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ISTVÁN FÁRY

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Absolute Subcomplexes.

ISTVÁN FÁRY (*)

dedicated to Hans Lewy

1. – Let X be the space formed by the union of two different circles A, B in the plane which are tangent at the point a (« figure eight curve »). We will say that a is an absolute vertex of X, as it is a vertex in every triangulation of X. Clearly, $\{\emptyset, A, B, A \cap B, A \cup B\}$ is the lattice of all absolute subcomplexes of X. We will refer to this figure eigth curve as Example 1.

EXAMPLE 2. Let X be now the union of two disjoint circles A, B plus the shortest segment $[a, b]$ connecting them ($a \in A$). Then $[a, b]$, A, B generate the lattice of all absolute subcomplexes of X.

By now the reader probably formulated the proper general definition of absolute subcomplexes, and can compare it with our Definition 1 in Section 2. The author believes that this concept is new, accordingly this paper will be an elementary development on basic facts. We will use homological methods to locate absolute subcomplexes in an arbitrary triangulable space. In both examples above it is sufficient to locate the absolute vertices to obtain the lattice of absolute subcomplexes. Our method will locate the open manifolds $A - a$, $B - a$ in Example 1, and $[a, b] - \{a, b\}$, $A - a$, $B - b$ in Example 2.

EXAMPLE 3. Let M be a triangulable space which is a homology manifold (see (47)) and a homology n-sphere, $n \geqslant 3$. We denote C the cone over M with vertex a. Let N be the space obtained by gluing together two copies of C along the boundary ∂C; a', a'' stand for the points of N obtained from $a \in C$.

(*) Department of Mathematics, University of California, Berkeley, California 94720.

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(Briefly stated, N is the suspension of M.) Clearly, N is a homology manifold and a homology $(n+1)$-sphere. We will find with our method that ∂C is an absolute subcomplex of C. We will not find absolute subcomplexes in N with the homological method. However, M may be a non-simply-connected C^0-manifold, and then it seems intuitively evident that $\{\emptyset, a', a'', \{a', a''\}, N\}$ is the lattice of all absolute subcomplexes of N. The homological method is then not adequate to find absolute subcomplexes. In this paper we will not go beyond the elementary homological methods, except for discussing this example, and using in this context the classical « fundamental group at a point ».

Stacks, *critical sets of stacks*, and the *Betti stack* will be defined in Section 11, 12 of this paper. We use these concepts in the theorem and corollary below, as they are quite essential in this context. However, the corollary can be understood without these concepts. In fact, its main statement is that the spaces (2) below can be defined so that each component of $C_i - C_{i+1}$ be a homology manifold (see (47)). We will see that Corollary 1 follows from the first statements of Theorem 1, hence the theorem lies deeper.

That the spaces (1) below are precisely the critical sets of the Betti stack as introduced by the author in [8] is noteworthy. It seems to us that this theorem would be a good enough justification to consider critical sets of the Betti stack, thus critical sets in general. In fact, critical sets of continuous stacks were introduced in [8] in view of critical sets of maps, as absolute subcomplexes were not known then. Let us state now the main result of this paper.

THEOREM 1. *If X is a triangulable space, every critical set B_i of the Betti stack $\mathcal{B}$ of X is an absolute subcomplex of X. The full sequence of critical sets of $\mathcal{B}$ is then*

$$B_0 = X \supset B_1 \supset \ldots \supset B_k \supset B_{k+1} = \emptyset\,, \tag{1}$$

where $\dim B_i > \dim B_{i+1}$, *thus* $k \leqslant \dim X$, *and* $B_i - B_{i+1}$ *is open and everywhere dense in* B_i, $i = 0, \ldots, k$. *Furthermore, each component of* $X - B_1$ *is a homology manifold, if integral coefficients are used for the Betti stack.*

COROLLARY 1. *For a given triangulable space X we define the subspaces C_i by induction: $C_0 = X$; if C_i has been defined, C_{i+1} is the first critical set of the Betti stack of C_i with integral coefficients. Then the spaces*

$$C_0 = X \supset C_1 \supset \ldots \supset C_l \supset C_{l+1} = \emptyset \tag{2}$$

are absolute subcomplexes of X, $\dim C_i > \dim C_{i+1}$, *thus* $l \leqslant \dim X$. $C_i - C_{i+1}$ *is open, everywhere dense in* C_i, *and each component of* $C_i - C_{i+1}$ *is a homology manifold,* $i = 0, \ldots, l$. *Finally* $C_1 = B_1$ (*see* (1)), *but the other* C_i's, $i \geqslant 2$, *may be different from the spaces in* (1).

In Example 1, $B_1 = C_1 = \{a\}$, $B_2 = C_2 = \emptyset$. In Example 2, $B_1 = C_1 = \{a, b\}$, $B_2 = C_2 = \emptyset$. In Example 3 for the cone C we find $B_1 = C_1 = \partial C$, $B_2 = C_2 = \emptyset$. For N we have $B_1 = C_1 = \emptyset$, no matter how M was selected. Thus the points a', a'' do not appear in (1), or (2).

We will formulate some more results after the definitions.

2. – Following [7], p. 60, a triangulation $\{f, (K, L)\}$ of a pair of spaces (X, A) is a simplicial pair (K, L) and a homeomorphism $f\colon (|K|, |L|) \to (X, A)$ of the first pair onto the second. We prefer to use single spaces instead of pairs, if possible; this corresponds to the case $A = \emptyset$. Accordingly, *a* **triangulation**

$$\{f, K, X\} \tag{3}$$

of a space X is a **finite** *simplicial complex* K ([18], p. 108) plus a homeomorphism $f\colon |K| \to X$ of the space $|K|$ of K onto the given space X. If a triangulation (3) exists, the space X is called *triangulable*. Clearly, a triangulable space is separable, metric and compact. Subcomplex ([18], p. 110) means for us « closed » subcomplex; we will not use the expression « open subcomplex ».

To the best of the author's knowledge the following definition is new.

DEFINITION 1. *A subspace* Y *of a triangulable space* X *is called an* **absolute subcomplex** *of* X, *if for each triangulation* $\{f, K, X\}$ *of* X, $f^{-1}Y$ *is the space of a subcomplex of* K.

DISCUSSION. Let be given a triangulation (3) of X. When S ranges through the spaces of the simplices of K, the compact sets $f(S)$ cover X, and acquire a linear structure via f. Hence a triangulation (3) of X can be thought of as a decomposition of X into simplices, or as a « simplicial decomposition of X ». We may say then that *an absolute subcomplex* Y *of* X *is the space of a subcomplex in every simplicial decomposition of* X, whence its name. For example, an absolute vertex of X is a vertex in every simplicial decomposition of X. Intuitively speaking, our definition requires then that whenever X is decomposed into simplices, Y be also *ipso facto* properly decomposed into simplices. Of course, Y may also have simplicial decompositions which are not induced this way. Rigorously speaking, there

may be triangulations

$$\{g, L, Y\} \tag{4}$$

of Y, such that g is not the restriction of an f in (3). Using the terminology of [7] as indicated above, Y is an absolute subcomplex of the triangulable space X, if $\{f, (K, L)\}$ is a triangulation of the pair (X, Y) whenever (3) is a triangulation of X, and L is properly selected. Alternatively, we may say that any triangulation (3) of X restricts to a triangulation (4) of Y.

Let us state some basic and elementary properties of absolute subcomplexes. Most proofs can be omitted, because they are evident.

PROPERTY 1. If Y is an absolute subcomplex of a triangulable space, then Y is a triangulable space.

PROPERTY 2. If Y is an absolute subcomplex of X, and Z is an absolute subcomplex of Y, then Z is an absolute subcomplex of X.

REMARKS. The reader will note the following. If $X \supset Y \supset Z$ and Y, Z are absolute subcomplexes of X, it does not follow that Z is an absolute subcomplex of Y. We can have, in general, a triangulation (4) of Y which is not obtained by restricting a triangulation (3) of X, consequently $g^{-1}Z$ need not be the space of a subcomplex of L. For example, if we set $Y = A$, $Z = \{a\}$ in Example 1, then Y, Z are absolute subcomplexes of X, but Z is not an absolute subcomplex of Y.

PROPERTY 3. If Y is an absolute subcomplex of X, and Z is a component of Y, then Z is an absolute subcomplex of X. Conversely, if all components of Y are absolute subcomplexes of X, then Y is an absolute subcomplex of X.

PROPERTY 4. If X is triangulable, $\emptyset$, X are absolute subcomplexes of X. Absolute subcomplexes form a *lattice* of subspaces under $\subset$, $\cap$, $\cup$ (hence this is a ring of sets). *This lattice is finite.* Connected absolute subcomplexes form a set of generators.

PROPERTY 5. A component of a triangulable space is an absolute subcomplex of that space.

PROPERTY 6. Given a point x of a triangulable space X, there is a unique, *minimal absolute subcomplex* Y of X which contains x.

PROOF. X itself is an absolute subcomplex containing x, thus the family of all absolute subcomplexes containing x is a non-empty, finite family of

sets. The intersection of this family is then the unique, minimal absolute subcomplex of X containing x.

REMARK. For « most points » this subcomplex is just a component of X; the important and useful case is when it is strictly smaller. In Example 1, it is A or B for $x \neq a$, and a for $x = a$. In Example 3, it is N except for $x = a', a''$.

PROPERTY 7. Let X be a triangulable space, and $g: X \to X$, $gX = X$, a homeomorphism. If Y is an absolute subcomplex of X, then gY is an absolute subcomplex of X.

PROOF. We have to prove that if (3) is a triangulation of X, then $f^{-1}(gY)$ is the space of a subcomplex of K. Let us consider the triplet $\{g^{-1}f, K, X\}$, which is clearly a triangulation of X. As Y is an absolute subcomplex of X, $(g^{-1}f)^{-1}Y = f^{-1}gY$ is the space of a subcomplex of K. This completes the proof of the property.

PROPERTY 8. If Y is the minimal absolute subcomplex of X which contains the point x, and $g: X \to X$, $gX = X$ is a homeomorphism such that $gx = x$ (or even $gx \in Y$), then $gY = Y$.

PROOF. This follows from Property 7 and from the uniqueness stated in Property 6.

We denote Homeo(X) the group of all homeomorphisms of X onto X; this is a transformation group of X. We do not introduce a topology on Homeo(X).

PROPERTY 9. Homeo(X) acts as a permutation group on the finite set of absolute subcomplexes of X. This is a group of automorphisms of the finite lattice of all absolute subcomplexes of X.

3. – The three examples discussed in the introduction show that absolute subcomplexes do exist. Still, Definition 1 may seem to be too restrictive, and one may have the feeling that absolute subcomplexes do not exist « in general ». One may argue that any given triangulation of X can be « slightly » modified around a given subspace Y, thereby preventing this subspace from being an absolute subcomplex. Our results will show that the presence of « singularities » in a space make such « slight modifications » impossible, thus this idea is not workable in general. However, the idea of « slight modifications » can be used for manifolds, and we will obtain the following result.

THEOREM 2. *If X is a triangulable space and a connected C^0-manifold without boundary, then X has no absolute subcomplexes besides $\emptyset$ and X.*

In this theorem we wanted to describe the case of spaces which are « locally free of singularities », and choose C^0-manifolds. Once this is accepted as basic hypothesis, all other conditions are plainly necessary. If X is not triangulable, the question of absolute subcomplexes is meaningless. If X is triangulable, not connected, each component is an absolute subcomplex, albeit trivially so. We will see below that the boundary is also an absolute subcomplex.

Applying Theorem 1 and Theorem 2, we get the next result in which only rather trivial subcomplexes will appear.

THEOREM 3. If X is a triangulable space and a C^0-manifold with boundary, the lattice of absolute subcomplexes is generated by the « obvious » elements: components of X and components of the boundary ∂X. In this case $k \leqslant 1$ in (1) and $B_1 = \partial X$, thus the components of B_0 and B_1 (see (1)) generate the lattice of absolute subcomplexes.

