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Hyperfunction Cohomology Classes and Their Boundary Values (*).

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dedicated to Jean Leray

Introduction.

The point of this paper is to extend the results of [1], [2], and [3] to the real analytic and hyperfunction categories. Namely we consider a general first order complex of linear partial differential operators with real analytic coefficients acting on either real analytic or hyperfunction sections of real analytic vector bundles over a real analytic manifold X, and a real analytic hypersurface S in an open set Ω of X having two sides. Assuming that S is non-characteristic for the complex under consideration, we show that it induces a boundary complex on S, consisting of partial differential operators tangential to S which act on real analytic or hyperfunction sections of a real analytic vector bundle over S. The cohomology spaces on Ω, S, and the two sides, taken with respect to either real analytic or hyperfunction sections, are then shown to be related by certain fundamental diagrams: the Mayer-Vietoris sequence and the ladder diagram. We also consider hyperfunction cohomology classes taken with respect to an arbitrary family of supports.

We hope that these results will provide a general formalism, for complexes with real analytic coefficients, in which one can better view a variety

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of natural questions about overdetermined systems of partial differential equations.

Obviously in our use of the hyperfunctions of Sato ([14], [15]) we have been very much influenced by the modern Japanese school of analysis. But for our simple needs we have employed the treatment of Martineau [12], as expounded by Schapira [17]. For a different approach to the study of hyperfunction boundary values for general elliptic systems, see Komatsu-Kawai [8], Kashiwara [7], and Sato-Kashiwara-Kawai [16].

The C^∞ and distribution categories for operators with C^∞ coefficients were discussed in [3], but we have not mentioned here the homology analogue which was taken up in [3]. The special case of the Dolbeault complex is treated in [1] in the C^∞ category, and in Stormark [18] and Polking-Wells [13] the Mayer-Vietoris sequence for the Dolbeault complex is discussed in the distribution and hyperfunction categories.

Our main results are Theorem 2.1 (Mayer-Vietoris sequence in the real analytic category), Theorems 3.1 and 3.3 (Mayer-Vietoris sequence in the hyperfunction category), and Theorems 4.1 and 4.4 (the ladder diagram in the two categories). In order to indicate how these theorems can be used, we give some examples in section 5. (These are merely the results of [1] in different categories.)

1. – Preliminaries.

(a) Ω is an open subset of a real analytic manifold X, of dimension n and countable at infinity. For each j, $j = 0,1,2,\ldots$, E^j is a real analytic vector bundle, $\mathcal{A}^j$ is the corresponding sheaf of germs of real analytic sections, and $\mathcal{A}^j(\omega)$ is the space of real analytic sections over ω where $\omega \subset \Omega$ is open. For each $x \in X$, E^j_x is the fiber of E^j at x and $\mathcal{A}^j_x$ is the stalk of $\mathcal{A}^j$ at x. If F is any subset of X, $\mathcal{A}^j(F)$ is the space of sections of E^j over F which have a real analytic extension to some open neighborhood of F. Thus $\mathcal{A}^j(F)$ is the inductive limit $\varinjlim_{\omega\supset F} \mathcal{A}^j(\omega)$, where the open sets $\omega \supset F$ are partially ordered by inclusion.

We shall consider complexes of linear partial differential operators

$$0 \to \mathcal{A}_0 \xrightarrow{D^0} \mathcal{A}^1 \xrightarrow{D^1} \cdots \tag{1.1}$$

with real analytic coefficients and locally constant orders. We shall assume also that the orders of the operators are all one, since this assumption includes all the applications we have in mind and allows significant simplifications in the proofs. Since (1.1) is a complex, $D^{j+1}\circ D^j = 0$ for $j \geqslant 0$.

For economy of notation, we consider $E = \bigoplus E^j$ a graded vector bundle, $\mathcal{A}$ a graded sheaf, and $D\colon \mathcal{A} \to \mathcal{A}$ a graded operator of degree one.

(*b*) Let S be a real analytic hypersurface locally closed in Ω with two sides. Thus $\Omega \setminus S = \mathring{\Omega}^+ \cup \mathring{\Omega}^-$ where $\mathring{\Omega}^+$ and $\mathring{\Omega}^-$ are disjoint open sets and $\Omega^\pm = \mathring{\Omega}^\pm \cup S$. By letting F above be respectively Ω, S, and $\Omega^\pm$ we obtain the complexes

$$\mathcal{A}(\Omega)\colon\ 0 \to \mathcal{A}^0(\Omega) \xrightarrow{D^0} \mathcal{A}^1(\Omega) \xrightarrow{D^1} \cdots$$

$$\mathcal{A}(S)\colon\ 0 \to \mathcal{A}^0(S) \xrightarrow{D^0} \mathcal{A}^1(S) \xrightarrow{D^1} \cdots$$

and

$$\mathcal{A}(\Omega^\pm)\colon 0 \to \mathcal{A}^0(\Omega^\pm) \xrightarrow{D^0} \mathcal{A}^1(\Omega^\pm) \xrightarrow{D^1} \cdots$$

with cohomologies $H^*(\mathcal{A}(\Omega))$, $H^*(\mathcal{A}(S))$, and $H^*(\mathcal{A}(\Omega^\pm))$.

That is, for each $j \geqslant 0$,

$$H^j(\mathcal{A}(\Omega)) = \ker\{D^j\colon \mathcal{A}^j(\Omega) \to \mathcal{A}^{j+1}(\Omega)\}/D^{j-1}(\mathcal{A}^{j-1}(\Omega))$$

where $\mathcal{A}^{-1}(\Omega) = 0$; and similarly for the other complexes.

Let ${}^*E$ be the dual bundle of E and let ${}^tE = {}^*E \otimes \Lambda^n CT^*(X)$, where $CT^*(X)$ is the complexified cotangent bundle of X. If $\mathcal{E}(\omega)$ is the space of smooth sections of E over ω, and if ${}^t\mathcal{D}(\omega)$ is the space of smooth sections of ${}^tE$ with compact support contained in ω, there is a bilinear pairing

$$\langle\cdot,\cdot\rangle\colon {}^t\mathcal{D}(\omega) \times \mathcal{E}(\omega) \to C_0^\infty(\omega, \Lambda^n CT^*)\,.$$

Corresponding to D there is a formal transpose operator ${}^tD$, graded with degree -1, such that

$$\int_\omega \langle\varphi, Du\rangle = \int_\omega \langle{}^tD\varphi, u\rangle$$

for every $\varphi \in {}^t\mathcal{D}(\omega)$ and $u \in \mathcal{E}(\omega)$.

DEFINITION 1.1. A section $u \in \mathcal{A}(\omega)$ (or $\mathcal{E}(\omega)$) has *zero Cauchy data on S if*

$$\int_{\omega^\pm} \langle\varphi, Du\rangle = \int_{\omega^\pm} \langle{}^tD\varphi, u\rangle$$

for all $\varphi \in {}^t\mathcal{D}(\omega)$, where $\omega^\pm = \omega \cap \Omega^\pm$. The space of all sections in $\mathcal{A}(\omega)$ with zero Cauchy data on S is $\mathfrak{J}(\omega, S)$, and $\mathfrak{J}(\omega^\pm, S) = \mathfrak{J}(\omega, S)|_{\omega^\pm}$.

