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Steady Solutions of the Navier-Stokes Equations in Unbounded Channels and Pipes (*).

CHARLES J. AMICK (**)

dedicated to Jean Leray

Summary. – *This paper is concerned with the steady flow of a viscous incompressible fluid in channels and pipes (in two and three dimensions respectively) which are cylindrical outside some compact set K. (See Figure* 1. *In this paper, « cylinder » is used to mean « strip » for the two-dimensional case.) The existence of a weak (or generalized) solution to the steady Navier-Stokes equations is shown for all Reynolds numbers $R < R_0$ (or equivalently, for all values of the kinematic viscosity $\nu > \sigma$) where R_0* does not depend *on that part of the domain within K but only on the cylindrical parts of the domain upstream and downstream. Moreover, R_0 is determined by a variational problem formulated on an infinite cylinder, and can be computed without difficulty for some cross-sections; indeed, the critical value R_0 (typically in the range* 100 *to* 300) *is familiar in the nonlinear theory of stability of parallel flows in the infinite cylinder in question.*

1. – Introduction.

Since Leray's fundamental paper [1] in 1933, the extensive work on steady solutions of the Navier-Stokes equations has centered on flow in two types of domain: interior and exterior domains in $\boldsymbol{R}^N$ (where $N = 2$ or 3 always). The two cases correspond to flow (a) inside a bounded domain Ω, and (b) in the complement of a bounded set; in both cases, $\partial\Omega$, the boundary of Ω, is compact. The present work concerns a class of domains of a third type distinguished from the other two by non-compact boundaries.

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Let $\Omega \subset \boldsymbol{R}^N$ be a domain (an open, connected set). The steady flow of a viscous incompressible fluid with density $\varrho = 1$, kinematic viscosity ν, velocity $u = (u_1, \ldots, u_N)$, pressure p and subject to an external force f satisfies the steady Navier-Stokes equations:

$$\left.\begin{aligned} &(1.1) \qquad -\nu\Delta u + (u\cdot\nabla)u = f - \nabla p\,, \\ &(1.2) \qquad \operatorname{div} u = 0 \end{aligned}\right\} \text{ in } \Omega\,.$$

In addition, we have the boundary condition

$$(1.3) \qquad u = g \quad \text{on } \partial\Omega\,,$$

where g is required to satisfy $\int_{\partial\Omega}(g\cdot n) = 0$ when Ω is bounded because of (1.2). We assume throughout this paper that the force f is derivable from a scalar potential; that is, $f = -\nabla P$. A sufficient condition for this is for Ω to be simply-connected and $\nabla \times f \equiv \operatorname{curl} f = 0$ in Ω. We now write p for the effective pressure, previously $p + P$.

If Ω is bounded, the problem of solving (1.1) to (1.3) for (u, p) is said to be of Type 1. If $\boldsymbol{R}^N \setminus \Omega$ is bounded, and the velocity is required to approach a given constant vector at infinity, the problem is said to be of Type 2.

The problems of Type 1 and 2 have been examined extensively in recent years (see [2], [3], and [4]) and the existence of solutions has been proved for all $\nu > 0$ and for suitably restricted data f, g and $\partial\Omega$. An exception is the problem of Type 2 for $N = 2$, for which an additional restriction to sufficiently large values of ν is required. The existence proofs make crucial use of the fact that $\partial\Omega$ is compact in these two cases. Type 3 problems are those for which $\partial\Omega$ is not compact.

An example of such a domain Ω is a cylinder (when $N = 3$, the cross-section is not necessarily circular) which is of the form $\boldsymbol{R} \times A$, where A is either an open interval $(-d, d)$, $d \in (0, \infty)$, for $N = 2$ or a simply-connected bounded domain in the plane for $N = 3$. For any such domain, there is a relatively simple solution, called Poiseuille flow, representing a velocity field parallel to the axis and the same in every cross-section. Let $N = 3$, let the axis of the cylinder be the x_1 axis, and set $g = 0$ (corresponding to a fixed pipe). Then there is a solution of (1.1) to (1.3) of the form $u = (u_1(x_2, x_3), 0, 0)$ and $p = -Cx_1$, provided that

$$(1.4) \qquad -\nu\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)u_1 = C \quad \text{in } A\,,$$

$$(1.5) \qquad u_1 = 0 \quad \text{on } \partial A\,.$$

The constant C is to be such that the flux condition

$$\int_A u_1 \, dx_2 \, dx_3 = M > 0 \tag{1.6}$$

is satisfied.

A unique solution of (1.4) to (1.6) exists for all $\nu > 0$, under mild conditions on ∂A. The solution (u, p) is the « Poiseuille » flow, and we define the Reynolds number to be $R = M/(\nu l)$ where $l = (|A|/\pi)^{\frac{1}{2}}$ and $|S|$ denotes the area of a measurable set $S \subset \boldsymbol{R}^2$.

For $N = 2$, we define

$$R = M/\nu = \int_{-d}^{d} u_1 \, dx_2/\nu \, .$$

For a general domain Ω, the Reynolds number R is of the form $\bar{u}\bar{L}/\nu$ where $\bar{u}$ is a constant representative of the velocity field u satisfying (1.1) to (1.3), $\bar{L}$ is a characteristic length depending on the geometry of Ω, and ν is the kinematic viscosity.

Another widely-known example of a problem of Type 3 is Jeffery-Hamel flow in the plane. Here the domain is $\Omega = \{(r, \theta) : r > 0, \theta \in (-\alpha, \alpha)\}$ where (r, θ) are plane polar coordinates, and the solutions $u = u(\theta; R, \alpha)$ are given in terms of Jacobian elliptic functions.

The final example of a problem of Type 3 concerns a symmetrical channel $\Omega \in \boldsymbol{R}^2$ with slowly curving walls and such that the product of local channel half-width and local wall curvature is bounded by a small parameter $\varepsilon > 0$. Fraenkel showed in [5] and [6] that under certain restrictions on R and α, a formal approximation in powers of ε to the stream function ψ (where $u = (u_1, u_2, 0) = \nabla \times (0, 0, \psi)$) is in fact a strict asymptotic expansion (for $\varepsilon \to 0$) of an exact solution to the steady Navier-Stokes equations.

The problem in this paper concerns steady viscous incompressible flow in domains of the following type.

DEFINITION 1.1. *A domain $\Omega \subset \boldsymbol{R}^N$ ($N = 2$ or 3) will be called admissible (Figure 1) if $\partial\Omega$ is of class C^∞, Ω is simply-connected and Ω is the union of three disjoint subsets as follows (note that Ω_2 is not open).*

(1) *$\Omega_1 = (-\infty, 0) \times A_1$, where $A_1 = (-d, d)$, $d \in (0, \infty)$, for the case of a channel ($N = 2$), or A_1 is a simply-connected bounded domain in the plane, with ∂A_1 of class C^∞, for a pipe ($N = 3$).*

(2) *In a different coordinate system, $\Omega_3 = (0, \infty) \times A_3$, where A_3 has the same properties as A_1. (However, A_3 need not equal A_1.)*

(3) *$\Omega_2 = \Omega \setminus (\Omega_1 \cup \Omega_3)$ is bounded.*

Throughout this paper, superscripts will denote labels and not exponents unless the contrary is explicitly stated.

Let the flux $M > 0$ through an admissible domain Ω be prescribed; and let q^1 and q^3 be the Poiseuille velocities for Ω_1 and Ω_3, respectively, corresponding to flux M. *We seek a solution* (u, p) *of the steady Navier-Stokes equations*

$$\left.\begin{aligned} &-\nu\Delta u + (u\cdot\nabla)u = -\nabla p\,, && (1.7)\\ &\operatorname{div} u = 0 && (1.8)\end{aligned}\right\} \textit{in } \Omega\,,$$

such that

$$u = 0 \quad \textit{on } \partial\Omega\,, \tag{1.9}$$

and

$$u \to q^j \textit{ as } |x| \to \infty \quad \textit{in } \Omega_j \ (j = 1 \textit{ or } 3)\,. \tag{1.10}$$

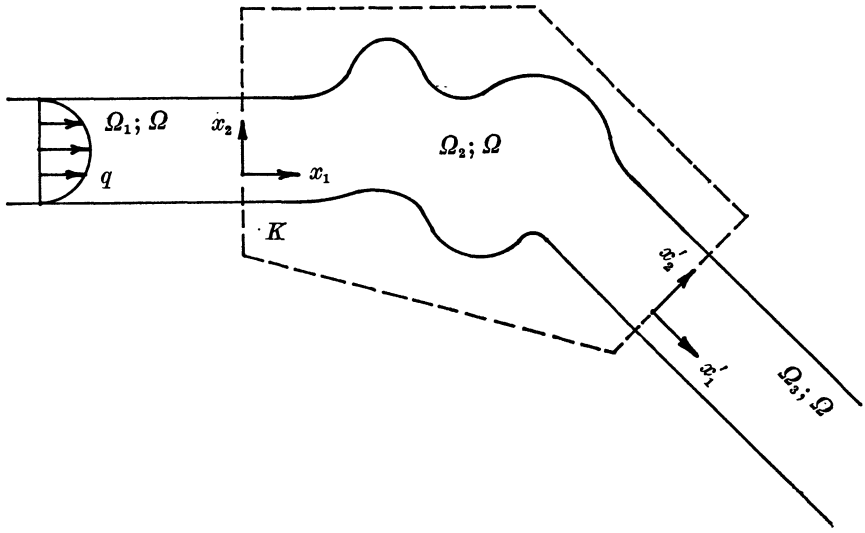

Figure 1. – Notation for an admissible domain $\Omega \subset \boldsymbol{R}^N$ ($N = 2$ or 3):

$$\Omega_1 = \{x \in \Omega : x_1 < 0\}\,, \qquad \Omega_3 = \{x \in \Omega : x_1' > 0\}\,, \qquad \Omega_2 = \Omega \setminus (\Omega_1 \cup \Omega_3)\,.$$

Note: for $N = 3$, Ox_1 and Ox_1' are not necessarily coplanar.

This problem was proposed (I believe) by Leray to Ladyzhenskaya, who in [7] attempted an existence proof under no restrictions on the viscosity ν. The problem is also mentioned by Finn in a review paper ([3], p. 150).

We shall seek a solution of (1.7) to (1.10) of the form

$$u = q + w,$$

where the velocity field q is to be such that

(1.11) (*a*) $\int_X (q \cdot n) = M$ whenever a cross-section X of Ω is defined,

(*b*) $\operatorname{div} q = 0$ in Ω, $q = 0$ on $\partial\Omega$, and

(*c*) $q = q^j$ in Ω_j $(j = 1$ or $3)$.

Thus q, which will be constructed a priori, is an « extended Poiseuille velocity field » that satisfies the boundary conditions. It follows that w is to be such that $q + w$ satisfies (1.7) and

(1.12) (*a*) $\int_X (w \cdot n) = 0$,

(*b*) $\operatorname{div} w = 0$ in Ω, $w = 0$ on $\partial\Omega$, and

(*c*) $w \to 0$ as $|x| \to \infty$ in Ω_j $(j = 1$ or $3)$.

(*d*) In addition, we require w to have finite Dirichlet norm, i.e.

$$\int_\Omega |\nabla w|^2 \equiv \int_\Omega \sum_{i,j} (\partial w_i / \partial x_j)^2 < \infty .$$

The paper is organized as follows.

In section 2, we give some notations, definitions, and preliminary lemmas; in section 3, the existence of a weak solution u of the problem (1.7) to (1.10) is proved for all values of the viscosity $\nu > \sigma$, where σ does not depend on Ω_2. In terms of the (non-dimensional) Reynolds number, the condition $\nu > \sigma$ becomes $R < R_0$, and some numerical values of the critical Reynolds number R_0 are given. In addition, we prove that a weak solution exists in certain domains which asymptotically approach cylinders as $|x| \to \infty$. Section 4 consists of theorems, concerned with the constant σ, which are needed in section 3.

If the admissible domain is symmetric about some axis, then many of the results for general admissible domains are improved. These results appear as corollaries to the main results.

2. – Preliminaries.

Let $x = (x_1, \dots, x_n)$ denote points in $\boldsymbol{R}^n$, and use the standard inner product $x \cdot y = \sum_i (x_i y_i)$. Let $\boldsymbol{R}^+ \subset \boldsymbol{R}$ be given by $\boldsymbol{R}^+ = \{x : x > 0\}$ and let $\boldsymbol{R}^-$ be defined analogously. All integrals in this paper are in the sense of Lebesgue.

Recall that the domain $\Omega \subset \boldsymbol{R}^N$ (where $N = 2$ or 3 always) is a union of three disjoint subsets (Figure 1)

$$\begin{cases} \Omega_1 = \{x \in \Omega : x_1 < 0\}\,, \\ \Omega_3 = \{x \in \Omega : x_1' > 0\}\,, \\ \Omega_2 = \Omega \setminus (\Omega_1 \cup \Omega_3)\,, \end{cases} \tag{2.1}$$

where $(x_1, \dots, x_N)$ and $(x_1', \dots, x_N')$ are distinct coordinate systems such that Ox_1 and Ox_1' are axes of the cylindrical domains Ω_1 and Ω_3, respectively.

2.1. *Function spaces.*

Let U be an arbitrary domain in $\boldsymbol{R}^n$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$, with each α_i a non-negative integer, be a multi-index of order $|\alpha| = \alpha_1 + \dots + \alpha_n$ and let

$$D_j = \partial / \partial x_j\,, \qquad D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}\,.$$

Write $V \subset\subset U$ when V is compact and $V \subset U$. The support $\operatorname{supp} v$ of a function $v : U \to \boldsymbol{R}^n$ is the closure of $\{x \in U : v(x) \neq 0\}$. Thus, v is said to have compact support in U if $\operatorname{supp} v \subset\subset U$.

The set of functions $C^\infty(U \to \boldsymbol{R}^n)$ denotes those functions defined on U with image in $\boldsymbol{R}^n$ and having all (partial) derivatives continuous. The set $C^\infty(\overline{U} \to \boldsymbol{R}^n)$ consists of those functions in $C^\infty(U \to \boldsymbol{R}^n)$ such that all derivatives can be extended to be bounded and continuous on $\overline{U}$. We introduce two sets of functions commonly called « test functions »:

$$C_0^\infty(U \to \boldsymbol{R}^n) = \{\varphi \in C^\infty(U \to \boldsymbol{R}^n) : \operatorname{supp} \varphi \subset\subset U\}\,,$$

$$J(U \to \boldsymbol{R}^n) = C_0^{\infty,\mathrm{sol}}(U \to \boldsymbol{R}^n) = \{\varphi \in C_0^\infty(U \to \boldsymbol{R}^n) : \operatorname{div} \varphi = 0 \text{ in } U\}\,.$$

(The « sol » superscript denotes solenoidal vector fields.)