Summing up, we may say that for C^0-manifolds the question of absolute subcomplexes is trivial. Example 3 shows that the case of homology manifolds is radically different. In this case the homological method does not give « interesting » absolute subcomplexes: again we find only components of X and components of ∂X, albeit there may be others, as stated in the next result.

THEOREM 4. *Let X be a triangulable space and a connected homology manifold without boundary. If X is not a C^0-manifold, then it contains an absolute subcomplex Y, $0 \leqslant \dim Y \leqslant \dim X - 2$ (thus $Y \neq \emptyset, X$).*

The following result is partly based on homology theory partly on a direct reasoning.

COROLLARY 2. *A triangulable space X has no absolute subcomplexes besides $\emptyset$, X, if and only if it is a connected C^0-manifold without boundary.*

At the beginning of this section we considered the suggestion that, in general, there are no absolute subcomplexes. Contrary to this, we may say now that only manifolds in the strictest sense are free of absolute subcomplexes.

4. – The question of determination of the full lattice of absolute subcomplexes of a space certainly involves homotopy properties, as shown by Example 3, and seems to be difficult. We will treat the more modest question

of existence of absolute subcomplexes, and utilization of classical invariants in the existence proofs.

The simplest classical invariant which can be used successfully is the dimension of X, or rather

$$d(X; x) = \text{dimension of } X \text{ at } x \in X, \tag{5}$$

whose definition will be recalled below.

EXAMPLE 4. Let X be the union of a 2-sphere A and a segment $[a, b]$, $A \cap [a, b] = a$. Then $d(X; x) = 1$, if $x \in [a, b]$, $x \neq a$, and $d(X; x) = 2$, if $x \in A$, including $x = a$. Thus $d(X; x)$ shows that A, $[a, b]$, a are absolute subcomplexes of X.

EXAMPLE 5. Let a be the limit point in R^∞ of a sequence of n-spheres, $n = 1, 2, \ldots$, which are pairwise disjoint and do not contain a. Let X be the union of the spheres and the point a. Then X is separable, metric, compact, but is not triangulable. Now $d(X; a) = 0$, and $d(X; x)$, $x \neq a$, is the dimension of the sphere containing x. The complement of any neighborhood of a is finite dimensional, but $\dim X = \infty$.

For the reader's convenience we quote verbatim the definition of [12] of (5) (one could hardly do better), but, of course, we must refer to [12] for discussion and elementary development. In the following definition « space » means « separable, metric space » (see [12], p. 153).

The empty space and only the empty space has dimension -1.

A space X has dimension $\leqslant n$ $(n \geqslant 0)$ at a point $x \in X$, if x has a fundamental system of *open* neighborhoods whose boundaries have dimensions $\leqslant n - 1$.

X has *dimension* $\leqslant n$, $\dim X \leqslant n$, if X has dimensions $\leqslant n$ at each of its points.

X has dimension n at a point x, denoted $d(X; x) = n$, if it is true that X has dimension $\leqslant n$ at x and it is false that X has dimension $\leqslant n - 1$ at x.

X has *dimension* n if $\dim X \leqslant n$ is true and $\dim X \leqslant n - 1$ is false.

X has dimension ∞ if $\dim X \leqslant n$ is false for each n.

REMARKS. If $\dim X$ is defined in a different way, for example with coverings (see [18], p. 152), or if it is defined inductively without involving (5) explicitly, then this function can be introduced as follows:

$d(X; x) = n$, *if every neighborhood of x contains an* **open** *neighborhood of x whose boundary is of dimension $\leqslant n - 1$, and the boundary of every sufficiently small open neighborhood of x is of dimension $\geqslant n - 1$.*

Using the limit superior, lim sup, of a sequence of numbers, and any definition of dimension, we have then

$$d(X; x) = \limsup_{n\to\infty} \{\min\{1 + \dim(\overline{U} - U) \colon U \subset V_n\}\} \tag{6}$$

where V_n is a fundamental sequence of neighborhoods of x, and U stands for open neighborhoods of x.

Dimension theory can be divided into two parts: the elementary part *not* using homology theory, and the more advanced part *based* on homology theory. That $\dim S^n \geqslant n$ belongs to the more advanced part structurally (whereas $\dim S^n \leqslant n$ is trivial, by induction), but in [12] this question is treated with a minimum of machinery, and we will not need more advanced results in this section.

LEMMA 1. Let K be a simplicial complex, and $x \in |K|$. Then $d(|K|; x) = p$, if and only if x belongs to the space of a p-simplex of K, but does not belong to the space of a q-simplex, $q > p$.

TERMINOLOGY AND NOTATIONS. To simplify the style we will say in the future that « *S is a simplex of K* » *if S is the* **space** *of a simplex of K*; thus S is a compact subset of $|K|$. This cannot lead to confusions as the vertices are well determined by the linear structure of $|K|$. In this terminology: $d(|K|; x) = p$ means that x belongs to a p-simplex of K, but does not belong to a q-simplex, $q > p$. Or even: $d(|K|; x)$ is the maximum of the dimensions of the simplices of K which contain x.

PROOF. Let us suppose that x belongs to a p-simplex S of K, but does not belong to a q-simplex, $q > p$. Clearly, x has then a fundamental system of polyhedral neighborhoods whose boundaries are of dimension $p-1$, consequently $d(|K|; x) \leqslant p$. Such neighborhoods can be obtained by taking the interior in $|K|$ of the star of x, $|\mathrm{St}(x)|$, and sets $\lambda\,|\mathrm{St}(x)|$, $0 < \lambda < 1$; this will be discussed later. Alternatively one may take the stars of x in subdivisions of K. To prove the opposite inequality, let us denote S a p-simplex of K containing x; the set $\mathrm{op}(S)$ is defined in (20) below. Any open neighborhood U of x intersects $\mathrm{op}(S)$ in an open set. Now the boundary of U in $|K|$ contains the boundary of $U \cap \mathrm{op}(S)$ in $\mathrm{op}(S)$ which is of dimension $\geqslant p-1$ by Corollary 2 on p. 46 of [12], provided that the complement is not everywhere dense in $\mathrm{op}(S)$. This will be the case for all sufficiently small neighborhoods U of x. This shows $d(|K|; x) \geqslant p$, thus the proof is complete.

THEOREM 5. If X is a triangulable space, and we set $D_p = \{x \in X \colon d(X; x) \geqslant p\}$ then D_p is an absolute subcomplex of X for all $p \geqslant 0$.

REMARKS. The subspace D_p is of course defined for any separable metric space X, but Example 5 shows that it need not be closed in X. For a triangulable space it is an absolute subcomplex thus a compact subspace.

PROOF. Let (3) be a given triangulation of X, and let us identify $|K|$ to X via f in the course of the following reasoning. Let $x \in D_p$ be given, and $q = d(X; x)$, thus $q \geqslant p$. Then there is a q-simplex S of K which contains x. By Lemma 1, $S \subset D_p$, thus D_p contains a simplex of K which contains x. This proves that D_p is the space of a subcomplex of K. As the triangulation (3) was arbitrary, this concludes the proof of the theorem.

COROLLARY 3. If X is a triangulable space, and $d(X; x)$ is not the same number for every $x \in X$, then X has an absolute subcomplex Y, $Y \neq \emptyset$, $Y \neq X$.

PROOF. If $d(X; x_1) = p < d(X; x_2) = q$, then $D_p \supset D_q$, $D_p \neq D_q$, and $D_q \neq \emptyset$, thus $Y = D_q$ has the properties stated.

5. – Clearly Corollary 2 is stronger than Corollary 3 but it is reasonable to try to obtain results on absolute subcomplexes with simple means. In this spirit, we state some more existence theorems based exclusively on properties of $d(X; x)$. The proofs are nearly as simple as the ones above, but we delay them to a later section, as more remarks on the geometry of simplicial complexes will be needed.

A particular consequence of Lemma 1 is that $d(S; x) = p$ for every point x of a p-simplex S. Before stating the next definition and result it is useful to remark the following (see Example 5 in this context).

LEMMA 2. If X is a triangulable space, any $x \in X$ has a neighborhood U in X, such that

$$d(X; y) \leqslant d(X; x), \quad \text{if } y \in U. \tag{7}$$

Consequently, either $d(X; y)$ is locally constant near x, thus continuous at x, or else, in every neighborhood of x there is a z such that

$$d(X; z) < d(X; x) \tag{8}$$

holds true.

DEFINITION 2. Let X be a separable, metric space, and $d(X; x)$ the dimension of X at the point $x \in X$. We introduce

$$E(X) = \{y \in X : d(X; x) \textit{ is discontinuous at } y\}. \tag{9}$$

We set $E_0(X) = X$, $E_1(X) = E(X)$, and $E_{m+1}(X) = E(E_m(X))$ for $m \geqslant 1$.

THEOREM 6. If X is a triangulable space, then

$$E_0(X) = X \supset E_1(X) \supset \ldots \supset E_\lambda(X) \supset E_{\lambda+1}(X) = \emptyset \tag{10}$$

is a sequence of absolute subcomplexes of X. Here the dimension of each space is strictly less than the dimension of the preceeding space, thus $\lambda \leqslant \dim X$. In fact,

$$d(E_{i+1}(X); x) < d(E_i(X); x) \qquad (x \in E_{i+1}(X)) \tag{11}$$

holds true for $i = 0, \ldots, \lambda - 1$; these inequalities cannot be improved in general.

In the first three Examples $\lambda = 0$ and $E(X) = \emptyset$, thus we do not get non-trivial absolute subcomplexes. In Example 4, $E(X) = \{a\}$, and the theorem states that this is an absolute vertex of X.

There is another way to form absolute subcomplexes starting with X, $E(X)$. At each point of $E(X)$ the function $d(X; x)$ is discontinuous, by definition. However, the *restriction* of this function to the subspace $E(X)$ *may be continuous at some points* (intuitively speaking, the discontinuity occurs « across » $E(X)$ and not « along » $E(X)$, where $E(X)$ is the set of original discontinuities). Thus the set of discontinuities of the restricted function may be strictly smaller than $E(X)$. Accordingly, let us introduce the following definition.

DEFINITION 3. Let X be a separable, metric space and $d(X; x)$, $E(X)$ as in Definition 2 above. We set: $F_0(X) = X$, $F_1(X) = E(X)$ (see (9)). If $F_i(X)$ has already been defined, we set $d(x) = d(X; x)$ and introduce

$$F_{i+1}(X) = \{\textit{set of discontinuities of } d|F_i(X)\} \tag{12}$$

for the next integer.

THEOREM 7. The sequence $F_i(X)$ of subspaces of a triangulable space has formally the same properties as (10). Specifically,

$$F_0(X) = X \supset F_1(X) \supset \ldots \supset F_\mu(X) \supset F_{\mu+1}(X) = \emptyset\,, \tag{13}$$

$$d(F_{i+1}(X); x) < d(F_i(X); x) \qquad (x \in F_{i+1}(X))\,, \tag{14}$$

hold true, thus $\mu \leqslant \dim X$. All spaces (13) are absolute subcomplexes of X. The sequences of spaces (10), (13) are, in general, different.

The sequences of spaces (10) and (13) are formed in a way as we will form (2) and (1), and this is one reason to discuss these elementary constructions. In many respects the two pairs of sequences of subspaces behave similarly. In fact, we can state now the following additional results on the critical sets of the Betti stack.

THEOREM 8. We have $d(B_{i+1}; x) < d(B_i; x)$, $x \in B_{i+1}$, $i = 0, \ldots, k-1$ for the *critical sets* B_i *of the Betti stack* of a triangulable space X (see (1)). Consequently, $d(C_{i+1}; x) < d(C_i; x)$, $x \in C_{i+1}$ for the spaces in (2), $i = 0, \ldots, l-1$.

REMARKS. We have stated in Theorem 1 that $B_i - B_{i+1}$ is open, everywhere dense in B_i. The inequalities of Theorem 8 follow trivially from this, as B_{i+1} is an absolute subcomplex of B_i. Nevertheless, it is useful to state these inequalities, as in an arbitrary separable, metric space the boundary of an open set need not be of strictly smaller dimension than the set (albeit this is true in R^n, see [12], p. 44).

