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On Signorini's Problem and Variational Problems with Thin Obstacles.

JENS FREHSE (*)

dedicated to Hans Lewy

0. - Introduction.

In this paper we study the question of the continuity of the first derivatives of variational problems or variational inequalities with obstacles at the boundary or thin interior obstacles, i.e. problems of the type:

(0.1) minimize $J(u) = \int F(x, u, \nabla u)\, dx,\ u \in \mathbf{K};$

or

(0.2) find $u \in \mathbf{K}$ such that

$$\sum_i \int F_i(x, u, \nabla u)\, \partial_i(u - v)\, dx \leqslant 0 \qquad \text{for all } v \in \mathbf{K},\ (i = 0, \ldots, n)\,.$$

$\mathbf{K}$ is one of the sets (0.3) or (0.4)

(0.3) « Obstacles at the boundary ». $\mathbf{K} = \{v \in H^{1,p}(\Omega);\ v \geqslant \psi \text{ on } \partial\Omega\}$

(0.4) « Interior thin obstacles ». $\mathbf{K} = \{v \in g + H_0^{1,p};\ v \geqslant \psi \text{ on } L\}\,.$

Here Ω is a bounded open subset of $\mathbf{K}^n \cdot H^{1,p}$, $(H_0^{1,p})$ is the usual Sobolev space over Ω (with zero boundary conditions) (see [18]). The relation $v \geqslant \psi$ is to be understood in the sense of $H^{1,p}$, see [17], p. 155. L is an oriented $(n-1)$-

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dimensional submanifold of Ω (with or without boundary), for example a line, and the compatibility condition

$$g \geqslant \psi \quad \text{on } \partial\Omega \cap L \tag{0.5}$$

holds where $g \in H^{1,p}$ represents the boundary condition. The functions F and F_i satisfy the following assumption:

(0.6) F and $F_i(x, u, \eta)$, $i = 0, \ldots; n$, are continuous in $(u, \eta) \in R^{1+n}$ and measurable in $x \in \bar{\Omega}$.

Case (0.3) is known in the literature as Signorini's problem. For its physical meaning and for older literature on this subject, see [7], 391-424. The study of case (0.4) was initiated by Hans Lewy [15], [16]. Meanwhile, both cases have been studied by many authors, e.g. Beirao da Veiga, Brezis, Giaquinta-Modica, Giusti, Kinderlehrer and Nitsche, see the bibliography. Roughly speaking, under natural conditions these authors obtain the existence of Lipschitz solutions for (0.1) or (0.2) including important cases such as the minimal surface case and the harmonic case. For a short description of these results we refer to [5].

Concerning higher regularity for the solutions u of (0.1) or (0.2), one expects the continuity of the first derivatives of u taken along the direction tangential to L or $\partial\Omega$ and the one-sided continuity of the corresponding normal derivative of u if the data is smooth.

The first positive result on this question is due to Hans Lewy who considered the two dimensional Dirichlet integral in the interior obstacle case. There are some indications that it is not possible to obtain an a-priori-estimate for the Hölder exponent of $\partial_j u$ depending on the data in a simple way, see the remark at the end of the paper and, in fact, in [5], the author was only able to prove the continuity of the first «tangential» derivatives $\partial_t u$ of u and the one-sided continuity of the «normal» derivatives $\partial_n u$ of u with a logarithmic modulus of continuity, i.e. the oscillation $\operatorname{osc}_R \partial_t u$ of $\partial_t u$ over balls of radius R can be estimated by $K_q |\ln R|^{-q}$ for any q. In [5] we had to confine ourselves to two-dimensional problems. In this paper we treat the case $n \geqslant 2$ and obtain the continuity of the first derivatives of u taken along the direction tangential to L or $\partial\Omega$. For $n \geqslant 3$, however, we obtain only the estimate $\operatorname{osc}_R \partial_t u \leqslant K |\ln R|^{-2/(nn-2n)}$.

We repeat the proof for the case $n = 2$, since in [5] the calculation of the powers q was not precise. The technique for the proof is essentially developed in our paper [5], although we try here to obtain the optimal constants in the auxiliary lemmata at certain points.

Finally let us mention that not much is known on the structure of the coincidence set I except Hans Lewy's beautiful result that I is a finite union of intervals in the case of the Dirichlet integral and the interior obstacle case with analytic data.

As in [5], we immediately begin with Lipschitz solutions u of (1) or (2) since from the above references we know many situations in which Lipschitz solutions exist. Thus we need only the following conditions for the functions F_i:

(0.7) $F_i(x, u, \eta)$ is Lipschitz in $x \in \Omega$ and continuously differentiable in $(u, \eta) \in \mathbf{R}^{1+n}$, $i = 1, \ldots, n$.

(0.8) $F_0(x, u, \eta)$ is measurable in $x \in \Omega$ and continuous in (u, η).

(0.9) The functions F_i, $i = 0, 1, \ldots, n$ and their derivatives F_{ix}, F_{ik}, $i = 1, \ldots, n$, $k = 0, 1, \ldots, n$ are uniformly bounded on compact subsets of $\bar{\Omega} \times \mathbf{R}^{1+n}$.

(0.10) Elipticity. The matrix $(F_{ik})_{i,k=1}^{n}$ of the derivatives of $F_i(x, u, \eta)$ with respect to $\eta \in \mathbf{R}^n$ is uniformly positive definite on compact subsets of $\bar{\Omega} \times \mathbf{R}^{1+n}$.

We shall assume that $\partial\Omega$ and L are $H^{2,\infty}$-surfaces, i.e. for every $z \in \partial\Omega$ or $z \in L$ there exists an open neighbourhood U of z_0 and an $H^{2,\infty}$-diffeomorphism $f_z \colon U \to B_1(0)$ onto the unit ball $B = B_1(0) \subset R^n$ such that $\partial\Omega \cap B$ resp. $L \cap B$ is mapped into the hyperplane $H = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | x_n = 0\}$.

Moreover, in the case of the interior obstacle we assume $U \subset \Omega$ and in the case of Signorini's problem, if $z \in \partial\Omega$, $\Omega \cap U$ is mapped onto $B \cap H_1$ where H_1 is the upper half space $H_1 = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | x_n > 0\}$. H_2 denotes the lower half space.

We do not prove regularity at the points where L meets $\partial\Omega$.