The point is that the expected boundary term on $S \cap \omega$ vanishes if u has zero Cauchy data. An easy calculation shows $D\colon \mathfrak{I}(\omega, S) \to \mathfrak{I}(\omega, S)$ since $D \circ D = 0$. Therefore there are complexes

$$\mathfrak{I}(\Omega, S)\colon\ 0 \to \mathfrak{I}^0(\Omega, S) \xrightarrow{D^0} \mathfrak{I}^1(\Omega, S) \xrightarrow{D^1} \cdots$$

and

$$\mathfrak{I}(\Omega^{\pm}, S)\colon 0 \to \mathfrak{I}^0(\Omega^{\pm}, S) \xrightarrow{D^0} \mathfrak{I}^1(\Omega^{\pm}, S) \xrightarrow{D^1} \cdots$$

with cohomologies $H^*\big(\mathfrak{I}(\Omega, S)\big)$ and $H^*\big(\mathfrak{I}(\Omega^{\pm}, S)\big)$ respectively.

It is clear that the maps $\omega \to \mathfrak{I}(\omega, S)$ constitute a presheaf which is also a sheaf, denoted by $\mathfrak{I}$. If $\omega \cap S = \emptyset$, $\mathfrak{I}(\omega, S) = \mathcal{A}(\omega)$.

(*c*) The tangential or boundary complex along S

$$\mathfrak{C}(\omega, S)\colon 0 \to \mathfrak{C}^0(\omega, S) \xrightarrow{D_S^0} \mathfrak{C}^1(\omega, S) \xrightarrow{D_S^1} \cdots$$

is the quotient complex

$$0 \to \mathfrak{I}(\omega, S) \to \mathcal{A}(\omega) \to \mathfrak{C}(\omega, S) \to 0$$

so that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathfrak{I}^{j-1}(\omega, S) \arrow[r, "D^{j-1}"] \arrow[d] & \mathfrak{I}^{j}(\omega, S) \arrow[r, "D^{j}"] \arrow[d] & \mathfrak{I}^{j+1}(\omega, S) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{A}^{j-1}(\omega) \arrow[r, "D^{j-1}"] \arrow[d] & \mathcal{A}^{j}(\omega) \arrow[r, "D^{j}"] \arrow[d] & \mathcal{A}^{j+1}(\omega) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathfrak{C}^{j-1}(\omega, S) \arrow[r, "D_S^{j-1}"] \arrow[d] & \mathfrak{C}^{j}(\omega, S) \arrow[r, "D_S^{j}"] \arrow[d] & \mathfrak{C}^{j+1}(\omega, S) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

commutes and has exact columns. The cohomology of the tangential complex $\mathfrak{C}(\Omega, S)$ is denoted by $H^*\big(\mathfrak{C}(S)\big)$. The space $\mathfrak{C}(\omega, S)$ is to be interpreted as

Cauchy data for D on $\omega \cap S$. Since $\mathcal{I}(\omega, S) = \mathcal{A}(\omega)$ if $\omega \cap S = \emptyset$, it follows that $\mathcal{C}(\omega, S)$ is concentrated on $\omega \cap S$, and that we may write $\mathcal{C}(\omega \cap S)$ for $\mathcal{C}(\omega, S)$ and $\mathcal{C}(S)$ for $\mathcal{C}(\Omega, S)$.

(*d*) In important cases, $\mathcal{C}(\omega \cap S)$ is the space of real analytic sections over $\omega \cap S$ of analytic vector bundles on S.

Recall that the principal symbol of D at (x, df), where $x \in \Omega$ and $df \in \boldsymbol{C}T_x^*$, is the linear map $\sigma_{df}(D)\colon E_x \to E_x$ defined by $\sigma_{df}(D)(u(x)) = D(fu)(x) - f(x)Du(x)$. If $df = dg$, then $\sigma_{df}(D) = \sigma_{dg}(D)$, so that the symbol is well defined. If ξ is a real analytic one-form, the symbol maps on the fibers piece together to give a real analytic vector bundle morphism of degree one $\sigma_\xi(D)\colon E \to E$. Since $D \circ D = 0$, it follows that for each ξ, $\sigma_\xi(D) \circ \sigma_\xi(D) = 0$.

DEFINITION 1.2. A cotangent vector ξ in $\boldsymbol{C}T_x^*$ is *noncharacteristic* for the complex

$$0 \to \mathcal{A}_0 \xrightarrow{D^0} \mathcal{A}^1 \xrightarrow{D^1} \cdots$$

iff the principal symbol complex

$$0 \to E_x^0 \xrightarrow{\sigma_\xi(D^0)} E_x^1 \xrightarrow{\sigma_\xi(D^1)} \cdots$$

is exact. A submanifold S of codimension one in Ω is noncharacteristic at $x \in S$ iff the cotangent vector $(x, d\varrho(x))$ is noncharacteristic, where ϱ locally defines S. S is noncharacteristic if it is noncharacteristic at each point.

REMARK. The functions $x \to \dim(\ker \sigma_{d\varrho}(D^j))_x$ and $x \to \dim(\operatorname{im} \sigma_{d\varrho}(D^{j-1}))_x$ are lower and upper semicontinuous, and, if S is noncharacteristic, they are equal for $x \in S$. In that case, there exists an open set ω with $S \subset \omega \subset \Omega$ such that $\dim(\ker \sigma_{d\varrho}(D^j))$ is locally constant on ω, and so that $\ker \sigma_{d\varrho}(D^j)$ is a real analytic vector bundle on ω.

Henceforth we assume S is noncharacteristic.

Since the operator D is first order, integration by parts shows that a section $u \in \mathcal{A}(\omega)$ has zero Cauchy data for D iff $\sigma_{d\varrho}(D)u$ vanishes identically on $\omega \cap S$, so $\mathcal{I}(\omega, S)$ is the space of sections in $\mathcal{A}(\omega)$, the restrictions of which to S lie in the vector sub-bundle $\ker \sigma_{d\varrho}(D)|_{S \cap \omega}$. Thus $\mathcal{C}(\omega, S)$ may be identified with sections of the quotient bundle $(E/\ker \sigma_{d\varrho}(D))|_{S \cap \omega}$. It follows that the maps $\omega \to \mathcal{C}(\omega, S)$ form a presheaf which is a sheaf supported on S which we denote by $\mathcal{C}$.

2. – The Mayer-Vietoris sequence in the analytic category.

The following theorem relates the cohomologies $H^*(\mathcal{A}(\Omega))$,$H^*(\mathcal{A}(\Omega^\pm))$, and $H^*(\mathcal{C}(S))$.

THEOREM 2.1. *If S is noncharacteristic, the Mayer-Vietoris sequence*

$$0 \to H^0(\mathcal{A}(\Omega)) \to H^0(\mathcal{A}(\Omega^+)) \oplus H^0(\mathcal{A}(\Omega^-)) \to H^0(C(S)) \to \ldots$$
$$\ldots \to H^j(\mathcal{A}(\Omega)) \to H^j(\mathcal{A}(\Omega^+)) \oplus H^j(\mathcal{A}(\Omega^-)) \to H^j(\mathcal{C}(S)) \to \ldots$$

is exact.

PROOF. The proof is a sequence of lemmas.

LEMMA 2.2. *The sequence*

$$0 \to \mathcal{A}(\Omega) \to \mathcal{A}(\Omega^+) \oplus \mathcal{A}(\Omega^-) \to \mathcal{A}(S) \to 0$$

is exact.