The symbols E_i, F_j in (9), (12) can be considered as functional signs for maps whose domain and range are the finite family of absolute subcomplexes of a triangulable space X. The maps generate, by compositions, a family of such maps. Specifically, if we denote Γ the free monoid on the symbols E_i, F_j (see [6], p. 4), then every word $w \in \Gamma$ determines such a map: If Y is an absolute subcomplex of X, $w(Y)$ is obtained by induction. If the last symbol of w is E_i, we replace Y by $E_i(Y)$, which is again an absolute subcomplex of X, thus w', that is w with the last symbol E_i omitted, is defined for this space. If the last symbol of w is F_j we replace Y by $F_j(Y)$, etc. At each step we obtain an absolute subcomplex of the preceding space, thus $w(Y)$ is defined by induction. The length $l(w)$ of a word w can be introduced in the usual way $(l(E_i) = l(F_i) = i)$, and it is clear that $w(Y) = \emptyset$, if $l(w) > \dim Y$.

COROLLARY 4. The free monoid Γ with generators E_i, F_j operates on the lattice of all absolute subcomplexes of a triangulable space X.

We can enlarge the set of generators of Γ by adding $D_p(X) = \{x \in X : d(X; x) \geqslant p\}$ as a generator (see Theorem 5). The picture, however, gets complicated, as we were not yet able to establish simple « universal relations » for these operations.

6. – PROOF OF THEOREM 2. Let $D^n = \{x \in R^n : \|x\| \leqslant 1\}$ be the closed unit n-disc, and let us construct a group G of homeomorphisms $g: D^n \to D^n$, $gD^n = D^n$, with the following properties:

$$g0 = 0\,; \quad g \text{ is the identity on } S^{n-1} = \partial D^n\,; \tag{15}$$

$$\text{if } 0 < \|a\| < 1\,, \quad 0 < \|b\| < 1\,, \quad \text{there is } g \in G,\ ga = b\,. \tag{16}$$

Such a G can be quickly obtained as follows. In an appropriate neighborhood U of the neutral element of SO_n (special orthogonal group, thus orthogonal matrices with determinant $+1$), we introduce « canonical coordinates of the first kind » (see [16], p. 290). In these coordinates every system of equation $h^i(t) = c^i t$, $|t| \leqslant \alpha$, $\sum (c^i)^2 = 1$, gives a one-parameter subgroup $h(t)$, $|t| \leqslant \alpha$. We denote $h(c^i t)$ the group element with coordinates $c^i t$. For given (c^i, t), $(c^i t) \in U$, we consider the map

$$gx = h\big(c^i(1 - \|x\|)\, t\big)(x) \qquad (x \in D^n)\,. \tag{17}$$

On a sphere of D^n centered to 0 (17) is a rotation; as the radius of the sphere tends to 1, the rotation tends to the identity. The group generated by the homeomorphisms (17) is transitive on each sphere of radius <1 centered to 0; (15) holds true. Let us also consider the homeomorphisms

$$x \mapsto \Big(1 + \lambda(1 - \|x\|)\Big)x \qquad (x \in D^n,\ \lambda \in R)\,; \tag{18}$$

these maps satisfy (15), thus this condition holds true in the group G generated by the maps (17) and (18). Let us prove that G also satisfies (16). Given a, b as in (16) we solve the equation $(1 + \lambda(1 - \|a\|)\|a\| = \|b\|$ for λ and consider (18) with this λ. Then $a \to b'$, where $\|b'\| = \|b\|$, thus there is an element of G sending b' into b. This proves (16). We have thus shown the existence of a group G of homeomorphisms of D^n onto D^n for which (15), (16) hold true.

Let us suppose that X is a triangulable space and a C^0-manifold without boundary; for the moment we do not suppose that X is connected. Let $Y \neq \emptyset$ be an absolute subcomplex of X; we will show that Y is open in X, thus it is a union of components of X. We select $x \in Y$, and suppose that Y is the unique minimal absolute subcomplex of X containing x (see Property 6). We must show now that Y is a component of X.

If x is an isolated point of Y, then $Y = x$ by Property 3 and the definition of the minimal absolute subcomplex. In this case we select a D^n in X containing x, so that $x \neq 0$, and we consider the group G introduced above; we extend the maps of G by the identity to $X - D^n$. By Property 7, $y = gx$ is an absolute subcomplex of X for every $y \in D^n - \{0\}$, but this is a contradiction as the set of absolute subcomplexes is finite. Thus x cannot be an isolated point of Y.

We select now a D^n centered to x, and construct the group G with maps extended to X. As x is not an isolated point of Y, we have a $y \in Y \cap D^n$, $y \neq x$. By Property 8 $gy \in Y$ for every g, consequently $D^n \subset Y$, by (16). This proves that Y is open in X thus a component of X. This completes the proof of Theorem 2.

PROOF OF THEOREM 3. Let us suppose that X is connected and $B = \partial X \neq \emptyset$. We anticipate a later result showing that B is an absolute subcomplex of X (in fact $B = B_1$ in (1)). Let Y be an arbitrary absolute subcomplex of X. Now B is a triangulable space and a C^0-manifold without boundary, hence by the proof above $Y \cap B$ is empty or a union of components of B.

If $Y \cap B = \emptyset$, we select $x \in Y$, and we consider a disc D^n centered to x, or containing x. Repeating the appropriate arguments from the proof of Theorem 2, we obtain a contradiction. (If x would be isolated, we would have infinitely many absolute vertices, if x would not be isolated, Y would be $X - B$, thus we have contradictions in both cases.)

If $Y \cap B$ is a union of components of B, then either $Y \subset B$, or there is an $x \in Y$, $x \notin B$. Repeating the arguments above, we find that x cannot be isolated, thus $Y \cap (X - B)$ is open, thus $Y = X$. This completes the proof of Theorem 3.

7. – We must recall some well known, elementary facts and notations concerning polyhedra, to prepare the proofs of the other results. Let $a_0, \ldots, a_p$ be affinely independent points in R^n $(n \geqslant p)$. We set

$$S = \left\{x \in R^n : x = \sum_{i=0}^{p} t_i a_i,\ t_i \geqslant 0,\ i = 0, \ldots, p,\ \sum_{i=0}^{p} t_i = 1\right\} \tag{19}$$

$$\mathrm{op}(S) = \{\ldots, t_i > 0, \ldots\} \tag{20}$$

The compact, convex set (19) is called *affine p-simplex* with vertices $a_0, \ldots, a_p$. The vertices determine S and S (given with its linear structure) determines the vertices, hence we will not distinguish in our notations between S and the simplex or the simplicial complex thus defined. The set (19) determines (20) and vice versa. If $p = 0$, $\mathrm{op}(S) = S$, otherwise $\mathrm{op}(S) \subset S$, $\mathrm{op}(S) \neq S$. If $p = n$, $\mathrm{op}(S)$ is the interior of S in R^n; in general, $\mathrm{op}(S)$ is the interior of S in the affine subspace of R^n spanned by the vertices. If K is a simplicial complex the compact sets (19) give a covering of $|K|$, and the sets (20) form a partition of the set $|K|$ as S ranges through the simplices of K (see Terminology and Notations after (6)). Thus every $x \in |K|$ belongs to a unique $\mathrm{op}(S)$ and the corresponding S is called *minimal simplex* of K containing x. The unique minimal simplex is the intersection of all simplices of K containing x, or the smallest dimensional simplex containing x. We say that S is a *top simplex* of K, if it is not the proper face of a simplex of K. Every $x \in |K|$ is contained in at least one top simplex, but may be contained in more than one, and they need not have the same dimension.

Let $S_1, \ldots, S_r$ be all the top simplices of K containing x, and numbered so that

$$\dim S_1 \geqslant \ldots \geqslant \dim S_r \,. \tag{21}$$

Then the union

$$|\mathrm{St}(x)| = S_1 \cup \ldots \cup S_r \tag{22}$$

is the space of a subcomplex $\mathrm{St}(x)$ of K called star of x; this is well defined for every $x \in |K|$. $|\mathrm{St}(x)|$ is a compact neighborhood of x in $|K|$.

Proof of Lemma 2. We select a triangulation (3) of X, and identify X to $|K|$ via f. We introduce the notations used in (21). Then $d(|K|; x) = \dim S_1 = p$ by Lemma 1. If U denotes the interior of $|\mathrm{St}(x)|$ in $|K|$, (7) follows: If $\dim S_r = p$, $d(|K|; y) = p$ for $y \in U$, thus $d(|K|; y)$ is constant in U. If $\dim S_r < p$, (8) hold true for $z \in \mathrm{op}(S_r)$. This completes the proof of the lemma.

Proof of Theorem 6. Let us prove that $E(X)$ in (9) is an absolute subcomplex of X. We select a triangulation (3) of X, and we identify $|K|$ to X via f. It will be sufficient to prove that the topologically defined $E(X) = E(|K|)$ is the space of a subcomplex of K.

For given $x \in X$, $x \notin E(X)$, let us carry out the construction (21), (22); we find then the open neighborhood $U = U_x$, interior of $|St(x)|$ in $|K|$. The complement, $X - U_x$ is the space of a subcomplex of K by construction. Let us prove

$$E(X) = \bigcap \{X - U_x \colon x \notin E(X)\} \,. \tag{23}$$

Every $X - U_x$ on the right of (23) contains $E(X)$, by construction. In fact, as $x \notin E(X)$, we have $\dim S_1 = \ldots = \dim S_r$ in (21), hence $U_x \cap E(X) = \emptyset$, as $d(X; y)$ is constant thus continuous in U_x. This shows that $E(X)$ is contained in the right-hand side of (23). Vice versa, given $x \notin E(X)$, we have the corresponding U_x on the right-hand side, $x \in U_x$, thus $x \notin X - U_x$, hence x is not contained in the intersection on the right-hand side of (23). The main contention of Theorem 6 is thus proved, as (3) was an arbitrary triangulation.

The other statements of this theorem are now easy to establish. We have the inclusions in (10) by the definition of these spaces. By the above $E_{i+1}(X)$ is an absolute subcomplex of $E_i(X)$ thus an absolute subcomplex of X by Property 2. Let us prove (11). This will follow by induction from

$$d(E(X); x) < d(X; x) \qquad (x \in E(X)) \,. \tag{24}$$

Referring to the triangulation (3) used before, we carry out the construction (21), (22) for the given x. As $x \in E(X)$, we have $r \geqslant 2$, $\dim S_1 > \dim S_r$, and $E(X) \cap U_x$ is contained in

$$\bigcup \{S_i \cap S_j : 1 \leqslant i < j \leqslant r\} = T, \tag{25}$$

as in the neighborhoods of the other points $d(X; y)$ is locally constant. Now $d(T; y) < d(X; y)$ is evident for every $y \in T$, thus (24) follows by elementary results on dimension. The inequality (11) follows then trivially by induction. This completes the proof of Theorem 6.

Proof of Theorem 7. We keep the notations introduced above. The first step is to establish that $F_2(X)$ is a subcomplex of $E(X)$. We select a point $x \in E(X)$ and carry out the construction (21), (22); we have then the S_i's in (21) and the U. As x is a point of discontinuity of

$$d(y) = d(X; y) \qquad (y \in U) \tag{26}$$

we have $r \geqslant 2$ and $\dim S_1 > \dim S_r$ in (19). Let us prove now

$$E(X) \cap U = \bigcup \{S_i \cap S_j \cap U : 1 \leqslant i < j \leqslant r, \dim S_i > \dim S_j\}. \tag{27}$$

It is clear that at any point of the right hand side of (27) the function (26) is discontinuous. If, on the other hand, $z \in U$ does not belong to the right hand side of (27) then (26) is locally constant near y, hence $z \notin E(X)$. This proves the equality of the two sides of (27).

Let us restrict the function (26) to $E(X) \cap U$, and use (27). We say that the restricted function is continuous on the set

$$\bigcup \{S_i \cap S_j \cap U : \ldots\} - \bigcup \{S_i \cap S_j \cap S_k \cap U : \ldots\} \tag{28}$$

where the first union is the same as the right-hand side of (27), and the condition for the second union (indicated with three dots in (28)) is: $1 \leqslant i < j < k$, $\dim S_i > \dim S_j > \dim S_k$. In fact, if a point z belongs to (28), there is a smallest i such that $z \in S_i$, and then $d(y) = \dim S_i$ for y's in this set and near z. Consequently, $X - U$ plus the second union in (28) is the space of a subcomplex of K, and the intersection of these subcomplexes is precisely $F_2(X)$.