With these conventions our result is:

THEOREM. *Let $\partial\Omega$ and L be $(n-1)$-dimensional regular $H^{2,\infty}$-surfaces and $\psi \in H^{2,\infty}(\bar{\Omega})$. Assume* (0.7)-(0.10) *for F_i and let u be a Lipschitz continuous solution of* (0.1) *or* (0.2). *Then, for any $z \in \partial\Omega$ or $\in L$, $(z \notin L \cap \partial\Omega)$, the function $v = u(f_z^{-1}(\cdot))$ defined on $B \cap H_1$ (Signorini's problem) or B (interior obstacle case) respectively has the following properties*:

(i) $\partial_j v \in C(B \cap H_1)$ *or* $\in C(B)$ *respectively,* $j = 1, \ldots, n-1$;

(ii) $\partial_n v \in C(B \cap H_1)$ *or* $\in C(B \cap H_i)$, $i = 1, 2$, *respectively,* $n = 2$;

(iii) $\operatorname{osc}_R \partial_j v \leqslant K |\ln R|^{-\alpha(n)}$, $j = 1, \ldots, n-1$, *where* $q(n) < 2/(n^2 - 2n)$ *if* $n \geq 3$ *and* $q(n)$ *is any constant* $q \geq 1$ *if* $n = 2$. osc_R *denotes the oscillation of* $\partial_j v$ *over* $B_R \cap H_1$ *or* B_R *respectively.*

(iv) ${}_i\operatorname{osc}_R \partial_n v \leq K|\ln R|^{-q}$ *for* $n = 2$, ${}_i\operatorname{osc}_R$ *denotes the oscillation of* $\partial_n v$ *over* $B_R \cap H_1$ *or* $B_R \cap H_i$, $i = 1, 2$, *respectively.*

The constant K *depends only on the data* (*including the Lipschitz constant for* u) *and* q.

The structure of the proof can be seen from § 3. § 1 and § 2 present auxiliary lemmata. In all estimates, K is a constant altering its value.

1. – Continuity tests.

In this chapter we prove some lemmata which guarantee the continuity of an H^1-function satisfying certain integral relations. They serve as tools for the proof of theorem 1 and might also be useful for other considerations.

The following lemma is a « logarithmic » analogue of the well known lemma of Morrey, [18], theorem 3.5.2.

LEMMA 1.1. *Suppose* $u \in H^{1,p}[B(x_0, R)]$, $1 \leq p \leq n$, *and suppose that there are constants* $\mu > 1$ *and* $L > 0$ *such that*

$$(1.1) \qquad \int_{B(x,r)} |\nabla u|^p \, dx \leq L^p (r/\delta)^{n-p} |\ln (r/\delta)|^{-\mu p}, \qquad 0 < r < \delta = R - |x - x_0|$$

for every $x \in B(x_0, R)$. *Then* $u \in C[B(x_0, r)]$ *for* $r < R$ *and*

$$|u(\xi) - u(x)| \leq CL\delta^{1-n/p} |\ln (|\xi - x|/\delta)|^{1-\mu} \qquad \text{for } |\xi - x| \leq \delta/2$$

where

$$C = 4\Gamma_n^{-1/p} (\mu - 1)^{-1},$$

Γ_n *being the volume of the unit ball in* $\mathbf{R}^n$.

PROOF. We copy Morrey's proof in [18], Theorem 3.5.2.

By approximation, we may assume $u \in C^1(B_R)$.

Let $x, \varrho \in B_R$ and $\varrho = |\xi - x|/2$, $\bar{x} = (\xi + x)/2$ and $\eta \in B(\bar{x}, \varrho)$. By simple arguments, Morrey obtains (see [18], (3.5.6))

$$|\bar{u}_{B(\bar{x},\varrho)} - u(\xi)| \leq 2\varrho |B(\bar{x}, \varrho)|^{-1} \int_{B(\bar{x},\varrho)} \left[\int_0^1 |\nabla u[\xi + t(\eta - \xi)]| \, dt \right] d\eta \, .$$

Here $\bar{u}_{B(\bar{x},\varrho)}$ is the mean value of u taken over $B(\bar{x}, \varrho)$.

We interchange the order of integration and set $y = \xi + t(\eta - \xi)$. Then y ranges over $B(\bar{x}_t, t\varrho)$, where $\bar{x}_t = (1-t)\xi + t\bar{x}$ and we obtain

$$|\bar{u}_{B(\bar{x},\varrho)} - u(\xi)| \leqslant 2\varrho |B_\varrho|^{-1} \int_0^1 \Big[\int_{B(\bar{x}_t, t\varrho)} |\nabla u(y)| dy \Big] t^{-n} dt := A .$$

Using Hölder's inequality and then (1.1) we estimate

$$A \leqslant 2\varrho |B_\varrho|^{-1} \int_0^1 |B_{t\varrho}|^{1-1/p} L(\varrho t/\delta)^{n/p-1} |\ln(\varrho t/\delta)|^{-\mu} t^{-n} dt =$$

$$= 2\varrho |B_\varrho|^{-1/p} \varrho^{n/p-1} L \delta^{1-n/p} \int_0^1 t^{-1} |\ln(\varrho t/\delta)|^{-\mu} dt =$$

$$= 2\Gamma_n^{-1/p} L \delta^{1-n/p} \int_0^{\varrho/\delta} \tau^{-1} |\ln \tau|^{-\mu} dt = 2\Gamma_n^{-1/p} (\mu - 1)^{-1} L \delta^{1-n/p} |\ln(\varrho/\delta)|^{1-\mu} .$$

Using the same result for x instead of ξ we obtain an estimate for the modulus of continuity and thus the theorem.

The following lemma is used for the proof of lemma 1.3. The proof relies on Moser's idea, see [18], chap. 5.3.

LEMMA 1.2. *Choose* $p_0 \geqslant 2$ *and define* $p_{i+1} = \gamma p_i + 2$, $i = 0, 1, 2, \ldots$, *where* $\gamma = n/(n-2)$ *for* $n \geqslant 3$, $\gamma = 2$ *for* $n = 2$.

Let Φ_i *be non-negative numbers such that*

$$\Phi_i \leqslant L p^s_{p_{i-1}} R^{-s\gamma} \Phi^\gamma_{i-1} \tag{1.2}$$

where $r \in \mathbf{R}$, $s = 2$ *for* $n \geqslant 3$, *and* $s = 1$ *for* $n = 2$.

Then

$$\limsup \Phi_i^{1/p_i} \leqslant K \hat{L} [R^{-n} \Phi_0]^E , \quad i \to \infty , \tag{1.3}$$

where

$$E = 1/(p_0 + n - 2) \quad \textit{for } n \geqslant 3$$

and

$$E = 1/(p_0 + 2) \quad \textit{for } n = 2 ,$$

$\hat{L} = L^H$, $H = 1/\big(p_0(\gamma - 1) + 2\big)$; *$K$ does not depend on* p_0, L, R.

PROOF. By recursion

$$p_i = \gamma^i p_0 + 2(\gamma^i - 1)/(\gamma - 1)$$

and

$$\Phi_i \leqslant C_i(LR^{-s\gamma})^{(\gamma^i-1)/(\gamma-1)}\Phi_0^{\gamma^i}, \qquad i = 1, 2, \dots, \tag{1.4}$$

where $C = \prod_{j=0}^{i-1} p_j^{r\gamma^{i-j-1}}$.