PROOF. The first map is restriction to Ω^+ and Ω^-; the second is the difference of the restrictions (of germs) to S. It is clear that the sequence is exact except possibly at $\mathcal{A}(S)$. Exactness there follows easily if we can show that for any open sets U^+ and U^- in the complexification $\tilde{X}$ of X with $\Omega^+ \subset U^+$ and $\Omega^- \subset U^-$, there exist sets, open in $\tilde{X}$, $\tilde{U}^+$ and $\tilde{U}^-$ with $\Omega^+ \subset \tilde{U}^+ \subset U^+$ and $\Omega^- \subset \tilde{U}^- \subset U^-$ such that

$$\tilde{\mathcal{A}}(\tilde{U}^+) \oplus \tilde{\mathcal{A}}(\tilde{U}^-) \to \tilde{\mathcal{A}}(\tilde{U}^+ \cap \tilde{U}^-) \to 0 \tag{2.1}$$

is exact, where $\tilde{\mathcal{A}}$ is the complexification of $\mathcal{A}$. By a theorem of Grauert [5], $\tilde{U}^+$ and $\tilde{U}^-$ may be chosen to be Stein and then modified so that $\tilde{U}^+ \cup \tilde{U}^-$ is Stein as well. By Cartan's Theorem B, the sheaf cohomology $H^1(\tilde{U}^+ \cup \tilde{U}^-, \tilde{\mathcal{A}}) = 0$ so by Leray's theorem on Stein covers, [9], $H^1(\mathfrak{U}, \tilde{\mathcal{A}}) = 0$ where $\mathfrak{U} = \{\tilde{U}^+, \tilde{U}^-\}$ is a Stein cover. In particular this implies that (2.1) is exact, which completes the proof of the lemma. Note that the lemma is really a consequence of the fact that the sheaf cohomology $H^1(\Omega, \mathcal{A}) = 0$.

The restriction and difference maps commute with all differential operators, so

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & & \mathcal{A}^{j-1}(\Omega) & \xrightarrow{D^{j-1}} & \mathcal{A}^{j}(\Omega) & \xrightarrow{D^{j}} & \mathcal{A}^{j+1}(\Omega) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & & \mathcal{A}^{j-1}(\Omega^{+}) & \to & \mathcal{A}^{j}(\Omega^{+}) & \to & \mathcal{A}^{j+1}(\Omega^{+}) & \to \cdots \\
 & & \oplus & & \oplus & & \oplus & \\
\cdots \to & & \mathcal{A}^{j-1}(\Omega^{-}) & \to & \mathcal{A}^{j}(\Omega^{-}) & \to & \mathcal{A}^{j+1}(\Omega^{-}) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & & \mathcal{A}^{j-1}(S) & \to & \mathcal{A}^{j}(S) & \to & \mathcal{A}^{j+1}(S) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0 &
\end{array}
$$

commutes and has exact columns.

The corresponding long exact cohomology sequence is

$$
\begin{aligned}
&0 \to H^0(\mathcal{A}(\Omega)) \to H^0(\mathcal{A}(\Omega^+)) \oplus H^0(\mathcal{A}(\Omega^-)) \to H^0(\mathcal{A}(S)) \to \cdots \\
&\cdots \to H^j(\mathcal{A}(\Omega)) \to H^j(\mathcal{A}(\Omega^+)) \oplus H^j(\mathcal{A}(\Omega^-)) \to H^j(\mathcal{A}(S)) \to \cdots
\end{aligned}
$$

The theorem is consequently reduced to

LEMMA 2.3. *If S is noncharacteristic, then for each $j \geq 0$,*

$$H^j(\mathcal{A}(S)) \cong H^j(\mathcal{C}(S)) .$$

REMARK. This lemma is a statement of the Cauchy-Kowalewski theorem for complexes of differential operators. See [10], and [11] for a description of how solvability of the Cauchy problem reduces to a statement of this form. Note that the operator $D_S\colon \mathcal{C}(S) \to \mathcal{C}(S)$ is a tangential operator; that is, it does not differentiate in the direction normal to S. On the other hand, objects of $\mathcal{A}(S)$ are germs on S of sections in $\mathcal{A}(\Omega)$ and have normal derivatives of all orders, and $D\colon \mathcal{A}(S) \to \mathcal{A}(S)$ differentiates in the normal directions. The passage from one case to the other is by solving for normal derivatives in terms of tangential ones on a noncharacteristic surface.

PROOF OF LEMMA 2.3. Let $\mathfrak{J}(S) = \varinjlim_{\Omega \supset \omega \supset S} \mathfrak{J}(\omega, S)$. The sequence $0 \to \mathfrak{J}(S) \to \mathcal{A}(S) \to \mathcal{C}(S) \to 0$ is exact. The corresponding long exact cohomology sequence is

$$0 \to H^0(\mathfrak{J}(S)) \to H^0(\mathcal{A}(S)) \to H^0(\mathcal{C}(S)) \to H^1(\mathfrak{J}(S)) \to \dots$$

If $H^j(\mathfrak{J}(S)) = 0$ for each j, the lemma follows. The theorem is now reduced to proving the following in the case $K = S$.

LEMMA 2.4. *If S is noncharacteristic, then for each $K \subset S$ and for each $j \geqslant 0$, $H^j(\mathfrak{J}(K)) = 0$.*

PROOF. The proof is basically due to Guillemin and is very similar to several proofs which appear in the literature, ([6], [11], [3]), and so we shall only outline it here.

Guillemin gives a decomposition of (1.1) over any open set $\omega \subset S$ such that the principal symbol complex

$$0 \to E^0 \xrightarrow{\sigma_{d\varrho}(D^0)} E^1 \xrightarrow{\sigma_{d\varrho}(D^1)} \dots$$

is exact over ω. There are bundles E_0^j such that $E_0^0 = E^0$; $E^j \cong E_0^j \oplus E_0^{j-1}$; and such that (1.1) decomposes into

$$\begin{array}{ccccccc}
\mathcal{A}_0^0 & \xrightarrow{-D_0^0} & \mathcal{A}_0^1 & \xrightarrow{-D_0^1} & \mathcal{A}_0^2 & \to & \dots \\
\uparrow{\scriptstyle D_1^0} & & \uparrow{\scriptstyle D_1^1} & & \uparrow{\scriptstyle D_1^2} & & \\
\mathcal{A}_0^0 & \xrightarrow{D_0^0} & \mathcal{A}_0^1 & \xrightarrow{D_0^1} & \mathcal{A}_0^2 & \to & \dots
\end{array}$$

Here $\mathcal{A}_0^j$ is the sheaf of germs of real analytic sections of E_0^j.

Thus if $u \in \mathcal{A}^j$ is written $u = (u_0, u_1)$ with $u_0 \in \mathcal{A}_0^j$ and $u_1 \in \mathcal{A}_0^{j-1}$, $D^j u = (D_0^j u_0, D_1^j u_0 - D_0^{j-1} u_1) \in \mathcal{A}_0^{j+1} \oplus \mathcal{A}_0^j$. Furthermore, each D_0^j is tangential to S (does not differentiate in the normal direction) and $\sigma_{d\varrho}(D_1^j)$ is the identity on E_0^j.