The other statements of the theorem can be similarly established by induction. We will not give the details of this. However, we will indicate informally an alternate way to complete the argument; in this method we do not need the precise description of the $F_i(X)$'s.

If we restrict d to $F_i(X)$, it is clear that the function will be locally constant in $\mathrm{op}(S)$, $S \subset F_i(X)$; taking top simplices of $F_i(X)$ it is clear that $d' = d|F_i(X)$ is locally constant in an everywhere dense open set. If $z \in \mathrm{op}(S)$ is a point of discontinuity of d', so is any other point of $\mathrm{op}(S)$ (this part of the argument can either be established directly or deduced using Lemma 7 to be stated below), and as the set of discontinuities is closed, S itself belongs to this set. This proves that the points of discontinuities of d' is a closed subcomplex of $F_i(X)$ and the proof of the theorem is complete by this second method.

8. – We will need some more elementary results on polyhedra. What follows can be read later, when these statements are applied.

LEMMA 3. *Let K be a simplicial complex, $|K|$ its space, and U an open subset of $|K|$. Then $U = |K| - |L|$, thus U is the complement of the space of a subcomplex L of K, if and only if the statement*

$$\mathrm{op}(S) \cap U \neq \emptyset \quad \textit{implies} \ \mathrm{op}(S) \subset U \tag{29}$$

holds true for every simplex S of K.

PROOF. *Necessity of* (29). Let us suppose that $U = |K| - |L|$ holds true for the appropriate subcomplex L of K. Given $x \in U$, x is contained in a unique $\mathrm{op}(S)$, S simplex of K, and not a simplex of L, thus $\mathrm{op}(S) \cap |L| = \emptyset$, hence $\mathrm{op}(S) \subset U$.

Sufficiency of (29). Let U be an open subset of $|K|$ such that (29) holds true. We set $C = |K| - U$, thus C is a compact subset of $|K|$. Given $y \in C$ there is a unique simplex S of K such that $y \in \mathrm{op}(S)$. Then $\mathrm{op}(S) \cap \cap U = \emptyset$, as otherwise we would have $\mathrm{op}(S) \subset U$ by (29), thus $y \in U$ contrary to the choice of y. Thus $\mathrm{op}(S) \subset C$, consequently $S \subset C$. Now L is defined by these simplices S, and C is the space of L. This completes the proof of the lemma.

In (22) we formed the space of the star of $x \in |K|$ as the union of the top simplices of K which contain x. We will define $\mathrm{St}(A)$ as a subcomplex of K for any $A \subset |K|$. Let $\sigma: I \to |K|$ be a map of the interval $I = [0, 1]$ into $|K|$ such that $\sigma(I) \subset S$ for some simplex S of K and that σ be linear in the affine structure of S ($\sigma(t) = (1-t)a + tb$, a, $b \in S$, $0 \leqslant t \leqslant 1$, see (19)). Then we say that σ is a segment in $|K|$. We leave to the reader to verify

the following equations

$$|\mathrm{St}(A)| = \bigcup \{\sigma(I) : \sigma \textit{ segment in } |K|,\ \sigma(0) \in \bar{A}\} \tag{30}$$

$$= \bigcup \{S : \quad S \textit{ simplex of } K,\ S \cap \bar{A} \neq \emptyset\} \tag{31}$$

$$= \bigcup \{S : \quad S \textit{ top simplex of } K,\ S \cap \bar{A} \neq \emptyset\} \tag{32}$$

$$= \bigcap \{|L| : L \textit{ subcomplex of } K,\ \bar{A} \subset \operatorname{int} |L|\} \tag{33}$$

where $\operatorname{int} B$ means the interior of $B \subset |K|$ in $|K|$. If $A = \{x\}$ is a single point, (30) is the definition of $|\mathrm{St}(x)|$ in [17]. We used (32) in (22). (33) shows that $|\mathrm{St}(A)|$ is the space of a subcomplex of K denoted $\mathrm{St}(A)$. Clearly, $\mathrm{St}(A) = \mathrm{St}(\bar{A})$.

If x is a point of $|K|$, the simplices of $\mathrm{St}(x)$ whose space does not contain x form a complex $\mathrm{Ln}(x)$ called link of x. Except for some special cases ($\mathrm{St}(x) = K$, for example), we have

$$|\mathrm{St}(x)| - \operatorname{int} |\mathrm{St}(x)| = |\mathrm{Ln}(x)|\,. \tag{34}$$

By (30), $|\mathrm{St}(x)|$ is contractible. The following elementary but important result is proved in [17] (see p. 122, Satz I): Let X be a topological space (not necessarily triangulable), $x \in X$, and let us suppose that x has a neighborhood in X which is a triangulable space. If $|\mathrm{St}_i(x)|$, $i = 1, 2$ are the stars of x in two triangulations of two neighborhoods of x, and $|\mathrm{Ln}_i(x)|$ the corresponding links, then these links are of the same homotopy type. We will not use the links of arbitrary sets although they could be defined (see (34)); such links would not be the same homotopy type, in general. (Take a two dimensional simplex as simplicial complex with space S, a circle A in $\mathrm{op}(S)$, and then a sufficiently fine subdivision of S. In the first case the link is the boundary of S thus connected, in the second case it is not connected.)

9. – Let $s_i : \Delta^p \to X$ be a continuous map of the standard p-simplex $\Delta^p = \{(t_0, \ldots, t_p) \in R^{p+1} : t_j \geqslant 0,\ \sum t_j = 1\}$ into a topological space X, and $\alpha_i \in G$, where G is a given abelian group called coefficient group, $i = 1, \ldots, r$. Then the formal sum

$$t = \sum_{i=1}^{r} \alpha_i s_i \tag{35}$$

is called a *singular chain* (with coefficients in G). If s_i, s_j are different maps for $1 \leqslant i < j \leqslant r$, and $\alpha_i \neq 0$, $i = 1, \ldots, r$, then t is called reduced. Every sin-

gular chain is equal to a reduced chain. The **support** of a reduced chain (35) is

$$S(t) = \bigcup_{i=1}^{r} s_i(\Delta^p)\,, \tag{36}$$

by definition. This is a compact subset of X. The group of singular p-chains of X is denoted $\Delta_p(X; G) = \Delta_p(X)$; $\Delta_*(X; G) = \Delta(X)$ stands for the *direct sum* over all p's, $p \geqslant 0$. We do not consider augmentation. We denote $H_p(X, A; G)$ or $H_p(X, A)$, the relative p-th homology group, and we write

$$H_*(X, A; G) = \sum_{p=0}^{\infty} H_p(X, A; G) \tag{37}$$

(direct sum) for the total group. The following lemma states a well-known excision property.

LEMMA 4. If A is a closed subset of $|K|$, the inclusion

$$(|\mathrm{St}(A)|, |\mathrm{St}(A)| - A) \subset (|K|, |K| - A)$$

induces

$$H_*(|\mathrm{St}(A)|, |\mathrm{St}(A)| - A) \xrightarrow{\cong} H_*(|K|, |K| - A)\,, \tag{38}$$

thus an isomorphism.

If S is a simplex of K, it has a linear structure, hence we can consider affine simplices S' in $\mathrm{op}(S)$; these, of course, are *not* simplices of K. If $x \in S'$,

$$|\mathrm{St}(x)| = |\mathrm{St}(S')| \qquad (S'\text{: } \textit{affine simplex in } \mathrm{op}(S))\,, \tag{39}$$

by (32).

LEMMA 5. Let S be a simplex of K, S' an affine simplex in $\mathrm{op}(S)$, and $N = |\mathrm{St}(S')|$. For all $x \in S'$, the inclusion map i_x: $(N, N - S') \to (N, N - x)$ induces an isomorphism,

$$i_{x*}\colon H_*(N, N - S') \xrightarrow{\cong} H_*(N, N - x) \tag{40}$$

for all points $x \in S'$.

PROOF. We can imbed N in a Euclidean space and carry out the following construction in that space. We note that N is star shaped with respect to any point $x \in S'$ (see (30)). Consequently, a ray issued from x

through a point of N intersects N, S, S' in Euclidean segments which are also segments in $|K|$. Clearly, there is a similarity map $y \to y'$, $y' - x = \lambda(y - x)$, $0 < \lambda < 1$, such that the image N' of N contains S'; we fix such a λ. We denote M the closure of the difference space $N - N'$. Then there is a projection $p: N - \{x\} \to M$ along the rays issued from x which is a retraction of $N - \{x\}$ onto M. To this retraction corresponds a deformation operator $D: \Delta(N - x) \to \Delta(N - x)$, such that

$$p_\# t - t = \partial D t + D \partial t \tag{41}$$

for any chain $t \in \Delta(N - x)$.

i_{x*} in (40) *is an epimorphism.* Let $h \in H_*(N, N - x)$ be given. We will find $h' \in H^*(N, N - S')$, $i_{x*} h' = h$. Let t in (35) be a chain in $\Delta(N)$ representing h, *i.e.* such that $t + \Delta(N - x)$ be a cycle of $\Delta(N)/\Delta(N - x)$ in the class h. Then $\partial t \in \Delta(N - x)$. We will use now the construction of D above (see (41)) and set $u = t + D\partial t$. Then $u = p_\# t - \partial D t$. Hence $\partial u = \partial p_\# t$, thus $u + \Delta(N - S')$ is a cycle determining a class h'. Then $i_{x*} h'$ is represented by

$$u + \Delta(N - x) = t + D\partial t + \Delta(N - x) = t + \Delta(N - x)\,, \qquad \text{as } D\partial t \in \Delta(N - x)\,.$$

Thus $i_{x*} h' = h$, proving that (40) is an epimorphism.

i_{x*} in (40) *is a monomorphism.* Let $k \in H_*(N, N - S')$ be given so that $i_{x*} k = 0$ for some x, which is then fixed in what follows. Let us suppose that t in (35) represents k, thus $t + \Delta(N - S')$ is a cycle of $\Delta(N)/\Delta(N - S')$ in the class k. Then $\partial t \in \Delta(N - S')$. As $t + \Delta(N - x)$ is in the class 0 in view of $i_{x*} k = 0$, we have $t + \Delta(N - x) = \partial u + \Delta(N - x)$ for some chain $u \in \Delta(N)$. Hence $t = \partial u + v$, where $v \in \Delta(N - x)$. Thus $\partial t = \partial v$, consequently $\partial v \in \Delta(N - S')$. Using the operator D in (41), $t = \partial u + p_\# v - \partial D v - D\partial v = \partial(u - Dv) + p_\# v - D\partial v$. This gives $t + \Delta(N - S') = \partial(u - Dv) + \Delta(N - S')$, as $p_\# v$, $D\partial v$ are in $\Delta(N - S')$. Thus $k = 0$, and the proof is complete.

The following result is a simple construction of a simplicial map, but it has strong consequences hence we give a detailed proof. The result itself is similar to a construction used in the proof of Theorem 2 (see (15), (16)) in fact this lemma could have been used there.

LEMMA 6. *Let S_0 be a p-face of the Euclidean q-simplex S_1, $p \leqslant q$, and b, b' be given points of* $\mathrm{op}(S_0)$. *Then there is a piecewise linear homeomorphism $g: S_1 \to S_1$, $gb = b'$, which is the identity on the subcomplex C constructed as follows: Let us denote A the $q - p$ simplex of ∂S_1 not intersecting ∂S_0. Then the top simplices of C are spanned by A and the $(p-1)$-simplices of ∂S_0:*

The construction below gives a well determined g with the following additional property of « coherence »: if $S_0 \subset S_1'$ and $S_0 \subset S_2''$ are given, so that $S_1' \cap S_2''$ is a face on both simplices, then the g' constructed for S_1' coincides on $S_1' \cap S_1''$ with the g'' constructed on S_1'', provided that b, b' are the same in both cases.