We have the following convergence relations

$$\limsup C_i^{1/p_i} \leqslant K, \qquad (i \to \infty),$$

$$[(\gamma^i - 1)/(\gamma - 1)]/p_i \to 1/[p_0(\gamma - 1) + 2] \quad (i \to \infty),$$

$$\gamma^i/p_i \to 1/(p_0 + 2(\gamma - 1)^{-1}), \qquad (i \to \infty).$$

Since

$$1/[p_0 + 2(\gamma - 1)^{-1}] = 1/[p_0 + 2] \qquad \text{for } n = 2$$

and

$$= 1/[p_0 + n - 2] \quad \text{for } n \geqslant 3,$$

and since

$$s\gamma/[p_0(\gamma - 1) + 2] = 2/[p_0 + 2] \qquad \text{for } n = 2$$

and

$$= n/[p_0 + n - 2] \quad \text{for } n \geqslant 3$$

we obtain from (1.4)

$$\limsup \Phi_i^{1/p_i} \leqslant K\hat{L}[R^{-n}\Phi_0]^E, \qquad i \to \infty. \qquad \text{q.e.d.}$$

LEMMA 1.3. *Let* $z \in H^{1,2}(B_\varrho(x_0)) \cap L^\infty$ *and* $c \in \mathbf{R}$ *with* $|c| \leqslant \|z\|_\infty$ *and suppose*

$$\int |\nabla z|^2 |z - c|^{p-2}\tau^2\,dx \leqslant K\int |\nabla z| \cdot |z - c|^{p-2}(|\nabla\tau| + 1)\,\tau\,dx \tag{1.5}$$

for all $\tau \in H^{1,\infty}(B_\varrho)$, $\tau \geqslant 0$ *which vanish on* $\partial B_\varrho(x_0)$ *and all* $p \geqslant 2$.

Then, for any $R \leqslant \varrho/2$, $R > 0$, *the following inequalities hold*:

for $n \geqslant 3$,

$$\|z - c\|_{\infty;B_R} \leqslant K\Big[R^{-n}\int_* |z - c|^{2n/(n-2)}\,dx\Big]^{(n-2)n^{-2}} + KR^\alpha \tag{1.6}$$

with $\alpha = 2(n-2)n^{-2}$ *and* $* = B_{2R}(x_0) - B_R(x_0)$;

and for $n = 2$

$$\|z - c\|_{\infty;B_R} \leqslant K\Big[R^{-2}\int_* |z - c|^t\,dx\Big]^{1/(t+4)} + K\Big[\int_{2R} |z - c|^t\,dx\Big]^{1/(t+4)}. \tag{1.7}$$

Here $\int_{2R}$ denotes integration over $B_{2R}(x_0)$.

PROOF. Let $B_i \supset B_{i+1}$, $i = 1, 2, \ldots$, be a sequence of concentric balls with center x_0 and radii

$$R_i = \left(2 - 6\pi^{-2}\sum_{j=1}^{i} j^{-2}\right)R, \qquad i = 1, 2, \ldots,$$
$$R_0 = 2R.$$

Choose $\tau_i \in H^{1,\infty}(B_\varrho)$ such that $\operatorname{supp}\tau_i \subset B_i$, and $\tau_i = 1$ on B_{i+1}, $\tau \geqslant 0$, and $|\nabla\tau_i| \leqslant (R_i - R_{i+1})^{-1} \leqslant K(i+1)^2$.

By (1.5)

$$\int |\nabla[\tau(z-c)^{p/2}]|^2\,dx \leqslant p^2 K \int |z-c|^{p-2}\big(|\nabla\tau|^2 + \chi_\tau\big)\,dx, \tag{1.8}$$

χ_τ being the characteristic function of $\operatorname{supp}\tau$.

Here K may depend on $\|z\|_\infty$.

We consider the cases $n \geqslant 3$ and $n = 2$ separately:

(i) $n \geqslant 3$. By Sobolev's inequality with $\gamma = n/(n-2)$,

$$\int |z-c|^{p\gamma}\tau^{2\gamma}\,dx \leqslant Kp^{2\gamma}\left(\int |z-c|^{p-2}\big(|\nabla\tau|^2 + \chi_\tau\big)\,dx\right) \tag{1.9}$$

where K does not depend on $\operatorname{supp}\tau$.

Now set $\tau = \tau_i$ and let $\int_i$ denote integration over B_i.

Let p_i be defined as in lemma 1.2 and set

$$\Phi_i = \int_i |z-c|^{p_i-2}\,dx, \qquad i \geqslant 1$$

and

$$\Phi_0 = \int |z-c|^{p_0-2}\big(\chi(B_{2R} - B_R) + R^2\chi(B_{2R})\big)\,dx$$

where $\chi(M)$ is the characteristic function of a set M.

We estimate $|\nabla\tau_i| \leqslant Kp_iR^{-1}$, use (1.9) with $\tau = \tau_i$ and $p = p_i$, and thereby obtain

$$\Phi_{i+1} \leqslant Kp_i^{4\gamma}R^{-2\gamma}\Phi_i^\gamma, \qquad i \geqslant 0.$$

(For $i = 0$, we have used the fact that $\nabla\tau = 0$ on B_R.)

By lemma 1.2, we obtain

$$\|z-c\|_{\infty;B_R} \leqslant K[R^{-n}\Phi_0]^E, \qquad E = 1/(p_0 + n - 2).$$

We finally set $p_0 = 2 + 2n/(n-2)$ and obtain inequality (1.6), since $E = 1/[2n/(n-2) + n] = (n-2)/n^2$.

(ii) The case $n = 2$. By Sobolev's inequality

$$(1.10) \qquad \int |z-c|^{2p}\tau^4\,dx \leqslant K\Big(\int |\nabla(\tau|z-c|^{p/2})|^{4/3}\,dx\Big)^3 \leqslant$$

$$\leqslant K\Big(\int |\nabla(\tau|z-c|^{p/2})|^2\,dx\Big)^2\cdot|\text{supp}\,\tau|\,.$$

Again K does not depend on supp τ.

From (1.8) and (1.10) we obtain

$$\int |z-c|^{2p}\tau^4\,dx \leqslant K|\text{supp}\,\tau|p^4\Big(\int |z-c|^{p-2}(|\nabla\tau|^2+\chi_\tau)\,dx\Big)^2.$$

Choosing τ_i, p_i and Φ_i as in case (i)—with $\gamma = 2$—we conclude

$$\Phi_{i+1} \leqslant K p_i^4 R^{-\gamma}\Phi_i^\gamma\,, \qquad i \geqslant 0\,.$$

By lemma 1.2

$$\|z-c\|_\infty \leqslant K[R^{-2}\Phi_0]^E\,, \qquad E = 1/(p_0+2)\,.$$

Setting $p_0 = t+2$, we obtain (1.7).