The following norm is used to define various Sobolev spaces:

$$|v|_{W_p^j} = \Big(\sum_{|\alpha| \leq j} \int_U |D^\alpha v^p| \Big)^{1/p} \tag{2.2}$$

for non-negative integers j and $p \in [1, \infty)$. Denote by $W_p^j(U \to \boldsymbol{R}^n)$ the completion of

$$\{v \in C^\infty(U \to \boldsymbol{R}^n) : |v|_{W_p^j} < \infty\}$$

in the norm (2.2). Similarly, define $\mathring{W}_p^j$ and $\mathring{W}_p^{j,\mathrm{sol}}$ as the completion in (2.2) of C_0^∞ and J, respectively. W_p^j, $\mathring{W}_p^j$, and $\mathring{W}_p^{j,\mathrm{sol}}$ are Banach spaces (they are Hilbert spaces for $p = 2$ with the obvious inner product) and $\mathring{W}_p^{j,\mathrm{sol}} \subset \mathring{W}_p^j \subset W_p^j$. Some of the properties of W_p^j and $\mathring{W}_p^j$ are given in [9] and [11].

For normed spaces A and B, we write $A \hookrightarrow B$ when the identity map $f \mapsto f$ is a bounded mapping from A into B. The space A is said to be imbedded in B.

2.2. *The space* $H(\Omega \to \boldsymbol{R}^N)$.

As stated in the Introduction, we seek a solution (u, p) of the steady Navier-Stokes equations (1.7) to (1.10) of the form $u = q + w$ where q is a known function satisfying (1.11) (the construction of q appears in section 3.1) and w is to satisfy (1.12). A natural setting for w is as an element of the Hilbert-Sobolev space $H(\Omega \to \boldsymbol{R}^N)$ ($N = 2$ or 3).

For an arbitrary domain $U \subset \boldsymbol{R}^n$, $H(U \to \boldsymbol{R}^n)$ is the completion of $J(U \to \boldsymbol{R}^n)$ in the norm implied by the inner product

$$\langle f, g \rangle_H = \int_U \nabla f : \nabla g = \sum_{i,j=1}^n \int_U (D_j f_i)(D_j g_i) . \tag{2.3}$$

The norm for H is

$$|f|_H = \langle f, f \rangle_H^{\frac{1}{2}}$$

and will be referred to as the Dirichlet norm. In addition, for a Lebesque measurable set $V \subset U$ define

$$\|f\|_V = \Big(\int_V |\nabla f|^2 \Big)^{\frac{1}{2}} . \tag{2.4}$$

Similarly, we define $E(U \to \boldsymbol{R}^n)$ to be the completion of C_0^∞ in the Dirichlet norm; vector fields in E need not be solenoidal.

We now give some properties of H.

(2.5) (*a*) If U is such that $E(U) \hookrightarrow L_2(U)$, then E is equivalent to $\mathring{W}_2^1$ and H to $\mathring{W}_2^{1,\mathrm{sol}}$.

(*b*) $\int_U (f \operatorname{div} g) = 0$, $\forall f \in L_2(U)$ and $g \in H(U)$.

(*c*) Let ∂U be of class C^∞ ([18], pp. 9-10). We can define in the usual manner a trace operator $T_{\partial U}: H(U) \to L_{2,\mathrm{loc}}(\partial U)$, where $T_{\partial U}$ is a bounded linear map and $T_{\partial U}(\varphi) = \varphi|_{\partial U} = 0 \quad \forall \varphi \in J(U)$ since $\operatorname{supp} \varphi \subset\subset U$. Hence $T_{\partial U}$ is the zero map and

$$\int_{\partial U} |f|^2 = 0 \qquad \forall f \in H(U),$$

where the integral is defined in terms of the local coordinates of ∂U ([11], pp. 231-232).

In fact, for the domains Ω in this paper there exists a large class of unbounded manifolds Γ for which we can define a trace operator $T_\Gamma: H(\Omega) \to \to L_2(\Gamma)$ and not merely into $L_{2,\mathrm{loc}}(\Gamma)$.

(*d*) Assume $H(U) \hookrightarrow L_2(U)$ and let X be a cross-section of U. (We define a cross-section $X \subset \boldsymbol{R}^{n-1}$ of a domain $U \subset \boldsymbol{R}^n$ to be a bounded open set of the form $X = U \cap P$ where P is an $(n-1)$-dimensional hyperplane.) Then elements in $H(U)$ carry no flux across X, i.e.,

$$\int_X (f \cdot n) = 0 \qquad \forall f \in H(U).$$

A conventional estimate, beginning with integration along a normal to any point $x \in X$, shows that

$$\int_X |f(x)|^2 \leqslant 2 |f|_{L_2(U)} |f|_{H(U)} \qquad \forall f \in H(U).$$

Since X is bounded, we have $f, (f \cdot n) \in L_1(X)$ and $|f \cdot n|_{L_1(X)} \leqslant \mathrm{const}\, |f|_{H(U)}$. We may define a trace operator $T_X: H(U) \to L_1(X)$ where T_X is a bounded linear map and $T_X(\varphi) = (\varphi \cdot \eta)|_X \quad \forall \varphi \in J(U)$. We now claim that T_X is identically zero, and it suffices to prove that $T_X(\varphi) = 0$ for an arbitrary $\varphi \in J(U)$. Since $\varphi \in J(U) \subset J(\boldsymbol{R}^N)$, we can apply the divergence theorem to a half-ball B bounded by the hyperplane P and a hemisphere Γ so large that $\operatorname{supp} \varphi \cap \Gamma = 0$. Then

$$0 = \int_B (\operatorname{div} \varphi) = \int_P (\varphi \cdot n) + \int_\Gamma (\varphi \cdot n) = \int_X (\varphi \cdot n).$$

(*e*) We shall need the following result in sections 3 and 4 (Lions [12], pp. 67-68; Heywood [13]). Assume that ∂U is of class C^2, then

$$\{\varphi \in H(U \to \boldsymbol{R}^n) : \operatorname{supp} \varphi \text{ is bounded}\} =$$
$$= \{\varphi \in \mathring{W}_2^1(U \to \boldsymbol{R}^n) : \operatorname{supp} \varphi \text{ is bounded, } \operatorname{div} \varphi = 0 \text{ in } U\}.$$

We now state two lemmas concerning the domain U of functions in $H(U) \subset E(U)$. The results are of critical importance for admissible domains.

LEMMA 2.1. (*a*) *Let* $L > 0$. *Then*

$$(2.6)(a) \qquad \int_0^L (v)^2 \leqslant \left(\frac{L}{\pi}\right)^2 \int_0^L \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{L}{\pi}\right)^2 |v|_E^2 \qquad \forall v \in E((0, L) \to \boldsymbol{R}).$$

(*b*) *Let* $U \subset \boldsymbol{R}^n$ *be a domain lying between parallel planes* (*i.e.* $(n-1)$-*dimensional hyperplanes*) *a distance* L *apart. Then*

$$(2.6)(b) \qquad \int_U |v|^2 \leqslant \left(\frac{L}{\pi}\right)^2 |v|_E^2 \qquad \forall v \in E(U \to \boldsymbol{R}^n).$$

(*c*) *Let* $U \subset \boldsymbol{R}^n$ *be a domain with* $U = \bigcup_{i=1}^{m} U_i$ $(m \geqslant 1)$ *and such that each domain* U_i *lies between parallel planes a distance* L *apart. Then*

$$(2.6)(c) \qquad \int_U |v|^2 \leqslant \text{const}\, |v|_E^2 \qquad \forall v \in E(U \to \boldsymbol{R}^n),$$

where the constant depends only on the geometry of U:

PROOF. It suffices to prove the lemma for all $v \in C_0^\infty$. Part (*a*) follows from a standard result of the calculus of variations [10] and (*b*) and (*c*) follow from (*a*).

The following form of certain Sobolev inequalities is due to Nirenberg ([19], p. 125).

LEMMA 2.2 *Let a domain* $U \subset \boldsymbol{R}^n$ *be such that* $E(U \subset \boldsymbol{R}^n) \hookrightarrow L_2(U \to \boldsymbol{R}^n)$. *Then* $E \hookrightarrow L_s$

(2.7) (*a*) *for* $2 \leqslant s < \infty$, *if* $n = 2$,

(*b*) *for* $2 \leqslant s \leqslant 6$, *if* $n = 3$, *and*

(*c*) *for* $2 \leqslant s \leqslant 2n/(n-2)$, *if* $n \geqslant 4$.

Since $H(U) \subset E(U)$, Lemma 2.1 also holds for functions $v \in H(U)$. If we apply Lemma 2.1(*c*) to an admissible domain Ω, then it follows that $H(\Omega) \hookrightarrow L_2(\Omega)$ and Lemma 2.2 holds with $E(\Omega)$ replaced by $H(\Omega)$.

We remind the reader that Lemmas 2.1 and 2.2 are applicable to an admissible domain Ω only because Ω_1 and Ω_3 can be individually bounded by parallel planes.

It is now clear why we seek a solution $u = q + w$ of the problem (1.7) to (1.10) with $w \in H(\Omega \to \boldsymbol{R}^N)$. Indeed, then w satisfies (1.12) in a generalized sense; i.e. w has finite Dirichlet norm $|w|_H$; (2.5)(b) implies $\operatorname{div} w = 0$ almost everywhere in Ω; (2.5)(c) gives $w = 0$ almost everywhere on $\partial\Omega$; and (2.5)(d) states that w carries no flux. However, (1.12)(c) may be satisfied only in a generalized sense since elements in $H(\Omega)$ need not go to zero pointwise as $|x| \to \infty$ in Ω.

2.3. *Boundary-layer integrals.*

For a domain $U \subset \boldsymbol{R}^n$ and $\varepsilon > 0$, we define $\alpha(x) = \operatorname{dist}(x, \partial U)$ and $U_\varepsilon = \{x \in U : \alpha(x) < \varepsilon\}$. If ∂U is sufficiently smooth, then we can introduce « boundary-layer » coordinates (s, t) in U_ε for small ε ([18], p. 38). Here s denotes a surface coordinate on ∂U, and t distance from ∂U along an inward normal.

LEMMA 2.3. *Let U be a domain in $\boldsymbol{R}^N$ ($N = 2$ or 3) with compact ∂U of class C^∞ or an admissible domain in $\boldsymbol{R}^N$. If $\varepsilon > 0$ is sufficiently small, then*

(a) $\alpha \in C^\infty(\overline{U}_\varepsilon \to \boldsymbol{R})$, *and*

(b) *every point $x_0 \in \partial U$ is the center of a ball $B_0 = B(x_0, r_0)$, with radius r_0 independent of ε, such that $x \mapsto (s, t)$ is a C^∞ diffeomorphism from $\overline{B}_0 \cap \overline{U}_\varepsilon$ to some compact subset of $\boldsymbol{R}^N$.*

PROOF. If ∂U is compact, then the result is standard ([18], p. 38). The proof for an admissible domain is analogous and uses the fact that U is cylindrical outside some bounded set.

LEMMA 2.4. *Let $v \in C^\infty([0, L] \to \boldsymbol{R})$ with $v(0) = 0$. Then*

$$\int_0^L \left(\frac{v}{t}\right)^2 dt \leqslant 4 \int_0^L \left(\frac{dv}{dt}\right)^2 dt .$$

PROOF. An integration by parts gives:

$$\int_0^L \left(\frac{v}{t}\right)^2 dt = -\frac{(v(L))^2}{L} + 2\int_0^L \frac{v}{t}\frac{dv}{dt} dt,$$

and the lemma follows after an application of the Schwarz inequality.

LEMMA 2.5. *Let U be as in Lemma 2.3. If $\varepsilon > 0$ is sufficiently small, then*

$$\int_{U_\varepsilon} \left|\frac{w}{\alpha}\right|^2 \leqslant (4 + O(\varepsilon)) \int_{U_\varepsilon} |\nabla w|^2 \qquad \forall w \in E(U \to \boldsymbol{R}^n) .$$

PROOF. If ∂U is compact, then the result is standard ([2], pp. 106-110) and makes use of Lemma 2.4. The proof for an admissible domain is analogous.

Particularly useful tools in the study of partial differential equations are mollifiers $\mu \in C^\infty(\boldsymbol{R} \to [0, 1])$ with $\operatorname{supp}(d\mu/dt) \subset\subset \boldsymbol{R}$. The following mollifier ([3], [8], [14]) $\mu = \mu(t; \varepsilon)$ is important in section 3.

LEMMA 2.6. *For every $\varepsilon > 0$, there exists a mollifier $\mu(\cdot\,; \varepsilon) \in C^\infty(\boldsymbol{R} \to [0, 1])$ (see Figure 2a) with $\operatorname{supp} \mu_t \subset (0, \varepsilon]$ and such that $\mu(0; \varepsilon) = 1$, $\mu(\varepsilon; \varepsilon) = 0$ and*

$$\mu(t; \varepsilon), \qquad \left|\frac{\partial}{\partial t}\mu(t; \varepsilon)\right| \leqslant \frac{\varepsilon}{t} \qquad \textit{for } t > 0 .$$

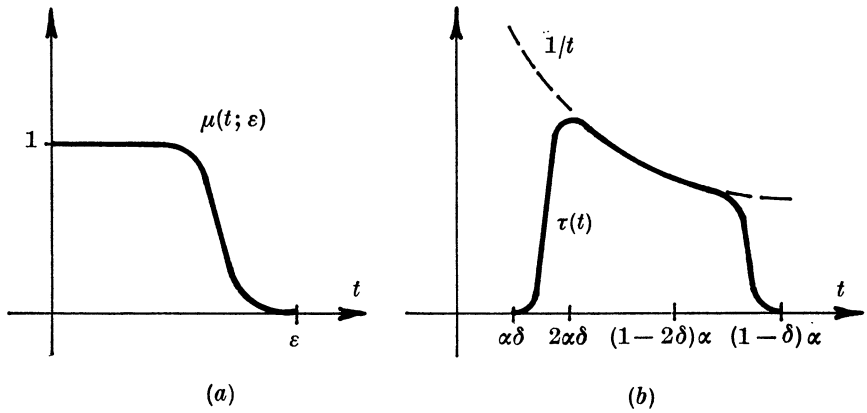

Figure 2. – (*a*) The mollifier $\mu = \mu(\cdot\,; \varepsilon) \in C^\infty(\boldsymbol{R} \to [0, 1])$. (*b*) The function τ used in the construction of μ.