Since $D^{j+1} \circ D^j = 0$, we have

$$D_0^{j+1} \circ D_0^j = 0 \tag{2.2}$$

and

$$D_1^{j+1} \circ D_0^j = D_0^j \circ D_1^j . \tag{2.3}$$

A section $u=(u_0, u_1)$ represents a class in $\mathfrak{J}^j(K)$ if $u \in \mathcal{A}^j(\omega)$ for some open set ω containing K and if $\sigma_{d\varrho}(D^j)u|_s = 0$. But since $\sigma_{d\varrho}(D_0^j)=0$ and $\sigma_{d\varrho}(D_1^j)=I$, we have $u\in\mathfrak{J}^j(K)$ iff $u_0|_s\equiv 0$. If u represents a cohomology class in $H^j(\mathfrak{J}(K))$, then $D^j u = 0$ or equivalently,

$$D_0^j u_0 = 0 \tag{2.4}$$

and

$$D_1^j u_0 = D_0^{j-1} u_1 \,. \tag{2.5}$$

Since $u_0|_s = 0$, u_0 has zero Cauchy data for the determined operator D_1^j and so by the classical Cauchy-Kowalewski theorem there is an open set ω' with $K\subset\omega'\subset\omega$ on which there is a solution v_0 of the equation $D_1^{j-1}v_0 = u_1$ such that v_0 vanishes on S.

We claim, furthermore, that $D_0^{j-1}v_0 = u_0$. For $0 = D_0^{j-1}(D_1^{j-1}v_0 - u_1) = D_1^j D_0^{j-1} v_0 - D_1^j u_0$ by (2.3) and (2.5). Thus $w = D_0^{j-1}v_0 - u_0$ satisfies the equation $D_1^j w = 0$ and w vanishes on S (recall D_0^{j-1} is tangential). By the uniqueness portion of the Cauchy-Kowalewski theorem, $w=0$ on ω' so $D_1^{j-1}v_0 = u_1$ and $D_0^{j-1}v_0 = u_0$ on ω'. This means that if $v=(v_0, 0)$, then $D^{j-1}v = u$ and $v\in\mathfrak{J}^{j-1}(\omega')$. Thus $H^j(\mathfrak{J}(K)) = 0$.

This completes the proof of Lemma 2.4 and of Theorem 2.1.

3. – The Mayer-Vietoris sequence in the hyperfunction category.

(*a*) There are two alternative definitions of hyperfunctions, namely that of Sato and that of Martineau. We shall use the definition of Martineau, as expounded by Schapira [17]. Although he considers only hyperfunctions on $\boldsymbol{R}^n$, the methods clearly generalize to define hyperfunction sections of an analytic vector bundle over a real analytic manifold countable at infinity.

We assume the existence of Hermitian inner products on the bundle E. The seminorms $\varrho_K(u) = \sup\limits_{x\in K}|u(x)|$ for $K\subset\subset\omega$ give $\mathcal{A}(\omega)$ a locally convex linear space topology, which is a Frechet-Schwartz (FS) topology. If K is a compact subset of X, we assign $\mathcal{A}(K)$ the topology of the inductive limit $\varinjlim\limits_{\omega\supset K}\mathcal{A}(\omega)$. This has the topology of a strong dual of a Frechet-Schwartz (DFS) space. The space of analytic functionals on $\mathcal{A}$ with support in K is by definition the strong dual $(\mathcal{A}(K))'$, which is an FS space. The space of hyperfunction sections of E with support in K coincides with $({}^t\mathcal{A}(K))'$.

(*b*) The formal transpose complex of (1.1) is

$$0 \leftarrow {}^t\mathcal{A}^0 \xleftarrow{{}^tD^0} {}^t\mathcal{A}^1 \xleftarrow{{}^tD^1} \cdots$$

The principal symbol complex of the formal transpose complex is the transpose of the original principal symbol complex; in particular, if S is noncharacteristic for the original complex, it is for the transpose complex as well.

Let ${}^t\mathfrak{J}^j(\omega, S)$ be the space of sections in ${}^t\mathcal{A}^j(\omega)$ which have zero Cauchy data for the operator ${}^tD^{j-1}$. As before,

$$0 \leftarrow {}^t\mathfrak{J}^0(\omega, S) \xleftarrow{{}^tD^0} {}^t\mathfrak{J}^1(\omega, S) \xleftarrow{{}^tD^1} \cdots$$

is a complex, and we may define ${}^t\mathcal{C}^j(\omega, S)$ by the exact sequence

$$0 \to {}^t\mathfrak{J}^{j+1}(\omega, S) \to {}^t\mathcal{A}^{j+1}(\omega) \to {}^t\mathcal{C}^j(\omega, S) \to 0\,.$$

The superscript in ${}^t\mathcal{C}^j(\omega, S)$ is in fact j and not $j+1$ as might be expected, for with this choice of superscript,

$${}^t\mathcal{C}^j(\omega, S) \xleftarrow{{}^tD^j_s} {}^t\mathcal{C}^{j+1}(\omega, S)$$

is the formal transpose of

$$\mathcal{C}^j(\omega, S) \xrightarrow{D^j_s} \mathcal{C}^{j+1}(\omega, S)$$

when S is noncharacteristic [3, § 9].

(*c*) Recall that for a bounded open set ω, $\mathfrak{B}(\omega)$, the space of hyperfunctions of E over ω, is defined to be $({}^t\mathcal{A}(\bar{\omega}))'/({}^t\mathcal{A}(\partial\omega))'$ and that $\mathfrak{B}$ is the sheaf associated to the presheaf

$$\begin{cases} \omega \to \mathfrak{B}(\omega) & \text{if } \omega \text{ is bounded} \\ \omega \to 0 & \text{if } \omega \text{ is unbounded}\,. \end{cases}$$

The sheaf $\mathfrak{B}$ is flabby.

Similarly, for a bounded open set ω define $\mathfrak{C}(\omega)$ to be $({}^t\mathcal{C}(\bar{\omega}))'/({}^t\mathcal{C}(\partial\omega))' = ({}^t\mathcal{C}(S \cap \bar{\omega}))'/({}^t\mathcal{C}(S \cap \partial\omega))'$ and let $\mathfrak{C}$ be the associated sheaf. Then $\mathfrak{C}^j$ is the sheaf of hyperfunction sections over S of the quotient bundle $(E^j/\ker \sigma_{d\varrho}(D^j))|_S$; and $\mathfrak{C}$ is flabby.

Let $\mathfrak{J}$ be the quotient sheaf $\mathfrak{B}/\mathfrak{C}$ so that for each j,

$$0 \to \mathfrak{C}^{j-1} \to \mathfrak{B}^j \to \mathfrak{J}^j \to 0$$

is an exact sheaf sequence. Since $\mathfrak{C}$ and $\mathfrak{B}$ are flabby, $\mathfrak{J}$ is also flabby by [4, Thm. 3.1.2, Cor.]. For any family of supports Φ, let $\Gamma_\Phi(\mathfrak{F}, \Omega)$ denote the sections of $\mathfrak{F}$ over Ω with support in Φ. Since $\mathfrak{C}$, $\mathfrak{B}$, and $\mathfrak{J}$ are flabby,

$$0 \to \Gamma_\Phi(\mathfrak{C}, \Omega) \to \Gamma_\Phi(\mathfrak{B}, \Omega) \to \Gamma_\Phi(\mathfrak{J}, \Omega) \to 0 \tag{3.1}$$

is exact [4, Thm. 3.1.3].

Let $\mathfrak{B}_s$ (resp. $\mathfrak{F}_s$) be the sheaf of germs of sections of $\mathfrak{B}$ (resp. $\mathfrak{J}$) with support contained in S. That is, $\mathfrak{B}_s$ is the sheaf associated with the presheaf $\omega \to \mathfrak{B}_s(\omega)$ where $\mathfrak{B}_s(\omega)$ is the space of hyperfunctions in $\mathfrak{B}(\omega)$ with support in S.