LEMMA 1.4. *Under the hypotheses of lemma 1.3 there holds for* $n \geqslant 3$

$$(1.11) \qquad \text{osc}\{z(x)\,|\,x\in B_R(x_0)\} \leqslant K\Big[R^{2-n}\int_* |\nabla z|^2 dx\Big]^{1/n} + KR^\alpha\,,$$

and for $n = 2$

$$(1.12) \qquad \text{osc}\,\{z(x)\,|\,x\in B_R(x_0)\} \leqslant K\Big[\int_* |\nabla z|^2\,dx\Big]^{\frac{1}{2}-2/(t+4)} +$$

$$+ KR^{2/(t+4)}\Big[\int_{2R} |\nabla z|^2\,dx\Big]^{\frac{1}{2}-2/(t+4)} + KR^\alpha\,.$$

PROOF. We choose the constant c in lemma 1.3 equal to the mean value of z taken over $B_{2R}(x_0) - B_R(x_0)$. This yields for $n \geqslant 3$

$$\Big(\int_* |z-c|^{2n/(n-2)}\,dx\Big)^{(n-2)/n} \leqslant K\int_* |\nabla z|^2\,dx$$

and inequality (1.11) follows using (1.6).

For $n = 2$ we have for $t > 2$

$$\int_* |z-c|^t\,dx \leqslant K\Big(\int_* |\nabla z|^{2t/(t+2)}\,dx\Big)^{(t+2)/2} \leqslant KR^2\Big(\int_* |\nabla z|^2\,dx\Big)^{2/t}$$

and we obtain (1.12).

In order to treat the regularity up to the boundary, we need an analogue for lemma 1.3 and 1.4 in half spaces. For this purpose let

$$H_1 = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | x_n > 0\}.$$

LEMMA 1.5. *Let $z \in H^{1,2}(B_\varrho(x_0) \cap H_1) \cap L^\infty$ and assume condition* (1.5) *the integration $\int$ being carried out only over $B_\varrho(x_0) \cap H_1$. Then there holds for $n \geq 3$*

$$\text{(1.13)} \qquad \operatorname{osc}\{z(x) | x \in B_R(x_0) \cap H_1\} \leq K\Big[\int_{**} |\nabla z|^2 dx\Big]^{1/n} + KR^\alpha$$

and for $n = 2$

$$\text{(1.14)} \qquad \operatorname{osc}\{z(x) | x \in B_R(x_0) \cap H_1\} \leq K\Big[\int_{**} |\nabla z|^2 dx\Big]^{\frac{1}{2} - 2/(t+4)} + KR^{2/(t+4)}\Big[\int_{*R} |\nabla z|^2 dx\Big]^{\frac{1}{2} - 2/(t+4)} + KR^\alpha .$$

Here $\int_{**}$ denotes integration over $(B_{2R}(x_0) - B_R(x_0)) \cap H_1$ and $\int_{*R}$ integration over $B_{2R} \cap H_1$.

The proof is similar to that of lemma 1.4 and 1.5. One has to choose the constant c of lemma 1.4 to be equal to the mean value of z over $(B_{2R} - B_R) \cap H_1$. One must also observe that we may still apply Sobolev's inequality to the function $\tau^2 |u - c|^{p/2}$, $\tau \in H_0^{1,\infty}(B_{2R})$ since the set of zeros of τ in Ω is large enough.

LEMMA 1.6. *Let $\Phi\colon [0, R_0] \to \mathbf{R}^1$ be an increasing nonnegative function such that*

$$\Phi(R) \leq K[\Phi(\sigma R) - \Phi(R)]^{1-1/q} + L|\ln(R_0/R)|^{-t}, \qquad 0 \leq R \leq \sigma^{-1} R_0,$$

with constants $\sigma > 1$, $q > 1$, R_0, $K > 0$, $t > 0$ and *$L \geq 0$.*
Then, for every $s \geq \max\{2, 2t\}$ there is a constant $C = C(\sigma, q, s, t)$ such that

$$\Phi(R) \leq C|\ln(R_0/R)|^{-s}\Phi(R_0) + CK^q|\ln(R_0/R)|^{1-q}|\ln\ln(R_0/R)|^{q-1} + CL|\ln(R_0/R)|^{-t}|\ln\ln(R_0/R)|, \qquad \text{for all } R \leq \sigma^{-1}R_0 .$$

The constant C does not depend on K, L, R_0.

REMARK. Let $\Phi(R) = |\ln R|^{-1}$, $q = 2$, $L = 0$.

Then the above hypothesis is fulfilled and we see that the power $|\ln(R_0/R)|^{1-q}$ in the estimate cannot be improved.

PROOF. We use Young's inequality

$$ab \leqslant q^{-1}a^q + p^{-1}b^p, \qquad p = q/(q-1)$$

with

$$a = K(\varepsilon/p)^{1-1/q}, \qquad b = (\varepsilon/p)^{1/q-1}[\Phi(\sigma R) - \Phi(R)]^{1-1/q}, \qquad \varepsilon > 0,$$

and obtain

$$\Phi(R) \leqslant \varepsilon^{-1}\big(\Phi(\sigma R) - \Phi(R)\big) + CK^q\varepsilon^{q-1} + L|\ln(R_0/R)|^{-t}. \tag{1.15}$$

Here and in the following, the constant C depends only on σ, q, s, t and may change its value. By (1.15)

$$\Phi(R) \leqslant (1+\varepsilon)^{-1}\Phi(\sigma R) + CK^q\varepsilon^q(1+\varepsilon)^{-1} + L\varepsilon(1+\varepsilon)^{-1}|\ln(R_0/R)|^{-t} \tag{1.16}$$

Applying this inequality with $R = R_j = \sigma^{-j}R_0$, $j = 1, \dots, N$, we obtain by iteration

$$\Phi(R_N) \leqslant (1+\varepsilon)^{-N}\Phi(R_0) + CK^q\varepsilon^q(1+\varepsilon)^{-1}\sum_{j=0}^{N-1}(1+\varepsilon)^{-j} + \tag{1.17}$$

$$+ L\varepsilon(1+\varepsilon)^{-1}|\ln\sigma|^{-t}\sum_{j=0}^{N-1}(1+\varepsilon)^{-j}(N-j)^{-t} \leqslant (1+\varepsilon)^{-N}\Phi(R_0) + CK^q\varepsilon^{q-1} + CL\varepsilon A$$

with $A = \sum_{j=0}^{N-1}(1+\varepsilon)^{-j}(N-j)^{-t}$.