PROOF. For any $\alpha > 0$ and $\delta \in (0, \frac{1}{4})$, let $\tau(t) = \tau(t; \alpha, \delta)$ be a C^∞ mollifier as in Figure 2*b*. The function τ has the properties: (*a*) $0 \leqslant \tau(t) \leqslant 1/t$ everywhere, (*b*) $\tau(t) = 1/t$ on $[2\alpha\delta, (1-2\delta)\alpha]$, (*c*) $\tau(t) = 0$ for $t \leqslant \alpha\delta$ and $t \geqslant (1-\delta)\alpha$. Let $T = \int_0^\alpha \tau(s)\, ds$ and define

$$\mu(t; \alpha, \delta) = 1 - \frac{1}{T}\int_0^t \tau(s)\, dS .$$

Since $T > \int_{2\alpha\delta}^{\alpha(1-2\delta)} ds/s = \log((1-2\delta)/2\delta)$, define δ by $\log((1-2\delta)/2\delta) = 1/\varepsilon$ so that $1/T < \varepsilon$ and $|\mu_t| = \tau(t)/T \leqslant \varepsilon/t$. Choose $\alpha(\varepsilon) = \varepsilon/(1-\delta)$ and let $\mu(t;\varepsilon) = \mu(t;\alpha(\varepsilon),\delta(\varepsilon))$. It follows that $\operatorname{supp}\mu_t \subset (0,\varepsilon]$ and if $t \in \operatorname{supp}\mu_t$, then $\varepsilon/t \geqslant 1 \geqslant \mu(t;\varepsilon)$.

If we combine the properties of μ with Lemma 2.5, then we obtain an estimate essentially due to Leray ([1], pp. 38-47), that is crucial in problems of Type 1 and 2 when one proves the existence of weak solutions for all $\nu > 0$. A similar inequality for admissible domains will be applied in Theorem 3.6 to the problem (1.7)-(1.10).

THEOREM 2.7. *Let U be as in Lemma 2.3 and let $Q \in C^\infty(\overline{U} \to \boldsymbol{R}^n)$ with $\nabla \times Q = 0$ on ∂U. For $\varepsilon > 0$, define $g(x;\varepsilon) = \nabla \times \{\mu(\alpha(x);\varepsilon)Q(x)\}$. If ε is sufficiently small, then*

$$\int_U |g\cdot(w\cdot\nabla)w| = \int_{U_\varepsilon} |g\cdot(w\cdot\nabla)w| \leqslant \text{const}\,\varepsilon \int_{U_\varepsilon} |\nabla w|^2 \qquad \forall w \in E(U \to \boldsymbol{R}^N),$$

and the constant is independent of ε.

PROOF. The properties of μ in Lemma 2.6 show that $\operatorname{supp} g \cap U = U_\varepsilon$ and $|g(x;\varepsilon)| \leqslant \text{const}\,\varepsilon/\alpha(x)$. Using the Schwarz inequality and then Lemma 2.5, we have

$$\int_{U_\varepsilon} |g\cdot(w\cdot\nabla)w| \leqslant \text{const}\,\varepsilon \int_{U_\varepsilon} \frac{|w|}{\alpha}|\nabla w| \leqslant \text{const}\,\varepsilon \left|\frac{w}{\alpha}\right|_{L_2(U_\varepsilon)} \|w\|_{U_\varepsilon} \leqslant \text{const}\,\varepsilon \|w\|^2_{U_\varepsilon}.$$

3. – Existence of a weak solution.

Before giving the definition of a weak (or generalized) solution of the problem (1.7) to (1.10), we construct the velocity field q satisfying (1.11).

3.1. *Construction of the « extended Poiseuille velocity field » q.*

Since $\partial\Omega$ is of class C^∞ and the desired function q is to coincide with the Poiseuille velocity field outside Ω_2 (hence $q \in C^\infty(\overline{\Omega \setminus \Omega_2})$, it is reasonable to require that $q \in C^\infty(\overline{\Omega} \to \boldsymbol{R}^N)$.

We recall that, for given $M > 0$, q is to satisfy:

(3.1) (*a*) $q \in C^\infty(\overline{\Omega} \to \boldsymbol{R}^N)$ ($N = 2$ or 3);

(*b*) $\operatorname{div} q = 0$ in Ω, $q = 0$ on $\partial\Omega$; and

(*c*) $q = q^j$ in Ω_j ($j = 1$ or 3), where q^j is the Poiseuille velocity field for the cylinder Ω_j and carries flux M.

We wish to construct the extended Poiseuille velocity field in the form $q = \nabla \times Q$ because then one can multiply the vector potential Q by a mollifier and retain solenoidality. As a first step in this construction, we shall need

LEMMA 3.1. *Let $U \subset \boldsymbol{R}^N$ be a cylinder of the form $\boldsymbol{R} \times A$ where A is an open interval $(-d, d)$, $d \in (0, \infty)$, for $N = 2$, and A is a simply-connected bounded domain in the plane, with ∂A of class C^∞, for $N = 3$. Let $v \in C^\infty(\overline{U} \to \boldsymbol{R}^N)$ with $v = (v_1(x_2, x_3), 0, 0)$. Then there exists $\psi \in C^\infty(\overline{U} \to \boldsymbol{R}^N)$ with $\psi = (0, \psi_2(x_2, x_3), \psi_3(x_2, x_3))$ such that $\nabla \times \psi = v$.*

PROOF. For $N = 2$, set $\psi_2 = 0$ and define

$$\psi_3(x_2) = \int_{-d}^{x_2} v_1(t)\, dt - \tfrac{1}{2} M .$$

For $N = 3$, $\nabla \times \psi = v$ implies that we need $D_2 \psi_3 - D_3 \psi_2 = v_1$ in A. Let ψ_2 be any solution of $(D_2^2 + D_3^2)\psi_2 = -D_3 v_1$ in A, with $\psi_2 \in C^\infty(\overline{A} \to \boldsymbol{R})$ (standard theory shows this to be true if $\psi_2|_{\partial A}$ is chosen of class C^∞). Define $\psi_3 \in C^\infty(\overline{A} \to \boldsymbol{R})$ by

$$\psi_3(x_2, x_3) = \int_{(p_2, p_3)}^{(x_2, x_3)} \{(v_1 + D_3 \psi_2)\, dt_2 - D_2 \psi_2\, dt_3\} \qquad \forall (x_2, x_3) \in \overline{A} ,$$

where (p_2, p_3) is some arbitrary (fixed) point in $\overline{A}$. It follows that ψ_3 is single-valued and of class C^∞ on $\overline{A}$ since v_1 and ψ_2 are. If we let $\psi = (0, \psi_2, \psi_3)$, then $\nabla \times \psi = v$ in $\overline{U}$, and in addition $\operatorname{div} \psi = 0$ in $\overline{U}$. The lemma is proved.

For $N = 2$, the Poiseuille velocity field in $\boldsymbol{R} \times (-d, d)$ carrying flux M is given by $(q_1(x_2), 0)$, where

$$q_1(x_2) = \frac{3M}{4d}\left\{1 - \left(\frac{x_2}{d}\right)^2\right\}.$$

It follows that a corresponding vector potential $\psi = (0, 0, \psi_3)$ (ψ_3 is often termed the stream function) is given by

$$\psi_3(x_2) = \frac{3M}{4}\left\{\frac{x_2}{d} - \frac{1}{3}\left(\frac{x_2}{d}\right)^3\right\}. \tag{3.2}$$

For $N = 3$ and a circular cross-section A of radius a, the Poiseuille velocity field in $\boldsymbol{R} \times A$ is given by $(q_1(x_2, x_3), 0, 0)$, where

$$q_1(x_2, x_3) = \frac{2M}{\pi a^2}\left\{1 - \left(\frac{x_2}{a}\right)^2 - \left(\frac{x_3}{a}\right)^2\right\}.$$

A corresponding vector potential $\psi = (0, \psi_2, \psi_3)$ is given by

$$(3.3) \qquad \psi_2(x_2, x_3) = -\frac{Mx_3}{2\pi a^2}\left\{2 - \left(\frac{x_2}{a}\right)^2 - \left(\frac{x_3}{a}\right)^2\right\}, \quad \psi_3(x_2, x_3) = \frac{Mx_2}{2\pi a^2} \cdot \left\{2 - \left(\frac{x_2}{a}\right)^2 - \left(\frac{x_3}{a}\right)^2\right\}.$$

The following lemma is a step in the construction of q and is a slight generalization of a result due to Finn ([15], pp. 206-208); a detailed and somewhat different treatment is also given in [16].

LEMMA 3.2. *Let U be an open set in $\boldsymbol{R}^N$ with ∂U of class C^∞ and such that ∂U consists of a finite number of compact components ∂U_i, $i = 1, 2, \ldots, m$. If a given function $g_0 \in C^\infty(\partial U \to \boldsymbol{R}^N)$ satisfies $\int_{\partial U_i} g_0 \cdot n = 0$ on each boundary component ∂U_i, then there exists a vector potential $\psi_b \in C^\infty(\overline{U} \to \boldsymbol{R}^N)$ with $\nabla \times \psi_b = g_0$ on ∂U.*

We assume without loss of generality that the admissible domain Ω is cylindrical for $x_1 \leqslant 1$ and $x_1' \geqslant -1$ (one may always translate the axes in Figure 1). For future reference, define

$$\Omega_a = \{x \in \Omega : x_1 < 1\},$$

and

$$\Omega_c = \{x \in \Omega : x_1' > -1\}.$$

The Poiseuille velocity fields q^1 and q^3 are defined in these respective domains. By Lemma 3.1, there exist vector potentials ψ_a and ψ_c such that

$$(3.4)(a) \qquad q^1 = \nabla \times \psi_a \quad \text{in} \quad \overline{\Omega}_a \quad (\supset \overline{\Omega}_1) \qquad \text{with } \psi_a \in C^\infty(\overline{\Omega}_a \to \boldsymbol{R}^N),$$

and

$$(3.4)(b) \qquad q^3 = \nabla \times \psi_c \quad \text{in} \quad \overline{\Omega}_c \quad (\supset \overline{\Omega}_3) \qquad \text{with } \psi_c \in C^\infty(\overline{\Omega}_c \to \boldsymbol{R}^N).$$

Let $\Omega_4 \subset \Omega_2$ be a domain with $\partial\Omega_4$ of class C^∞ and with $\partial\Omega_4 = \partial\Omega_2$ except for boundary points with $x_1 \in (0, \frac{1}{2})$ or $x_1' \in (-\frac{1}{2}, 0)$; in these regions, $\partial\Omega_4 \subset \Omega$ (see Figure 3). Define $g_0 : \partial\Omega_4 \to \boldsymbol{R}^N$ by

$$g_0(x) = \begin{cases} q^1(x) & \text{for } x \in \partial\Omega_4, \quad 0 < x_1 \leqslant \frac{1}{2}, \\ q^3(x) & \text{for } x \in \partial\Omega_4, \ -\frac{1}{2} \leqslant x_1' < 0, \\ 0 & \text{elsewhere on } \partial\Omega_4. \end{cases}$$

Since q^1 and q^3 are of class C^∞ in $\bar{\Omega}_a$ and $\bar{\Omega}_c$, respectively, and $\partial\Omega_4$ is is of class C^∞, it follows that $g_0 \in C^\infty(\partial\Omega_4 \to \boldsymbol{R}^N)$. We also note that

$$\int_{\partial\Omega_4} (g_0 \cdot n) = 0 .$$

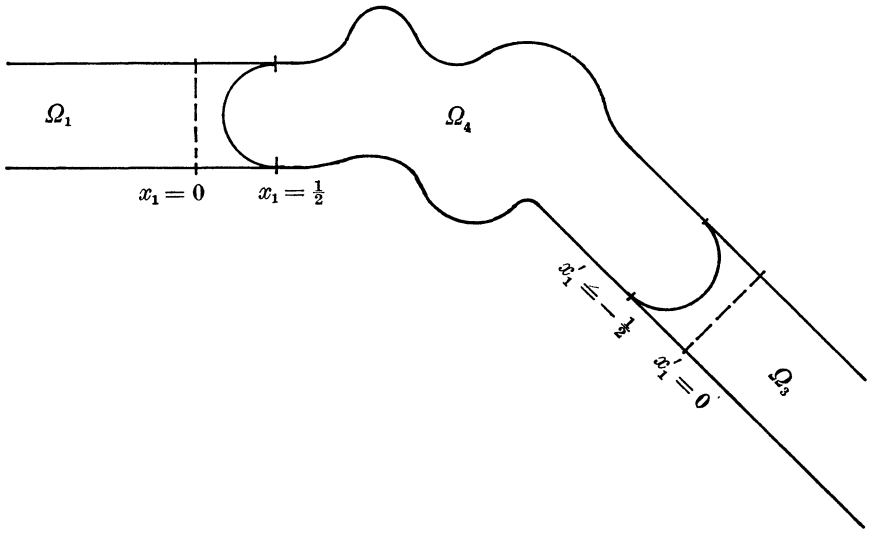

Figure 3. – The domain Ω_4 used in the construction of the extended Poiseuille velocity field.

Hence, the domain Ω_4 and the boundary function g_0 satisfy the hypotheses of Lemma 3.2, and so there exists a vector potential ψ_b with

$$(3.4)(c) \qquad \psi_b \in C^\infty(\bar{\Omega}_4 \to \boldsymbol{R}^N) \quad \text{and} \quad \nabla \times \psi_b = g_0 \qquad \text{on } \partial\Omega_4 .$$

THEOREM 3.3. *Let Ω be an admissible domain (in the sense of Definition* 1.1). *Then there exists a vector potential $Q \in C^\infty(\bar{\Omega} \to \boldsymbol{R}^N)$ ($N = 2$ or 3) such that*

(*a*) $\nabla \times Q = 0$ *on* $\partial\Omega$, *and*

(*b*) $Q = \psi_a$ *in* Ω_1 *and* $Q = \psi_c$ *in* Ω_3, *where* ψ_a *and* ψ_c *are as in* (3.4). *Thus,* $q = \nabla \times Q$ *is an extended Poiseuille velocity field and satisfies* (3.1).

PROOF. We shall give the proof for $N = 3$ since that for $N = 2$ is analogous.

