2 \to H_2$ is compact, we infer that the sequence $\{P_2 u_n\}$ is relatively compact in H_2. We may thus assume (for a subsequence)

$$P_2 u_n \to z \qquad (n \to \infty)$$

in H_2 and a.e. in Ω. Moreover there exists a function $y \in H$ such that, for some further subsequence,

$$|P_2 u_n(x)| \leqq y(x) \qquad \forall n \in \boldsymbol{N} , \text{ a.e. } x \in \Omega .$$

(This useful fact occurs as an intermediate step in the standard proof of completeness of L^p-spaces.) Since H_1 is finite-dimensional, we may also assume

$$w_n \to w \qquad (n \to \infty)$$

in H_1 and a.e. in Ω, with $\|w\| = 1$. Hence $w(x) \neq 0$ for a.e. $x \in \Omega$, by hypothesis (I), and consequently

$$(12) \qquad u_n \to \begin{cases} +\infty \\ -\infty \end{cases} \quad \text{a.e. on the sets} \quad \begin{matrix} \{w > 0\}, \\ \{w < 0\}. \end{matrix}$$

We now split up the first integral in (11):

$$\begin{aligned} \int_\Omega (g(u_n) - g_+)(nw_n)^+ \, dx &= \int_{\{u_n \geqq \delta + y\}} \ldots + \int_{\{u_n < \delta + y\}} \ldots = \\ &= \int_{\{u_n \geqq \delta + y\}} (g(u_n) - g_+)\, u_n \, dx - \int_{\{u_n < \delta + y\}} (g(u_n) - g_+)\, P_2 u_n \, dx + \\ &+ \int_{\{u_n < \delta + y\}} (g(u_n) - g_+)(nw_n)^+ \, dx = I_{1,n} - I_{2,n} + I_{3,n}\,, \qquad \text{say.} \end{aligned}$$

The behaviour as $n \to \infty$ is now studied for each of the three integrals separately. In the following χ_ω denotes the characteristic function of the set $\omega \subset \Omega$.

a) $I_{1,n} = \int_\Omega \chi_{\{u_n \geqq \delta + y\}} (g(u_n) - g_+)\, u_n \, dx\,.$

Since the integrand is non-negative, we obtain by (12) and the Fatou lemma that

$$\liminf_{n \to \infty} I_{1,n} \geqq \mu(\{w > 0\}) \cdot \gamma_+\,.$$

Here $\mu(\{w > 0\})$ is the Lebesgue measure of the subset $\{w > 0\}$ of Ω.

b) $I_{2,n} = \int_\Omega \chi_{\{u_n \geqq \delta + y\}} (g(u_n) - g_+)\, P_2 u_n \, dx\,.$

The integrand converges to 0 a.e. in $\{w > 0\}$ and $\{w < 0\}$ and is majorized by some multiple of the function $y \in H$; hence

$$I_{2,n} \to 0 \qquad (n \to \infty)$$

by Lebesgue's theorem.

c) $I_{3,n} = \int_\Omega \chi_{\{u_n < \delta + y\}} (g(u_n) - g_+)(nw_n)^+ \, dx\,.$

Since $\delta + y(x) > u_n(x) = nw_n(x) + P_2 u_n(x) \Rightarrow nw_n(x) < \delta + 2y(x)$, the function $|\chi_{\{u_n<\delta+y\}}(nw_n)^+|$ is bounded by $\delta+2y\in H$. Moreover the integrand converges to 0 a.e. in $\{w>0\}$ and $\{w<0\}$. Again by Lebesgue's theorem,

$$I_{3,n}\to 0 \qquad (n\to\infty)\,.$$

Similarly one treats the second integral in (11) and concludes that in both cases (α) and (β),

$$0\geqq \liminf \int_\Omega \big[\big(g(u_n)-g_+\big)(nw_n)^+ + \big(g(u_n)-g_-\big)\big(-(nw_n)^-\big)\big]\,dx \geqq$$

$$\geqq \mu\big(\{w>0\}\big)\gamma_+ + \mu\big(\{w<0\}\big)\gamma_- > 0\,.$$

This contradiction proves the existence of $n_0\in\boldsymbol{N}$ such that (5) holds.

