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# Some Critical Point Theorems and Applications to Semilinear Elliptic Partial Differential Equations(\*).

PAUL H. RABINOWITZ (\*\*)

*dedicated to Jean Leray*

## Introduction.

Let  $E$  be a real Banach space and  $I$  a continuously differentiable map from  $E$  to  $\mathbf{R}$ , i.e.  $I \in C^1(E, \mathbf{R})$ . We say  $I$  satisfies the Palais-Smale condition (PS) if any sequence  $(u_m)$  such that  $I(u_m)$  is bounded and  $I'(u_m) \rightarrow 0$  is precompact. In [1] and [2], by imposing various qualitative conditions on  $I$  near 0 and  $\infty$ , Ambrosetti and the author obtained several results concerning the existence of critical points of  $I$  and applied them to semilinear elliptic boundary value problems to get existence theorems in that setting.

The purpose of this paper is to extend the theory of [1] and [2]. To be more precise, let  $B_r = \{x \in E \mid \|x\| < r\}$ . It was shown in [1, Theorem 2.1] that if  $I(0) = 0$  and

1) There are constants  $\varrho, \alpha > 0$  such that  $I > 0$  in  $B_\varrho \setminus \{0\}$  and  $I > \alpha$  on  $\partial B_\varrho$ , and

2) There is an  $e \in E$ ,  $e \neq 0$  such that  $I(e) = 0$ ,

then  $I$  has a critical value  $c \geq \alpha$ .

It was further shown in [1, Theorem 2.21], [2, Theorem 3.37] that if  $I$  is even and 2) is replaced by the requirement that  $I$  be negative at infinity

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in an appropriate sense (see condition  $(I_3)$  of § 1), then 1) can be weakened to hold in  $B_\rho \cap \hat{E}$  where  $\hat{E}$  is a subspace of  $E$  of finite codimension. Moreover for this case  $I$  has an unbounded sequence of positive critical values.

Our main results here show how one can still obtain critical points of  $I$  under weakened versions of 1) without requiring that  $I$  be even. The abstract critical point theorems will be given in § 1 and some applications to semilinear elliptic partial differential equations will be carried out in § 2.

## 1. – The critical point theorems.

In this section we shall extend the results of [1] and [2] mentioned in the Introduction. Some additional variants will be presented for the special case of  $E = \mathbf{R}^n$ . Lastly a case where we only require 1) for  $B_\rho$  intersected with a finite dimensional subspace of  $E$  will be treated. As in [1], [2], our arguments are based on minimax characterizations of critical values of  $I$ .

Our main result is

**THEOREM 1.1.** *Let  $E$  be a real Banach space and let  $I \in C^1(E, \mathbf{R})$  and satisfy (PS). Suppose that  $E = E_k \oplus \hat{E}$  where  $E_k$  is  $k$  dimensional and  $I$  satisfies*

$$(I_1) \quad I|_{E_k} \leq 0,$$

$$(I_2) \quad \text{There are constants } \rho, \alpha > 0 \text{ such that } I \geq 0 \text{ in } B_\rho \cap \hat{E} \text{ and } I \geq \alpha \text{ on } \partial B_\rho \cap \hat{E}.$$

$$(I_3) \quad \text{For each finite dimensional subspace } \tilde{E} \text{ of } E, \text{ there is a constant } R = R(\tilde{E}) \text{ such that } I \leq 0 \text{ on } \tilde{E} \setminus B_{R(\tilde{E})}.$$

*Then  $I$  has a positive critical value,  $c$ , characterized by*

$$(1.2) \quad c = \inf_{h \in \Gamma} \max_{u \in \bar{B}_r \cap E_{k+1}} I(h(u))$$

*where  $E_{k+1} = E_k \oplus \text{span}\{\varphi\}$  for some fixed  $\varphi \in \tilde{E} \setminus \{0\}$ ,  $r = R(E_{k+1})$ , and*

$$\Gamma = \{h \in C(\bar{B}_r \cap E_{k+1}, E) | h(u) = u \quad \text{if } I(u) \leq 0\}.$$

A finite dimensional version of Theorem 1.1 was given in [5] and it motivated this paper. Two preliminary results are required for the proof of Theorem 1.1. The first is a standard lemma from the calculus of variations. Let  $A_s = \{u \in E | I(u) \leq s\}$  and  $K_s = \{u \in E | I(u) = s \text{ and } I'(u) = 0\}$ .