We claim that $\mathfrak{B}_s$ is flabby, since any section in $\mathfrak{B}_s(\omega)$ extends by zero to $\Omega\setminus S$, and then, since $\mathfrak{B}$ is flabby, it extends to a section of $\mathfrak{B}_s(\Omega)$. The same proof also shows $\mathfrak{F}_s$ is flabby.

We have that if ω is a bounded set, then $\mathfrak{J}_s(\omega) \cong ({}^t\mathfrak{I}(S \cap \bar{\omega}))'/({}^t\mathfrak{I}(S \cap \partial\omega))'$. The proof consists of considering several sequences and diagrams. For each compact set K, the sequence

$$0 \to {}^t\mathfrak{I}^j(K) \to {}^t\mathcal{A}^j(K) \to {}^t\mathcal{C}^{j-1}(K) \to 0$$

is an exact sequence of DFS spaces so that the strong dual sequence

$$0 \to ({}^t\mathcal{C}^{j-1}(K))' \to ({}^t\mathcal{A}^j(K))' \to ({}^t\mathfrak{I}^j(K))' \to 0$$

is exact; which is to say

$$0 \to \mathfrak{C}_K^{j-1}(\Omega) \to \mathfrak{B}_K^j(\Omega) \to ({}^t\mathfrak{I}^j(K))' \to 0$$

is exact, where for each sheaf $\mathfrak{F}$, $\mathfrak{F}_K(\Omega)$ denotes sections of $\mathfrak{F}$ over Ω with support in K.

If Φ is the family of closed subsets of K, (3.1) says that the sequence

$$0 \to \mathfrak{C}_K^{j-1}(\Omega) \to \mathfrak{B}_K^j(\Omega) \to \mathfrak{F}_K^j(\Omega) \to 0$$

is exact, and so $({}^t\mathfrak{I}^j(K))' \cong \mathfrak{J}_K^j(\Omega)$. Since for a bounded set ω $\mathfrak{C}_s(\omega) = \mathfrak{C}_{s\cap\bar{\omega}}(\Omega)/\mathfrak{C}_{s\cap\partial\omega}(\Omega)$ and $\mathfrak{B}_s(\omega) = \mathfrak{B}_{s\cap\bar{\omega}}(\Omega)/\mathfrak{B}_{s\cap\partial\omega}(\Omega)$ it follows that $\mathfrak{F}_s(\omega) = \mathfrak{F}_{s\cap\bar{\omega}}(\Omega)/\mathfrak{J}_{s\cap\partial\omega}(\Omega) \cong ({}^t\mathfrak{I}(S \cap \bar{\omega}))'/({}^t\mathfrak{I}(S \cap \partial\omega))'.$, (*d*) We are now prepared to prove the hyperfunction analogue of Theorem 2.1. Denote the cohomologies of

$$\mathfrak{B}(\Omega):\ 0 \to \mathfrak{B}^0(\Omega) \xrightarrow{D^0} \mathfrak{B}^1(\Omega) \xrightarrow{D^1} \dots$$

$$\mathfrak{B}(\mathring{\Omega}^\pm):\ 0 \to \mathfrak{B}^0(\mathring{\Omega}^\pm) \xrightarrow{D^0} \mathfrak{B}^1(\mathring{\Omega}^\pm) \xrightarrow{D^1} \dots$$

$$\mathfrak{B}_s(\Omega):\ 0 \to \mathfrak{B}^0_s(\Omega) \xrightarrow{D^0} \mathfrak{B}^1_s(\Omega) \xrightarrow{D^1} \dots$$

$$\mathfrak{C}(S):\ 0 \to \mathfrak{C}^0(S) \xrightarrow{D^1_s} \mathfrak{C}^1(S) \xrightarrow{D^1_s} \dots$$

and

$$\mathfrak{I}_s(\Omega):\ 0 \to \mathfrak{I}^0_s(\Omega) \xrightarrow{D^0} \mathfrak{I}^1_s(\Omega) \xrightarrow{D^1} \dots$$

by $H^*(\mathfrak{B}(\Omega))$, $H^*(\mathfrak{B}(\mathring{\Omega}^\pm))$, and so on.

We have

THEOREM 3.1. *If S is noncharacteristic, the Mayer-Vietoris sequence*

$$0 \to H^0(\mathfrak{B}(\Omega)) \to H^0(\mathfrak{B}(\mathring{\Omega}^+) \oplus H^0(\mathfrak{B}(\mathring{\Omega}^-)) \to H^0(\mathfrak{C}(S)) \to \dots$$

$$\dots \to H^j(\mathfrak{B}(\Omega)) \to H^j(\mathfrak{B}(\mathring{\Omega}^+) \oplus H^j(\mathfrak{B}(\mathring{\Omega}^-)) \to H^j(\mathfrak{C}(S)) \to \dots$$

is exact.

PROOF. As before, we have a sequence of lemmas.

LEMMA 3.2. *The sequence*

$$0 \to \mathfrak{B}_s(\Omega) \to \mathfrak{B}(\Omega) \to \mathfrak{B}(\mathring{\Omega}^+) \oplus \mathfrak{B}(\mathring{\Omega}^-) \to 0$$

is exact.

PROOF. The maps are inclusion and restriction. The sequence is clearly exact except possibly at the last position, where it is exact by flabbiness. That is, any section on $\mathring{\Omega}^+ \cup \mathring{\Omega}^-$ extends to a section over Ω.

The long exact cohomology sequence corresponding to this is

$$0 \to H^0(\mathfrak{B}_s(\Omega)) \to H^0(\mathfrak{B}(\Omega)) \to H^0(\mathfrak{B}(\mathring{\Omega}^+)) \oplus H^0(\mathfrak{B}(\mathring{\Omega}^-))$$
$$\to H^1(\mathfrak{B}_s(\Omega)) \to \dots$$

The theorem is reduced to showing

$$H^{j+1}(\mathfrak{B}_s(\Omega)) \cong H^j(\mathfrak{C}(S)) \qquad \text{for each } j\,.$$

As a special case of (3.1) where Φ is the family of closed subsets of S, we have

$$0 \to \mathfrak{C}^j(S) \to \mathfrak{B}_s^{j+1}(\Omega) \to \mathfrak{F}_s^{j+1}(\Omega) \to 0$$

is exact for $j \geqslant 0$, and

$$0 \to \mathfrak{B}_s^0(\Omega) \to \mathfrak{F}_s^0(\Omega) \to 0$$

is exact. The corresponding long exact sequence is

$$0 \to H^0(\mathfrak{B}_s(\Omega)) \to H^0(\mathfrak{F}_s(\Omega)) \to H^0(\mathfrak{C}(S)) \to H^1(\mathfrak{B}_s(\Omega)) \to \ldots$$

Again, the problem reduces to showing

$$H^j(\mathfrak{F}_s(\Omega)) = 0 \qquad \text{for each } j \geqslant 0\,;$$

that is, we must show

$$0 \to \mathfrak{F}_s^0(\Omega) \xrightarrow{D^0} \mathfrak{F}_s^1(\Omega) \xrightarrow{D^1} \ldots$$

is exact. Since the sheaf $\mathfrak{F}_s$ is flabby, this follows if the sheaf sequence

$$0 \to \mathfrak{F}_s^0 \xrightarrow{D^0} \mathfrak{F}_s^1 \xrightarrow{D^1} \ldots$$

is exact [4, Thm. 3.1.3]. In fact, we can show that the sequence

$$0 \to \mathfrak{F}_s^0(\omega) \xrightarrow{D^0} \mathfrak{F}_s^1(\omega) \xrightarrow{D^1} \ldots \tag{3.2}$$

is exact for any bounded ω.