If $N/2 \leqslant j \leqslant N-1$, then $(1+\varepsilon)^{-j}(N-j)^{-t} \leqslant (1+\varepsilon)^{-N/2}$ and if $0 \leqslant j \leqslant N/2$, then $(1+\varepsilon)^{-j}(N-j)^{-t} \leqslant 2^tN^{-t}$.

Thus

$$A \leqslant N(1+\varepsilon)^{-N/2} + 2^tN^{-t+1}. \tag{1.18}$$

Set $\varepsilon = \alpha N^{-1}$, $\alpha > 0$. Since $(1+M^{-1})^{M+1} \geqslant e$, $M \geqslant 1$, there holds

$$(1+M^{-1})^{-M} \leqslant e^{-1}(1+M^{-1}),$$

and, with $M = N\alpha^{-1}$,

$$(1+\varepsilon)^{-N} = (1+\alpha N^{-1})^{-N} = (1+M)^{-\alpha M} \leqslant e^{-\alpha}(1+\alpha N^{-1})^{\alpha} \leqslant e^{-\alpha}e^{1/N} \leqslant 2e^{-\alpha}.$$

Setting $\alpha = s\ln N$, $N \geqslant 2$, we obtain

$$(1+\varepsilon)^{-N} \leqslant 2N^{-s}. \tag{1.19}$$

This holds, if $M \geqslant 1$, $N \geqslant 2$, i.e. $N \geqslant N_s$.

Enlarging the constant at the right hand side of (1.19) we obtain

$$(1+\varepsilon)^{-N} \leqslant CN^{-s}, \qquad N \in \mathbf{N},\ N \geqslant 2 .$$

By (1.18) and the condition $s \geqslant \max\{2, 2t\}$

$$A \leqslant CN^{-t+1}$$

and by (1.17)

$$\Phi(R_N) \leqslant CN^{-s}\Phi(R_0) + CK^q N^{-q+1}|\ln N|^{q-1} + CLN^{-t}\ln N , \qquad N \geqslant 2 . \tag{1.20}$$

Since $N = |\ln \sigma|^{-1} ln(R_0/R)$, we obtain the statement of the lemma for $R = R_N$, $N \geqslant 2$. By enlarging C in an admissible way, we obtain it for $R = R_1$ using (1.16). The general case $R \in [R_N, R_{N-1}]$, $N \geqslant 1$ follows easily.

Lemmata 1.1 and 1.6 have an interesting consequence

LEMMA 1.7. *Let $z \in H^1(B_1)$ and $\int |\nabla z|^2 |x - x_0|^{2-n} dx \leqslant K'$, $x_0 \in B_{\frac{1}{2}}$. Suppose*

$$\int_R |\nabla z|^2 |x - x_0|^{2-n} dx \leqslant K\Big(\int_{**} |\nabla z|^2 |x - x_0|^{2-n} dx\Big)^\beta + KR^\alpha$$

with constants K and α, $\beta \in]0, 1]$.

$\int_R$ and $\int_{**}$ denote integration over $B_R(x_0) \subset B_1$, $B_{2R}(x_0) - B_R(x_0) \subset B_1$.

Then z is continuous in B_r, $r < \frac{1}{2}$, if $\beta > \frac{2}{3}$.

2. – Admissible variations and properties of the solution.

In the following, let $\mathbf{K}$ be one of the sets

(i) $\mathbf{K} = \{v \in H^{1,\infty}(\Omega),\ v \geqslant \psi \text{ on } \partial\Omega\}$ or

(ii) $\mathbf{K} = \{v \in H_0^{1,\infty}(\Omega),\ v \geqslant \psi \text{ on } L\}$

where L is an $(n-1)$-dimensional manifold and $\psi \in H^{1,\infty}(\bar{\Omega})$.

We assume that for a ball $B_\varrho \subset R^n$ the set $B_\varrho \cap \partial\Omega$ or $B_\varrho \cap L$, respectively, is a portion of a hyperplane, i.e. $B_\varrho \cap \partial\Omega = B_\varrho \cap H$ etc. where $H = \{(x_1, \ldots, x_n) \in \mathbf{R}^n,\ x_n = 0\}$. In case (ii), let $B_\varrho \subset \Omega$.

Furthermore let $[\xi]^p = |\xi|^{p-1}\xi$ and

$$D_i^{\pm h} w(x) = \pm h^{-1}[w(x \pm e_i h) - w(x)] , \qquad E_i^{\pm h} w(x \pm e_i h) ,$$

e_i being the i-th unit vector.

LEMMA 2.1. *Let* $u \in \mathbf{K}$ *and* $\varphi \in H_0^{1,\infty}(B_\varrho)$, $\operatorname{supp}\varphi \subset\subset B_\varrho$, $\varphi \geqslant 0$. *Then, for each* $h \in]0, \operatorname{dist}(\partial\Omega, \partial B_\varrho)[$, $p \geqslant 1$, *and* $c \in \mathbf{R}$, *there is an* $\varepsilon > 0$ *such that the functions*

$$u_\varepsilon := u + \varepsilon D_j^{-h}[\varphi D_i^h(u - \psi)]\,, \quad j = 0, 1, \ldots, n-1$$

and

$$u_{\varepsilon_p} = u + \varepsilon\varphi D_j^{-h}[D_i^h(u - \psi) - c]^p\,, \quad j = 0, 1, \ldots, n-1$$

are in $\mathbf{K}$.

PROOF. A simple calculation shows that $u_\varepsilon \in \mathbf{K}$ for $0 < \varepsilon \leqslant \frac{1}{2}h^2$, see e.g. [5]. For the inclusion $u_{\varepsilon_p} \in \mathbf{K}$, we observe that $[\cdot]^p$ is monotone increasing and thus, for $x \in B_\varrho \cap H$,

$$[D_i^h(u - \psi) - c]^p(x) \geqslant [h^{-1}(u - \psi)(x) - c]^p$$
$$- E_i^{-h}[D_j^h(u - \psi) - c]^p(x) \geqslant -[-h^{-1}(u - \psi)(x) - c]^p\,.$$

We have used the fact that $(u - \psi)(x \pm e_i h) \geqslant 0$.

Thus

$$u_{\varepsilon_p}(x) \geqslant u(x) + \varepsilon h^{-1}\varphi(x)\{[h^{-1}(u - \psi)(x) - c]^p - [-h^{-1}(u - \psi)(x) - c]^p\}$$

or

$$u_{\varepsilon_p}(x) \geqslant f\big(u(x) - \psi(x)\big) + \psi(x)$$

with

$$f(\zeta) = \zeta + \varepsilon h^{-1}\varphi(x)\{[h^{-1}\zeta - c]^p - [h^{-1}\zeta - c]^p\}\,.$$

The function f is monotone increasing in any fixed interval, if we choose $\varepsilon > 0$ small enough (calculate the first derivatives!). Thus $f\big(u(x) - \psi(x)\big) \geqslant f(0) = 0$ for $x \in B_\varrho \cap H$ and $u_{\varepsilon_p}(x) \geqslant \psi(x)$. The conclusion follows.