**LEMMA 1.3.** *Let  $I \in C^1(E, \mathbf{R})$  and satisfy (PS). Let  $c \in \mathbf{R}$ ,  $\mathcal{O}$  be any neighborhood of  $K_c$ , and  $\bar{\varepsilon} > 0$ . Then there is an  $\varepsilon \in (0, \bar{\varepsilon})$  and an  $\eta \in C([0, 1] \times E, E)$  such that*

- 1)  $\eta(t, x) = x$  if  $I(x) \notin [c - \varepsilon, c + \varepsilon]$ ,
- 2)  $\eta(1, A_{c+\varepsilon} \setminus \mathcal{O}) \subset A_{c-\varepsilon}$ ,
- 3) If  $K_c = \emptyset$ ,  $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$ .

See e.g. [2] or [3] for a proof of this lemma. Next we need a topological lemma. For  $j \leq m$ , let  $\mathbf{R}^j = \{x = (x_1, \dots, x_m) | x_i = 0, j+1 \leq i \leq m\}$  and  $(\mathbf{R}^j)^\perp$  the orthogonal complement of  $\mathbf{R}^j$  in  $\mathbf{R}^m$ .

**LEMMA 1.4.** *Let  $b_R = \mathbf{R}^{k+1} \cap \bar{B}_R$  and let  $b_R^+ = \{x \in \bar{B}_R | x_{k+1} \geq 0\}$ . If  $g \in C(b_R^+, \mathbf{R}^m)$ ,  $m > k$ ,  $\varrho < R$ , and there is a homotopy  $G \in C([0, 1] \times \partial b_R^+, \mathbf{R}^m \setminus (\partial B_\varrho \cap (\mathbf{R}^k)^\perp))$  such that  $G(0, x) = x$  and  $G(1, x) = g(x)$ , then  $g(b_R^+) \cap \partial B_\varrho \cap (\mathbf{R}^k)^\perp \neq \emptyset$ .*

**PROOF.** A proof of Lemma 1.4 due to E. Fadell using intersection theory [4, p. 197] can be found in [5, Lemma A-2]. For the convenience of the reader, we include it here. The problem is normalized by taking  $R = 1$  and  $\varrho < 1$ . Let  $D = \bar{B}_\varrho \cap (\mathbf{R}^k)^\perp$ ,  $\hat{D} = (\partial B_\varrho) \cap (\mathbf{R}^k)^\perp$ ,  $b^+ = b_1^+$ , and  $\partial b^+ = \{x \in b^+ | x_{k+1} = 0 \text{ or } |x| = 1\}$ . The intersection pairing:

$$H_{m+k}(D, \hat{D}) \times H_k(\partial b^+) \rightarrow \mathbf{Z}$$

yields  $+1$  as intersection number for appropriately chosen generators in the homology groups above. Furthermore naturality yields the diagram

$$\begin{array}{ccc} H_{m+k}(D, \hat{D}) \times H_k(\partial b^+) & \rightarrow & \mathbf{Z} \\ \downarrow id \times j_* & & \uparrow \\ H_{m+k}(D, \hat{D}) \times H_k(\mathbf{R}^m \setminus \hat{D}) & & \end{array}$$

where  $j: \partial b^+ \subset \mathbf{R}^m \setminus \hat{D}$  is inclusion. If  $g \in C(b^+, \mathbf{R}^m \setminus \hat{D})$  and  $G$  is as in the statement of the lemma, then  $j_*$  would be the trivial homomorphism and the intersection number would be 0, a contradiction.