Let $\lambda \in C^\infty(\boldsymbol{R} \to [0, 1])$ be a mollifier with $\lambda(x_1) = 1$ for $x_1 \leqslant \frac{1}{2}$ and $\lambda(x_1) = 0$ for $x_1 \geqslant 1$. Then for ψ_a, ψ_b, ψ_c as in (3.4), define $Q: \bar{\Omega} \to \boldsymbol{R}^N$ by

$$(3.5) \qquad Q(x) = \begin{cases} \lambda(x_1)\psi_a + \{1 - \lambda(x_1)\}\psi_b + \Psi & \text{for } x \in \bar{\Omega}_a, \\ \lambda(-x_1')\psi_c + \{1 - \lambda(-x_1')\}\psi_b + \Psi & \text{for } x \in \bar{\Omega}_c, \\ \psi_b(x) \quad \text{elsewhere in } \bar{\Omega}, \end{cases}$$

where Ψ is to be determined by the condition $\nabla \times Q = 0$ on $\partial\Omega$. (For the case $N = 2$, the function Ψ in (3.5) may be taken as identically zero.) Thus, $\Psi = (\psi_1, 0, 0)$ is to be such that

$$(3.6) \qquad \left.\begin{cases} \dfrac{\partial \psi_1}{\partial x_3} - \lambda'(x_1)(\psi_{a3} - \psi_{b3}) = 0, \\ -\dfrac{\partial \psi_1}{\partial x_2} + \lambda'(x_1)(\psi_{a2} - \psi_{b2}) = 0 \end{cases}\right\} \text{ for } \tfrac{1}{2} \leqslant x_1 \leqslant 1, \ (x_2, x_3) \in \partial A_1,$$

and similarly for $-1 \leqslant x_1' \leqslant -\frac{1}{2}$, $(x_2', x_3') \in \partial A_3$.

We construct ψ_1 as follows. Let (s, t) be « boundary-layer coordinates »: s denotes distance along ∂A_1, measured from a line in $[\frac{1}{2}, 1] \times \partial A_1$ parallel to Ox_1, and in the direction that makes ∂A_1 positively oriented; moreover, s is constant on each normal to ∂A_1, while t denotes distance along the inward normal to ∂A_1. By Lemma 2.3, the map $(x_2, x_3) \mapsto (s, t)$ is one-to-one and of class C^∞ for sufficiently small positive values of t, say for $0 \leqslant t \leqslant t_0$.

Let $\mu \in C^\infty(\boldsymbol{R} \to [0, 1])$ be a mollifier with $\mu(t) = 1$ for $t \leqslant t_0/2$ and $\mu(t) = 0$ for $t \geqslant t_0$. We define

$$(3.7) \qquad \psi_1(x_1; s, t) = \mu(t)\{g(x_1, s) + tf(x_1, s)\},$$

where

$$(3.8)(a) \qquad f(x_1, s) = \lambda'(x_1)\left\{\left(0, -\frac{\partial x_3}{\partial s}, \frac{\partial x_2}{\partial s}\right) \cdot (\psi_a - \psi_b)\right\}\Bigg|_{\partial A_1},$$

$$(3.8)(b) \qquad g(x_1, s) = \lambda'(x_1)\int_0^s \left\{(\psi_a - \psi_b) \cdot \left(0, \frac{\partial x_2}{\partial s'}, \frac{\partial x_3}{\partial s'}\right)\right\}\Bigg|_{\partial A_1} ds'.$$

Note that g is single-valued since

$$\int_{\partial A_1} (\psi_a - \psi_b) \cdot (0, dx_2, dx_3) = \int_{A_1} \{(D_2\psi_{a3} - D_3\psi_{a2}) - (D_2\psi_{b3} - D_3\psi_{b2})\}\, dx_2\, dx_3$$
$$= \int_{A_1} (q_1^1 - q_1^2)\, dx_2\, dx_3 = M - M = 0,$$

where $q^2 = \nabla \times \psi_b$.

It follows immediately from (3.7) that ψ_1 satisfies (3.6). Equation (3.8) gives: $\Omega_a \cap \operatorname{supp} \psi_1 \subset (\operatorname{supp} \lambda') \times A_1 \subset [\frac{1}{2}, 1] \times A_1$. Hence, $\Psi = 0$ in $\bar{\Omega}_1$ and from (3.5) we have $Q = \psi_a$ in $\bar{\Omega}_1$.

An analogous argument holds in Ω_c, and the theorem is proved.

For $N = 2$ and certain admissible domains Ω, we can give an explicit representation of Q. Assume that $f \in C^\infty(\boldsymbol{R} \to \boldsymbol{R}^2)$, $\varphi_+, \varphi_- \in C^\infty(\boldsymbol{R} \to \boldsymbol{R})$, and that the map $\gamma: \boldsymbol{R} \times [-1, 1] \to \bar{\Omega}$ is a C^∞ homeomorphism given by

$$x = \gamma(s, t) = f(s) + \tfrac{1}{2} n\{t(\varphi_+ + \varphi_-) + (\varphi_+ - \varphi_-)\}\,, \tag{3.9}$$

where $n = (-f_2'(s), f_1'(s))/|\nabla f|$ is the unit normal to the curve $x = f(s)$, and $\boldsymbol{R} \times \{1\}$, $\boldsymbol{R} \times \{-1\}$ are mapped onto the « upper » and « lower » components of $\partial\Omega$, respectively. The curve $\{f(s): s \in \boldsymbol{R}\}$ is a generalized axis for Ω, and φ_+ is the distance along the normal from this axis to the « upper » boundary and similarly for φ_-.

A vector potential satisfying Theorem 3.3 is given by $Q = (0, 0, Q_3)$, where

$$Q_3(x_1, x_2) = \tfrac{3}{4} M\big[t(x_1, x_2) - \tfrac{1}{3}\{t(x_1, x_2)\}^3\big]\,.$$

3.2. *The weak solution.*

DEFINITION 3.1. *Let q satisfy* (3.1). *The function $u = q + w$ is a weak* (*or generalized*) *solution of the problem* (1.7) *to* (1.10) *if $w \in H(\Omega \to \boldsymbol{R}^N)$* ($N = 2$ *or* 3) *and*

$$\nu \int_\Omega (\nabla\varphi : \nabla u) + \{\varphi, u, u\} = 0 \qquad \forall \varphi \in J(\Omega) \tag{3.10}$$

(*where* $J(\Omega) = C_0^{\infty,\mathrm{sol}}(\Omega \to \boldsymbol{R}^N)$), *or, equivalently*,

$$\nu\langle \varphi, w\rangle_H + \{\varphi, q + w, w\} + \{\varphi, w, q\} = -\nu \int_\Omega (\nabla\varphi : \nabla q) - \{\varphi, q, q\} \qquad \forall \varphi \in J(\Omega)\,, \tag{3.11}$$

where

$$\{\varphi, \psi, \chi\} = \int_\Omega \varphi \cdot (\psi \cdot \nabla)\chi\,.$$

Since we are interested in $w \in H(\Omega)$, we shall work almost entirely with (3.11). Note that if (u, p) is a classical solution (i.e. has sufficiently many derivatives) of the problem (1.7) to (1.10), then, upon dot-multiplying (1.7) by $\varphi \in J(\Omega)$ and integrating over Ω, we recover (3.11), since

$$-\int_\Omega (\varphi \cdot \nabla p) = \int_\Omega (p \operatorname{div} \varphi) = 0\,.$$

Accordingly, if (u, p) is a classical solution with $\int_\Omega |\nabla(u-q)|^2 < \infty$, then $u-q=w\in H$ satisfies (3.11). The converse, that a weak solution is classical, is known for problems of Type 1 and 2 with smooth data, and will be shown for the present case in a later paper.

We now give some properties of the triple product $\{\varphi, \psi, \chi\}$ which will be used throughout this section.

(*a*) If φ, ψ, $\chi\in H(\Omega)$, then

$$|\{\varphi, \psi, \chi\}| \leqslant \text{const}\, |\varphi|_{L_4}|\psi|_{L_4}|\chi|_H \leqslant \text{const}\, |\varphi|_H|\psi|_H|\chi|_H \tag{3.12(a)}$$

by the Schwarz inequality and the imbedding $H(\Omega)\hookrightarrow L_4(\Omega)$ of Lemma 2.2. Integrating by parts gives

$$\{\varphi, \psi, \chi\} = -\{\chi, \psi, \varphi\}\,. \tag{3.12(b)}$$

(*b*) If $\psi=q$ and φ, $\chi\in H$ or $\chi=q$ and φ, $\psi\in H$, then (3.12)(*b*) remains valid, and an analogous version of (3.12) (*a*) holds.

We first show that Ω_1 and Ω_3 do not contribute to the right-hand side of (3.11), which can therefore be bounded, and then extend (3.11) to all $\varphi\in H(\Omega)$.

LEMMA 3.4. *There exists an element $r\in H(\Omega\to \mathbf{R}^N)$ such that $u=q+w$ is a weak solution if and only if*

$$\nu\langle\varphi, w\rangle_H + \{\varphi, q+w, w\} + \{\varphi, w, q\} = \langle\varphi, r\rangle_H \qquad \forall\varphi\in H(\Omega)\,. \tag{3.13}$$

PROOF. We shall prove that the expression

$$-\nu\int_\Omega (\nabla\varphi:\nabla q) - \{\varphi, q, q\}$$

defines a continuous linear functional on J which may be extended by continuity to H.

Let $\varphi\in J$, then

$$\int_\Omega (\nabla\varphi:\nabla q) = -\int_\Omega (\varphi\cdot\Delta q)\,. \tag{3.14}$$

Since q is the Poiseuille velocity field in Ω_1, it follows by (1.4) that $\Delta q=(C_1, 0, 0)$ in Ω_1, where C_1 is a constant. The function φ carries no

flux, and so, by (2.5)(d),

$$\int_{\Omega_1}(\varphi\cdot\Delta q) = C_1\int_{-\infty}^{0}dx_1\int_{A_1}\varphi_1 = 0\,,$$

and similarly for Ω_3. Use of this result in (3.14) gives

$$\int_{\Omega}(\nabla\varphi:\nabla q) = -\int_{\Omega_2}(\varphi\cdot\Delta q)\leqslant \text{const}\,|\varphi|_{L_2}\leqslant \text{const}\,|\varphi|_H\,, \tag{3.15}$$

since $H\hookrightarrow L_2$ by Lemma 2.1(c), and the constant is independent of φ.

If $x\in\Omega_1$, then $q(x) = \big(q_1(x_2, x_3), 0, 0\big)$ and so $(q\cdot\nabla)q = 0$ (similarly in Ω_3). Hence,

$$\{\varphi, q, q\} = \int_{\Omega_1}\{\varphi\cdot(q\cdot\nabla)q\}\leqslant \text{const}\,|\varphi|_H, \tag{3.16}$$

and the constant is independent of φ.

Equations (3.15) and (3.16) show that the right-hand side of (3.11) is a continuous linear functional on J; we extend the functional to H by continuity. Therefore, the Riesz representation theorem ensures existence of a unique element $r\in H$ such that

$$-\nu\int_{\Omega}(\nabla\varphi:\nabla q) - \{\varphi, q, q\} = \langle\varphi, r\rangle_H \qquad \forall\varphi\in H(\Omega)\,.$$

If $u = q + w$ is a weak solution, then it follows easily by (3.12) that the individual terms of the left-hand side of (3.11) define continuous linear functionals on J, which we then extend by continuity to H, and so (3.13) is satisfied

Conversely, if (3.13) holds, then by restricting φ to J, it follows that w satisfies (3.11) and so $u = q + w$ is a weak solution. The lemma is proved.

We consider an expanding sequence of bounded domains U_m such that $U_m\to\Omega$ as $m\to\infty$ and ∂U_m is of class C^∞ for $m = 1, 2, \ldots$. Denote the surfaces (for $N = 3$) or arcs (for $N = 2$) $\partial U_m\cap\Omega_j$ by Γ_m^j, $j = 1$ or 3 (Figure 4). Since Ω_1 is cylindrical, we assume that the Γ_m^1 are identical for $m = 1, 2, \ldots$ in the sense that Γ_m^1 is a translation parallel to Ox_1 of Γ_1^1 (similarly Γ_m^3 is the translation parallel to Ox_1' of Γ_1^3).

For reasons to be explained presently, we now construct velocity fields $g^m\in C^\infty(\overline{U}_m\to\boldsymbol{R}^N)$ with $g^m = q = \nabla\times Q$ on ∂U_m. Let

$$\alpha_m(x) = \text{dist}\,(x, \partial U_m)\,;$$

and let $\mu(t;\varepsilon)$ be the mollifier in Lemma 2.6. Let ε be sufficiently small in the sense of Lemma 2.3 for the domain U_1. Then, by the translation property of Γ_m^j, the same ε serves for all U_m. Now define $G_m \in C^\infty(\overline{U}_m \to \boldsymbol{R}^N)$ by

$$G_m(x;\varepsilon) = \mu(\alpha_m(x);\varepsilon)Q(x), \tag{3.17(a)}$$

and let

$$g^m(x;\varepsilon) = \nabla \times G_m. \tag{3.17(b)}$$

The function g^m depends on m near Γ_m^1, and Γ_m^3, but is independent of m elsewhere on $\overline{U}_k$ for $m \geqslant k$; its support is in a layer of width ε adjacent to ∂U_m.

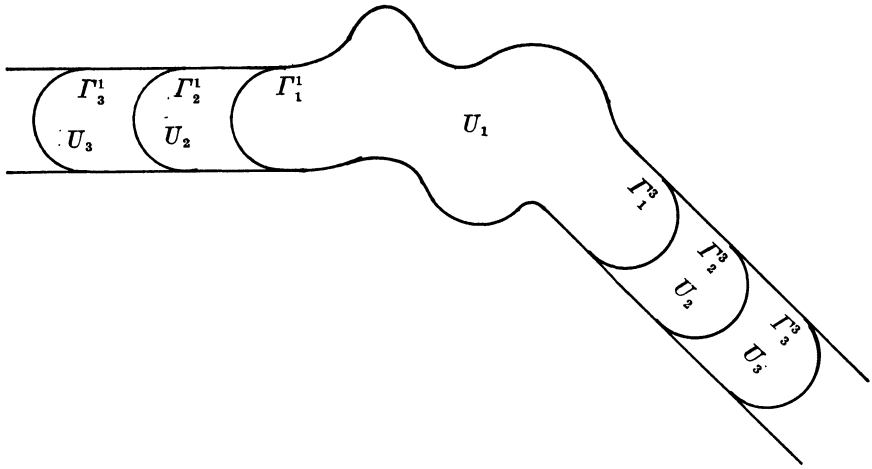

Figure 4. – The expanding sequence of domains $\{U_m\}$. $\Gamma_m^j = \Omega_j \cap \partial U_m$ is a translation of Γ_1^j.

In each domain U_m, we seek a solution (u^m, p^m) of the steady Navier-Stokes equations (1.1) to (1.3) with $u^m = g^m$ on ∂U_m. Since U_m is bounded, this is a problem of Type 1 and the following result is known (see Finn [3]).