In the following, we assume that H_1 and H_2 are the two half spaces which are separated by the hyperplane

$$H = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | x_n = 0\}\,.$$

In the case (i)—the boundary obstacle case—we assume that

$$B_\varrho(x_0) \cap \Omega = B_\varrho(x_0) \cap H_1\,.$$

LEMMA 2.2. *Let* $u \in \mathbf{K}$ *be a solution of* (1) *or* (2) *and assume the conditions* (0.7)-(0.10). *Then* $u \in H^2\big(B_{\varrho/2}(x_0) \cap H_j\big)$, $j = 1$ *in case* (i) *and* $j = 1, 2$

in case (ii). *Moreover* $\int |\nabla^2 u|^2 |x-x_0|^{2-n} dx < \infty$ *the integration being carried out over* $B_\varrho(x_0) \cap H_j$.

(Note that $\nabla^2 u$ exists only in $\Omega \cap H_j$, not necessarily in Ω).

PROOF. (*a*) We insert the function u_ε of lemma 2.1 into the variational inequality and obtain

$$(2.1) \qquad -\sum_{i=0}^{n} \big(F_i(\cdot, u, \nabla u), \varepsilon \partial_i \big(D_j^{-h}(\varphi D_j^h(u-\psi))\big)\big) \leqslant 0 , \qquad j = 1, \dots, n-1 .$$

Cancelling $\varepsilon > 0$ and setting $\varphi = \tau^2$, $\tau \in C_0^\infty\big(B_\varrho(x_0)\big)$ we obtain by standard arguments a uniform bound for $D_j^h \nabla u$ in L^2 as $h \to 0$, $j = 1, \dots, n-1$, and thus $\partial_j \nabla u \in L^2\big(B_\varrho(x_0) \cap H_j\big)$.

In the interior of $\Omega \cap H_j$, $j = 1, 2$, the differential equation holds and thus $\partial_n \nabla u \in L^2_{\mathrm{loc}}\big(B_{\varrho/2}(x_0)\big)$. By inspecting the differential equation we observe that $\partial_n \nabla u$ is bounded by $\partial_j \nabla u$, $j = 1, \dots, n-1$, and lower order terms. This yields $\partial_n \nabla u \in L^2\big(B_{\varrho/2}(x_0) \cap H_j\big)$.

(*b*) Let $n \geqslant 3$. By (2.1) and the information just obtained

$$(2.2) \qquad \sum_{i=0}^{n} \big(\partial_j F_i(\cdot, u, \nabla u), \partial_i(\varphi(\partial_j u - \psi)\big) \leqslant 0 .$$

Let F_{ik} be the partial derivative of F_i at $\big(x, u(x), \nabla u(x)\big)$ with respect to the argument which has been evaluated at $\partial_k u$.

Then $\sum_{i,k=1} \partial_i F_{ik} \partial_k v$ is a linear elliptic operator in v with measurable coefficients. For $h > 0$, let $\delta_h \in L^\infty\big(B_\varrho(x_0)\big)$ have the properties

$$\delta_h \geqslant 0 , \qquad \int \delta_h \, dx = 1 , \qquad \operatorname{supp} \delta_h \subset B_h(x_0) \cap \Omega .$$

Let $G_h \in H_0^1\big(B_\varrho(x_0) \cap \Omega\big)$ such that

$$\sum_{i,k=1}^{n} (F_{ik} \partial_k v, \partial_i G_h) = (\delta_h, v) \qquad \text{for all } v \in H_0^1\big(B_\varrho(x_0) \cap \Omega\big) .$$

It is known (see [6]), that G_h is uniformly bounded in $H_0^{1,q}\big(B_\varrho(x_0) \cap \Omega\big)$, $q = n/(n-1) - \varepsilon_0$ as $h \to 0$, that $G_h \geqslant 0$ and that $G_h \to G$ in $H^{1,q}$ where G has the property

$$m|x-x_0|^{2-n} \leqslant G(x) \leqslant m^{-1}|x-x_0|^{2-n}$$

for some constant $m > 0$.

Now in inequality (2.2), set $\varphi = \tau^2 G_h$ where $\tau \in C_0^\infty(B_\varrho(x_0))$, $\tau \geqslant 0$, and $\tau = 1$ on $B_{\varrho/2}(x_0)$.

Then we obtain that

$$\int F_{ik}\partial_k\partial_j u\cdot\partial_i\partial_j u G_h\tau^2\,dx + \int F_{ik}\partial_k\partial_j u\cdot\partial_j u\partial_i G_h\tau^2\,dx$$

(summation convention $i, k = 1, \dots, n$) is bounded uniformly from above as $h \to 0$, $j = 1, \dots, n-1$.

Since the second integral

$$\int F_{ik}\partial_k\partial_j u\cdot\partial_j u\,\partial_i G_h\tau^2\,dx = \tfrac{1}{2}\int F_{ik}\partial_k\big(\tau^2(\partial_j u)^2\big)\partial_i G_h\,dx + B = \tfrac{1}{2}\big(\delta_h(\partial_j u)^2\big) + B \leqslant B$$

where B stands for lower order terms, we obtain—using ellipticity—that

$$\int |\partial_k\partial_j u|^2 G_h\tau^2\,dx$$

is bounded for $h \to 0$, $j = 1, \dots, n-1$.

For $j = 1, \dots, n-1$, the conclusion now follows by a lower semicontinuity argument. For $j = n$, the conclusion follows by inspecting the differential equation and using the boundedness of

$$\int |\partial_k\partial_l u|^2 G\,dx \qquad \text{for } l = 1, \dots, n-1\,.$$

LEMMA 2.3. *Assume the hypothesis of lemma* 2.2 *and suppose* $\partial_j u$, $j = 1, \dots, n-1$, *are continuous. Then*

$$\int_R |\nabla\partial_j(u-\psi)|^2|x-x_0|^{2-n}\,dx \leqslant K\big(\operatorname{osc}_{2R}\{\nabla\partial_j(u-\psi)\}\big)^2 + KR^\alpha, \qquad j = 1, \dots, n-1,$$

for any $\alpha \in]0, 1[$ *and* $R \in]0, \varrho/2[$.
Here $\int_R$ *denotes integration over* $B_R(x_0)$ *and* osc_{2R} *oscillation over* $B_{2R}(x_0)$, *resp. over* $B_{2R}(x_0) \cap H_i$, etc.