(ii) Let now $f\in H$ be such that $f_1\in \mathcal{S}_{f_2}\setminus\mathcal{S}$. By the proof of Theorem 1(i) there exists a rectangle $\mathfrak{B}_{n_0}$ in H such that $\deg\big(\mathcal{K}(1,\cdot),\mathfrak{B}_{n_0},0\big)=1$. Further there is $\overline{w}\in N(L)$ such that

$$(f_1,\overline{w})>\int_\Omega \big(g_+(\overline{w})^+ - g_-(\overline{w})^-\big)\,dx\,. \tag{13}$$

Since the integral on the right side in (13) is nonnegative, $f_1\neq 0$ and thus $f\notin R(L)$. Let the constant $K\geqq 0$ be such that the equation

$$Lu+G(u)=(1+K)f$$

has no solution in H (note that G has bounded range in H). We consider the homotopy mapping

$$\mathcal{K}(t,u)=u+(L+P_1)^{-1}\big(G(u)-P_1u-(1+t)f\big)\,,$$

$t\in[0,K]$, $u\in H$. There exists a constant $c>b$ such that

$$\mathcal{K}(t,u)=0 \qquad \text{for } t\in[0,K]\,,\ u\in H\Rightarrow \|P_2u\|<c\,. \tag{14}$$

For $n\in\boldsymbol{N}$ let

$$\mathcal{C}_n=\big\{u\in H\colon \|P_1u\|<n,\ \|P_2u\|<c\big\}\,.$$

We assert that for some $n_1>n_0$,

$$\mathcal{K}(t,u)\neq 0 \qquad \forall t\in[0,K]\,,\ \forall u\in\partial\mathcal{C}_{n_1}\,. \tag{15}$$

For suppose, again to the contrary, that to each $n > n_0$ there exist $t_n \in [0, K]$ and $u_n \in \partial \mathcal{C}_n$ such that

$$\mathcal{K}(t_n, u_n) = 0 .$$

Then

$$Lu_n + G(u_n) = (1 + t_n) f . \tag{16}$$

We may assume $t_n \to t$ $(n \to \infty)$. By (14) we have $\|P_1 u_n\| = n$.

Arguing as in the proof of assertion (i), we infer that $u_n(x) \to \pm \infty$ a.e. on the sets $\{w \gtrless 0\}$ (cf. (12)).

Taking the inner product of (16) with the function $\overline{w}$ of (13) we obtain

$$\big(G(u_n), \overline{w}\big) = (1 + t_n)(f_1, \overline{w}) ;$$

in the limit it follows

$$\int_{\{w>0\}} g_+ \overline{w}\, dx + \int_{\{w<0\}} g_- \overline{w}\, dx = (1 + t)(f_1, \overline{w}) \geqq (f_1, \overline{w})$$

(the second inequality sign holding since $(f_1, \overline{w}) > 0$ by (13)). However

$$\int_{\Omega} \big(g_+(\overline{w})^+ - g_-(\overline{w})^-\big)\, dx \geqq \int_{\{w>0\}} g_+(\overline{w})^+\, dx - \int_{\{w>0\}} g_+(\overline{w})^-\, dx + \int_{\{w<0\}} g_-(\overline{w})^+\, dx - \int_{\{w<0\}} g_-(\overline{w})^-\, dx .$$

We arrive at a contradiction to (13).

Thus by homotopy invariance of the degree,

$$0 = \deg\big(\mathcal{K}(K, \cdot), \mathcal{C}_{n_1}, 0\big) = \deg\big(\mathcal{K}(0, \cdot), C_{n_1}, 0\big) = \deg\big(\mathcal{H}(1, \cdot), \mathcal{C}_{n_1}, 0\big) .$$

We conclude by (6) and the additivity of the degree that

$$\deg\big(\mathcal{H}(1, \cdot), \mathcal{C}_{n_1} \setminus \mathrm{cl}\,(\mathfrak{B}_{n_0}), 0\big) = -1 ;$$

hence there exists a second solution of (1) in the set $\mathcal{C}_{n_1} \setminus \mathrm{cl}\,(\mathfrak{B}_{n_0})$. This proves Theorem 1.

3. – Proof of Theorem 2.

Let $\{f^n\}$ be a sequence of functions in H such that (1) admits solutions for each f^n, and suppose $f^n \to f$ in H. Writing $f = f_1 + f_2$, with $f_1 \in H_1$,

$f_2 \in H_2$, we distinguish between two cases:

(a) $f_1 \in \mathcal{S}$. Then (1) is solvable for this f by Theorem 1(i).

(b) $f_1 \notin \mathcal{S}$. There exists $\overline{w} \in N(L)$ such that

$$(f_1, \overline{w}) > \int_\Omega \left(g_+(\overline{w})^+ - g_-(\overline{w})^-\right) dx . \tag{17}$$

We claim that the solutions u_n of the equations

$$Lu_n + G(u_n) = f^n \tag{18}$$

remain bounded in H, as $n \to \infty$. Clearly $\|P_2 u_n\| \leqq d$, $\forall n \in \boldsymbol{N}$, with some constant d. Assuming that $\|P_1 u_n\| \to \infty$ $(n \to \infty)$, we derive as in the proof of Theorem 1(i) that

$$u_n(x) \to \pm \infty \qquad \text{for a.e. } x \in \Omega .$$

Taking the inner product of (18) with $\overline{w}$ and passing to the limit $n \to \infty$ we obtain as in the proof of Theorem 1(ii) that

$$(f_1, \overline{w}) \leqq \int_\Omega \left(g_+(\overline{w})^+ - g_-(\overline{w})^-\right) dx ,$$

contradicting (17).