THEOREM 3.5. *For every $\nu > 0$, there exists a solution*

$$(u^m, p^m) \in C^\infty(\overline{U}_m \to \boldsymbol{R}^N) \times C^\infty(\overline{U}_m \to \boldsymbol{R})$$

of the steady Navier-Stokes equations (1.1) to (1.3) *such that $u^m = g^m = q$ on ∂U_m.*

Define $w^m = u^m - q$. Then $w^m = 0$ on ∂U_m and $\operatorname{div} w^m = 0$ in U^m, and it follows by (2.5)(e) that $w^m \in H(U_m \to \boldsymbol{R}^N) \subset H(\Omega \to \boldsymbol{R}^N)$ (set $w^m = 0$ out-

side U_m) and satisfies (cf. (3.11))

$$\nu\langle\varphi, w^m\rangle_H + \{\varphi, q + w^m, w^m\} + \{\varphi, w^m, q\} = \langle\varphi, r\rangle_H \qquad \forall\varphi \in H(U_m)\,, \tag{3.18}$$

where $r \in H(\Omega)$ is as in Lemma 3.4.

If we can show that the sequence $\{w^m\}$ is bounded in $H(\Omega)$ independently of m, then, since H is a Hilbert space, a suitable subsequence of $\{w^m\}$ will converge weakly in H to an element $w \in H$. We shall then prove that this w satisfies (3.13), so that $u = q + w$ is the weak solution of our problem.

Remarks on the velocity fields q and g^m.

(a) A function g closely related to g^m is needed in Theorem 3.6 for the crucial estimate (3.25). In addition, certain properties of the function g^m and q show why the methods for problems of Type 1 and 2 fail here.

(b) The proof of Theorem 3.5 depends on a representation $u^m = g^m + v^m$, $v^m \in H(U_m)$, and on Theorem 2.7, i.e.,

$$|\{g^m(\cdot\,;\varepsilon), \varphi, \varphi\}| \leqslant \text{const}\,\varepsilon|\varphi|^2_{H(U_m)} \qquad \forall\varphi \in H(U_m)\,, \tag{3.19(a)}$$

where $\varepsilon > 0$ is sufficiently small and the constant is independent of ε and m. For a problem of Type 1 or 2, it is the arbitrary small parameter ε in (3.19)(a) which is essential in the proof of the existence of a solution (u^m, p^m) for all $\nu > 0$ (see [3], pp. 131-132).

(c) It follows that $v^m \in H(U_m) \subset H(\Omega)$ (set $v^m = 0$ outside U_m) satisfies (cf. (3.11))

$$\begin{aligned}\nu\langle\varphi, v^m\rangle_H + \{\varphi, g^m + v^m, v^m\} + \{\varphi, v^m, g^m\} &= \\ = -\nu\int_{U_m}(\nabla\varphi : \nabla g^m) - \{\varphi, g^m, g^m\} &\qquad \forall\varphi \in H(U_m)\,.\end{aligned} \tag{3.19(b)}$$

However, the usual procedure for bounding $\{v^m\}$ fails here. For, setting $\varphi = v^m$ in (3.19)(b) gives, in view of (3.12) and (3.19)(a),

$$\begin{aligned}\nu|v^m|^2_H &= \{g^m, v^m, v^m\} - \nu\int_{U_m}(\nabla v^m : \nabla g^m) - \{v^m, g^m, g^m\} \\ &\leqslant \text{const}\,\varepsilon|v^m|^2_H - \nu\int_{U_m}(\nabla v^m : \nabla g^m) - \{v^m, g^m, g^m\}\,,\end{aligned}$$

where the constant is independent of ε and m, but the non-compactness of $\partial\Omega$ makes it appear impossible to bound the last two terms independently of m.

(*d*) The function q satisfying (3.1) is inferior to g^m in that it has only a feebler analogue of (3.19)(*a*) which gives

$$|\{q, \varphi, \varphi\}| \leqslant c|\varphi|_H^2 \qquad \forall \varphi \in H(U_m)\,,$$

where the constant c is independent of m. On the other hand, Lemma 3.4 shows q is superior to g^m in that the term $-\nu\int_{U_m}(\nabla\varphi : \nabla q) - \{\varphi, q, q\}$ can be bounded by const $|\varphi|_H$, and the constant is independent of m. Moreover, the condition $u \to q$ in (3.10) forces us to consider $u - q$ and motivates Definition 3.1.

THEOREM 3.6. *Let S_j be the cylinder $\boldsymbol{R} \times A_j$ of which Ω_j forms a part ($j = 1$ or 3). Let $M > 0$ and define*

$$\sigma_j = \sigma_j(S_j;\, M) = \sup_{\varphi \in H(S_j)} \frac{\{q^j, \varphi, \varphi\}_{S_j}}{|\varphi|^2_{H(S_j)}} \qquad (|\varphi|_{H(S_j)} \neq 0) \tag{3.20}$$

where

$$\{q^j, \varphi, \varphi\}_{S_j} = \int_{S_j} q^j \cdot (\varphi \cdot \nabla)\varphi\,,$$

and q^j is the Poiseuille velocity field in S_j carrying flux $M > 0$. Let $\sigma = \sigma(S_1, S_2;\, M) = \max(\sigma_1, \sigma_3)$.

If $\nu > \sigma$ and w^m satisfies (3.18), *then w^m is bounded in H independently of m.*

REMARKS. We emphasize that the constant σ in Theorem 3.6 is *independent* of Ω_2 and is determined only by the cross-section A_j of Ω_j. The constants σ_j are familiar in the nonlinear theory of hydrodynamic stability. If $N = 2$, then a calculation shows that $\sigma_1 = \sigma_3$.

PROOF OF THEOREM 3.6. We shall give the proof for the case $N = 3$ since that for $N = 2$ is analogous.

The choice $\varphi = w^m$ in (3.18) gives, in view of (3.12)(*b*),

$$\nu|w^m|_H^2 = \{q, w^m, w^m\} + \langle w^m, r\rangle_H\,. \tag{3.21}$$

Although $q \notin H(U_m \to \boldsymbol{R}^N)$ (because elements in H carry no flux by (2.5)(*d*) while q carries flux M), we shall construct a function $s \in H(U_m)$ for m sufficiently large such that $\{s, w^m, w^m\}$ is an approximation to $\{q, w^m, w^m\}$ in a certain sense. This is important since the choice $\varphi = s$ in (3.18) shows that $\{s, w^m, w^m\}$ can increase only *linearly* with $|w^m|_H$.

Let ε be as in the definition (3.17) of g^m. Let $\alpha(x) = \operatorname{dist}(x, \partial\Omega)$ and let

$$g(x;\varepsilon) = \nabla \times G(x;\varepsilon) \qquad \text{where } G(x;\varepsilon) = \mu(\alpha(x);\varepsilon)Q(x).$$

Now let $\delta > 0$, assume that m is sufficiently large and define the function s by [1]

$$(3.22) \qquad s(x;\varepsilon,\delta) = \begin{cases} q(x) - g(x) & \text{in } \Omega_2, \\ \nabla \times [\{Q(0, x_2, x_3) - G(0, x_2, x_3)\}\{1 - \theta(\delta x_1)^2\}] & \text{in } \Omega_1, \\ \nabla \times [\{Q(0, x_2', x_3') - G(0, x_2', x_3')\}\{1 - \theta(-\delta x_1')^2\}] & \text{in } \Omega_3, \end{cases}$$

where $\theta \in C^\infty(\boldsymbol{R} \to [0, 1])$ is a mollifier such that $\theta(t) = 1$ for $t \leqslant -1$ and $\theta(t) = 0$ for $t \geqslant 0$. Thus $s = 0$ for $x_1 \leqslant -1/\delta$ in Ω_1 and $x_1' \geqslant 1/\delta$ in Ω_3; in due course we shall choose ε and δ to be small positive constants independent of m. Therefore, we can take m so large that $\operatorname{supp} s \subset U_m$ and then $s \in J(U_m) \subset H(U_m)$.

(i) The contribution of Ω_2 to the difference of the triple products is

$$\begin{aligned} \{q, w^m, w^m\}_{\Omega_2} - \{s, w^m, w^m\}_{\Omega_2} &= \{g, w^m, w^m\}_{\Omega_2} \\ &\leqslant \int_{\Omega_2} |g \cdot (w^m \cdot \nabla) w^m| \\ &\leqslant \int_{\Omega_\varepsilon} |g \cdot (w^m \cdot \nabla) w^m| \end{aligned}$$

because $\operatorname{supp} g \cap \Omega_2 \subset \Omega_\varepsilon$. Since $Q \in C^\infty(\bar{\Omega} \to \boldsymbol{R}^N)$ and $\nabla \times Q = 0$ on $\partial\Omega$, we have, by Theorem 2.7,

$$(3.23) \qquad \{q, w^m, w^m\}_{\Omega_2} - \{s, w^m, w^m\}_{\Omega_2} \leqslant \text{const}\, \varepsilon \|w^m\|^2_{\Omega_\varepsilon} \leqslant \text{const}\, \varepsilon |w^m|^2_H$$

and the constant is independent of ε and m.

(ii) Let $\theta_\delta(x)$ stand for $\theta(\delta x_1)$ in Ω_1 and for $\theta(-\delta x_1')$ in Ω_3. It follows from (3.22) that the components of q and s are related in Ω_1 by

$$q_1 - s_1 = q_1\theta_\delta^2 + g_1(1 - \theta_\delta^2); \qquad q_2 = q_3 = 0; \qquad |s_2|, |s_3| \leqslant \text{const}\, \delta,$$

[1] In (3.22) and throughout this proof, $\theta(\cdot)^2 = \{\theta(\cdot)\}^2$, the superscript denoting a square.

so that

$$\{q, w^m, w^m\}_{\Omega_1} - \{s, w^m, w^m\}_{\Omega_1} - \{q\theta_\delta^2, w^m, w^m\}_{\Omega_1} =$$
$$= \int_{\Omega_1} \{(1-\theta_\delta^2)(g_1, 0, 0)\cdot(w^m\cdot\nabla)w^m\} - \int_{\Omega_1} \{(0, s_2, s_3)\cdot(w^m\cdot\nabla)w^m\}\,.$$

Estimating the first integral on the right as in (3.23) and the second by means of the Schwarz inequality and Lemma 2.1(c), we obtain

$$\{q, w^m, w^m\}_{\Omega_1} - \{s, w^m, w^m\}_{\Omega_1} - \{q\theta_\delta^2, w^m, w^m\}_{\Omega_1} \leqslant \text{const}\,(\varepsilon+\delta)|w^m|_H^2, \tag{3.24}$$

and the constant is independent of ε, δ, and m. A similar result holds in Ω_3:

(iii) From (3.23) and (3.24), we have the estimate

$$\{q, w^m, w^m\} \leqslant \{s, w^m, w^m\} + \{q\theta_\delta^2, w^m, w^m\}_{\Omega_1\cup\Omega_3} + \text{const}\,(\varepsilon+\delta)|w^m|_H^2\,. \tag{3.25}$$

The choice $\varphi = s$ in (3.18) yields, since s has compact support,

$$\begin{aligned}\{s, w^m, w^m\} &= -\nu\langle s, w^m\rangle_H - \{s, q, w^m\} - \{s, w^m, q\} + \langle s, r\rangle_H \\ &\leqslant k_0(\varepsilon, \delta) + k_1(\varepsilon, \delta)|w^m|_H\end{aligned} \tag{3.26}$$

for certain (large) functions k_0 and k_1 independent of w^m and m. Use of (3.25) and of (3.26) in (3.21) gives

$$\begin{aligned}\nu|w^m|_H^2 &\leqslant \{s, w^m, w^m\} + \{q\theta_\delta^2, w^m, w^m\}_{\Omega_1\cup\Omega_3} + \text{const}\,(\varepsilon+\delta)|w^m|_H^2 + |r|_H|w^m|_H \\ &\leqslant \{q\theta_\delta^2, w^m, w^m\}_{\Omega_1\cup\Omega_3} + k_0(\varepsilon, \delta) + k_1(\varepsilon, \delta)|w^m|_H + C(\varepsilon+\delta)|w^m|_H^2 + |r|_H|w^m|_H;\end{aligned} \tag{3.27}$$

where the constant C is independent of ε, δ, w^m, and m.

Define

$$\Gamma(\delta) = \sup_{v\in H(\Omega)} \frac{\{q\theta_\delta^2, v, v\}_{\Omega_1\cup\Omega_3}}{|v|_{H(\Omega)}^2} \qquad (|v|_{H(\Omega)} \neq 0); \tag{3.28}$$

we refer to Theorem 4.3 for the proof that $\lim_{\delta\to 0^+} \Gamma(\delta) = \sigma = \sigma(S_1, S_3; M)$ (where σ is described in the statement of the present theorem).

Now let $\nu > \sigma$ and choose ε and δ so small that $C(\varepsilon+\delta) \leqslant \frac{1}{4}(\nu-\sigma)$ and $\Gamma(\delta) - \sigma \leqslant \frac{1}{4}(\nu-\sigma)$, then from (3.27) we obtain

$$\tfrac{1}{2}(\nu-\sigma)|w^m|_H^2 \leqslant k_0(\varepsilon, \delta) + \{k_1(\varepsilon, \delta) + |r|_H\}|w^m|_H\,,$$

and so w^m is bounded in H independently of m. The theorem is proved.

The following standard lemma ([9], pp. 84-85) is needed in Theorem 3.8 to ensure that if w is the weak limit in H of a certain subsequence of $\{w^m\}$, then w satisfies (3.13).

LEMMA 3.7. *Let $U \subset \boldsymbol{R}^N$ be a bounded domain with ∂U of class C^1, then the imbedding $W_2^1(U \to \boldsymbol{R}^N) \hookrightarrow L_s(U \to \boldsymbol{R}^N)$ is compact*

$$\text{for } 1 \leqslant s < \infty, \quad \text{if } N = 2, \text{ and}$$
$$\text{for } 1 \leqslant s < 6, \quad \text{if } N = 3.$$

THEOREM 3.8. *If $\nu > \sigma$ (σ as in the statement of Theorem 3.6), then there exists a weak solution $u = q + w$ of the problem (1.7) to (1.10), i.e., there exists $w \in H(\Omega \to \boldsymbol{R}^N)$ satisfying*

$$\nu\langle \varphi, w\rangle_H + \{\varphi, q + w, w\} + \{\varphi, w, q\} = \langle \varphi, r\rangle_H \qquad \forall \varphi \in H(\Omega), \tag{3.13}$$

where q satisfies (3.1) and $r \in H(\Omega)$ is as in Lemma 3.4.

PROOF. Since the sequence $\{w^m\}$ is bounded in the Hilbert space $H(\Omega)$, it contains a weakly convergent subsequence; say

$$w^{m_i} \to w \quad \text{weakly in } H(\Omega) \text{ as } i \to \infty.$$

It suffices to prove that w satisfies (3.13) for all $\varphi \in J$ since J is dense in H.