PROOF. We number the vertices $a_0, \dots, a_q$ of S_1 so that $a_0, \dots, a_p$ be the vertices of S_0; if S_1', S_2'' are given, we number the union of all vertices this way. We suppose b, b' given

$$b = \sum_{i=0}^{p} \beta_i a_i\,, \qquad \beta_i > 0\,, \qquad i = 0, \dots, p, \ \sum \beta_i = 1\,; \tag{42}$$

$$b' = \sum_{i=0}^{p} \beta_i' a_i\,, \qquad \beta_i' > 0\,, \qquad i = 0, \dots, p, \ \sum \beta_i' = 1\,. \tag{43}$$

For a point

$$a = \sum_{v=0}^{q} \alpha_i a_i\,, \qquad \alpha_i \geqslant 0, \ \sum \alpha_i = 1 \tag{44}$$

we set $\eta(a) = \eta = \min\{\alpha_i/\beta_i : i = 0, \dots, p\}$, and $\xi_i(a) = \xi_i = \alpha_i - \eta\beta_i$, $i = 0, \dots, p$, $\xi_j = \alpha_j$, $j = p+1, \dots, q$ We note $\sum \xi_i + \eta = 1$, and that there is a smallest $k = k(a)$, $0 \leqslant k \leqslant p$, such that $\xi_k = 0$. Thus

$$a = \sum_{\substack{i=0 \\ i \neq k}}^{q} \xi_i a_i + \eta b \tag{45}$$

is a barycentric representation of a in a q-simplex whose vertices are q vertices of S_1 plus b; if $a \in S_0$, we have p vertices of S_0 plus b. We note at this point that the numbering of the vertices was introduced in order to simplify the notations, and (45) is independent of it. We define

$$ga = \sum \xi_i a_i + \eta b'\,. \tag{46}$$

It is then easy to verify that $a \mapsto ga$ is a map with properties stated in the lemma whose proof is thus complete. Let us add that g is a simplicial map in subdivisions of S_1: b and b' are the only « new » vertices in the subdivisions of the domain and the range respectively; $gb = b'$, and $gx = x$ for the other vertices.

LEMMA 7. *Let K be a simplicial complex, S a simplex of K, $N = |\mathrm{St}(S)|$. If b, b' are given points of* $\mathrm{op}(S)$, *then there is a piecewise linear homeomor-*

phism $g\colon |K| \to |K|$, $gb = b'$, *which maps every simplex of* K *onto itself, consequently preserves all subcomplexes, and which is the identity on* $|K| - N$.

PROOF. We imbed N in a Euclidean space. We carry out the construction of Lemma 6 for all pairs $S_0 = S \subset S_1$ where S_1 ranges through the top simplices of K containing S_0. For each such pair a map (46) is defined and these maps agree on intersections, thus $N \to N$ is obtained. Furthermore, the map $N \to N$ is the identity on the boundary of N in $|K|$ (see the definition of C in Lemma 6), hence we can extend it by the identity to $|K|$. This completes the proof of the lemma.

10. – There is an extensive theory of topological spaces which are in some sense homological manifolds (see [22], [3]). We will only consider triangulable spaces and open subsets of these spaces in which case these definitions coincide, and singular homology can be used. If X is an *open subset of a triangulable space* we call it **homology manifold** (without boundary), if

$$(47) \qquad H_p(X, X - x; Z) = 0\,, \quad \text{if } p \neq n\,, \quad \text{and} \quad H_n(X, X - x; Z) \cong Z$$

holds true for every point $x \in X$; we note that integral coefficients Z were used, and this is quite essential. These spaces need not be compact. If X is also compact, then it is compact and open in a triangulable space, consequently it is a triangulable space. Summing up: a *compact homology manifold* is, by definition, a triangulable space for which (47) holds true for every point.

We consider a triangulation (3) of X, identify $|K|$ to X via f and formulate some results in this simplicial structure of X.

We suppose that X is connected. It is known that $H_n(X; Z)$ is either 0 or Z.

CASE $H_n(X; Z) \cong Z$: *compact, connected, orientable homology manifold (without boundary), which is the space of* K. We have the chain group of oriented simplices of K (see [18], p. 158). There is a quotient group of a subgroup of $\Delta(X)$ which is naturally isomorphic to the group of oriented chains and this morphism induces an isomorphism on the homology groups. If t in (35), $G = Z$, is an n-cycle whose class is a generator of $H_n(X; Z)$, then the oriented chain corresponding to it is

$$(48) \qquad \hat{t} = \sum \varepsilon_i \hat{S}_i$$

where $\hat{t}$ is reduced, S_i ranges through all top simplices of K, $\hat{S}_i$ denoting the simplex with orientation, and $\varepsilon_i = \pm 1$. Changing the orientation of S_i,

if $\varepsilon_i = -1$, will give then

$$t = \sum \hat{S}_i \tag{49}$$

where the chain is reduced and all top simplices appear with a definite orientation.

For a given $x_0 \in X$, we set $N = |\mathrm{St}(x_0)|$ and denote M the boundary of N in X; M is the space of the link of x_0: The projection from x_0 is a deformation retraction of $N - x_0$ onto M, hence the morphism induced by the inclusion

$$H_p(M; Z) \xrightarrow{\cong} H_p(N - x_0; Z) \tag{50}$$

is an isomorphism. Taking the exact sequence of the pair $(N, N - x_0)$ in singular homology, and using (50), we have

$$H_p(N, N - x_0; Z) \xrightarrow{\cong} H_{p-1}(M; Z) \qquad (p \geqslant 2, n \geqslant 2)\,. \tag{51}$$

Similarly, as N is contractible we have

$$H_p(N, M; Z) \xrightarrow{\cong} H_{p-1}(M; Z)\,. \tag{52}$$

It is known that the isomorphism between singular homology and the homology based on oriented simplices maps a generator of $H_n(N, M; Z)$ onto a class represented by

$$t' = \sum \{\hat{S}_i \colon S_i \ n\text{-simplex of } \mathrm{St}(x_0)\} \tag{53}$$

where the orientations of (49) are kept. We have thus proved the following result.

LEMMA 8. Let X be a connected, triangulable space, and a homology n-manifold with $H_n(X; Z) \simeq Z$. Then $(X, \emptyset) \to (X, X - x)$ induces an isomorphism

$$H_n(X; Z) \xrightarrow{\cong} H_n(X, X - x; Z)\,. \tag{54}$$

If (35) is an n-cycle whose class is a generator of $H_n(X; Z)$, then $t + \Delta(X - x)$ is an n-cycle whose class is a generator of $H_n(X, X - x; Z)$.

CASE $H_n(X; Z) = 0$: *compact, connected, non-orientable homology n-manifold (without boundary) which is the space of K.* We use the notations $N \supset M$ introduced above (see (50)). It is known that M is a connected, homology

$(n-1)$-manifold, and a homology $(n-1)$-sphere (see [17], p. 239, Satz II), albeit it may be non-simply connected (see Example 3 and Section 14). Even if it is simply connected it may be $\neq S^{n-1}$. The suspension SM is then a connected, compact, homology n-manifold (see [18], p. 41; we take non-reduced suspension). We identify N to « one half » of the suspension SM. From a singular n-cycle of SM whose class is the generator of $H_n(SM; Z)$ (taking subdivision and an appropriate deformation operator) we obtain an integral singular n-chain $t \in \Delta(N)$ such that $\partial t \in \Delta(M)$, and such that the class of $t + \Delta(N-x)$ be a generator of $H_n(N, N-x; Z)$ for all $x \in N - M$. Summing up, we have the following result.

LEMMA 9. Let X be a connected, triangulable space which is a homology n-manifold. Given $x_0 \in X$, there is a closed neighborhood N of x_0 with boundary M in X, such that the injection $j_x\colon (N, M) \to (N, N-x)$ induces

$$j_{x*}\colon H_n(N, M; Z) \xrightarrow{\cong} H_n(N, N-x; Z) \tag{55}$$

for all point $x \in N - M$. Consequently there is an n-chain (35), $\partial t \in \Delta(X - \operatorname{int} N)$, such that $t + \Delta(X-x)$ is a cycle whose class is a generator of $H_n(X, X-x; Z)$, for every $x \in \operatorname{int} N$.

11. – Before turning to the proof of Theorem 1 and stating the related results, we will recall some general concepts on stacks, define the Betti stack, and apply the general results to the Betti stack.

If P is a partially ordered set, we call **stack** F over P a rule associating with every $A \in P$ an abelian group $F(A)$ and with every inequality $A \geqslant B$ a group morphism $f_{BA}\colon F(A) \to F(B)$ in such a way that f_{AA} be the identity, and that $f_{CA} = f_{CB} f_{BA}$ hold true for $A \geqslant B \geqslant C$ (see [11]). We will consider here only the case when P is the family of all closed subspaces of a topological space X, $\geqslant$ is $\supset$, and to simplify the terminology we say that F is defined over X (see [13]).

We say that the stack F defined over X is **continuous**, if the following two conditions are satisfied:

(C1) *Given $a \in F(A)$ there is a closed neighborhood V of A in X, and an $a' \in F(V)$ such that $f_{AV}(a') = a$.*

(C2) *If $f_{BA}(b) = 0$, $b \in F(A)$, $A \supset B$, then B has a closed neighborhood W in A, such that $f_{WA}(b) = 0$.*

We may refer informally to (C1) by saying that a extends to a neighborhood V of A. Similarly, we may say that the b in (C2) is zero in the neighborhood W of B.

The two conditions together can be stated using the concept of direct limit, denoted $\varinjlim$. The family $\{W\}$ of closed neighborhoods W of B in A forms an ordered and directed set, thus $\{F(W): f_{WV}\}$ is a directed system, hence the direct limit is defined ([12], p. 132). These remarks apply to every stack F over X. The stack F is continuous, if and only if

$$\varinjlim F(W) = F(B) \qquad (W \textit{ ngd of } B \textit{ in } A) \tag{56}$$

holds true for every pair $A \supset B$ of closed sets of X. Thus equation (56) could be used to define continuous stacks (for a more precise result, see [11]).

The stack F being essentially a function, we may take restrictions and extensions. In particular, if F is a stack over X, and Y is a closed subspace of X, we may take the stack G over Y such that $G(A) \to G(B)$ is f_{AB} if A, B are closed in Y (and, of course, G is defined only for closed subsets of Y). G is then the restriction of F to Y. If F is continuous over X, then G is continuous over Y. In other contexts it may be useful to consider other restrictions, but only the type just described will be used in the sequel.

In what follows all spaces will be locally compact, Hausdorff spaces, as these are the only spaces we need in our applications here.

For continuous stacks we can introduce the concepts of *critical points* and *non-critical points*. These concepts were defined in [8]; our present definition will be different but equivalent to the former definition in the cases considered here. The two definitions are compared in Section 17 of this paper and until then it is not necessary to know the other definition.

We start with a *preliminary discussion* to lead up to the definition. We say that a point $x_0 \in X$ is non-critical for the continuous stack F defined over X, if $F(x)$ is isomorphic to a fixed group as the point x ranges through a neighborhood of x_0. The necessity to consider such non-critical points seems to be evident, but the definition would be clearly useless, if the isomorphisms would not be prescribed by the stack morphisms f_{BA}. For this reason, we may consider the diagram

$$\begin{cases} F(V) \to F(x) \\ \ker f_{xV} \qquad \operatorname{coker} f_{xV} \end{cases} \tag{57}$$

where V is a closed neighborhood of x_0, and x ranges through a (possibly smaller) neighborhood of x_0. If in (57) coker $f_{xV} = 0$, and ker f_{xV} is always the same group M we would have an isomorphism

$$F(V)/M \cong F(x) \qquad (x \in W \subset V) \tag{58}$$

for every $x \in W$. At the first sight this may seem as the right definition (see [9]), but it has a drawback: non-critical points may become critical, if we restrict F to a subspace (critical being the negation of non-critical). We will see later how important such restrictions are thus we cannot use presently this definition. It requires too much from non-critical points.