PROOF. (*a*) $n \geqslant 3$. Let $\tau \in H_0^{1,\infty}(B_{2R}(x_0))$, $\tau = 1$ on $B_R(x_0)$, $|\nabla\tau| \leqslant R^{-1}$, $\tau \geqslant 0$. In (2.2), we set $\varphi = \tau^2 G_h$ where the «regularized Green's function» G_h is defined as in the proof of lemma 2.2. Setting $z = \partial_j u - \partial_j\psi$, noting $\psi \in H^{2,\infty}$, $z \in L^\infty$ and the hypothesis for the F_i we obtain

$$(2.3)\qquad \int F_{ik}\partial_k z\partial_i z\tau^2 G_h\,dx + \int F_{ik}\partial_k z\,\partial_i(\tau^2 G_h)z\,dx \leqslant$$
$$\leqslant K\int |\nabla(\tau^2 G_h z)|\,dx + K\int |\nabla z| G_h\tau^2|z|\,dx + K\int G_h\tau^2|z|\,dx$$

(summation convention $i, k = 1, \dots, n$).

In the case $u > \psi$ on $B_{2R}(x_0) \cap H$, inequality (2.3) holds with

$$z = \partial_j u - \partial_j \psi - c \tag{2.4}$$

where c is any constant with $|c| \leqslant |\nabla u - \nabla \psi|$.

In fact, since $u > \psi$ on $B_{2R}(x_0)$ the function u satisfies the differential equation and we may choose $u + D_j^{-h}\big(\varphi(D_j^h(u - \psi) - c)\big)$ admissible variation which leads to (2.2) and (2.3) with z defined by (2.4).

We observe that

$$\sum_{ik} \int F_{ik}\, \partial_k(z^2 \tau^2)\, \partial_i G_k \, dx \geqslant 0 \qquad (i, k = 1, \dots, n)$$

and thus

$$\sum_{ik} \int F_{ik}\, \partial_k z\, \partial_i(\tau^2 G_h)\, z \, dx \geqslant -K\int |\nabla z||\nabla \tau|\tau|z| G_k \, dx - K\int z^2 |\nabla \tau||\nabla G_k| \, dx \,. \tag{2.5}$$

Using ellipticity, Young's and Hölder's inequality we obtain from (2.3) and (2.5)

$$\int |\nabla z|^2 G_h \tau^2 \, dx \leqslant K\int |z|^2\big(|\nabla \tau|^2 G_h + |\nabla \tau||\nabla G_h|\big)\, dx + KD \tag{2.7}$$

where $D = \int |z| G_h \tau^2 \, dx + \int |\nabla(\tau^2 G_h) z| dx$.

It is well known, see e.g. [22] or [5], that

$$0 \leqslant G_h(x) \leqslant K|x - x_0|^{2-n} \tag{2.8}$$

uniformly as $h \to 0$ and it follows (see [22] or [5]), that ∇G_h is uniformly bounded in L^q where q is any number $\in [1, n/(n-1)[$. Using Hölder's inequality we conclude

$$D \leqslant KR^\alpha, \qquad 0 < \alpha < 1 \,.$$

Since $\nabla \tau = 0$ on $B_{2R} - B_R$ we obtain from (2.8)

$$|\nabla \tau|^2 G_h \leqslant KR^{-n}$$

and, via the equation

$$\int F_{ik}\, \partial_i G_h\, \partial_k \zeta \, dx = \zeta(x_0)\,, \qquad \int |\nabla \tau|^2 |\nabla G_h|^2 \, dx \leqslant KR^{-n} \,.$$

So we arrive at the inequality

$$\int |\nabla z|^2 G_h \tau^2 \, dx \leqslant K \max\{z(x) | x \in B_{2R}(x_0)\}^2 + KR^\alpha \,.$$

Passing to the limit $G_k \rightharpoonup G$ weakly in $H^{1,q}$, using a lower semicontinuity argument and the estimate $G(x) \geqslant m|x-x_0|^{2-n}$, $m>0$ (see [21]), we obtain

$$\int |\nabla z|^2 |x-x_0|^{2-n} \tau^2 \, dx \leqslant K \max \{z(x) | x \in B_R(x_0)\}^2 + KR^\alpha . \tag{2.9}$$

If $u(x) > \psi(x)$ for all $x \in B_{2R}(x_0) \cap H$, we choose the constant c in (2.4) equal to $c = \min\{(\partial_j u - \partial_j \psi)(x) | x \in B_R(x_0)\}$ and obtain

$$\|z\|_\infty \leqslant \operatorname{osc}\{(\partial_j u - \partial_j \psi)(x) | x \in B_R(x_0)\} . \tag{2.10}$$

If there is a $y \in \operatorname{int} B_{2R}(x_0) \cap H$ with $u(y) = \psi(y)$, however, we have to set $z = \partial_j u - \partial_j \psi$. Since $u \geqslant \psi$ on $B_{2R}(x_0) \cap H$ and $\partial_j u - \partial_j \psi$ is continuous we conclude $z(y) = (\partial_j u - \partial_j \psi)(y) = 0$ and thus

$$\max\{z(x) | x \in B_{2R}(x_0)\} \leqslant \operatorname{osc}\{z(x) | x \in B_{2R}(x_0)\} . \tag{2.11}$$

The lemma for $n \geqslant 3$ follows from (2.9)-(2.11).

(*b*) $n = 2$. We repeat the formulas and considerations of case (*a*) replacing G_h by the constant 1. This completes the proof of lemma 2.3.

3. – Proof of the theorem.

The solution u of the variational inequality satisfies its corresponding partial differential equation in $\Omega - L$ (interior obstacle case) or in Ω (boundary obstacle case).

Thus by our hypotheses on the data, the solution is regular there and we need to prove the theorem only in a neighbourhood of each $y_0 \in L \cap \Omega$ or $y_0 \in \partial\Omega$, resp. Since L and $\partial\Omega$ are $(n-1)$-dimensional $H^{2,\infty}$-manifolds, there is an $H^{2,\infty}$-diffeomorphism f from a neighbourhood U of $y_0 \in L \cap \Omega$ or $y_0 \in \partial\Omega$ onto a ball $B_P(0) \subset \mathbf{R}^n$ such that the points of $\partial\Omega \cap U$ or $L \cap \Omega \cap U$ are mapped onto the hyperplane $H = \{x \in \mathbf{R}^n, x^n = 0\}$. In the boundary obstacle case, $\bar{\Omega} \cap U$ is mapped into the upper halfspace $H_1 = \{x \in \mathbf{R}^n | x_n \geqslant 0\}$. Setting $\tilde{u} = u(f^{-1}(\cdot))$, it suffices to prove the theorem for $\tilde{u}$, which satisfies a local variational inequality in $B_P \cap H_1$ of type (0.2) with functions $\tilde{F}_i$ and obstacles $\tilde{\psi}$ on H. In the following, we omit the $\sim$ over u, F_i, ψ. Thus we are in the situation studied in § 2. We may choose $x_0 \in B_{P/2}(0) = B_{P/2}$, $0 < \varrho < P/2$, and begin the essential part of the proof.