Let $\varphi \in J$ and choose k such that $\operatorname{supp} \varphi \subset\subset U_k$. Then for all $m_i \geqslant k$, $\varphi \in H(U_{m_i})$ and w^{m_i} satisfies (3.18), i.e.

$$\nu\langle \varphi, w^{m_i}\rangle_H + \{\varphi, q + w^{m_i}, w^{m_i}\} + \{\varphi, w^{m_i}, q\} = \langle \varphi, r\rangle_H. \tag{3.18}$$

Since $w^{m_i} \to w$ weakly in H and $H \hookrightarrow L_2$,

$$\nu\langle \varphi, w^{m_i}\rangle_H + \{\varphi, q, w^{m_i}\} + \{\varphi, w^{m_i}, q\} \to \nu\langle \varphi, w\rangle_H + \{\varphi, q, w\} + \{\varphi, w, q\}$$

as $i \to \infty$ because, for fixed $\varphi \in H(\Omega)$, the terms on the left-hand side are bounded linear functionals on $H(\Omega)$ with argument w^{m_i}.

Now consider $\{\varphi, w^{m_i}, w^{m_i}\}$. By Lemma 3.7, $W_2^1(U_k)$ is imbedded compactly in $L_4(U_k)$, so that w^{m_i} converges strongly in $L_4(U_k)$ and, by use of (3.12)(a), we have

$$\begin{aligned} |\{\varphi, w^{m_i}, w^{m_i}\} - \{\varphi, w, w\}| &= |\{\varphi, w^{m_i} - w, w^{m_i}\} - \{w^{m_i} - w, w, \varphi\}| \\ &\leqslant \text{const}\, |w^{m_i} - w|_{L_4(U_k)} \to 0 \quad \text{as } i \to \infty \end{aligned}$$

because of the strong convergence in $L_4(U_k)$.

REMARKS

(*a*) The constants σ_j ($j = 1$ or 3) in Theorem 3.6 occur in the nonlinear theory of hydrodynamic stability. Let $S_j = \boldsymbol{R} \times A_j$ be an admissible cylindrical domain. If $\nu > \sigma_j$, then the Poiseuille velocity field q^j is the unique solution of the steady Navier-Stokes equations for the domain S_j, among all functions u such that $u - q^j \in H(S_j)$. If $\nu > \sigma_j$ and $v(x, t)$ is a solution of the time-dependent Navier-Stokes equations in S_j, with initial velocity $v(x, 0)$ such that $v(x, 0) - q^j(x) \in H(S_j)$, then

$$\lim_{t \to \infty} |v(\cdot, t) - q^j|_H = 0$$

(Serrin [17]), provided that v exists for all time.

(*b*) The restrictions on Ω can be relaxed to allow a finite number of smooth bounded bodies V_i to be in the interior of Ω. In this case, the coordinate systems are chosen so that $V_i \cap \Omega_j = \emptyset$ for $j=1$ and 3 and $i = 1, 2, \ldots$. Lemma 3.2 allows the construction in Theorem 3.3 of a velocity field q satisfying (3.1). The existence theorems 3.6 and 3.8 also hold after slight modification in their proofs.

(*c*) Instead of an admissible domain Ω with one cylindrical domain Ω_1 « upstream » and one Ω_3 « downstream », we could have j disjoint cylinders $\{U_i\}_{i=1}^{j}$ upstream, where $U_i = \boldsymbol{R} \times A_i$ in some coordinate system, and k disjoint cylinders $\{U_i\}_{i=j+1}^{j+k}$ downstream. As usual, we assume that $\Omega \setminus \left(\bigcup_{i=1}^{j+k} U_i\right)$ is bounded. If q^i is the Poiseuille velocity field in U_i carrying flux M_i and $\sum_{i=1}^{j} M_i = \sum_{i=j+1}^{j+k} M_i$, then we can seek a solution of (1.7) to (1.9) with $u(x) \to q^i(x)$ as $|x| \to \infty$ in U_i. A slight modification in the proof of the theorems of this section gives the existence of a weak solution u for all $\nu > \max(\sigma_1, \ldots, \sigma_{j+k})$ where σ_l is defined as in the statement of Theorem 3.6.

(*d*) The assumption, in the Introduction, that the external force f be derivable from a scalar potential P (that is, $f = -\nabla P$) can be relaxed. It suffices for $\int_\Omega (\varphi \cdot f)$ to define a bounded linear functional on $H(\Omega)$; in this case, there exists by the Riesz representation theorem a unique element $\tilde{f} \in H(\Omega)$ such that

$$\int_\Omega (\varphi \cdot f) = \langle \varphi, \tilde{f} \rangle_H \qquad \forall \varphi \in H(\Omega),$$

and $\tilde{f}$ may then be absorbed into the term r of Lemma 3.4.

3.3. *Existence of a weak solution for a symmetric admissible domain.*

If the admissible domain Ω is symmetric about Ox_1, then the results of Theorem 3.8 can be improved.

We take cylindrical coordinates (x_1, r, θ) in Ω (the θ coordinate is omitted for $N=2$). It is possible to show the existence of $Q_s = Q_s(x_1, r)$ such that $q_s = \nabla \times Q_s$ satisfies (3.1) and is symmetric about Ox_1. Let $H_s(\Omega \to \boldsymbol{R}^N)$ denote the completion of $J_s = \{\varphi \in J\colon \varphi$ is symmetric about $Ox_1\}$ in the Dirichlet norm.

DEFINITION 3.2. *Let Ω be a symmetric admissible domain and let $q_s = q_s(x_1, r)$ satisfy* (3.1). *The function $u_s = q_s + w_s$ is a symmetric weak solution of the problem* (1.7) *to* (1.10) *if $w_s \in H_s(\Omega \to \boldsymbol{R}^N)$ ($N = 2$ or 3) and*

$$\nu\langle\varphi, w_s\rangle_H + \{\varphi, q_s + w_s, w_s\} + \{\varphi, w_s, q_s\} = \langle\varphi, r_s\rangle_H \qquad \forall\varphi \in H_s(\Omega)\,, \tag{3.29}$$

where $r_s \in H_s(\Omega)$ is as in Lemma 3.4.

A simple argument shows that if u_s is a symmetric weak solution of the problem, then it is also a weak solution in the sense of Definition 3.1.

COROLLARY 3.9. *Let $\Omega \subset \boldsymbol{R}^N$ be a symmetric admissible domain and let S_j be as in the statement of Theorem* 3.6. *Define*

$$\sigma_{j,s} = \sup_{\varphi \in H_s(S_j)} \frac{\{q_s^j, \varphi, \varphi\}_{S_j}}{|\varphi|^2_{H(S_j)}} \quad \left(|\varphi|_{H(S_j)} \neq 0\right),$$

where q_s^j is the symmetric Poiseuille velocity field in S_j carrying flux $\boldsymbol{M} > 0$. Let $\sigma_s = \max(\sigma_{1,s}, \sigma_{3,s})$. If $\nu > \sigma_s$, then there exists a symmetric weak solution of the problem (1.7) *to* (1.10).

REMARKS. We note that the cross-sections of S_j for $N=3$ are circular, and a simple calculation shows that $\sigma_{1,s} = \sigma_{3,s}$ for $N=2$.

PROOF OF COROLLARY 3.9. The corollary follows easily by certain observations.

(i) Lemma 3.4 holds with u replaced by u_s, q by q_s, w by w_s, H by H_s, and $r \in H$ by $r_s \in H_s$.

(ii) If the increasing bounded domains U_m are chosen to be symmetric about Ox_1, then the functions u^m in Theorem 3.5 can be chosen to be symmetric about Ox_1.

(iii) The functions $w^m = u^w - q_s$ are symmetric and the proof of Theorem 3.6 holds with the obvious change of notation.

(iv) As in Theorem 3.6, we leave to Corollary 4.4 the proof that $\lim_{\delta\to 0^+} \Gamma_s(\delta) = \sigma_s$ where

$$\Gamma_s(\delta) = \sup_{v\in H_s(\Omega)} \frac{\{q_s\theta_\delta^2, v, v\}_{\Omega_1\cup\Omega_3}}{|v|^2_{H(\Omega)}} \qquad (|v|_{(H\Omega)} \neq 0)\,. \tag{3.30}$$

(v) Theorem 3.8 holds with the obvious change of notation.

3.4. *The critical Reynolds number.*

DEFINITION 3.3. *Let Ω be an admissible domain in $\boldsymbol{R}^N$ ($N=2$ or 3) and let M, $\nu > 0$. We define the Reynolds number $R = R(\Omega, M, \nu)$ by*

$$R = \begin{cases} \dfrac{M}{\nu} & \text{for } N = 2\,, \quad \text{and} \\[2ex] \dfrac{M\pi^{\frac{1}{2}}}{\nu}\left(|A_1|^{-\frac{1}{2}}, |A_3|^{-\frac{1}{2}}\right) = (R_1, R_3) & \text{for } N = 3\,, \end{cases}$$

where $|A_j|$ is the two-dimensional measure of the cross-section A_j of S_j. For $N=3$, we write $R' < R$ if and only if $R_1' < R_1$ and $R_3' < R_3$.

The following theorem follows immediately from Theorems 3.6 and 3.8.

THEOREM 3.10. *Let $\Omega \subset \boldsymbol{R}^N$ be an admissible domain and let $M > 0$. Let σ_1, σ_3 and σ be as in Theorem 3.6 and define*

$$R_0 = \begin{cases} \dfrac{M}{\sigma} = \dfrac{M}{\sigma_1} = \dfrac{M}{\sigma_3} & \text{for } N = 2\,, \quad \text{and} \\[2ex] M\pi^{\frac{1}{2}}\left(\dfrac{|A_1|^{-\frac{1}{2}}}{\sigma_1}, \dfrac{|A_3|^{-\frac{1}{2}}}{\sigma_3}\right) & \text{for } N = 3\,. \end{cases}$$

If $R < R_0$, then there exists a weak solution u of the problem (1.7) *to* (1.10).

In the case that Ω is a symmetric admissible domain, a symmetric weak solution exists for all $R < R_{0,s}$ where $R_{0,s}$ is as in Theorem 3.10 with the replacement of σ, σ_1, and σ_3 by σ_s, $\sigma_{1,s}$, and $\sigma_{3,s}$, respectively.

We now give some numerical values of R_0 and $R_{0,s}$ for various types of admissible domains; these values follow immediately from the calculation of σ_j which will be discussed in a forthcoming paper. The values of R_0 and $R_{0,s}$ are known for the case $N=2$ and for certain simple geometries of Ω for $N=3$.

The cross-sections A_i of Ω_i ($i=1$ or 3) are not necessarily equal in any of the following domains.

(*a*) $N=2$ and Ω is an admissible domain:

$$R_0 = 116.5\,.$$

(*b*) $N=2$ and Ω is a symmetric admissible domain:

$$R_{0,s} = 194.6\,.$$

(*c*) $N=3$ and Ω is an admissible domain such that Ω_1 and Ω_3 have circular cross-sections:

$$R_0 = (127.9,\ 127.9)\,.$$

(*d*) $N=3$ and Ω is a symmetric admissible domain:

$$R_{0,s} = (282.6,\ 282.6)\,.$$

3.5. *Asymptotically cylindrical domains.*

In this section, we prove the existence of a weak solution of (1.7) to (1.10) for a domain $\Omega \subset \boldsymbol{R}^N$ which asymptotically approaches a cylinder S_j as $|x| \to \infty$ in Ω_j ($j=1$ or 3). We shall show that if the asymptotic approach is sufficiently fast, then there exists a weak solution for all $\nu > \sigma(S_1, S_3; M)$, where σ is as in the statement of Theorem 3.6. For simplicity, we shall consider only the case $N=2$, since that for $N=3$ is more tedious although analogous.

We shall be concerned with *asymptotically cylindrical domains* $\Omega \subset \boldsymbol{R}^2$ with $\partial\Omega$ of class C^∞ and of the form $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ where

(1) $\Omega_1 = \{(x_1, x_2) : x_1 < 0,\ x_2 \in (-d_1\varphi_-(x_1),\ d_1\varphi_+(x_1))\} \qquad (d_1 > 0)$,

where $\varphi_-,\ \varphi_+ \in C^\infty((-\infty, 0])$ and $\varphi_+(x_1),\ \varphi_-(x_1) \to 1$ as $x_1 \to -\infty$.

(2) $\Omega_3 = \{(x_1', x_2') : x_1' > 0,\ x_2' \in (-d_3\psi_-(x_1'),\ d_3\psi_+(x_1'))\} \qquad (d_3 > 0)$,

where $\psi_-,\ \psi_+ \in C^\infty([0, \infty))$ and $\psi_+(x_1'),\ \psi_-(x_1') \to 1$ as $x_1' \to \infty$.

(3) $\Omega_2 = \Omega \setminus (\Omega_1 \cup \Omega_3)$ is bounded.

We assume that the boundary functions $(\varphi_-, \varphi_+, \psi_-, \psi_+)$ are positive-valued. The domain Ω_1 approaches the cylinder $S_1 = \boldsymbol{R} \times (-d_1, d_1)$ as $x_1 \to -\infty$ in Ω_1 and similarly Ω_3 approaches $S_3 = \boldsymbol{R} \times (-d_3, d_3)$.

As in section 3.2, we seek a weak solution u of (1.7) to (1.10) in the form $u = q + w$ where q is to satisfy

(3.31) (a) $q \in C^\infty(\bar{\Omega} \to \boldsymbol{R}^2)$;

(b) $\operatorname{div} q = 0$ in Ω, $q = 0$ on $\partial\Omega$; and

(c) $q(x) \to q^j(x)$ uniformly as $|x| \to \infty$ in Ω_j, where q^j is the Poiseuille velocity field in the cylinder S_j carrying flux $M > 0$.