Let us consider now in place of (57) the following diagram:

$$\begin{cases} F(V) \longrightarrow F(x) \\ E \xrightarrow[e_x = f_{xV}|E]{} F(x) \end{cases} \tag{59}$$

where E is some subgroup of $F(V)$ and e_x is the restriction of f_{xV} to this subgroup. If, by an appropriate choice of E, the e_x morphism has constant kernel and zero cokernel, we have the desired isomorphism

$$E/M \simeq F(x) \qquad (M = \ker e_x = \text{constant}) \tag{60}$$

for the groups $F(x)$. Furthermore, V may be replaced by a smaller neighborhood W, if we take $f_{WV}E$ in $F(W)$, see below. This discussion leads then to the following definition.

DEFINITION 4. *Let F be a continuous stack over the locally compact, Hausdorff space X. We say that a point $x_0 \in X$ is non-critical for F, if x_0 has a closed neighborhood V for which $F(V)$ contains a subgroup E, such that, if we set $e_x = f_{xV}|E$, then* $\ker e_x$ *is constant (same subgroup for all $x \in V$), and* $\operatorname{coker} e_x = 0$, $x \in V$ (see (59)). *If these conditions are not satisfied, we say that x_0 is a critical point of F.*

PROPERTY 1. The set of non-critical points is an open subspace of X (it may be empty), and its complement is the closed subspace of critical points.

PROOF. If x_0 is a non-critical point for F, we have V, E with properties stated in the definition above. If x_1 is in the interior int V of V, then the same data V, E show that F is non-critical at x_1 (expression also used in place of « x_1 is non-critical for F » or « x_1 is a non-critical point of F »). Thus int V is contained in the set of non-critical points which is thus open. Consequently, the complement, which is precisely the set of critical points, is closed.

PROPERTY 2. Let F be a continuous stack, and x_0 a non-critical point for F. If Y is a closed subspace of X, $x_0 \in Y$, and F' is the restriction of F to Y, then x_0 is a non-critical point for F'.

PROOF. We have V, E, and e_x of the definition for F and x_0. We set $V' = V \cap Y$, $E' = f_{V'V}(E)$, $e'_x = f_{xV'}|E'$. Then: V' is a neighborhood of x_0 in Y. $E' \subset F'(V')$. We have $e_x = e'_x(f_{V'V}|E)$ in the diagram

$$\begin{tikzcd} E \arrow[r, "e_x"] \arrow[d, "f_{V'V}|E"'] & F(x) \\ E' \arrow[ur, "e'_x"'] & \end{tikzcd} \tag{61}$$

Consequently, $\operatorname{coker} e'_x = 0$, and $\ker e'_x = f_{V'V}(\ker e_x)$ is a constant subgroup of E'. This proves that x_0 is a non-critical point for F'.

If we restrict the stack F to the closed subspace of all of its critical points, the restricted stack may or may not have critical points. We will see examples, where the restricted stack has non-critical points. In such cases the following definition is useful.

DEFINITION 5. If F is a continuous stack over the locally compact, Hausdorff space X, we define the α-th critical set C_α of F for transfinite ordinals α by transfinite induction: (0) $C_0 = X$, by definition; (1) if $\beta = \alpha + 1$, and C_α has already been defined, C_β is the set of all critical points of the restriction of F to C_α; (2) if β is a limit number, and C_α has already been defined for all $\alpha < \beta$, we set

$$C_\beta = \bigcap \{C_\alpha : \alpha < \beta\} . \tag{62}$$

The intersection of all critical sets is called last or smallest critical set.

Justification. We have to prove by induction that C_α is closed in X. Now C_0 is closed, by definition. If $\beta = \alpha + 1$, we can apply Property 1 of the critical points, and see that C_β is closed. Supposing that C_α is closed for all $\alpha < \beta$, β limit number, (62) shows that C_β is closed. The term « last critical set » will be justified below; this set may be empty.

PROPERTY 1. $\{C_\alpha\}$, where α is an ordinal, is a decreasing transfinite sequence of closed subsets of X: if $\alpha < \beta$, $C_\alpha \supset C_\beta$. There is a first ordinal Ω such that

$$C_{\Omega+1} = C_\Omega \tag{63}$$

and then C_Ω is the smallest critical set, that is, the intersection of all critical sets. If $C_\alpha = \emptyset$ for some α, then $\emptyset$ is the smallest critical set, and vice versa.

We will not prove this and similar other elementary properties of critical sets, as in our applications the sequence of critical sets will be finite (see (1)).

12. – In an appropriate systematic development of homology theory the *Betti complex* in the sense of Leray and the *Betti stack* appear naturally and necessarily (see [13], [2], [11]). Let us say here only the following. If we take the singular chain group $\Delta(X)$ with the supports (36) we obtain a complex in the sense of Leray. Consequently, we form the stack

$$A \mapsto \Delta(X)/\Delta(X-A) \tag{64}$$

$$(A \supset B) \mapsto \big(\Delta(X)/\Delta(X-A) \to \Delta(X)/\Delta(X-B)\big) \tag{65}$$

where $\Delta(U)$ is identified to a subgroup of $\Delta(V)$, if $U \subset V$. Still following the theory of Leray we form the derived stack of (64), (65): it associates to A the homology group of the chain group in (64), and it associates to $A \supset B$ the homeomorphism induced by the morphism indicated in (65).

We give now a direct definition of the Betti stack which is independent of the previous remarks.

DEFINITION 6. *If X is a locally compact, Hausdorff space, the Betti stack $\mathfrak{B}$ of X associates with every closed set A of X the group*

$$\mathfrak{B}(A) = H_*(X, X-A; G) \tag{66}$$

(total singular homology, see (37)) *and with every inclusion $A \supset B$ of closed sets the natural morphism*

$$H_*(X, X-A; G) \to H_*(X, X-B; G) \tag{67}$$

induced by the inclusion map $(X, X-A) \subset (X, X-B)$. The stack $\mathfrak{B}_p$, $p \geqslant 0$, is formed with p-th homology groups. $\mathfrak{B}$ is considered to be the direct sum of the $\mathfrak{B}_p$'s.

PROPERTY 1. The Betti stack of X is continuous.

PROOF; *an outline.* In order to prove condition (C1) of continuity, we suppose $a \in \mathfrak{B}_p(A)$ given. This is a homology class represented by a chain t in (35), $t \in \Delta(X)$, such that $\partial t \in \Delta(X-A)$. As A is closed in X and $S(\partial t)$ is compact (see (36)), there is a closed neighborhood V of A in X, such that $S(\partial t)$ and V are also disjoint, thus $\partial t + \Delta(X-V)$ is a cycle of $\Delta(X)/\Delta(X-V)$ which determines a class $a' \in \mathfrak{B}(V)$. Then a' is mapped into a under the natural morphism, thus condition (C1) of continuity is proved. The proof of (C2) is similar.

REMARKS. We will use the Betti stack only when X is a triangulable space. In a more general setting this definition is probably not very helpful (see [1]). Let us add that it is important for us that $\mathcal{B}(A)$ is defined for an arbitrary closed set, not just for triangulable A.

13. – We will apply the concepts introduced above to triangulable spaces, and here we will use the lemmas stated in Sections 8, 9, 10.

LEMMA 10. For the space $X = |K|$ of a simplicial complex, we have

$$\mathcal{B}(A) = H_*(|\mathrm{St}(A)|, |\mathrm{St}(A)| - A; G) \tag{68}$$

for every closed set A of X (see (30)-(33)). In particular

$$\mathcal{B}(x) = H_*(|\mathrm{St}(x)|, |\mathrm{St}(\mathrm{x})| - x; G) \tag{69}$$

holds true. The morphisms (67) are also expressed with the help of these groups in the sense that the diagram

$$\begin{array}{ccc} H_*(X, X-A) & \longrightarrow & H_*(X, X-B) \\ \cong\uparrow & & \cong\uparrow \\ H_*(|\mathrm{St}(A)|, |\mathrm{St}(A)| - A) & \to & H_*(|\mathrm{St}(A)|, |\mathrm{St}(A)| - B) \\ & & \cong\uparrow \\ & & H_*(|\mathrm{St}(B)|, |\mathrm{St}(B)| - B) \end{array} \tag{70}$$

is commutative.

This is simply Lemma 4 with new notations, and added excision isomorphisms. *We note that* (69) *is the classical « homology group at a point » and that the Betti stack organizes these various homology groups at points into a whole.*

THEOREM 9. *Let X be a triangulable space and $\mathcal{B}$ the Betti stack of X (see* (66), (67)). *The set of non-critical points of $\mathcal{B}$ is an open, everywhere dense subspace of X, and the set B_1 of critical points is an absolute subcomplex of X. If x_0 is a non-critical point for $\mathcal{B}$, and $p = d(X; x_0)$, then*

$$\mathcal{B}_q(x_0) = \begin{cases} G, & \text{if } q = p; \\ 0, & \text{if } q \neq p, \end{cases} \tag{71}$$

thus $\mathfrak{B}(x_0) = \mathfrak{B}_p(x_0) \cong G$ *holds true, where* G *is the coefficient group of the homology. Consequently, if* $G = Z$, *each component of* $X - B_1 = B_0 - B_1$ *is a homology manifold. We have* $d(B_1; x) < d(X; x)$ *if* $x \in B_1$, *in particular* $\dim B_1 < \dim X$.

PROOF. Let (3) be a triangulation of X and let us identify $|K|$ to X via f. We select a top simplex S of K, and $x_0 \in \mathrm{op}(S)$. Then $\mathrm{St}(x_0)$ is the complex of the simplex S, $|\mathrm{St}(x_0)| = S$, and $\mathfrak{B}(x_0) = H_*(S, S - x_0; G)$ by (69), thus (71) follows. Let S' be an affine p-simplex in $\mathrm{op}(S)$, $\dim S' = p$, for which x_0 is an inner point. By Lemma 5, x_0 is a non-critical point of $\mathfrak{B}$, as in Definition 4 we can select $V = S'$, $E = \mathfrak{B}(S')$, we have thus for e_x the isomorphism (40) of Lemma 5. This proves that the set of non-critical points of $\mathfrak{B}$ contains

$$\bigcup \{\mathrm{op}(S) \colon S \text{ top simplex of } K\} \tag{72}$$

which is everywhere dense in $|K|$. We note in passing that all isolated points of X are in the set (72).

Let x_0 be a non-critical point of $\mathfrak{B}$, V a closed neighborhood of x_0, and $E \subset \mathfrak{B}(V)$ a subgroup so that the conditions of Definition 4 be satisfied. Then the intersection of the interior of V with the set (72) contains a point x_1 to which the first paragraph of this proof applies, and as $\mathfrak{B}(x_0) \cong \mathfrak{B}(x_1)$, we conclude that (71) also holds for these points x_0, thus in general for all non-critical points.

As in the theorem, we denote B_1 the first critical set of $\mathfrak{B}$, *i.e.*, the set of all critical points (see Definitions 4, 5). To prove that B_1 is the space of a subcomplex of K, we will show that $U = X - B_1$ satisfies (29) in Lemma 3.

Given $x_0 \in U$, thus a non-critical point, it is contained in a unique minimal simplex S of K, and as S is minimal, $x_0 \in \mathrm{op}(S)$. We must show $\mathrm{op}(S) \subset U$, hence that all points of $\mathrm{op}(S)$ are noncritical for $\mathfrak{B}$. We select a closed neighborhood V of x_0 in $|K|$ and an $E \subset \mathfrak{B}(V)$ so that the conditions of Definition 4 be satisfied. By Property 2 of the non-critical points (see (61)), we may suppose $V \subset N = |\mathrm{St}(\mathrm{op}\, S)|$. Given another point $x_1 \in \mathrm{op}(S)$ we can apply Lemma 7: there is a piecewise linear homeomorphism $g \colon |K| \to |K|$, $gN = N$, $gx_0 = x_1$, g the identity on the boundary of N in $|K|$. Let us consider now the following diagram

$$\begin{array}{ccc} H_*(N, N - V) & \to & H_*(N, N - V') \\ \downarrow & & \downarrow \\ H_*(N, N - x) & \to & H_*(N, N - x') \end{array} \tag{73}$$

where $V' = gV$, $x' = gx$, the horizontal arrows are induced by g, and the vertical arrows are induced by inclusions. As the horizontal arrows are isomorphisms in (73), g^*E is a subgroup of $\mathfrak{B}(V')$ for which the conditions of Definition 4 are satisfied. Consequently, x_1 is a non-critical point of $\mathfrak{B}$, hence $\mathrm{op}(S) \subset U$. By Lemma 3 (see (29)) B_1 is the space of a subcomplex of K. As the triangulation (3) chosen was arbitrary this proves that B_1 is an absolute subcomplex of X.