By lemma 2.2, $\partial_j u \in H^1(B_{P/2})$, $j = 1, \ldots, n-1$ and

$$\int |\nabla \partial_j u|^2 |x-x_0|^{2-n} \, dx < \infty \tag{3.1}$$

the integration carried out over $B_\varrho(\cap H_1)$.

By lemma 2.1, the function u_{ε_p} defined there is an admissible variation which yields (with $\varphi = \tau^2$)

$$\sum_i \Big((D_j^h F_i(\cdot, u, \nabla u)), \partial_i(\tau^2 D_j^h[u-\psi-c]^{p-1}) \Big) \leqslant 0 \qquad (i = 0, \ldots, n)$$

Passing to the limit $h \to 0$, performing the differentiation ∂_j, $j = 1, \ldots, n-1$, invoking ellipticity and Hölder's inequality and the Lipschitz-assumptions for the data and for u, we obtain

$$\int |\nabla z|^2 |z-c|^{p-2} \tau^2 \, dx \leqslant K \int |\nabla z||z-c|^{p-2}(|\nabla \tau| + 1)\, \tau \, dx ,$$

where $z = \partial_j u - \partial_j \psi$.

Thus we may apply lemma 1.4 (whose proof is based on the first part of Moser's technique [18], 5.3) and obtain for $R \leqslant \varrho/2$

$$\text{(3.2)} \qquad \operatorname{osc}_R z \leqslant K\Big[R^{2-n} \int_* |\nabla z|^2 dx\Big]^{1/n} + KR^\alpha , \qquad n \geqslant 3 ,$$

$$\operatorname{osc}_R z \leqslant K\Big[\int_* |\nabla z|^2 dx\Big]^{\frac{1}{2}-2/(t+4)} + KR^{2/(t+4)}\Big[\int_{2R} |\nabla z|^2 dx\Big]^{\frac{1}{2}-2/(t+4)} + KR^\alpha , \qquad n = 2 .$$

Here, $\operatorname{osc}_R z = \operatorname{osc}\{z(x) | x \in B_R(x_0)\}$ and $\int_*$ and $\int_{2R}$ denote integration over $B_{2R}(x_0) - B_R(x_0)$ and $B_{2R}(x_0)$. In the boundary obstacle case the oscillation and integrations $\int_*$ and $\int_{2R}$ are taken only over

$$B_R(x_0) \cap H_1 , \qquad (B_{2R}(x_0) - B_R(x_0)) \cap H_1 , \qquad B_{2R}(x_0) \cap H_1$$

respectively.

Since $R^{2-n} \leqslant K|x-x_0|^{2-n}$ on $B_{2R} - B_R$ we obtain by (3.1) that

$$R^{2-n} \int_* |\nabla z|^2 dx \leqslant K \int_* |\nabla z|^2 |x - x_0|^{2-n} dx$$

is small if R is small. With (3.2) this gives us the *continuity* of $z = \partial_j u - \partial_j \psi$.

Setting

$$\Phi(R) = \int_R |\nabla z|^2 |x-x_0|^{2-n} dx$$

where $\int_R$ denotes integration over B_R (or over $B_R \cap H_1$ in the boundary obstacle case) we have by lemma 2.3

$$\Phi(R) \leqslant K(\mathrm{osc}_{2R} \nabla z)^2 + KR^\alpha$$

and by (3.2)

$$\Phi(R) \leqslant K[\Phi(4R) - \Phi(R)]^{2/n} + KR^{2\alpha}, \quad n \geqslant 3, \tag{3.3}$$

and with $k = t + 4$

$$\Phi(R) \leqslant K[\Phi(4R) - \Phi(R)]^{1-1/k} + KR^\gamma, \quad \gamma = \gamma(k),\ n = 2. \tag{3.4}$$

We apply lemma 1.6 with $\sigma = 4$, $R_0 = \varrho/2$, $q = n/(n-2)$ for $n \geqslant 3$ and $q = k$ for $n = 2$, noting that $|\ln(R_0/R)|^{-t} \geqslant KR^{2\alpha}$ or $KR^{2\gamma}$.

This yields (choosing the parameters s and t in lemma 1.6 large enough) for $n \geqslant 3$, $R < 4^{-1}R_0$

$$\Phi(R) \leqslant K \ln\ln(R_0/R)[\ln(R_0/R)]^{-2/(n-2)}$$

and for $n = 2$

$$\Phi(R) \leqslant K \ln\ln(R_0/R)[\ln(R_0/R)]^{1-k} \quad \text{for any } k > 1 \tag{3.5}$$

Using inequality (3.2) again, we obtain for $n \geqslant 3$

$$\mathrm{osc}_R z \leqslant K[\Phi(2R)]^{1/n} + KR^\alpha \leqslant K \ln\ln(R_0/R)/\ln(R_0/R)^{q(n)}$$

with

$$q(n) = 2/(n^2 - 2n)$$

and for $n = 2$

$$\mathrm{osc}_R z \leqslant K[\ln(R_0/R)]^{-p} \quad \text{for any } p > 1.$$

Thus the theorem is proved for $n \geqslant 3$.

For $n = 2$ it remains further to prove that the normal derivatives $\partial_n u - \partial_n \psi$ are uniformly continuous in half spaces H_1 (and H_2). By inspecting the differential equation in $H_1 \cap \Omega$, we obtain from (3.5) a similar inequality for the quantity

$$\Phi(R) = \int_R |\partial_n u - \partial_n \psi|^2 \, dx$$

where $\int_R$ denotes integration over $B_R \cap H_1$ (or $B_R \cap H_2$).

Reflecting

$$\partial_n u - \partial_n \psi \quad \text{at} \quad H = \{x \in \mathbf{R}^n; x_n = 0\}$$

we may apply lemma 1.1 and obtain in the case $n = 2$ the one-sided continuity of the normal derivatives of u also with any power of $K[\ln(R_0/R)]^{-1}$ as estimate of

$$\operatorname{osc}\{\partial_n u(x) | x \in B_R \cap H_i\}, \quad i = 1, 2.$$

This completes the proof of the theorem.

REMARK. Recently Hildebrandt and Nitsche have studied two dimensional *parametric* minimal surfaces with boundary obstacles.

At branch points of odd order m, they obtain $C^{1+\alpha}$-regularity with $a = (2m+1)^{-1} \leqslant \frac{1}{3}$. This indicates that also in our case one should not expect an a-priori-estimate for the Hölder-exponent of $\partial_j u$ and that our logarithmic modulus of continuity is optimal in a certain sense. A non-parametric example of this type is due to Shamir see [4].

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