The function $u = q + w$, where $w \in H(\Omega \to \boldsymbol{R}^2)$, is to satisfy (Definition 3.1)

$$\nu\langle\varphi, w\rangle_H + \{\varphi, q + w, w\} + \{\varphi, w, q\} = \\ = -\nu\int_\Omega (\nabla\varphi : \nabla q) - \{\varphi, q, q\} \qquad \forall\varphi \in J(\Omega \to \boldsymbol{R}^2)\,. \tag{3.11}$$

The construction of a function q satisfying (3.31) is analogous to that in section 3.1. We can define Q_p such that $\nabla \times (0, 0, Q_p)$ satisfies (3.31) in $\bar{\Omega}_1 \cup \bar{\Omega}_3$. Let $M > 0$ and define

$$Q_p(x) = \tfrac{3}{4}M[t(x) - \tfrac{1}{3}\{t(x)\}^3] \qquad \text{for } x \in \bar{\Omega}_1 \cup \bar{\Omega}_3\,, \tag{3.32}$$

where

$$t(x) = \begin{cases} \{2x_2/d_1 - (\varphi_+ - \varphi_-)(x_1)\}/(\varphi_+ + \varphi_-)(x_1) & \text{in } \bar{\Omega}_1\,, \\ \{2x_2'/d_3 - (\psi_+ - \psi_-)(x_1')\}/(\psi_+ + \psi_-)(x_1') & \text{in } \bar{\Omega}_3\,. \end{cases}$$

It follows that for $x \in \bar{\Omega}_1$, we have

$$q(x) = \big(D_2 Q_p(x), -D_1 Q_p(x)\big) = \tfrac{3}{4}M[1 - \{t(x)\}^2]\big(D_2 t(x), -D_1 t(x)\big)\,,$$

where $D_2 t = 2/\{d_1(\varphi_+ + \varphi_-)\}$ and

$$D_1 t = \left\{\frac{-2x_2}{d_1} D_1(\varphi_+ + \varphi_-) + 2\varphi_+ D_1\varphi_- - 2\varphi_- D_1\varphi_+\right\}\Big/(\varphi_+ + \varphi_-)^2\,.$$

Since the Poiseuille velocity field q^1 in $S_1 = \boldsymbol{R} \times (-d_1, d_1)$ is given by

$$q^1(x) = \frac{3M}{4d_1}\left\{1 - \left(\frac{x_2}{d_1}\right)^2\right\}(1, 0)\,,$$

it follows that $q(x) \to q^1(x)$ uniformly as $|x| \to \infty$ in Ω_1 if

P(1)(*a*) $$D_1\varphi_+(x_1)\,, \quad D_1\varphi_-(x_1) \to 0 \quad \text{as } x_1 \to -\infty\,.$$

Similarly $q(x) \to q^3(x)$ uniformly as $|x| \to \infty$ in Ω_3 if

P(1)(*b*) $$D_1\psi_+(x_1')\,, \quad D_1\psi_-(x_1') \to 0 \quad \text{as } x_1' \to \infty\,.$$

Using the same arguments as in Theorems 3.1 and 3.3, we have the following result.

THEOREM 3.11. *Let $\Omega \subset \boldsymbol{R}^2$ be an asymptotically cylindrical domain and assume that the boundary functions φ_+, φ_-, ψ_+, ψ_- satisfy* P(1). *Then there exists $Q \in C^\infty(\bar{\Omega} \to \boldsymbol{R})$ such that*

(*a*) $\nabla \times (0, 0, Q) = 0$ *on* $\partial\Omega$, *and*

(*b*) $Q = Q_p$ *in* $\bar{\Omega}_1 \cup \bar{\Omega}_3$ (Q_p *as in* (3.32)).

Thus $q = \nabla \times (0, 0, Q_p)$ *satisfies* (3.31).

We now state conditions which ensure that

$$-\nu\int_\Omega (\nabla\varphi : \nabla q) - \{\varphi, q, q\}$$

defines a continuous linear functional on $J(\Omega)$ which can be extended by continuity to $H(\Omega)$. A calculation shows that this will be true if, in addition to P(1), we have

P(2)(*a*) $$(1-\varphi_+)\,, \quad (1-\varphi_-)\,, \quad D_1^2\varphi_+,\ D_1^2\varphi_- \in L_2(\boldsymbol{R}^-)\,;$$

P(2)(*b*) $$(1-\psi_+)\,, \quad (1-\psi_-)\,, \quad D_1^2\psi_+,\ D_1^2\psi_- \in L_2(\boldsymbol{R}_+)\,.$$

We use this result and follow the arguments of Lemma 3.4 to obtain

LEMMA 3.12. *Let $\Omega \subset \boldsymbol{R}^2$ be an asymptotically cylindrical domain and assume that the boundary functions φ_+, φ_-, ψ_+, ψ_- satisfy* P(1) *and* P(2). *Then there exists an element $r \in H(\Omega)$ such that $u = q + w$ is a weak solution of the problem* (1.7) *to* (1.10) *if and only if*

$$\nu\langle\varphi, w\rangle_H + \{\varphi, q+w, w\} + \{\varphi, w, q\} = \langle\varphi, r\rangle_H \qquad \forall\varphi \in H(\Omega \to \boldsymbol{R}^2)\,.$$

Before proceeding as in section 3.3, we need a preliminary result. For each positive integer n, decompose Ω by $\Omega = \Omega_1^n \cup \Omega_2^n \cup \Omega_3^n$ where

$$\Omega_1^n = \{x \in \Omega_1 : x_1 < -n\},$$
$$\Omega_3^n = \{x \in \Omega_3 : x_1' > n\},$$
$$\Omega_2^n = \Omega \setminus (\Omega_1^n \cup \Omega_3^n).$$

Recall that the mollifier $\theta \in C^\infty(\boldsymbol{R} \to [0, 1])$ used in the proof of Theorem 3.6 satisfies $\theta(t) = 1$ for $t < -1$ and $\theta(t) = 0$ for $t > 0$. For n a positive integer and $\delta > 0$, define

$$\theta_\delta^n(x) = \begin{cases} \theta(\delta(x_1 + n)) & \text{for } x \in \Omega_1^n, \\ \theta(-\delta(x_1' - n)) & \text{for } x \in \Omega_3^n. \end{cases}$$

LEMMA 3.13. *Let $\Omega \subset \boldsymbol{R}^2$ be an asymptotically cylindrical domain and assume that the boundary functions φ_+, φ_-, ψ_+, ψ_- satisfy* P(1). *For $\delta > 0$ and n a positive integer, let*

$$\Gamma_n(\delta) = \sup_{\varphi \in H(\Omega)} \frac{\{q(\theta_\delta^n)^2, \varphi, \varphi\}_{\Omega_1^n \cup \Omega_3^n}}{|\varphi|_{H(\Omega)}^2} \qquad (|\varphi|_{H(\Omega)} \neq 0), \tag{3.33}$$

where q satisfies (3.31). *Let $S_j = \boldsymbol{R} \times (-d_j, d_j)$ ($j = 1$ or 3) be the cylinder which Ω_j approaches as $|x| \to \infty$, and define*

$$\sigma_j = \sigma_j(S_j; M) = \sup_{\varphi \in H(S_j)} \frac{\{q^j, \varphi, \varphi\}_{S_j}}{|\varphi|_{H(S_j)}^2} \qquad (|\varphi|_{H(S_j)} \neq 0),$$

where q^j is the Poiseuille velocity field in S_j carrying flux $M > 0$. Then

$$\lim_{n \to \infty} \left(\lim_{\delta \to 0^+} \Gamma_n(\delta) \right) = \sigma(S_1, S_3; M) = \max(\sigma_1, \sigma_3).$$

PROOF. Let $\{U^n\}$ be a non-decreasing sequence of admissible domains contained in Ω with $\lim U^n = \Omega$ and such that $U^n = U_1^n \cup U_2^n \cup U_3^n$, where

$$U_1^n = (-\infty, n) \times (-d_1^n, d_1^n) \subset \Omega_1^n \qquad (d_1^n > 0),$$
$$U_3^n = (n, \infty) \times (-d_3^n, d_3^n) \subset \Omega_3^n \qquad (d_3^n > 0),$$
$$U_2^n = U^n \setminus (U_1^n \cup U_3^n).$$

Since $\lim U^n = \Omega$, we have necessarily that $\lim d_1^n = d_1$ and $\lim d_3^n = d_3$, where d_1 and d_3 are as in the definition of the asymptotically cylindrical domain Ω.

Let r^n denote the extended Poiseuille velocity field in U^n satisfying (3.1). By Theorem 4.3, we have

$$\lim_{\delta\to 0^+} \sup_{\varphi\in H(U^n)} \frac{\{r^n(\theta_\delta^n)^2, \varphi, \varphi\}_{U_1^n \cup U_3^n}}{|\varphi|^2_{H(U^n)}} = \sigma(\boldsymbol{R}\times(-d_1^n, d_1^n), \boldsymbol{R}\times(-d_3^n, d_3^n); M).$$

By construction $H(U^n)\subset H(\Omega)$ and $U_j^n \subset \Omega_j^n$ ($j=1$ or 3), so that

$$\sup_{\varphi\in H(U^n)} \frac{\{r^n(\theta_\delta^n)^2, \varphi, \varphi\}_{U_1^n \cup U_3^n}}{|\varphi|^2_{H(U^n)}} \leqslant \sup_{\varphi\in H(\Omega)} \frac{\{r^n(\theta_\delta^n)^2, \varphi, \varphi\}_{\Omega_1^n \cup \Omega_3^n}}{|\varphi|^2_{H(\Omega)}}.$$

Letting $\delta\to 0+$, we have

$$\sigma(\boldsymbol{R}\times(-d_1^n, d_1^n), \boldsymbol{R}\times(-d_3^n, d_3^n); M) \leqslant \lim_{\delta\to 0^+}\left(\sup_{\varphi\in H(\Omega)} \frac{\{r^n(\theta_\delta^n)^2, \varphi, \varphi\}_{\Omega_1^n \cup \Omega_3^n}}{|\varphi|^2_{H(\Omega)}}\right).$$

If we use property P(1) of the boundary functions, then

$$\lim_{n\to\infty} \sup_{x\in \Omega_1^n \cup \Omega_3^n} |q(x) - r^n(x)| = 0,$$

and since one can show easily that

$$\lim_{n\to\infty} \sigma(\boldsymbol{R}\times(-d_1^n, d_1^n), \boldsymbol{R}\times(-d_3^n, d_3^n); M) = \sigma(S_1, S_3; M),$$

we have

$$\sigma(S_1, S_3; M) \leqslant \lim_{n\to\infty}\left(\lim_{\delta\to 0^+} \Gamma_n(\delta)\right).$$

A similar argument using a non-increasing sequence of admissible domains $\{V^n\}$ each containing Ω with $\lim V^n = \Omega$ gives

$$\lim_{n\to\infty}\left(\lim_{\delta\to 0^+} \Gamma_n(\delta)\right) \leqslant \sigma(S_1, S_3; M),$$

and the lemma is proved.

Consider a non-decreasing sequence of bounded domains $\{U_m\}$ analogous to that of Figure 4 such that each U_m is contained in Ω and $\lim U_m = \Omega$. We assume that there exists a positive ε_0 independent of m and sufficiently small in the sense of Lemma 2.3 for all U_m. This assumption is valid if we have the following condition on the boundary functions:

P(3) $$\sup_{x_1<0} |D_1^2\varphi_+(x_1)|, \quad |D_1^2\varphi_-(x_1)| < \infty,$$

and

$$\sup_{x_1'>0} |D_1^2\psi_+(x_1')|, \quad |D_1^2\psi_-(x_1')| < \infty.$$

Let $w^m = u^m - q$ where u^m is the solution (cf. Theorem 3.5) of the steady Navier-Stokes equations (1.1) to (1.3) in U_m such that $u^m = q$ on ∂U_m. We have the following version of Theorems 3.6 and 3.8.

THEOREM 3.14. *Let $M > 0$. Let $\Omega \subset \boldsymbol{R}^2$ be an asymptotically cylindrical domain and $S_j = \boldsymbol{R} \times (-d_j, d_j)$ ($j = 1$ or 3) the cylinder which Ω_j approaches as $|x| \to \infty$ in Ω_j. Assume that the boundary functions satisfy* P(1), P(2), *and* P(3). *If $\nu > \sigma(S_1, S_3; M) = \max(\sigma_1(S_1; M), \sigma_3(S_3; M))$ (σ_j as in* (3.20)), *then*

(*a*) *w^m is bounded in $H(\Omega)$ independently of m;*

(*b*) *there exists a weak solution u of the problem* (1.7) *to* (1.10) *of the form $u = q + w$ where q satisfies* (3.31) *and $w \in H(\Omega)$.*

PROOF. (*a*) Let $\nu > \sigma$. By Lemma 3.13, we can choose a positive integer k so that $|\sigma - \lim_{\delta \to 0^+} \Gamma_k(\delta)| \leqslant \frac{1}{4}(\nu - \sigma)$. The proof of Theorem 3.6 holds with minor changes and Ω_1, Ω_2, Ω_3, θ_δ, and $\Gamma(\delta)$ replaced by Ω_1^k, Ω_2^k, Ω_3^k, θ_δ^k, and $\Gamma_k(\delta)$, respectively. The parameter ε must be restricted to the interval $(0, \varepsilon_0]$ by P(3).

(*b*) The proof of (*b*) is identical to that of Theorem 3.8.

4. – Comparison of the suprema σ and $\Gamma(\delta)$.

In this section, we complete the proof of Theorem 3.6 by showing in Theorem 4.3 that $\lim_{\delta \to 0^+} \Gamma(\delta) = \sigma$ where $\Gamma(\delta)$ is defined in (3.28) and σ in (3.20). As a corollary, for symmetrical admissible domains, we complete the proof of Corollary 3.9 by proving that $\Gamma_s(\delta) \to \sigma_s$ as $\delta \to 0+$.

Our plan is to shift the θ_δ from q to v in (3.28); unfortunately, $\theta_\delta v$ is not necessarily solenoidal and so we shall need the following lemma, which shows that $\theta_\delta v$ is close to a solenoidal function in a certain sense.

Throughout this section, Ω will be an admissible domain in $\boldsymbol{R}^N$ ($N = 2$ or 3) unless the contrary is stated.

LEMMA 4.1. *Denote by S_1 the cylinder $\boldsymbol{R} \times A_1$ of which Ω_1 forms a part. Let $E = E(S_1 \to \boldsymbol{R}^N)$ denote as before the completion of $C_0^\infty(S_1 \to \boldsymbol{R}^N)$ in the Dirichlet norm and let $H = H(S_1 \to \boldsymbol{R}^N)$. Given any $v \in H(\Omega \to \boldsymbol{R}^N)$ define $f \in E$ by*

$$f(x) = \theta(\delta x_1) v(x)$$

where the mollifier θ is as in (3.22). *Under the orthogonal decomposition*

$$f = g + h, \quad g \in H^{\perp}, \ h \in H,$$

we have

(*a*) $\operatorname{div} f = \operatorname{div} g \in E(S_1 \to \boldsymbol{R})$,

(*b*) $|g|_E \leqslant C\delta \|v\|_{\Omega_1}$,

where the constant C depends only on the cross-section A_1 of Ω_1.

PROOF. We shall give the proof for the case $N = 3$ since that for $N = 2$ is analogous.

(*a*) Since $\operatorname{div} v = 0$ in Ω (in a generalized sense), we have

$$\operatorname{div} f = \operatorname{div} g = \delta\theta'(\delta x_1) v_1(x) \in E(S_1 \to \boldsymbol{R}) \qquad \left(\theta'(t) = \frac{d\theta(t)}{dt}\right)$$

because $v \in H(\Omega \to \boldsymbol{R}^N)$.

(*b*) It suffices to prove (*b*) for $v \in J(\Omega \to \boldsymbol{R}^N)$, and this will be done in three steps.