We thus may assume, by the compactness of $(L + P_1)^{-1}$, that $u_n \to u$ in H. The passage to the limit $n \to \infty$ in (18) is now immediate and proves the solvability of (1) for the function f also in this case.

4. – Let $\omega \subset \boldsymbol{R}^N$ $(N \geqq 1)$ be a bounded domain with smooth boundary, and let us denote by $\mathcal{A}$:

$$\mathcal{A}u = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - \lambda u \qquad (\lambda \in \boldsymbol{R})$$

a formally selfadjoint, uniformly elliptic differential expression of second order, with real-valued coefficient functions $a_{ij} = a_{ji} \in C^1(\bar{\omega})$. Together with homogeneous Dirichlet boundary conditions, $\mathcal{A}$ induces a selfadjoint differential operator A in $L^2(\omega)$ by

$$D(A) = H_0^1(\omega) \cap H^2(\omega) ,$$

$$Au = \mathcal{A}u \quad (u \in D(A)) .$$

It is known that the eigenfunctions of $\mathcal{A}$, i.e. the functions in $N(A)$, have the unique continuation property (e.g. [8]).

a) The elliptic problem. Here we set $\Omega := \omega$, $H := L^2(\omega)$, and $L := \pm A$. Then L satisfies all the assumptions made in the paper.

b) The parabolic problem with periodicity condition in time. Let $T > 0$ be given, and set $H := L^2(0, T; L^2(\omega)) = L^2(\Omega)$, where Ω denotes the cylinder $(0, T) \times \omega$ in $\boldsymbol{R}^{1+N}$.

Let $\tilde{A}$ be the extension of the above introduced elliptic differential operator to H; it is defined by

$$v = \tilde{A}u \Leftrightarrow u, v \in H \;, \; u(t) \in D(A) \text{ and } v(t) = Au(t) \text{ for a.a. } t \in (0, T) \,.$$

$\tilde{A}$ is a selfadjoint operator in H.

Let further $d/dt\colon H \supset D(d/dt) \to H$ be the linear operator given by

$$D\left(\frac{d}{dt}\right) = \{u \in H\colon u' \in H, u(0) = u(T)\} \,, \qquad \frac{d}{dt}\, u = u' \qquad \left(u \in D\left(\frac{d}{dt}\right)\right).$$

Here the time-derivative is meant in the distributive sense. Note that $u, u' \in H$ implies that u is (perhaps after modification on a nullset in $[0, T]$) a continuous and a.e. differentiable mapping of $[0, T]$ into $L^2(\omega)$. From the relation

$$(u', v) + (u, v') = \big(u(T), v(T)\big)_{L^2(\omega)} - \big(u(0), v(0)\big)_{L^2(\omega)} \,,$$

which holds for all $u, v \in H$ with $u', v' \in H$, it follows that

$$\left(\frac{d}{dt}\right)^* = -\frac{d}{dt}$$

and thus

$$\left(\frac{d}{dt} u, u\right) = 0 \qquad \forall u \in D\left(\frac{d}{dt}\right).$$

We claim that the mappings $L = \pm\, d/dt \pm \tilde{A}$ (where all 4 combinations are allowed) satisfy the conditions imposed on L, with

$$N(L) = N(L^*) = N\left(\frac{d}{dt}\right) \cap N(\tilde{A}) = N(A) \,.$$

For the sake of definiteness suppose in the following that $L = d/dt + \tilde{A}$.

As in [3, Theorem 19] (where the initial-value problem is considered), one shows first that

$$R\left(\pm\frac{d}{dt}+\tilde{A}+(\lambda+1)I\right)=H \tag{19}$$

(note that $\tilde{A}+(\lambda+1)I$ is monotone and selfadjoint, hence a subdifferential). Further

$$\left(\frac{d}{dt}u,\tilde{A}u\right)=0 \qquad \forall u\in D\left(\frac{d}{dt}\right)\cap D(\tilde{A})\,. \tag{20}$$

From (19) it follows that

$$\left(\frac{d}{dt}+\tilde{A}\right)^{*}=-\frac{d}{dt}+\tilde{A}\,;$$

by (20) we then conclude the above assertions on the nullspaces of L and L^{*}. Finally $(d/dt+\tilde{A}+(\lambda+1)I)^{-1}\colon H\to H$ is compact by Aubin's lemma.

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