These preliminaries being completed, we can now give the

**PROOF OF THEOREM 1.1.** Assume for the moment that  $c \geq \alpha$ . If  $c$  is not a critical value of  $I$ , we can invoke Lemma 1.3 with  $\bar{\varepsilon} = \alpha/2$ . Let  $h \in \Gamma$  be such that

$$(1.5) \quad \max_{u \in \bar{B}_r \cap E_{k+1}} I(h(u)) \leq c + \varepsilon.$$

Since  $\eta(1, h) \in C(\bar{B}_r \cap E_{k+1}, E)$  and if  $I(u) \leq 0$ , then  $\eta(1, h(u)) = \eta(1, u) = u$  by 1) of Lemma 1.3 and our choice of  $\varepsilon$ , it follows that  $\eta(1, h) \in \Gamma$ . But then by (1.5) and 3) of Lemma 1.3,

$$(1.6) \quad \max_{u \in \bar{B}_r \cap E_{k+1}} I(\eta(1, h(u))) \leq c - \varepsilon$$

contrary to the definition of  $c$ .

To complete the proof, we must show  $c \geq \alpha$ . Using  $(I_2)$ , it suffices to show  $h(\bar{B}_r \cap E_{k+1}) \cap \partial B_\varrho \cap \hat{E} \neq \emptyset$  for all  $h \in \Gamma$ . Let  $K = h(\bar{B}_r \cap E_{k+1})$ . Since  $K$  is compact, by an approximation lemma of Leray-Schauder [9], for all  $\varepsilon > 0$ , there exists a finite dimensional subspace  $F_\varepsilon \subset E$  and a mapping  $i_\varepsilon \in C(K, F_\varepsilon)$  such that  $\|i_\varepsilon(u) - u\| \leq \varepsilon$  for all  $u \in K$ . We can assume  $F_\varepsilon \supset E_{k+1}$ . Let  $h_\varepsilon = i_\varepsilon \circ h$ . Then  $h_\varepsilon \in C(\bar{B}_r \cap E_{k+1}, F_\varepsilon)$  and  $h_\varepsilon(u) \rightarrow h(u)$  as  $\varepsilon \rightarrow 0$  uniformly for  $u \in \bar{B}_r \cap E_{k+1}$ . Observe that if  $u \in (\partial B_r \cap E_{k+1}) \cup E_k$ , by  $(I_1)$  and  $(I_3)$  we have

$$(1.7) \quad \|h_\varepsilon(u) - h(u)\| = \|h_\varepsilon(u) - u\| \leq \varepsilon.$$

Since  $E = E_k \oplus \hat{E}$ ,  $u \in E$  implies  $u = v + w$ , where  $v \in E_k$ ,  $w \in \hat{E}$ . We can assume  $\|u\|_E = (\|v\|_{E_k}^2 + \|w\|_{\hat{E}}^2)^{\frac{1}{2}}$ . Thus if  $u \in (\partial B_r \cap E_{k+1}) \cup E_k$ , it follows that

$$(1.8) \quad \|u - (\partial B_\varrho \cap \hat{E})\| \geq \beta \equiv \min(r - \varrho, \varrho).$$

Choosing  $\varepsilon < \beta$ , identifying  $F_\varepsilon$  with  $\mathbf{R}^m$ ,  $E_k, E_{k+1}$  with  $\mathbf{R}^k, \mathbf{R}^{k+1}$ , and defining  $G(t, u) = th_\varepsilon(u) + (1-t)u$ , it follows from (1.7)-(1.8) that  $G$  satisfies the hypotheses of Lemma 1.4. Hence  $h_\varepsilon(\bar{B}_r \cap E_{k+1}) \cap \partial B_\varrho \cap \hat{E} \neq \emptyset$ . Consequently there exists  $u_\varepsilon \in \bar{B}_r \cap E_{k+1}$  such that  $h_\varepsilon(u_\varepsilon) \in \partial B_\varrho \cap \hat{E}$ . As  $\varepsilon \rightarrow 0$ , along a subsequence we have  $u_\varepsilon \rightarrow u \in \bar{B}_r \cap E_{k+1}$  and

$$\|h_\varepsilon(u_\varepsilon) - h(u)\| \leq \|h_\varepsilon(u_\varepsilon) - h(u_\varepsilon)\| + \|h(u_\varepsilon) - h(u)\| \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ . Therefore  $h(u) \in \partial B_\varrho \cap \hat{E}$  and the proof is complete.