We keep the notation K introduced above to prove the statement on dimension. If $x \in B_1$, then x does not belong to the set (72) by the results above, hence the inequality $d(B_1; x) < d(X; x)$ follows by Lemma 1. This completes the proof of the theorem.

PROOF OF COROLLARY 1. We see by induction that C_{i+1} is an absolute subcomplex of C_i by Theorem 9, hence an absolute subcomplex of X by Property 2 of absolute subcomplexes. We note that

$$d(C_{i+1}; x) < d(C_i; x) \qquad (x \in C_{i+1};\ i = 0, \dots, l-1) \tag{74}$$

follows from the inequality in Theorem 9, in particular $\dim C_{i+1} < \dim C_i$. We have seen above that the set of non-critical points is everywhere dense, consequently $C_i - C_{i+1}$ is everywhere dense in C_i. Furthermore, (71) with $G = Z$ proves that each component of $C_i - C_{i+1}$ is a homology manifold, as integral coefficients are used here. This completes the proof of Corollary 1. We note that only Theorem 9 was used, which is part of the statement of Theorem 1 (except for the remark on $d(X; x)$).

Before proceeding to the proof of Theorem 1, we need two preliminary results.

LEMMA 11. Let K be a simplicial complex, and $\mathfrak{B}$ the Betti stack of $|K|$. If S is an arbitrary simplex of K, and $\mathfrak{B}'$ denotes the restriction of $\mathfrak{B}$ to S, then $\mathrm{op}(S)$ contains no critical point of $\mathfrak{B}'$.

PROOF. This is a trivial corollary of Lemma 5 (see (40)), but it will be useful to have it as a reference. Given $x_0 \in \mathrm{op}(S)$ we select an affine simplex $S' \subset \mathrm{op}(S)$, $\dim S' = \dim S$ containing x_0 in $\mathrm{op}(S')$; thus S' is a closed neighborhood of x_0 in $\mathrm{op}(S)$. Then (40) shows that the conditions of Definition 4 are satisfied for $\mathfrak{B}'$ and x_0 with $V = S'$ and $E = \mathfrak{B}'(S')$.

LEMMA 12. Let K be a simplicial complex, L a subcomplex of K, $\mathfrak{B}$ the Betti stack of $|K|$ and $\mathfrak{B}'$ the restriction of $\mathfrak{B}$ to $|L|$. Then there is a subcomplex M of L such that $|M|$ is the first critical set of $\mathfrak{B}'$. Also, $d(|M|; x) < d(|L|; x)$, if $x \in |M|$, consequently $\dim |M| < \dim |L|$.

PROOF. Let S be a top simplex of L. If we restrict $\mathcal{B}'$ to S, and denote the restriction $\mathcal{B}''$, then Lemma 11 states that $\mathcal{B}''$ has no critical point in $\mathrm{op}(S)$. Now $\mathrm{op}(S)$ is an open subspace of $|L|$ hence $\mathcal{B}'$ has no critical point in $\mathrm{op}(S)$: the same S' which is a neighborhood of a point $x \in \mathrm{op}(S)$ is a neighborhood of x in $|L|$. This shows that the set of non-critical points of $\mathcal{B}'$ contains the set $\bigcup \mathrm{op}(S)$, where S ranges through the top simplices of L. If we denote B_1' the set of critical points of $\mathcal{B}'$, we see that $|L| - B_1'$ is everywhere dense in $|L|$ and that $d(B_1'; x) < d(|L|; x)$ for $x \in B_1'$.

In order to prove that B_1' is the space of a subcomplex M of L we will use Lemma 3 (see (29)). Let be given a simplex S of L, and an $x_0 \in \mathrm{op}(S)$ which is non-critical for $\mathcal{B}'$. According to (29) we have to prove that every $x_1 \in \mathrm{op}(S)$ is non-critical for $\mathcal{B}'$. We denote N the star of $\mathrm{op}(S)$ in K. By Lemma 7 there is a piecewise linear homeomorphism $g\colon K \to K$, $gL = L$, $gS = S$, $gx_0 = x_1$ which is the identity outside N. Let V be a closed neighborhood of x_1 in $|L|$ and $E \subset \mathcal{B}'(V)$ such that the conditions of Definition 4 be satisfied. As we can take restrictions we can suppose $V \subset N$. We can then use diagram (73) with the present meaning of the symbols there. Then $V' = gV$ is a neighborhood of x_1 in $|L|$, as $g|L| = |L|$, and $E' = g_* E$ is a subgroup of $\mathcal{B}'(V')$ such that the conditions of Definition 4 are now satisfied for x_1. This shows that x_1 is a non-critical point of $\mathcal{B}'$. Thus $\mathrm{op}(S) \subset |L| - B_1'$. By Lemma 3, $B_1' = |M|$, space of a subcomplex of L. This completes the proof of the lemma.

THEOREM 10. *The i-th critical set B_i of the Betti stack $\mathcal{B}$ of a triangulable space X is an absolute subcomplex of X. If $i > \dim X$, $B_i = \emptyset$.*

PROOF. We will prove the first statement of the theorem by an induction on i. $B_0 = X$ is trivially an absolute subcomplex of X, and B_1 is an absolute subcomplex by Theorem 9. Also, $\dim B_1 < \dim B_0$. We suppose that the first statement of theorem is true for B_i and that $d(B_{j+1}; x) < d(B_j; x)$, if $x \in B_{j+1}$, and $j + 1 \leqslant i$. We select a triangulation (3) of X, and denote L the subcomplex of K for which $|L| = B_i$. By Definition 5, we have to take the restriction of $\mathcal{B}$ to $|L|$ and find the critical points of this restriction in order to obtain B_{i+1}. By Lemma 12, B_{i+1} is the space of a subcomplex of L. We have also the inequality for the dimension. As this holds true for every triangulation of X, the proof is complete.

PROOF OF THEOREM 1 AND THEOREM 8. We recall that by Theorem 9, B_1 is an absolute subcomplex of X, and that we already deduced from this Corollary 1. Now we see that all statements of Theorem 1 are covered by previously proved results. By Theorem 10 the B_i's are absolute subcomplexes of X. In the proof of Theorem 10 we have seen that, if we select a

triangulation (3) of X, then $\bigcup \mathrm{op}(S)$ where S runs through the top simplices of the given triangulation of B_i, is contained in $B_i - B_{i+1}$. On one hand this gives that $B_i - B_{i+1}$ is everywhere dense in B_i, on the other hand it gives the inequalities $d(x, B_{i+1}) < d(B_i, x)$, $x \in B_{i+1}$ in view of Lemma 1. This completes the proof of both theorems.

14. – Let us mention some applications of the former results to absolute vertices.

THEOREM 11. Let X be a triangulable space and x_0 a given point of X. If x_0 is not an absolute vertex of X, then for every neighborhood V of x_0 there is a homeomorphism $g: X \to X$, which is the identity in $X - V$, and which maps x_0 into a different point. If x_0 is an absolute vertex of X, then every homeomorphism $g: X \to X$ near the identity map keeps x_0 fixed; here «near the identity map» means that $d(x, gx) < \varepsilon$, for all $x \in X$, where $\varepsilon > 0$ depends only on the metric d used.

PROOF. Let us suppose first that x_0 is not an absolute vertex of X. We can then select a triangulation (3) of X such that $f^{-1}x_0$ is not a vertex of K. We may suppose then that $f\,|\mathrm{St}(f^{-1}x_0)| \subset V$, as otherwise we could take an appropriate subdivision of K (in which $f^{-1}x_0$ was not introduced as a vertex). Identifying now $|K|$ to X via f, x_0 is contained in minimal simplex S of K, and $x_0 \in \mathrm{op}(S)$, $\dim S \geqslant 1$. Then Lemma 7 states the existence of a hemeomorphism g with properties stated in the theorem.

Let us suppose now that x_0 is an absolute vertex of X. We select a metric d for X. The set of absolute vertices of X is finite and we can thus select an $\varepsilon > 0$ which is smaller than the mutual distances of the absolute vertices. If $g: X \to X$, $gX = X$, is a homeomorphism, and $d(x, gx) < \varepsilon$ for all x, then by Property 7 of absolute subcomplexes gx_0 will be an absolute vertex, $d(x_0, gx_0) < \varepsilon$, thus $gx_0 = x_0$. This completes the proof of the theorem.

Our results would imply some other properties of trajectories of points under homeomorphisms, but we will not formulate them presently.

DISCUSSION OF EXAMPLE 3. Let $SO(3)$ be the special orthogonal group in three variables, and I the subgroup leaving a regular icosahedron invariant (see [17], p. 218, [5], p. 56). Then $M = SO(3)/I$ is an analytic homology 3-sphere which is not simply connected. If N is the suspension of M as in Example 3, then the fundamental group at the points a', a'' (see [17], p. 177) is $\pi_1(M) \neq 0$, at other points it is zero. The fundamental group at a point is a topological invariant of the space ([17], p. 177), consequently, if $g: N \to N$, $gN = N$ is a homeomorphism, $ga' = a'$ and $ga'' = a''$,

or else, $ga' = a''$, $ga'' = a'$ holds true; clearly there are homeomorphisms of both types, the second possibility being realized by the symmetry of the suspension. By Theorem 11 these points are thus absolute vertices of N. As $N - \{a', a''\}$ is homeomorphic to $M \times R$, it is a C^0 manifold. The reasoning of the proof of Theorem 2 shows that N has no other nontrivial absolute subcomplexes, thus $\{\emptyset, a', a'', \{a', a''\}, N\}$ is the full lattice of absolute subcomplexes of N. The subcomplex Y appearing in Theorem 4 is now $\{a', a''\}$ which is itself a non-connected zero dimensional manifold. The author is thankful to Professor Kirby for pointing out that other similar constructions can be obtained using spaces constructed in [14].

15. – We used classical invariants to locate absolute subcomplexes in the discussion above. Theorem 4 is special in the sense that we can only affirm existence of the absolute subcomplex Y, but we cannot use any classical invariant to describe this space. The proof of this theorem is, however, easy; lemmas 3 and 7 will be useful once again.

PROOF OF THEOREM 4. We recall that we use the expression homology manifold in the following sense: it is an *open* subspace of a triangulable space, and (47) holds true for every point of the space. Our X is a triangulable space, hence a compact homology manifold (without boundary). We consider now the following family of subspaces

$$\{U\colon U \text{ open in } X,\ U \text{ is a } C^0\text{-manifold}\}\,; \tag{75}$$

the U's need not be connected. The family (75) is not empty: if (3) is a triangulation of X, and S is a top simplex of K, then $f(\mathrm{op}(S))$ belongs to (75). The union V of all elements of (75) belongs to this family of sets, hence it is the maximal C^0-manifold which is an open subset of X. We set $C = X - V$.

If C would be empty, X would be a C^0-manifold which was excluded. Consequently, $\dim C \geqslant 0$. We already noted that if (3) is a triangulation of X, and S a top simplex of K, then C does not intersect $f(\mathrm{op}(S))$, consequently $\dim C \leqslant n-1$. Better yet, it is known that in a homology n-manifold every $(n-1)$-simplex S' is the face of precisely two n-simplices (see [17], p. 237, II), thus $f(\mathrm{op}(S')) \subset V$ hence $\dim C \leqslant n-2$.