If K is any compact subset of S,

$$0 \to \mathfrak{F}_K^0(\Omega) \xrightarrow{D^0} \mathfrak{F}_K^1(\Omega) \xrightarrow{D^1} \ldots \tag{3.3}$$

is the strong dual of

$$0 \leftarrow {}^t\mathfrak{J}^0(K) \xleftarrow{{}^tD^0} {}^t\mathfrak{J}^1(K) \xleftarrow{{}^tD^1} \ldots \tag{3.4}$$

Since S is noncharacteristic for the original complex, it is for the transpose complex, and hence (3.4) is an exact sequence of DFS spaces by Lemma 2.4.

Therefore the strong dual complex (3.3) is exact. A simple diagram chase shows that since $\mathfrak{F}_s(\omega) \cong \mathfrak{F}_{s\cap\bar{\omega}}(\Omega)/\mathfrak{F}_{s\cap\partial\omega}(\Omega)$ it follows that (3.2) is exact, which proves the theorem.

(*e*) The preceding results can be extended to include cohomology with supports. Let Φ be any family of supports on Ω. Then there is a family of supports on S, induced by intersection with S, which we denote by $\Phi(S)$, and corresponding induced families on $\mathring{\Omega}^+$ and $\mathring{\Omega}^-$, which we continue to denote by Φ. For any sheaf $\mathfrak{F}$ we denote $\Gamma_\Phi(\Omega, \mathfrak{F})$ by $\mathfrak{F}_\Phi(\Omega)$ and $\Gamma_{\Phi(S)}(\Omega, \mathfrak{F})$ by $\mathfrak{F}_{\Phi(S)}(\Omega)$.

We consider the complexes $\mathfrak{B}_\Phi(\Omega)$, $\mathfrak{B}_\Phi(\mathring{\Omega}^\pm)$, $\mathfrak{B}_{\Phi(S)}(\Omega)$, $\mathfrak{C}_{\Phi(S)}(\Omega) = \mathfrak{C}_{\Phi(S)}(S)$, $\mathfrak{F}_{\Phi(S)}(\Omega)$, and their cohomologies. Corresponding to the usual Mayer-Vietoris sequence we have the Mayer-Vietoris sequence with supports in Φ.

THEOREM 3.3. *If S is noncharacteristic, the Mayer-Vietoris sequence*

$$0 \to H^0(\mathfrak{B}_\Phi(\Omega)) \to H^0(\mathfrak{B}_\Phi(\mathring{\Omega}^+)) \oplus H^0(\mathfrak{B}_\Phi(\mathring{\Omega}^-)) \to H^0(\mathfrak{C}_{\Phi(S)}(S)) \to \dots$$
$$\dots \to H^j(\mathfrak{B}_\Phi(\Omega)) \to H^j(\mathfrak{B}_\Phi(\mathring{\Omega}^+)) \oplus H^j(\mathfrak{B}_\Phi(\mathring{\Omega}^-)) \to H^j(\mathfrak{C}_{\Phi(S)}(S)) \to \dots$$

is exact.

The proof is practically the same as the proof of Theorem 3.1. The sequence in Lemma 3.2 is replaced by the sequence

$$0 \to \mathfrak{B}_{\Phi(S)}(\Omega) \to \mathfrak{B}_\Phi(\Omega) \to \mathfrak{B}_\Phi(\mathring{\Omega}^+) \oplus \mathfrak{B}_\Phi(\mathring{\Omega}^-) \to 0$$

which can be shown to be exact by the same method used to show $\mathfrak{B}_S$ is flabby. As a special case of (3.1)

$$0 \to \mathfrak{C}^j_{\Phi(S)}(S) \to \mathfrak{B}^{j+1}_{\Phi(S)}(\Omega) \to \mathfrak{F}^{j+1}_{\Phi(S)}(\Omega) \to 0$$

is exact, and since

$$0 \to \mathfrak{F}^0_S \xrightarrow{D^0} \mathfrak{F}^1_S \xrightarrow{D^1} \dots$$

is an exact sequence of flabby sheaves,

$$0 \to \mathfrak{F}^0_{\Phi(S)}(\Omega) \xrightarrow{D^0} \mathfrak{F}^1_{\Phi(S)}(\Omega) \xrightarrow{D^1} \dots$$

is exact [4]. Consideration of the exact cohomology sequences of these sequences suffices to prove the theorem.

4. – The ladder diagram.

(*a*) Now we return to the real analytic category. In what follows we assume that every connected component of Ω meets S.

THEOREM 4.1. *If S is noncharacteristic, then there is a commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{A}(\Omega)) \arrow[r] \arrow[d] & H^0(\mathcal{A}(\Omega^\pm)) \arrow[r] \arrow[d] & H^1(\mathfrak{J}(\Omega^\mp)) \arrow[r] \arrow[d] & \ldots \\
0 \arrow[r] & H^0(\mathcal{A}(\Omega^\mp)) \arrow[r] & H^0(\mathfrak{C}(S)) \arrow[r] & H^1(\mathfrak{J}(\Omega^\mp)) \arrow[r] & \ldots
\end{tikzcd}$$

$$\begin{tikzcd}
\ldots \arrow[r] & H^j(\mathfrak{J}(\Omega^\mp)) \arrow[r] \arrow[d] & H^j(\mathcal{A}(\Omega)) \arrow[r] \arrow[d] & H^j(\mathcal{A}(\Omega^\pm)) \arrow[r] \arrow[d] & H^{j+1}(\mathfrak{J}(\Omega^\mp)) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & H^j(\mathfrak{J}(\Omega^\mp)) \arrow[r] & H^j(\mathcal{A}(\Omega^\mp)) \arrow[r] & H^j(\mathfrak{C}(S)) \arrow[r] & H^{j+1}(\mathfrak{J}(\Omega^\mp)) \arrow[r] & \ldots
\end{tikzcd}$$

PROOF. The proof requires two lemmas.