**REMARK 1.9.** If  $E$  is infinite dimensional, in general  $I$  will not be bounded from above in contrast to the finite dimensional case where this is a consequence of  $(I_3)$ . By  $(I_2)$ ,  $I$  then has a positive maximum  $\bar{c}$  in  $B_r$ . In general  $c < \bar{c}$ . However if  $c = \bar{c}$ ,  $I$  possesses infinitely many critical points. Indeed (1.2) then shows  $I$  achieves its maximum on  $h(B_r \cap E_{k+1})$  for each  $h \in \Gamma$  and this implies  $I$  has infinitely many distinct critical points corresponding to  $c = \bar{c}$ . Our next result gives another characterization of critical values of  $I$  in the finite case and a more precise description of the degenerate situation.

**THEOREM 1.10.** *Let  $I \in C^1(\mathbf{R}^m, \mathbf{R})$  and satisfy  $(I_1)$ – $(I_3)$ . Then  $I$  possesses at least two positive critical values characterized by*

$$\bar{c} = \max_{u \in B_r} I(u), \quad r = r(\mathbf{R}^m)$$

and

$$(1.11) \quad b = \sup_{\substack{A \subset M \\ \text{cat}_M A = 2}} \min_{u \in A} I(u),$$

where  $M = \bar{B}_r \cap (\mathbf{R}^m \setminus \mathbf{R}^k)$ . If  $b = \bar{c}$ ,  $\text{cat}_M K_b = 2$ .

Before proving Theorem 1.10, a few remarks are in order. The notation  $\text{cat}_M A$  refers to the Ljusternik-Schnirelman category of the subset  $A$  of  $M$ . In (1.11) as admissible  $A$  we only consider  $A \subset M$  with  $A$  closed in  $\mathbf{R}^m$ . By definition  $\text{cat}_M A = 1$  if  $A$  is contractible to a point in  $M$  and  $\text{cat}_M A = j$  if  $j$  is the least integer such that  $A$  can be covered by  $j$  closed sets  $A_i$  with  $\text{cat}_M A_i = 1$ ,  $1 \leq i \leq j$ . It is clear that any admissible  $A \subset M$  is homotopic to a subset of the unit sphere  $S^{m-k}$  in  $(\mathbf{R}^k)^\perp$ . Moreover  $\text{cat}_M S^{m-k} = 2$  since if  $\text{cat}_M S^{m-k} = 1$ , any homotopy of  $S^{m-k}$  to a point in  $M$  would induce a homotopy of  $S^{m-k}$  to a point in  $S^{m-k}$ . However as is well known (see e.g. [6]),  $\text{cat}_{S^{m-k}} S^{m-k} = 2$  so no such homotopy can exist.

**PROOF OF THEOREM 1.10.** Define

$$(1.12) \quad c_i = \sup_{\substack{A \subset M \\ \text{cat}_M A \geq i}} \min_{u \in A} I(u), \quad i = 1, 2.$$

Then  $c_1 = \bar{c}$  since we can take  $A = \{x\}$  for any  $x \in M$  and  $c_2 = b \leq \bar{c}$ . Note further that by  $(I_2)$ ,  $I > 0$  on  $\partial S^{m-k}$  and therefore  $b > 0$ .  $(I_c)$  implies  $I$  satisfies (PS) on the set where  $I > 0$ . A slightly strengthened version of Lemma 1.3 and a standard argument (similar to the first paragraph of the proof of Theorem 1.1) shows that  $b$  is a critical value of  $I$  and if  $\bar{c} = b$ ,  $\text{cat}_M K_b = 2$ . (See e.g. [6], [7], or [2]).