We will prove now that C is an absolute subcomplex of X. Let be given a triangulation (3) of X, and let us identify $|K|$ to X via f. In order to prove that C is the space of a subcomplex of K, we will prove that V satisfies condition (29) of Lemma 3. Let S be an arbitrary simplex of K and $x_0 \in \mathrm{op}(S) \cap V$. If $\dim S = 0$, (29) is trivially satisfied. We suppose now

$\dim S \geqslant 1$, and $x_1 \in \mathrm{op}(S)$ given. By Lemma 7 there is a homeomorphism $g\colon |K| \to |K|$, $gx_0 = x_1$. As $x_0 \in V$, there is a neighborhood W of x_0 which is homeomorphic to an open n-ball. Then gW is a neighborhood of x_1 with the same property, thus $gW \subset V$ by the maximality of V. This proves condition (29) of Lemma 3. By this lemma, C is the space of a subcomplex of K. This being true for every triangulation (3) of X, C is an absolute subcomplex of X. Thus the statement of Theorem 4 holds true for any Y which is $\neq \emptyset$ and an absolute subcomplex of C, for example $Y = C$.

If X is a triangulable space and (3) is a triangulation of X then X is partioned into the sets $f(\mathrm{op}(S))$, where S runs through the simplices of K and $f(\mathrm{op}(S))$ is a C^0-manifold. This decomposition of X is useful in some construction, but clearly has no immediate connection with global geometric properties of X. In the following theorem a decomposition is obtained using absolute subcomplexes and it is expected that this decomposition has a meaning for the global and local properties of X. To obtain this decomposition we will use Theorem 1 and the ideas of the proof of Theorem 4 (in particular, lemmas 3 and 7).

THEOREM 12. *If X is a triangulable space there is a sequence*

$$A_0 = X \supset A_1 \supset \ldots \supset A_m \supset A_{m+1} = \emptyset \tag{76}$$

*of absolute subcomplexes of X with the following properties. $d(A_{i+1}; x) < d(A_i; x)$, $x \in A_{i+1}$, in particular $\dim A_{i+1} < \dim A_i$, thus $m \leqslant \dim X$. $A_i - A_{i+1}$ is open and everywhere dense in A_i, and each component of $A_i - A_{i+1}$ is a C^0-**manifold**, $i = 0, \ldots, m$. Furthermore $A_1 \supset B_1$ where B_1 is the first critical set of the integral Betti stack of X, thus $X - A_1 \subset X - B_1 = X - C_1$* (see (1), (2)).

REMARK. It does **not** follow from this theorem that every absolute subcomplex Y of X is necessarily contained in one of the A_i's, $i \geqslant 1$ (see Example 1).

PROOF. Let us define the A_i's by an induction on i. For this end we will describe A_1. Let B_1 be the first critical set of the integral Betti stack of X, so that each component of $X - B_1$ is a homology manifold and an open subset of X. Let

$$X - B_1 = \bigcup_{j=1}^{r} M_j \tag{77}$$

be the component decomposition, where each M_j is a connected homology manifold by Theorem 9. As M_j is not compact, Theorem 4 does not apply.

However, using the proof of this theorem, we will show presently that a $M_j - Y_j$ is a C^0-manifold for suitable Y_j.

We select and fix a component M_j from (77). We consider the family of sets

$$\{U\colon U \text{ open in } M_j;\ U \text{ a } C^0\text{-manifold}\} \quad (j \text{ fixed}). \tag{78}$$

As in the proof of Theorem 4 (see (75)), we see that if (3) is a triangulation of X, and S is a top simplex of K then $f(\mathrm{op}(S)) \subset X - B_1$, hence, if $f(\mathrm{op}(S))$ intersects M_j it is contained in M_j, and these sets belong to the familily (78). Also, the union of all elements of (78) is an element, thus we have a maximal set V belonging to (78). As in the proof of Theorem 4, we will show now that

$$Y_j = (M_j - V) \cup B_1 \tag{79}$$

is an absolute subcomplex of X.

We select a triangulation (3) of X and in the following arguments we identify $|K|$ to X via f. We note that $X - Y_j = W$ is an open subset of X. In fact, if $x \in W$, then $x \notin B_1$ thus x is in (77). As (77) is a component decomposition, either $x \in M_i$ for $i \neq j$, or $x \in M_j$. In the first case, M_i is a neighborhood of x in W, as M_i is an open subset of X. In the second case $x \in V$, and V being open in M_j it is also open in X, thus V is a neighborhood of x which is contained in W. We will show now that W satisfies condition (29) of Lemma 3. This is done in the same way as in the proof of Theorem 4 (see second paragraph after (75)). We select a simplex S of K and we suppose $x_0 \in \mathrm{op}(S) \cap W$. We suppose given an $x_1 \in \mathrm{op}(S)$ and we construct $g\colon X \to X$, $gx_0 = x_1$ using Lemma 7. The point x_0 has an open neighborhood $U' \subset V$, U' open in X, which is homeomorphic to a p-ball. As x_0 is in M_j and $gB_1 = B_1$, $gx_0 = x_1$ is also in M_j ((77) was the component decomposition of $X - B_1$ thus a connected set S cannot intersect different components), and we may suppose that U', gU' are in M_j (selecting U' smaller if necessary, which involves only the continuity of g). Thus gU' is a neighborhood of x_1 in X which is homeomorphic to a p-ball, and $gU' \subset M_j$. Thus $gU' \subset V$. This shows $x_1 \in \mathrm{op}(S) \cap W$ and by Lemma 3 we conclude that the complement Y_j of W is the space of a subcomplex of K. We also note

$$d(Y_j; x) < d(X; x) \quad (x \in Y_j) \tag{80}$$

resulting from this discussion (we have seen that Y_j does not contain top simplices of X). As (3) was an arbitrary triangulation of X we conclude that Y_j is an absolute subcomplex of X.

For every j, $j=1, \ldots, r$ (see (77)) we construct the corresponding Y_j, and we note $Y_i \cap Y_j = B_1$ if $i \neq j$. We set

$$A_1 = \bigcup_{j=1}^{r} Y_j \tag{81}$$

introducing thus the first set in (76). The construction above is then an assignment

$$X \to A_1 \tag{82}$$

defined for every triangulable space X. Iterating the map (82) we have $A_1 \mapsto A_2, \ldots, A_i \mapsto A_{i+1}$. All statements of the theorem follow then. We note finally that in the definition of A_2 the Betti stack of A_1 is important, and the Betti stack of this space cannot be obtained without the knowledge of Y_j's in (81), hence it is not directly related to the Betti stack of X.

PROOF OF COROLLARY 2. Given an arbitrary triangulable space, we have (76). If $\emptyset$ and X are the only absolute subcomplexes of X, $A_1 = \emptyset$ (as $A_1 = X$ is impossible because of $\dim A_1 < \dim X$). Thus $X = X - A_1$ is a C^0-manifold. Conversely, if a triangulable space is a C^0-manifold, it has only $\emptyset$, X as absolute subcomplexes by Theorem 2. This completes the proof of Corollary 2.

16. – We have now a number of results which can be interpreted as maps $X \mapsto Y$ associating with a triangulable space X an absolute subcomplex Y of X. For example, given an integer $i \geqslant 0$, a coefficient group G, Theorem 1 associates with X the absolute subcomplex $B_i = B_i(X, G)$. As in Corollary 4, we can construct a monoid Γ operating on the lattice of absolute subcomplexes of a fixed triangulable space X (in fact Γ now contains the monoid utilized in Corollary 4). Except for some trivial remarks on dimension we do not know about the relations in the action of this monoid.

17. – The question of absolute subcomplexes is clearly related to the subjects of [19], [20], [21]. Our topic is also relevant in connection with [2, 3, 5, 8, 9, 10, 15]. To keep this paper elementary and reasonably short, we avoided to make exhaustive references, however, we intend to come back to these questions in later publications. To finish, we will discuss the relation of Definition 4 with a former definition of the author. We will not repeat definitions and results of [8], but we will use only pp. 442-454 of that paper.

Let us recall that a space X is locally connected, if components of open sets of X are open in X. We say that X is locally connected *at* x_0, if x_0 has a fundamental system $\{U\}$ of compact neighborhoods U which are locally connected spaces.

LEMMA 13. Let X be a locally compact, Hausdorff space, and F a continuous stack over X. If x_0 is a non-critical point of F in the sense of Definition 4, and X is locally connected at x_0, then x_0 is non-critical for F in the sense of Definition 1 on p. 449 of [8].

REMARKS. If X is not locally connected at x_0, then x_0 is necessarily a critical point for every continuous stack F over X in the sense of Definition 1 on p. 449 of [8]. For this reason Definition 4 may be more useful in some cases as it was in this paper.

PROOF. In virtue of Definition 4, x_0 has a compact neighborhood V and there is a subgroup E of $F(V)$ such that $e_x = f_{xV}|E$ has a constant kernel and zero cokernel for all $x \in V$. As X is locally connected at x_0 we may suppose that V is connected and locally connected. Let us define the stack G over V as follows: $G(A) = f_{AV}(E)$, and $G(A) \to G(B)$ is the appropriate restriction of f_{BA} $(V \supset A \supset B)$. The stack G and the restriction of F to V are then pointwise isomorphic (« ponctuellement isomorphe » in [8]) according to Proposition 3 on p. 447 of [8], as $\operatorname{coker} e_x = 0$ for $x \in V$. We set $G''(A) = \{a \in G(A)\colon e_x(a) = 0, x \in V\}$, define $G''(A) \to G''(B)$, $A \supset B$ as the appropriate restriction of f_{BA}, and define the stack G' from the exact sequence $0 \to G'' \to G \to G' \to 0$. Then G' is clearly a constant stack, and G, G' are pointwise isomorphic in virtue of Proposition 3 on p. 447 of [8]. This completes the proof of the lemma.

LEMMA 14. Let X be a locally compact, Hausdorff space, F a continuous stack over X, and $x_0 \in X$. If x_0 is a non-critical point of F in the sense of Definition 1 on p. 449 of [8], and $F(x_0)$ is a finitely generated group, then x_0 is a non-critical point of F in the sense of Definition 4.

PROOF. By the hypothesis of local connectedness (see paragraph preceding Lemma 13 and Remarks after the lemma), we can find a compact, connected, locally connected neighborhood U of x_0, such that the restriction of F to U is pointwise isomorphic to the constant $\simeq F(x_0)$ stack; for simplicity this restriction of F will be denoted also F. Consequently the zero dimensional cohomology sheaf $\mathfrak{C}_F$ of F has the following property: the map $\sigma \mapsto \sigma(x)$, $\sigma \in \mathfrak{C}_F(A)$, $x \in A$, is an isomorphism of $\mathfrak{C}_F(A)$ onto $F(x)$ for any connected closed set A and $x \in A$, $A \subset U$. We select generators

$a_1, \ldots, a_m$ in $F(x_0)$; we have $\sigma_1, \ldots, \sigma_m \in \mathcal{C}_F(U)$, $\sigma_i(x_0) = a_i$, $i = 1, \ldots, m$, generators of the group of cross sections (« systèmes cohérents » in [8], see p. 446). By continuity of the cross sections, $\sigma_i(x) = xb_i$ in some neighborhoods of x_0, hence there is a compact, connected neighborhood V of x_0, such that these equations hold true for $b_i \in F(V)$, $i = 1, \ldots, m$. Let E be the subgroup of $F(V)$ generated by $b_1, \ldots, b_m$, and let us show that for these data $E \subset F(V)$ the conditions of Definition 4 are satisfied. If

$$x(\sum m_i b_i) = \sum m_i x b_i = \sum m_i \sigma_i(x) = 0$$

for a point $x \in V$, then the same holds true for all points of V. Consequently $\ker e_x$ is independent of x. We have also $\operatorname{coker} e_x = 0$ by construction, hence the conditions of Definition 4 are satisfied. This completes the proof of the lemma.

THEOREM 13. Let X be a locally compact, Hausdorff space, which is locally connected at x_0, and F a continuous stack over X for which $F(x_0)$ is finitely generated. Under these conditions, x_0 is non-critical for F in the sense of Definition 4, if and only if it is non-critical in the sense of Definition 1 on p. 449 of [8].

COROLLARY 5. *For the Betti stack of a triangulable space the two definitions of critical points and of critical sets coincide.*

REMARK. Even in a triangulable space there are continuous stacks for which the first critical set is not locally connected, and in that case the two definitions of critical sets may give different results.

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