LEMMA 4.2. *The restriction map induces an isomorphism*

$$\mathcal{A}(\Omega^\pm)/\mathcal{A}(\Omega) \cong \mathcal{A}(S)/\mathcal{A}(\Omega^\mp)\,.$$

LEMMA 4.3. *There is a commutative diagram with exact rows and columns:*

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathfrak{C}(S) \arrow[r] \arrow[u] & \mathfrak{C}(S) \arrow[r] \arrow[u] & 0 & \\
0 \arrow[r] & \mathcal{A}(\Omega^\mp) \arrow[r] \arrow[u, "\alpha"'] & \mathcal{A}(S) \arrow[r] \arrow[u] & \mathcal{A}(S)/\mathcal{A}(\Omega^\mp) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathfrak{J}(\Omega^\mp) \arrow[r] \arrow[u] & \mathfrak{J}(S) \arrow[r] \arrow[u] & \mathfrak{J}(S)/\mathfrak{J}(\Omega^\mp) \arrow[r] \arrow[u, "\beta"'] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{4.1}$$

According to Lemma 4.2 there is a commuting diagram

$$\begin{array}{ccccccccc}
0 \to & \mathcal{A}(\Omega) & \to & \mathcal{A}(\Omega^{\pm}) & \to & \mathcal{A}(\Omega^{\pm})/\mathcal{A}(\Omega) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle\alpha} & \\
0 \to & \mathcal{A}(\Omega^{\mp}) & \to & \mathcal{A}(S) & \to & \mathcal{A}(S)/\mathcal{A}(\Omega^{\mp}) & \to 0
\end{array}$$

with exact rows in which α is an isomorphism. Hence we have the corresponding long exact cohomology sequence

$$\begin{array}{ccccccccc}
\ldots \to & H^j(\mathcal{A}(\Omega)) & \to & H^j(\mathcal{A}(\Omega^{\pm})) & \to & H^j(\mathcal{A}(\Omega^{\pm})/\mathcal{A}(\Omega)) & \to & H^{j+1}(\mathcal{A}(\Omega)) & \to \ldots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\ldots \to & H^j(\mathcal{A}(\Omega^{\mp})) & \to & H^j(\mathcal{A}(S)) & \to & H^j(\mathcal{A}(S)/\mathcal{A}(\Omega^{\mp})) & \to & H^{j+1}(\mathcal{A}(\Omega^{\mp})) & \to \ldots
\end{array}$$

Since S is noncharacteristic we can use Lemma 2.3 to make the substitution $H^j(\mathcal{A}(S)) \cong H^j(\mathfrak{C}(S))$. Since $H^j(\mathfrak{I}(S)) = 0$ for $j \geqslant 0$, according to Lemma 2.4, the long exact cohomology sequence which corresponds to the bottom row in (4.1) yields:

$$\begin{aligned}
H^j(\mathcal{A}(\Omega^{\pm})/\mathcal{A}(\Omega)) &\cong H^j(\mathcal{A}(S)/\mathcal{A}(\Omega^{\mp})) \\
&\cong H^j(\mathfrak{I}(S)/\mathfrak{I}(\Omega^{\mp})) \\
&\cong H^{j+1}(\mathfrak{I}(\Omega^{\mp}))\,.
\end{aligned}$$

This completes the proof, except for the proof of the two lemmas.

PROOF OF LEMMA 4.2. We will show that $\mathcal{A}(\Omega^{+})/\mathcal{A}(\Omega) \cong \mathcal{A}(S)/\mathcal{A}(\Omega^{-})$. It suffices to observe that in the commuting diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{A}(\Omega^{-}) & \to & \mathcal{A}(\Omega^{-}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \to \mathcal{A}(\Omega) & \to & \mathcal{A}(\Omega^{+}) \oplus \mathcal{A}(\Omega^{-}) & \to & \mathcal{A}(S) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \to \mathcal{A}(\Omega) & \xrightarrow{\alpha} & \mathcal{A}(\Omega^{+}) & \to & \mathcal{A}(S)/\mathcal{A}(\Omega^{-}) & \to & 0\,, \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

the columns and top two rows are exact. Since α is injective, a diagram chase shows that the bottom row is also exact.

PROOF OF LEMMA 4.3. The essential point is to show that α is surjective. Namely, since the rows in (4.1) are obviously exact, the second column is exact, the first column is exact at $\mathfrak{J}(\Omega^{\mp})$ and $\mathcal{A}(\Omega^{\mp})$, and β is injective, a diagram chase shows that β is surjective if α is.

To complete the proof let $\mathfrak{a}(S)$ denote real analytic sections of $E|_s$ over S. Then the surjectivity of α follows from the commutative exact diagram

$$\begin{tikzcd}
 & 0 & \\
\mathcal{A}(\Omega^{\mp}) \arrow[r, "\alpha"] & \mathcal{C}(S) \arrow[u] & \\
\mathcal{A}(\Omega) \arrow[u] \arrow[r, "\alpha'"] & \mathfrak{a}(S) \arrow[u, "p_r"'] \arrow[r] & 0 \\
0 \arrow[u] & &
\end{tikzcd}$$

in which p_r is projection of sections of $E|_s$ onto sections of the quotient bundle $(E/\ker \sigma_{dp}(D))|_s$. The surjectivity of α' is a consequence of the real analytic version of the Oka extension principle, since the sheaf cohomology $H^1(\Omega, \mathcal{A}) = 0$; i.e. it is possible to solve the requisite Cousin problems to show that S can be globally defined and that any real analytic section of $E|_s$ over S has a real analytic extension to a section of E over all of Ω.

(*b*) The corresponding theorem in the hyperfunction category is

THEOREM 4.4. *If S is noncharacteristic, then there is a commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathfrak{B}_{\Omega^{\mp}}(\Omega)) \arrow[r] \arrow[d] & H^0(\mathfrak{B}(\Omega)) \arrow[r] \arrow[d] & H^0(\mathfrak{B}(\mathring{\Omega}^{\pm})) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & H^0(\mathfrak{B}_{\Omega^{\mp}}(\Omega)) \arrow[r] & H^0(\mathfrak{B}(\mathring{\Omega}^{\mp})) \arrow[r] & H^0(\mathfrak{C}(S)) \arrow[r] & \dots
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \arrow[r] & H^j(\mathfrak{B}_{\Omega^{\mp}}(\Omega)) \arrow[r] \arrow[d] & H^j(\mathfrak{B}(\Omega)) \arrow[r] \arrow[d] & H^j(\mathfrak{B}(\mathring{\Omega}^{\pm})) \arrow[r] \arrow[d] & H^{j+1}(\mathfrak{B}_{\Omega^{\mp}}(\Omega)) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^j(\mathfrak{B}_{\Omega^{\mp}}(\Omega)) \arrow[r] & H^j(\mathfrak{B}(\mathring{\Omega}^{\mp})) \arrow[r] & H^j(\mathfrak{C}(S)) \arrow[r] & H^{j+1}(\mathfrak{B}_{\Omega^{\mp}}(\Omega)) \arrow[r] & \dots
\end{tikzcd}$$

Here $\mathfrak{B}_{\Omega^{\mp}}(\Omega)$ is the space of hyperfunction sections over Ω with support in $\Omega^{\mp}$.

PROOF. The following short ladder diagram is commutative with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathfrak{B}_{\Omega^{\mp}}(\Omega) \arrow[r] \arrow[d] & \mathfrak{B}(\Omega) \arrow[r] \arrow[d] & \mathfrak{B}(\mathring{\Omega}^{\pm}) \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{B}_s(\Omega) \arrow[r] & \mathfrak{B}_{\Omega^{\mp}}(\Omega) \arrow[r] & \mathfrak{B}(\mathring{\Omega}^{\mp}) \arrow[r] & 0 &
\end{tikzcd}$$

This is obvious except perhaps for exactness at the last position in each row. The top row is exact at the last position since $\mathfrak{B}$ is flabby. Similarly a section in $\mathfrak{B}(\mathring{\Omega}^{\mp})$ extends by zero to a section in $\mathfrak{B}(\mathring{\Omega}^{+} \cup \mathring{\Omega}^{-})$ which, since $\mathfrak{B}$ is flabby, extends to a section in $\mathfrak{B}_{\Omega^{\mp}}(\Omega)$.