**REMARK 1.13.** We believe that  $b$  as defined in (1.11) equals  $c$  defined in (1.2).

An examination of the proof of Theorem 1.10 shows that the argument in fact gives the following result which interchanges the finite dimensional and finite codimensional hypotheses  $(I_1)$  and  $(I_2)$  of Theorem 1.1.

**THEOREM 1.14.** *Let  $E$  be a real Banach space,  $I \in C^1(E, \mathbf{R})$  and satisfy (PS).*

Suppose that  $E = E_k \oplus \hat{E}$  where  $k \geq 1$ ,  $E_k$  is  $k$ -dimensional, and  $I$  satisfies

$$(I_4) \quad I|_{\hat{E}} \leq 0,$$

$$(I_5) \quad \text{There are constants } \rho, \alpha > 0 \text{ such that } I \geq 0 \text{ in } B_\rho \cap E_k \text{ and } I \geq \alpha \text{ on } \partial B_\rho \cap E_k,$$

$$(I_6) \quad I \text{ is bounded from above.}$$

Then  $I$  possesses at least two positive critical values characterized by

$$(1.15) \quad b_i = \sup_{\substack{A \in \mathcal{M} \\ \text{cat}_M A \geq i}} \min_{u \in A} I(u) \quad i = 1, 2$$

where  $A$  is compact in  $E$  and  $M = E \setminus \hat{E}$ . Moreover if  $b_1 = b_2 = d$ ,  $\text{cat}_M K_d \geq 2$ .

PROOF. Immediate from that of Theorem 1.10 and the remark preceding it.

## 2. - Applications to partial differential equations.

In this section we shall show how Theorems 1.1 and 1.14 can be employed to obtain existence theorems for semilinear elliptic boundary value problems. Consider

$$(2.1) \quad \begin{cases} Lu \equiv - \sum_{i,j=1}^n (a_{ij}(x) u_{x_i})_{x_j} + c(x)u = a(x)u + p(x, u), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

where  $\Omega$  is a bounded domain in  $\mathbf{R}^n$  with a smooth boundary,  $L$  is uniformly elliptic in  $\bar{\Omega}$  with smooth coefficients,  $c \geq 0$  in  $\Omega$ ,  $a$  is smooth and positive in  $\bar{\Omega}$ , and  $p$  is smooth and satisfies

$$(p_1) \quad |p(x, z)| \leq a_1 + a_2 |z|^s, \quad s < \frac{n+2}{n-2}, \quad n > 2,$$

$$(p_2) \quad p(x, z) = o(|z|) \quad \text{at } z = 0.$$

There are constants  $M > 0$  and  $\theta \in (0, \frac{1}{2})$  such that

$$(p_3) \quad 0 < P(x, z) = \int_0^z p(x, t) dt \leq \theta z p(x, z) \quad \text{for } |z| \geq M.$$

If  $n \leq 2$ ,  $(p_1)$  can be considerably improved. See e.g. § 3 of [1].

Set

$$I(u) = \int_{\Omega} \left[ \frac{1}{2} \left( \sum_{i,j=1}^n a_{ij}(x) u_{x_i} u_{x_j} + c(x) u^2 - a(x) u^2 \right) - P(x, u) \right] dx.$$

Formally critical points of  $I$  in  $E = W_0^{1,2}(\Omega)$  are weak solutions of (2.1). The smoothness of  $L$ ,  $a$ ,  $p$ , and  $(p_1)$  and standard regularity results then imply such weak solutions are smooth functions. Thus we focus our attention on finding critical points of  $I$  in  $E$ .

Consider the linear eigenvalue problem

$$(2.3) \quad \begin{cases} Lv = \lambda av, & x \in \Omega \\ v = 0, & x \in \partial\Omega. \end{cases}$$

As is well known, (2.3) possesses an unbounded sequence of eigenvalues  $0 < \lambda_1 < \lambda_2 < \dots \leq \lambda_m \rightarrow \infty$  as  $m \rightarrow \infty$  with corresponding eigenfunctions  $v_1, \dots, v_m, \dots$ . It was shown in [1] that if  $p$  satisfies  $(p_1)$ - $(p_3)$  and  $\lambda_1 > 1$ , (2.1) possesses a positive solution (i.e.  $u > 0$  in  $\Omega$ ) and a negative solution. If  $\lambda_1 \leq 1$ , the argument of [1] fails unless  $p(x, z)$  is odd in  $z$ . We will show:

**THEOREM 2.4.** *If  $p$  satisfies  $(p_1)$ - $(p_3)$  and*

$$(p_4) \quad zp(x, z) \geq 0 \quad \text{for } x \in \Omega, z \in \mathbf{R},$$

*then (2.1) possesses a nontrivial solution.*

**PROOF.** Because of the result just mentioned, we can assume  $\lambda_1 \leq 1$ . In fact suppose  $\lambda_k \leq 1 < \lambda_{k+1}$ . Let  $E_k = \text{span}\{v_1, \dots, v_k\}$  and  $\hat{E} = E_k^\perp$ , the orthogonal complement of  $E_k$ . By  $(p_4)$ ,  $I \leq 0$  on  $E_k$ . It was further shown in [1] that  $(p_1)$ - $(p_3)$  imply that  $I \in C^1(E, \mathbf{R})$  and satisfies  $(I_2)$ ,  $(I_3)$  and (PS). Hence Theorem 2.4 follows immediately from Theorem 1.1.

**REMARK 2.5.** As was noted above for  $\lambda_1 > 1$ , (2.1) has a positive and a negative solution. In contrast we next show:

**COROLLARY 2.6.** *Under the hypotheses of Theorem 2.4, if  $\lambda_1 \leq 1$ , (2.1) does not have a positive (or negative) solution  $u(x)$  unless  $\lambda_1 = 1$ ,  $u(x)$  is a multiple of  $v_1(x)$  and  $p(x, u(x)) \equiv 0$ .*

**PROOF.** Suppose  $u$  is a positive solution of (2.1) with  $\lambda_1 \leq 1$ . Then.

$$(2.7) \quad \int_{\Omega} v_1 Lu \, dx = \int_{\Omega} v_1 (au + p(x, u)) \, dx = \int_{\Omega} uLv_1 \, dx = \lambda_1 \int_{\Omega} auv_1 \, dx.$$



Since we can assume  $v_1 > 0$  in  $\Omega$ , we have

$$(2.8) \quad 0 \geq (\lambda_1 - 1) \int_{\Omega} a u v_1 dx = \int_{\Omega} v_1 p(x, u) dx \geq 0$$

which is impossible unless  $p(x, z) \equiv 0$  for  $0 \leq z \leq \max u(x)$  and  $\lambda_1 = 1$  in which case  $u \equiv \beta v_1$ .

Next we illustrate how Theorem 1.14 can be used in similar situations. Consider (2.1) again where  $p$  satisfies  $(p_2)$  and e.g.

$$(p_4) \quad 0 \geq z^{-1} p(x, z) \rightarrow -\infty \quad \text{as } |z| \rightarrow \infty.$$

No growth conditions are needed for this case since  $(p_4)$  implies that  $az + p$  can be redefined to be independent of  $z$  for large  $|z|$  so the modified nonlinearity is uniformly bounded. Moreover solutions of the modified equation are still solutions of (2.1). (See e.g. Theorem 3.4 of [2]). It is easily verified that if  $\lambda_k < 1 \leq \lambda_{k+1}$ ,  $E_k$ ,  $\hat{E}$  are as in Theorem 2.4 and

$$J(u) = - \int_{\Omega} \left[ \frac{1}{2} \left( \sum_{i,j=1}^n a_{ij}(x) u_{x_i} u_{x_j} + c(x) u^2 \right) - \bar{P}(x, u) \right] dx$$

where  $\bar{P}$  is the primitive of the modification of  $az + p$ , then  $J$  satisfies the hypotheses of Theorem 1.14.

Since stronger results can be obtained for this problem using methods based on Leray-Schauder degree theory [8] we will not carry out the details here. However if  $L$  were replaced by a higher order divergence structure elliptic operator one could obtain results using Theorem 1.14 where degree theoretic methods would fail.

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