(i) Suppose that we can construct $g^0 \in E$ with bounded support such that

$$\text{(4.1)(a)} \qquad \operatorname{div} g^0 = \operatorname{div} f \quad \text{in} \quad S_1,$$

$$\text{(4.1)(b)} \qquad g^0 = 0 \quad \text{on } \partial S_1,$$

and

$$\text{(4.2)} \qquad |g^0|_E \leqslant C\delta \|v\|_{\Omega_1}.$$

Now $g - g^0 = f - h - g^0 \in H$ because $h \in H$, and because $\operatorname{div}(f - g^0) = 0$ in S_1, $(f - g^0) = 0$ on ∂S_1, and $\operatorname{supp}(f - g^0)$ is bounded, so that it follows by (2.5)(*e*) that $(f - g^0) \in H$. Hence $g_0 = g + (g^0 - g)$ with $g \in H^{\perp}$ and $(g^0 - g) \in H$, so that

$$|g^0|_E^2 = |g|_E^2 + |g^0 - g|_E^2 \quad \text{implies } |g|_E \leqslant |g^0|_E.$$

(ii) We seek $g^0 = (0, g_2^0, g_3^0)$ in the form

$$\text{(4.3)} \qquad g^0(x) = \delta\theta'(\delta x_1)\{\nabla'\tau + \nabla' \times (\psi, 0, 0)\}$$

where $\nabla' = (0, D_2, D_3)$. Recall that $\operatorname{supp} \theta'(\delta x_1) \subset [-1/\delta, 0]$.

We choose τ to be the solution of

$$(4.4)(a) \qquad (D_2^2 + D_3^2)\tau = v_1 \quad \text{in } (-1/\delta, 0)\times A_1,$$

$$(4.4)(b) \qquad \tau = 0 \quad \text{on } (-1/\delta, 0)\times \partial A_1,$$

and then construct ψ to make $g^0 = 0$ on ∂S_1; equation (4.4)(a) ensures that (4.1)(a) is satisfied.

Now (4.4) is the familiar Dirichlet problem, and $v_1(x_1, \cdot, \cdot) \in E(A_1 \to \boldsymbol{R})$ for each fixed $x_1 \in (-1/\delta, 0)$. Abbreviating this statement to $v_1 \in E(A_1)$, we know from standard results that $\tau \in W_2^3(A_1)$ and

$$(4.5)(a) \qquad |\tau|_{W_2^3(A_1)} \leqslant \text{const}\, |v_1|_{E(A_1)}.$$

Moreover, (4.4) holds with τ replaced by $D_1\tau$ and v_1 by $D_1 v_1$, and since $D_1 v_1 \in L_2(A_1)$, we have $D_1\tau \in W_2^2(A_1)$ and

$$(4.5)(b) \qquad |D_1\tau|_{W_2^2(A_1)} \leqslant \text{const}\, |D_1 v_1|_{L_2(A_1)}.$$

Integrating these estimates with respect to x_1, we find that

$$(4.6) \qquad |\delta\theta'\nabla'\tau|_E \leqslant \text{const}\, \delta \|v\|_{\Omega_1}.$$

(iii) To construct ψ, we need results concerning the trace of τ and its derivatives on ∂A_1. The trace maps $W_2^3(A_1) \to W_2^2(\partial A_1)$ and $W_2^2(A_1) \to \to W_2^1(\partial A_1)$ are bounded (Treves [11], p. 237), so that

$$(4.7)(a) \qquad |\tau|_{W_2^2(\partial A_1)} \leqslant \text{const}\, |\tau|_{W_2^3(A_1)} \leqslant \text{const}\, |v_1|_{E(A_1)},$$

$$(4.7)(b) \qquad |D_1\tau|_{W_2^1(\partial A_1)} \leqslant \text{const}\, |D_1\tau|_{W_2^2(A_1)} \leqslant \text{const}\, |D_1 v_1|_{L_2(A_1)},$$

and the constant depends only on A_1.

Since ∂A_1 is of class C^∞, we define boundary-layer coordinates (s, t) as in Theorem 3.3. Recall that the map $(x_2, x_3) \mapsto (s, t)$ is one-to-one and C^∞ for sufficiently small positive values of t, say for $0 \leqslant t \leqslant t_0$. In order that g^0 vanish on $(-1/\delta, 0)\times\partial A_1$, we demand in view of (4.3) that

$$(4.8) \qquad \left.\begin{aligned} \tau_s + \psi_t &= 0 \\ \tau_t - \psi_s &= 0 \end{aligned}\right\} \quad \text{for } t = 0,\ -1/\delta < x_1 < 0.$$

Using the fact that $\tau \in W_2^2(\partial A_1)$, define $F(x_1, \cdot) \in W_2^1(\partial A_1)$ by

$$F(x_1, s) = \left.\frac{\partial \tau}{\partial t}\right|_{t=0}.$$

For t_0 as above, let $\mu \in C^\infty(\boldsymbol{R} \to [0, 1])$ be a mollifier with $\mu(t) = 1$ for $t \leqslant t_0/2$ and $\mu(t) = 0$ for $t \geqslant t_0$. Define

$$\psi(x_1, s, t) = \mu(t) \int_0^s F(x_1, s')\, ds' . \tag{4.9}$$

The function ψ is well-defined because, by the zero-flux property of v noted in (2.5)(d),

$$\int_{\partial A_1} \frac{\partial \tau}{\partial t} = -\int_{A_1} (D_2^2 + D_3^2)\, \tau = -\int_{A_1} v_1 = 0 .$$

It remains to verify that

$$|\delta\theta' \nabla' \times (\psi, 0, 0)|_E \leqslant \text{const}\, \delta \|v\|_{\Omega_1} . \tag{4.10}$$

The relevant second derivatives of ψ are given by

$$\psi_{ss} = \tau_{ts}|_{t=0} \mu(t) , \qquad \psi_{st} = \tau_t|_{t=0} \mu'(t) ,$$

$$\psi_{tt} = \mu''(t) \int_0^s F(x_1, s')\, ds' , \quad \psi_{tx_1} = \mu'(t) \int_0^s D_1 F(x_1, s')\, ds' ,$$

$$\psi_{sx_1} = \mu(t)\, \tau_{tx_1}|_{t=0} .$$

By (4.7), we have

$$|\psi|_{W_2^2(A_1)} \leqslant \text{const}\, |\tau|_{W_2^2(\partial A_1)} \leqslant \text{const}\, |v_1|_{E(A_1)} ,$$

$$|D_1 \psi|_{W_2^1(A_1)} \leqslant \text{const}\, |D_1 \tau|_{W_2^1(\partial A_1)} \leqslant \text{const}\, |D_1 v_1|_{L_2(A_1)} ,$$

and the constant depends only on A_1.

Integrating these estimates with respect to x_1, we obtain (4.10) and the lemma is proved.

A corresponding result holds with Ω_1 replaced by Ω_3 in Lemma 4.1.

The lemma has a simple analogue if the admissible domain Ω is symmetrical about Ox_1. Let E_s and H_s denote the axisymmetric functions in $E(S_1 \to \boldsymbol{R}^N)$ and $H(S_1 \to \boldsymbol{R}^N)$, respectively.

COROLLARY 4.2. *Let Ω be axisymmetric and denote by S_1 the cylinder $\boldsymbol{R}\times A_1$ of which Ω_1 forms a part. Given any $v\in H_s(\Omega\to\boldsymbol{R}^N)$, define $f\in E_s$ by*

$$f(x)=\theta(\delta x_1)v(x)\,.$$

Under the orthogonal decomposition

$$f=g+h\,,\qquad g\in H_s^{\perp},\ h\in H_s\,,$$

we have

(*a*) $\operatorname{div} f=\operatorname{div} g\in E_s(S_1\to\boldsymbol{R})$,

(*b*) $|g|_{E_s}\leqslant \text{const}\,\delta\|v\|_{\Omega_1}$.

We are now able to complete the proof of Theorem 3.6.

THEOREM 4.3. *Let*

$$\Gamma(\delta)=\sup_{v\in H(\Omega)}\frac{\{q\theta_\delta^2,\,v,\,v\}_{\Omega_1\cup\Omega_3}}{|v|^2_{H(\Omega)}}\qquad\big(|v|_{H(\Omega)}\neq 0\big)$$

(*as in* (3.28)), *and let* σ_1, σ_3, *and* σ *be as in Theorem* 3.6. (*Recall that* $\theta_\delta^2=(\theta_\delta)^2$.) *Then*

$$\lim_{\delta\to 0^+}\Gamma(\delta)=\sigma\,.$$

PROOF. The theorem will be proved in several steps. We shall consider the case $N=3$ since that for $N=2$ is analogous.

(i) First we show that $\Gamma(\delta)\geqslant\sigma=\max(\sigma_1,\sigma_3)$. Given $\varphi\in J(S_1\to\boldsymbol{R}^N)$, we can form a corresponding function $v_\varphi\in J\big((-\infty,-1/\delta)\times A_1\big)\subset J(\Omega)$ by translation parallel to Ox_1 (thus $v_\varphi(x_1,x_2,x_3)=\varphi(x_1+k,x_2,x_3)$ for some $k\geqslant 0$). Then $q\theta_\delta^2=q^1$ on $\operatorname{supp} v_\varphi$, and

$$\Gamma(\delta)\geqslant\sup_{\varphi\in J(S_1)}\frac{\{q\theta_\delta^2,\,v_\varphi,\,v_\varphi\}_{\Omega_1}}{\|v_\varphi\|^2_{\Omega_1}}=\sup_{\varphi\in J(S_1)}\frac{\{q^1,\,\varphi,\,\varphi\}_{S_1}}{|\varphi|^2_{H(S_1)}}=\sigma_1\,,$$

and similarly $\Gamma(\delta)\geqslant\sigma_3$.

(ii) Next, we show that $\Gamma(\delta)\leqslant\sigma+o(\delta)$. Define for $j=1$ or 3:

$$\Gamma_j(\delta,v)=\frac{\{q\theta_\delta^2,\,v,\,v\}_{\Omega_j}}{\|v\|^2_{\Omega_j}}\qquad\text{and}\qquad\Gamma_j(\delta)=\sup_{v\in H(\Omega)}\Gamma_j(\delta,v)\,.$$

Since (for fixed $v \in J(\Omega)$)

$$\frac{\{q\theta_\delta^2, v, v\}_{\Omega_1 \cup \Omega_3}}{|v|^2_{H(\Omega)}} \leqslant \max\{\Gamma_1(\delta, v), \Gamma_3(\delta, v)\},$$

it suffices to show that

$$\Gamma_j(\delta) \leqslant \sigma_j + o(\delta), \quad j = 1 \text{ or } 3.$$

An integration by parts gives (with superscript 2 denoting a square and not a label)

$$\begin{aligned}(4.11)\qquad \Gamma_1(\delta, v) &= \int_{\Omega_1} \{q_1^1 \theta_\delta^2 (v_1 D_1 v_1 + v_2 D_2 v_1 + v_3 D_3 v_1)\} / \|v\|^2_{\Omega_1} \\ &= -\int_{\Omega_1} \{q_1^1 \theta_\delta \theta_\delta' \delta v_1^2\} / \|v\|^2_{\Omega_1} + \int_{\Omega_1} \{q_1^1 \theta_\delta^2 (v_2 D_2 v_1 + v_3 D_3 v_1\} / \|v\|^2_{\Omega_1} \\ &= o(\delta) + \int_{\Omega_1} \{q_1^1 \theta_\delta^2 (v_2 D_2 v_1 + v_3 D_3 v_1\} / \|v\|^2_{\Omega_1}\end{aligned}$$

since $|v|_{L_3(\Omega_1)} \leqslant \text{const}\, \|v\|_{\Omega_1}$.

(iii) Consider the last integral in (4.11); our plan is to shift the θ_δ^2 from q_1^1 to the v terms, and to show that the projection h of $\theta_\delta v$ into $H(S_1)$ is sufficiently close to $\theta_\delta v$ for our purposes.

Define $f \in E(\Omega_1) \subset E(S_1) = E$ by $f(x) = \theta_\delta(x_1) v(x)$. Then (4.11) becomes

$$\begin{aligned}(4.12)\qquad \Gamma_1(\delta, v) &= o(\delta) + \int_{\Omega_1} q_1^1 (f_2 D_2 f_1 + f_3 D_3 f_1) / \|v\|^2_{\Omega_1} \\ &= o(\delta) + \{q^1, f, f\}_{S_1} / \|v\|^2_{\Omega_1}\end{aligned}$$

because $q_1^1 f_1 D_1 f_1$ integrates to zero.

One can show that

$$\begin{aligned}\|f\|^2_{\Omega_1} &= \int_{\Omega_1} (\theta_\delta^2 |\nabla v|^2) + o(\delta) \|v\|^2_{\Omega_1} \\ &\leqslant \{1 + o(\delta)\} \|v\|^2_{\Omega_1}\end{aligned}$$

since $\theta(t) \in [0, 1]$ for all t.

We write $f = g + h$, where $h \in H(S_1)$ and $g \in H^\perp(S_1)$; by Lemma 4.1, we have

$$|g|_E \leqslant \text{const}\, \delta \|v\|_{\Omega_1}.$$

Moreover,

$$\|h\|_{S_1} = |h|_E \leqslant |f|_E \leqslant \{1 + o(\delta)\} \|v\|_{\Omega_1}.$$

Accordingly, setting $f = g + h$,

$$\begin{aligned}(4.13)\qquad \Gamma_1(\delta, v) &= o(\delta) + [\{q^1, g, g\}_{S_1} + \{q^1, g, h\}_{S_1} + \{q^1, h, g\}_{S_1} + \{q^1, h, h\}_{S_1}] / \|v\|^2_{\Omega_1} \\ &\leqslant o(\delta) + \{q^1, h, h\}_{S_1} / \|v\|^2_{\Omega_1} + \text{const}\,(|g|_E |h|_E + |g|^2_E) / \|v\|^2_{\Omega_1} \\ &\leqslant o(\delta) + \{q^1, h, h\}_{S_1} / \|h\|^2_{S_1}.\end{aligned}$$

By taking the supremum of (4.13) over $v \in J(\Omega)$, it follows that $\Gamma_1(\delta) \leqslant \leqslant \sigma_1 + o(\delta)$ and the theorem is proved.

We have the following corollary for an axisymmetric admissible domain.

COROLLARY 4.4. *For a symmetrical admissible domain Ω, let*

$$\Gamma_s(\delta) = \sup_{v \in H_s(\Omega)} \frac{\{q_s \theta^2_\delta, v, v\}_{\Omega_1 \cup \Omega_3}}{|v|^2_{H_s(\Omega)}} \qquad (|v|_{H_s(\Omega)} \neq 0)$$

(as in (3.30)*), and let $\sigma_{1,s}$, $\sigma_{3,s}$, and σ_s be as in Corollary* 3.9. *Then*

$$\lim_{\delta \to 0^+} \Gamma_s(\delta) = \sigma_s.$$

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