The corresponding cohomology ladder diagram is

$$\begin{tikzcd}
 & 0 \arrow[r] & H^0(\mathfrak{B}_{\Omega^{\mp}}(\Omega)) \arrow[r] \arrow[d] & H^0(\mathfrak{B}(\Omega)) \arrow[r] \arrow[d] & H^0(\mathfrak{B}(\mathring{\Omega}^{\pm})) \arrow[r] & \ldots \\
0 \arrow[r] & H^0(\mathfrak{B}_s(\Omega)) \arrow[r] & H^0(\mathfrak{B}_{\Omega^{\mp}}(\Omega)) \arrow[r] & H^0(\mathfrak{B}(\mathring{\Omega}^{\mp})) \arrow[r] & H^1(\mathfrak{B}_s(\Omega)) \arrow[r] & \ldots \\
\ldots \arrow[r] & H^{j-1}(\mathfrak{B}(\mathring{\Omega}^{\pm})) \arrow[r] & H^j(\mathfrak{B}_{\Omega^{\mp}}(\Omega)) \arrow[r] \arrow[d] & H^j(\mathfrak{B}(\Omega)) \arrow[r] \arrow[d] & H^j(\mathfrak{B}(\mathring{\Omega}^{\pm})) \arrow[r] & \ldots \\
\ldots \arrow[r] & H^j(\mathfrak{B}_s(\Omega)) \arrow[r] & H^j(\mathfrak{B}_{\Omega^{\mp}}(\Omega)) \arrow[r] & H^j(\mathfrak{B}(\mathring{\Omega}^{\mp})) \arrow[r] & H^{j+1}(\mathfrak{B}_s(\Omega)) \arrow[r] & \ldots
\end{tikzcd}$$

In § 3, the isomorphisms $H^j(\mathfrak{B}_s(\Omega)) \cong H^{j-1}(\mathcal{C}(S))$ for $j \geqslant 1$ and $H_0(\mathfrak{B}_s(\Omega)) = 0$ were proven. It remains to fill in the maps $H^j(\mathfrak{B}(\mathring{\Omega}^{\pm})) \to H^j(\mathfrak{C}(S)) = 0$ from the top row to the bottom. These maps are the same as the maps $H^j(\mathfrak{B}(\mathring{\Omega}^{\pm})) \to H^j(\mathfrak{C}(S))$ which appear in the bottom row of the diagram with $+$ and $-$ interchanged, but we omit the exercise of proving that the diagram commutes when the maps are filled in.

REMARK. Clearly there is a corresponding ladder diagram for hyperfunction cohomologies with a family of supports.

5. – Some consequences.

Consider, in the hyperfunction category, the following homomorphisms:

$$\text{I)} \quad r\colon H^0(\mathfrak{B}(\Omega)) \to H^0(\mathfrak{B}(\mathring{\Omega}^{+})),$$

$$\text{II)} \quad b\colon H^0(\mathfrak{B}(\mathring{\Omega}^{-})) \to H^0(\mathfrak{C}(S)).$$

These are induced respectively, by restriction and by taking that part of the boundary values on S, in the sense of hyperfunctions, which corresponds to the Cauchy data for D^0 there. One can ask when either of these maps is injective, or surjective, or is an isomorphism.

EXAMPLE 1. If X is a complex analytic manifold and (1.1) is the Dolbeault complex then I) and II) become

$$\text{I}')\quad r\colon \mathcal{O}(\Omega) \to \mathcal{O}(\mathring{\Omega}^+),$$
$$\text{II}')\quad b\colon \mathcal{O}(\mathring{\Omega}^-) \to CR(S, \mathfrak{B}),$$

where $\mathcal{O}(\omega)$ denotes holomorphic functions in the open set ω and $CR(S, \mathfrak{B})$ denotes hyperfunctions u on S which satisfy the tangential Cauchy-Riemann equations $\bar\partial_s u = 0$ on S. Here we have used the fact that hyperfunctions f which satisfy $\bar\partial f = 0$ on ω are just holomorphic functions in ω (see [17], [16]). Then an isomorphism in I$'$) would correspond to the classical Hartogs phenomenon of simultaneous holomorphic extension of all holomorphic functions from $\mathring{\Omega}^+$ to Ω, and an isomorphism in II$'$) would correspond to the Hans Lewy phenomenon of extension of CR functions on S (in the hyperfunction category) to holomorphic functions on $\mathring{\Omega}^-$ with the prescribed boundary values on S achieved in the sense of hyperfunctions.

In this connection a chase of the ladder diagram in Theorem 4.4 leads to the following results:

THEOREM 5.1.

A) The injectivity in I) or II) is equivalent to $H^0(\mathfrak{B}_{\Omega^-}(\Omega)) = 0$.

B) Surjectivity in II) always implies surjectivity in I.

C) Surjectivity in I) and II) are equivalent if either of the following equivalent conditions are satisfied:

a) $0 \to H^1(\mathfrak{B}(\Omega)) \to H^1(\mathfrak{B}(\mathring{\Omega}^-))$ *is exact.*

b) $H^0(\mathfrak{B}(\mathring{\Omega}^-)) \to H^1(\mathfrak{B}_{\Omega^+}(\Omega)) \to 0$ *is exact.*

Moreover a) and b) are implied by either

1) $H^1(\mathfrak{B}(\Omega)) = 0$,

or

2) $H^1(\mathfrak{B}_{\Omega^+}(\Omega)) = 0$.

EXAMPLE 2. If Ω is connected, $\mathring{\Omega}^- \neq \emptyset$, and D^0 is an elliptic operator with real analytic coefficients, then $H^0(\mathfrak{B}_{\Omega^-}(\Omega)) = 0$ by real analytic hypoellipticity in the hyperfunction category (see [17], [16]), so I) and II) are injective.

EXAMPLE 3. Going back to Example 1, we assume in what follows that Ω is connected and $\mathring{\Omega}^- \neq \emptyset$. Suppose that the sheaf cohomology $H^1(\Omega, \mathcal{O}) = 0$ (e.g. Ω could be a Stein manifold or, more generally, an $(n-2)$-complete manifold). Then condition 1) is satisfied; hence I$'$) is an isomorphism if and only if II$'$) is an isomorphism. Thus in such a situation the classical Hartogs extension phenomenon *is equivalent* to the Lewy extension phenomenon (for real analytic S, but then even for hyperfunction CR functions).

EXAMPLE 4. Let Ω^- be a compact domain in $\boldsymbol{C}^n (n \geqslant 2)$ with connected real analytic boundary S. Then the classical result of Hartogs that holomorphic functions in $\mathring{\Omega}^+ \equiv \boldsymbol{C}^n - \Omega^-$ extend holomorphically to $\boldsymbol{C}^n$ is equivalent to the following statement:

Each hyperfunction f on S such that $\bar{\partial}_S f = 0$ on S has a unique extension to a holomorphic function F on $\mathring{\Omega}^-$ which assumes the boundary values f on $S = \partial\Omega^-$ in the sense of hyperfunctions.

EXAMPLE 5. Or, still in the context Example 1, one could take a real analytic hypersurface S whose Levi form at a certain point p has at least one nonzero eigenvalue. Then the local Lewy extension phenomenon near p to one side (call it the Ω^- side) *is equivalent*, even for hyperfunction CR functions, to the classical E. E. Levi theorem (the Kontinuitätssatz) which says that there is local holomorphic extension from $\mathring{\Omega}^+$ to Ω across S.

We leave to the reader the task of formulating and proving the same results in the real analytic category. By using the results of [3], the same results can be proved in the distribution category.

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